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AFFINE CONES OVER FANO THREEFOLDS AND ADDITIVE GROUP ACTIONS

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Abstract

In this paper we address the following questions for smooth Fano threefolds of Picard number 1:

- *When does such a threefold X possess an open cylinder $U \simeq Z \times \mathbb{A}^1$, where Z is a surface*?
- *When does an affine cone over X admit an effective action of the additive group of the base field*?

A geometric criterion from [26] (see also [27]) says that the two questions above are equivalent. In [26] we found some interesting families of Fano threefolds carrying a cylinder. Here we provide new such examples.

Introduction

All varieties will be defined over $\mathbb{C}$. In particular, $\mathbb{A}^n$ stands for the affine n-space over $\mathbb{C}$. We say that a projective variety is *cylindrical* if it contains a Zariski open subset $U \simeq Z \times \mathbb{A}^1$ called a *cylinder*, where Z is a quasiprojective variety. There is also a more restrictive notion of *polar cylindricity* ([26, Section 3]) explained below. However, both notions coincide for varieties with Picard number 1.

A classification of cylindrical smooth Fano threefolds of Picard number 1 is tempting. In the present paper we make a further step in this direction providing in our main Theorem 0.1 two new families of examples. All Fano threefolds are assumed being smooth unless explicitly stated otherwise. Given a Fano threefold X of index $r = 1$, by a line on X we mean a smooth rational curve L on X such that $-K_X \cdot L = 1$. We let $\tau(X)$ denote the Fano scheme of X, that is, the component of the Hilbert scheme parameterizing the lines on X. In the setting of Theorem 0.1 the scheme $\tau(X)$ is generically reduced, but can contain non-reduced points (see Theorem 1.1 below). The reduced scheme $\tau(X)_{\mathrm{red}}$ is purely one dimensional.

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Theorem 0.1. *Let X be a Fano threefold of genus $g = 9$ or 10 with*

$$\mathrm{Pic}(X) = \mathbb{Z} \cdot (-K_X).$$

If the scheme $\tau(X)$ is non-smooth then X is cylindrical. The Fano threefolds with a non-smooth scheme $\tau(X)$ form a codimension one subvariety in the corresponding moduli space.

Presumably, a general point of the moduli space of Fano threefolds of genus $g = 9$ or 10 with Picard number 1 corresponds to a non-cylindrical rational Fano threefold. However, we do not dispose at the moment necessary tools for establishing this. Moreover, we do not know any example of a non-cylindrical rational Fano threefold.

Our interest to the cylindricity problem came from affine geometry, see Corollary 0.2 below. Given a projective embedding $X \hookrightarrow \mathbb{P}^N$, a natural question arises as to when the affine cone $\mathrm{AffCone}(X)$ over X admits an effective action of the additive group $\mathbb{G}_a$ of the base field. Indeed, the existence of such an action implies that the automorphism group $\mathrm{Aut}(\mathrm{AffCone}(X))$ is of infinite dimension, see e.g. [1]. For instance, the affine cone over a smooth del Pezzo surface S of degree $d \geq 4$ anti-canonically embedded in $\mathbb{P}^d$ admits three independent effective $\mathbb{G}_a$-actions [26, 3.3.2]. Moreover, the group $\mathrm{Aut}(\mathrm{AffCone}(S))$ acts on $\mathrm{AffCone}(S)$ infinitely transitively off the vertex [33].[1]

Applying Theorems 0.1 and 0.3 (see below) we deduce such a corollary.

Corollary 0.2. *Let X be a Fano threefold of genus $g = 9$ or 10 with $\mathrm{Pic}(X) \simeq \mathbb{Z}[-K_X]$ and with a non-smooth Fano scheme of lines $\tau(X)$. Then any affine cone over X admits a non-linear $\mathbb{G}_a$-action.*[2] *Consequently, the automorphism group of such a cone is infinite dimensional.*

Some explanations are in order. A linear group action on an affine cone over X induces such an action on X. However, any Fano threefold X as in Theorem 0.1 has finite automorphism group ([36]). In particular, it does not admit any effective $\mathbb{G}_a$-action. Thus the linear automorphism group of any affine cone over X is one-dimensional and such a cone does not admit any effective linear $\mathbb{G}_a$-action. This explains why the (effective) $\mathbb{G}_a$-action as in Corollary 0.2 is non-linear.

The proof of Corollary 0.2 is based on the geometric criterion of Theorem 0.3 below, which follows in turn from a more general Corollary 2.12 in [27]; cf. also [26, Theorem 3.1.9]. Let X be a projective variety polarized by an ample divisor H, and

[1] Such a cone is *flexible*, i.e. the tangent space in any smooth point is generated by tangent vectors to the orbits of $\mathbb{G}_a$-actions, see [1]. For the affine cones over projective varieties this property admits a characterization in terms of existence of a covering of the variety by a family of transversal cylinders, see [33]. See also [4, Conjecture 1.4] for a conjectural relation between unirationality of a variety and the existence of a flexible affine model in its stable birational class.

[2] Notice that any such affine cone is normal, see [19, Chapter 1, Proposition 4.9].

let $U = X \setminus \operatorname{supp} D$ be a cylinder in X. We say that U is *H-polar* if one can choose $D \in |dH|$ for some $d \in \mathbb{N}$ ([26]) or, equivalently, $[D] \in \mathbb{Q}_+[H]$ in $\mathrm{Pic}_{\mathbb{Q}}(X)$, see [27, Definition 0.2].

Theorem 0.3. *Let X be a normal projective variety of dimension ≥ 1, $H \in \mathrm{Pic}(X)$ be a very ample divisor, and $\mathrm{AffCone}_H(X)$ be the associated affine cone over X. Then $\mathrm{AffCone}_H(X)$ admits an effective $\mathbb{G}_a$-action if and only if X contains an H-polar cylinder.*

À priori the existence of an effective $\mathbb{G}_a$-action on $\mathrm{AffCone}(X)$ depends upon the polarization chosen. However, an H-polar cylinder in X is also H'-polar for any $H' \in \mathrm{Pic}(X)$ such that $pH \sim qH'$ for some positive integers p and q. Assuming that $\mathrm{Pic}(X) \simeq \mathbb{Z}$, all ample polarizations on X are proportional and so a cylinder polar with respect to one is polar with respect to any other.

Furthermore, shrinking suitably a given cylinder in X with $\mathrm{Pic}(X) \simeq \mathbb{Z}$ we can get an affine H-polar cylinder with respect to some ample polarization H of X. Using now Theorem 0.3 we conclude that all the affine cones over X simultaneously admit or do not admit an effective $\mathbb{G}_a$-action. Thus in this particular case the criterion of Theorem 0.3 becomes especially simple.

Corollary 0.4. *Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety with $\mathrm{Pic}(X) \simeq \mathbb{Z}$. Then $\mathrm{AffCone}(X)$ admits an effective $\mathbb{G}_a$-action if and only if X is cylindrical.*

The reason why we choose to deal with Fano varieties, and more specifically with rational Fano threefolds, is as follows. First of all, the existence of a non-linear $\mathbb{G}_a$-action on an affine cone over a projective variety X implies that X is birationally uniruled [26, Corollary 2.1.4]. Hence under the assumption $\mathrm{Pic}(X) \simeq \mathbb{Z}$ we have to restrict to Fano varieties, since otherwise X is not birationally uniruled and so the affine cones over X do not admit a nonlinear $\mathbb{G}_a$-action, see [26, 3.2.1].

Furthermore, any cylindrical Fano threefold X is rational due to the Castelnuovo rationality criterion for surfaces. Indeed, since X is cylindrical it is birationally equivalent to $Z \times \mathbb{A}^1$. Let $\tilde{Z}$ be any smooth projective birational model of Z. Since X is Fano, it does not admit any nonzero global holomorphic pluri-form, and the same holds for $\tilde{Z}$. In particular, $H^0(\tilde{Z}, 2K_{\tilde{Z}}) = 0$ and $H^1(\tilde{Z}, \mathcal{O}_{\tilde{Z}}) = H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}) = 0$. By the Castelnuovo rationality criterion the surface $\tilde{Z}$ is rational. Hence X is rational too.

The list of examples of cylindrical Fano varieties with Picard number 1 that we know so far looks as follows. Clearly, every Fano variety X which contains the affine space $\mathbb{A}^n$ as a Zariski open subset is cylindrical.[3] This applies e.g. to $\mathbb{P}^n$, the smooth quadric Q in $\mathbb{P}^{n+1}$, and the Fano threefold X_5 of index 2 and degree 5. Two more

[3] A well known Hirzebruch problem ([15]) asks to classify the complete complex varieties that contain $\mathbb{A}^n$ as a dense open subset; see [9] and the references therein for studies on this problem.

families of cylindrical Fano threefolds X with Picard number 1 are the smooth intersections $X_{2\cdot 2}$ of two quadrics in $\mathbb{P}^5$ and the Fano threefolds X_{22} of genus 12 (see [26, Propositions 5.0.1 and 5.0.2]). Note that a cylindrical projective variety does not need to contain an affine space as an open subset. For instance, the moduli space of the latter family has dimension 6, while the subfamily of compactifications of $\mathbb{A}^3$ is only 4-dimensional.

We established in [26, 3.2.2] the polar cylindricity of all the anticanonically polarized del Pezzo surfaces of degree ≥ 4. It occurs however that those of degree 1 and 2 do not fall in this class [28].

The geometric construction used in the proof of Theorem 0.1 involves a line L on X which corresponds to a non-smooth point of $\tau(X)$. We found further families of examples whose construction evokes instead a smooth point $[L] \in \tau(X)$ (see arXiv:`1106.1312`). We expect that the latter families are not contained in the former ones.

In Section 1 we give a brief overview on Fano threefolds, with a special accent on the property of being rational. Besides, we collect there some useful facts on the Hilbert scheme of lines in a Fano threefold. In Section 2 we describe two standard constructions, which give all Fano threefolds of genus 9 and 10. Sometimes the proofs are hardly accessible in the literature, so we provide them. The main Theorem 0.1 is proven in the last Section 3.

1. Generalities on Fano threefolds

We recall that a Fano variety is a smooth projective variety X with an ample anticanonical class $-K_X$. The Fano index $r = i(X)$ is defined via $-K_X = rH$, where $H \in \operatorname{Pic}(X)$ is a primitive ample divisor class. It is well known that $r \leq \dim X + 1$. We write $X = X_d$ for a Fano threefold of degree d, where $d = H^3$. The genus g of X is defined via $2g - 2 = -K_X^3$ $(= dr^3)$.

1.1. Rational Fano threefolds. Any Fano threefold X has index $r \leq 4$. Furthermore,

- if $r = 4$ then $X \simeq \mathbb{P}^3$;
- if $r = 3$ then $X \simeq Q$, where Q is a smooth quadric in $\mathbb{P}^4$.

  We assume in the sequel that $\operatorname{Pic}(X) \simeq \mathbb{Z}$.
- If $r = 2$ then the degree of X varies in the range $d = 1, \ldots, 5$. More precisely,

  (1) if $d = 1$ then X is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$. Such a threefold X is non-rational [42], [11];

  (2) if $d = 2$ then X is a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$. Such a threefold X is non-rational [43];

  (3) if $d = 3$ then X is a cubic hypersurface in $\mathbb{P}^4$, which is known to be non-rational [5];

(4) if $d = 4$ then $X = X_{2\cdot 2}$ is an intersection of two quadrics in $\mathbb{P}^5$. Such a threefold is rational [10], [22];

(5) if $d = 5$ then $X = X_5$ is a linear section (by $\mathbb{P}^6$) of the Grassmannian $G(2,5)$ under its Plücker embedding in $\mathbb{P}^9$. Such a threefold is rational and unique up to isomorphism [8], [19], [22].

- If $r = 1$ then the genus of X varies in the range $g = 2, \dots, 10$ and 12. More precisely,

(a) If $g = 2$, 3, 5, or 8, then the threefold X is non-rational (see [20], [23] for $g = 2$, [21], [20] for $g = 3$, [2] for $g = 5$, [20] and [5] for $g = 8$);

(b) if $g = 4$ or 6 then a general threefold X is non-rational [2], [23], [42];

(c) if $g = 7, 9, 10$, or 12 then X is rational [22].

We are interested in Fano threefolds which possess a cylinder. As we explained in the Introduction, due to the Castelnuovo rationality criterion for surfaces such a threefold must be rational. Of course, the projective space $\mathbb{P}^3$, a smooth quadric Q in $\mathbb{P}^4$, and the Fano threefold X_5 are cylindrical since they contain the affine space $\mathbb{A}^3$ as an open subset. A cylinder exists in every Fano threefold X_{22} or $X_{2\cdot 2}$ [26, Section 5]. In Theorem 3.1 below we describe families of Fano threefolds with a cylinder among the X_{16} $(g = 9)$ and the X_{18} $(g = 10)$.

The question remains whether *every* rational Fano threefold carries a cylinder. In particular, whether this is true for all the threefolds X_{12} $(g = 7)$, X_{16}, and X_{18}.

1.2. Families of lines on Fano threefolds. In the sequel we need the following facts.

Theorem 1.1 ([40], [37], [19, Chapter 3, Section 2], [22, Section 4.2], [41])**.** *Let $X = X_{2g-2}$ be a Fano threefold of genus $g \geq 3$ with* $\mathrm{Pic}(X) = \mathbb{Z}\cdot(-K_X)$, *anticanonically embedded in $\mathbb{P}^{g+1}$.*[4] *Then the following hold.*

(1) *There is a line L on X.*

(2) *For the normal bundle $\mathcal{N}_{L/X}$ there are the following possibilities*:

(α) $\mathcal{N}_{L/X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, *or*

(β) $\mathcal{N}_{L/X} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

(3) *The reduced scheme $\tau(X)_{\mathrm{red}}$ is purely one dimensional.*

(4) *The scheme $\tau(X)$ is smooth at a point $[L] \in \tau(X)$ if and only if the corresponding line L is of type* (α).

(5) *For $g \geq 7$ any line L on X meets at most a finite number of lines on X.*

REMARK 1.2. Let $g = 9$ or 10 and $\mathrm{Pic}(X) = \mathbb{Z}\cdot(-K_X)$. According to [35] and [12] every irreducible component of the scheme $\tau(X)$ is generically reduced. Thus for a Fano threefold X as in Theorem 0.1, the set of non-smooth points[5] of the scheme

[4] Such an embedding exists always if $g \geq 4$, and for general members of the family for $g = 3$.

[5] That is, of singular or non-reduced points.

$\tau(X)$ is at most finite. On the other hand, for a general Fano threefold X of this type the scheme $\tau(X)$ is a smooth, reduced, irreducible curve [34, Section 3.2], [16, Corollary 5.1.b].

2. Fano threefolds of genera 9 and 10

We need the following lemma.

Lemma 2.1. (a) *Any smooth curve Γ of degree 7 and genus 3 in $\mathbb{P}^3$ lies on a unique cubic surface $F = F(\Gamma)$ in $\mathbb{P}^3$, and this surface is irreducible.*
(b) *For any smooth, linearly non-degenerate curve Γ of degree 7 and genus 2 in $\mathbb{P}^4$, the quadrics containing Γ form a linear pencil, say, $\mathcal{Q}$. The base locus of this pencil is an irreducible quartic surface $F = F(\Gamma)$ in $\mathbb{P}^4$.*

Proof. We provide a proof of (b); that of (a) is similar. Let $\mathscr{I}_\Gamma$ be the ideal sheaf of $\Gamma \subseteq \mathbb{P}^4$. Using the exact sequence

$$0 \to \mathscr{I}_\Gamma(2) \to \mathscr{O}_{\mathbb{P}^4}(2) \to \mathscr{O}_\Gamma(2) \to 0$$

we obtain that $\dim H^0(\mathscr{I}_\Gamma(2)) \geq 2$ by the Riemann–Roch theorem. Hence there is a pencil of quadrics $\mathcal{Q}$ through Γ.

Assume to the contrary that there exist three linearly independent quadrics Q_1, Q_2, and $Q_3 \subseteq \mathbb{P}^4$ passing through Γ. Since Γ is linearly nondegenerate, these quadrics are irreducible. Moreover, $Q_1 \cap Q_2$ is an irreducible surface. Indeed, otherwise Γ is contained in an irreducible component S of $Q_1 \cap Q_2$ with $\deg S \leq 3$. Thus $\deg \Gamma \leq 2 \deg S \leq 6$, a contradiction.

Then $Q_1 \cap Q_2 \cap Q_3 = \Gamma + L$ (as a scheme), where L is a line. Consider the exact sequence

$$0 \to \mathscr{O}_{\Gamma \cup L} \to \mathscr{O}_\Gamma \oplus \mathscr{O}_L \to \mathscr{F} \to 0, \tag{2.1.1}$$

where the quotient sheaf $\mathscr{F}$ is supported on $\Gamma \cap L$. Since

$$\chi(\mathscr{O}_{\Gamma \cup L}) = -4$$

and

$$\chi(\mathscr{O}_\Gamma \oplus \mathscr{O}_L) = \chi(\mathscr{O}_\Gamma) + \chi(\mathscr{O}_L) = 0,$$

we obtain by (2.1.1)

$$\#(\Gamma \cap L) = \dim H^0(\mathscr{F}) = \chi(\mathscr{O}_\Gamma \oplus \mathscr{O}_L) - \chi(\mathscr{O}_{\Gamma \cup L}) = 4.$$

Thus L must be a 4-secant line of Γ. Hence the projection with center L would map Γ to a plane cubic, a contradiction.

Let us show finally that F is irreducible. Indeed, otherwise Γ would be contained in an irreducible surface F' of degree ≤ 3 in $\mathbb{P}^4$. Since Γ is assumed to be linearly non-degenerate, F' must be a linearly non-degenerate surface of degree 3. By [10, Chapter 4, Section 3], either F' is a cone over the twisted cubic or F' is isomorphic to the Hirzebruch surface $\mathbb{F}_1$. Proceeding as in the beginning of the proof, it is easily seen that in both cases $h^0(\mathscr{I}_\Gamma(2)) \geq h^0(\mathscr{I}_{F'}(2)) \geq 3$. Hence there is a two-dimensional family of quadrics passing through Γ, which leads to a contradiction as before. □

In the next proposition we list the possibilities for the surface F as in Lemma 2.1. Given a Hirzebruch surface $\mathbb{F}_n$ we let l denote a ruling of $\mathbb{F}_n$ and Σ the exceptional section if $n > 0$, or just a section if $n = 0$.

Proposition 2.2. *In the notation and assumptions as in* Lemma 2.1 (a) *we suppose in addition that* Γ *is not hyperelliptic, and we let* $W = \mathbb{P}^3$ *and* $g = 9$. *In case* (b) *of* Lemma 2.1 *we let* $W = Q \subseteq \mathbb{P}^4$ *be a smooth quadric containing* Γ *and* $g = 10$ (*such a quadric does exist, see* Lemma 2.4 *below*). *With these notation and assumptions, the surface* $F = F(\Gamma) \subseteq \mathbb{P}^{g-6}$ *belongs to one of the following classes.*

(1) $F \subseteq \mathbb{P}^{g-6}$ *is a normal del Pezzo surface with at worst Du Val singularities*; *or*

(2) $F \subseteq \mathbb{P}^{g-6}$ *is a non-normal scroll, whose singular locus* $\Lambda = \mathrm{Sing}(F)$ *is a double line. Furthermore, the normalization* F' *of* F *is a smooth scroll* F' *of the minimal degree* $g-6$ *in* $\mathbb{P}^{g-5}$, *and the normalization map* $\nu\colon F' \to F$ *is induced by the projection from a point* $P \in \mathbb{P}^{g-5} \setminus F'$. *The restriction* $\nu|_{\nu^{-1}(\Lambda)} : \nu^{-1}(\Lambda) \to \Lambda$ *is a ramified double cover. Letting* Γ' *be the proper transform of* Γ *in* F', *there are the following possibilities.*

(a) *If* $g = 9$ *then* $F' \simeq \mathbb{F}_1$, *the embedding* $\mathbb{F}_1 \simeq F' \hookrightarrow \mathbb{P}^4$ *is defined by the linear system* $|\Sigma + 2l|$ *on* $\mathbb{F}_1$, *and* $\nu^{-1}(\Lambda) \sim \Sigma + l$ *is a reduced conic on* $F' \subseteq \mathbb{P}^4$, *which is either smooth or degenerate. Furthermore,* $\Gamma' \sim 3\Sigma + 4l$ *on* F'. *If* $g = 10$ *then one of the following holds.*

(b) $F' \simeq \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, *the embedding* $\mathbb{F}_0 \simeq F' \hookrightarrow \mathbb{P}^5$ *is defined by the linear system* $|\Sigma + 2l|$ *on* $\mathbb{F}_0$, $\nu^{-1}(\Lambda) \sim \Sigma$ *is a smooth conic on* $F' \subseteq \mathbb{P}^5$, *and* $\Gamma' \sim 2\Sigma + 3l$ *on* F' *or*

(b′) $F' \simeq \mathbb{F}_2$, *the embedding* $\mathbb{F}_2 \simeq F' \hookrightarrow \mathbb{P}^5$ *is defined by the linear system* $|\Sigma + 3l|$ *on* $\mathbb{F}_2$, $\nu^{-1}(\Lambda) \sim \Sigma + l$ *is a reduced degenerate conic on* $F' \subseteq \mathbb{P}^5$, *and* $\Gamma' \sim 2\Sigma + 5l$ *on* F'.

The double line Λ *is a* $(13-g)$*-secant line of* Γ *i.e. a* 3*-secant if* $g = 10$ *and* 4*-secant if* $g = 9$.

Proof. Since F is a complete intersection, it is Gorenstein. By the adjunction formula $\omega_F \simeq \mathscr{O}_F(-1)$, i.e. F is (possibly non-normal) del Pezzo surface.

Let us show that (1) holds provided F is normal, and otherwise (2) holds.

If F is normal, then by [14] F is either a surface described in (1), or a cone over an elliptic curve $C \subseteq \mathbb{P}^{g-7}$ of degree $g-6$. Assume to the contrary that F is a cone. Let $\xi : \tilde{F} \to F$ be the blowup of the vertex. Then $\tilde{F}$ is a smooth ruled surface over

C. Let as before Σ and l be the exceptional section and a ruling, respectively, with $\Sigma^2 = -k$. Letting $M = \xi^*\mathscr{O}_F(1)$ and $\tilde{\Gamma}$ be the proper transform of Γ on $\tilde{F}$, we can write $M \equiv \Sigma + kl$ and $\tilde{\Gamma} \equiv a\Sigma + bl$. Then

$$0 = M \cdot \Sigma, \quad g - 6 = M^2 = k, \quad \Sigma^2 = -k = 6 - g,$$
$$7 = \tilde{\Gamma} \cdot M = b, \quad \text{and} \quad \tilde{\Gamma} \cdot \Sigma = a(6 - g) + 7 \geq 0.$$

Since $\tilde{\Gamma} \simeq \Gamma$ is not an elliptic curve, $a \geq 2$. This is only possible if $g = 9$, $a = 2$, and so $k = 3$. On the other hand, by adjunction

$$2g(\tilde{\Gamma}) - 2 = (\tilde{\Gamma} + K_{\tilde{F}}) \cdot \tilde{\Gamma} = 8,$$

a contradiction, since $g(\tilde{\Gamma}) = g(\Gamma) = 3$. Thus (1) holds.

If F is non-normal then by [32, Theorem 8], [39], [7, 9.2.1], F is a projection of a normal surface F' of the minimal degree $g - 6$ in $\mathbb{P}^{g-5}$. It is well known (see e.g., [10, Chapter 4, Section 3, p. 525]) that $F' \subseteq \mathbb{P}^{g-5}$ is either a Veronese surface $F'_4 \subseteq \mathbb{P}^5$, or the image of a Hirzebruch surface $\mathbb{F}_n$ under the map given by the linear system $|\Sigma + kl|$, where $k \geq n$ and $2k - n = g - 6$. The case of the Veronese surface is impossible because the degree of every curve on $F'_4 \subseteq \mathbb{P}^5$ is even. Thus we have a birational morphism $\mu \colon \mathbb{F}_n \to F'$. This is either an isomorphism or the contraction of the negative section. Let $\Gamma' \subseteq \mathbb{F}_n$ be the proper transform of Γ on $\mathbb{F}_n$. We can write $\Gamma' \sim a\Sigma + bl$, where $a \geq 2$ and $b \geq na$. Note that $a \geq 3$ if $g = 9$, since Γ is assumed being non-hyperelliptic. By adjunction

$$2g(\Gamma) - 2 = 2na - 2b - (n + 2)a - na^2 + 2ab, \quad \deg \Gamma = 7 = -na + b + ka,$$

where, as we have seen already,

$$\deg F = g - 6 = 2k - n, \quad k \geq n, \ a \geq 2, \ b \geq na, \quad \text{and} \quad a \geq 3 \quad \text{if} \quad g = 9.$$

The data as in (a), (b), and (b′), respectively, give three solutions of this system, and these are the only solutions. In all cases, $k > n$ and so $\mu \colon \mathbb{F}_n \to F$ does not contract the section Σ. Hence $F' \simeq \mathbb{F}_n$. It is easily seen that the remaining possibilities are as indicated in (2). See also [39] for a description of the inverse image $\nu^{-1}(\Lambda)$ of the singular locus Λ of F. □

In the following elementary lemma we describe some families of lines and conics on the surface F.

Lemma 2.3. *The surface F as in* (b) *and* (b′) *of* Proposition 2.2 *is linearly ruled, that is, through a general point of F passes a line, and this line meets the double line Λ.*

In case (b) F *contains just a one-parameter family of non-degenerate conics, and a general such conic does not meet the double line* Λ. *In case* (b′) F *contains no non-degenerate conic.*

Proof. Since in case (b) (resp. (b′)) the morphism $F' \to F \subseteq \mathbb{P}^4$ is defined by a subsystem of $|\Sigma + 2l|$ (resp. $|\Sigma + 3l|$) the image of any ruling $l \subseteq F'$ is a line on F meeting Λ. In case (b) the non-degenerate conics on F are just the images of the sections, say, $\Sigma_t \sim \Sigma$ on F'. These conics form a one-dimensional family, and the image $\nu(\Sigma_t)$ of a general section Σ_t does not meet Λ. Assuming in case (b′) that there is a non-degenerate conic C on F we obtain the relations $\nu^{-1}(C) \cdot (\Sigma + 3l) = 2$ and $\nu^{-1}(C) \cdot \Sigma \geq 0$. These imply that $\nu^{-1}(C) \cdot \Sigma = 2$ and $\nu^{-1}(C) \cdot l = 0$, which is impossible. □

Now we can strengthen part (b) of Lemma 2.1.

Lemma 2.4. *In case* (b) *of* Lemma 2.1 *the pencil* $\mathcal{Q}$ *contains a smooth quadric.*

Proof. Assume to the contrary that every quadric in the pencil $\mathcal{Q}$ is singular. Let $Q \in \mathcal{Q}$ be a general member. By the Bertini theorem Q is smooth outside F. Since F is a complete intersection, Q is smooth at the points of $F \setminus \mathrm{Sing}(F)$. If F has at worst isolated singularities, then so does Q. Moreover, in this case all these quadrics Q have a common singularity. Hence F must be a cone, which contradicts Proposition 2.2.

Thus under our assumption F has non-isolated singularities. Moreover, by Proposition 2.2 (2) F must be singular along a line Λ. If some quadric $Q \in \mathcal{Q}$ is singular along Λ, then F is again a cone, which is impossible. Thus we may assume that a general quadric $Q \in \mathcal{Q}$ has an isolated singular point $P = P_Q \in \Lambda$ and so $F = Q_1 \cap Q_2$, where Q_i $(i = 1, 2)$ is a quadratic cone with vertex $P_i \in \Lambda$ and $P_1 \neq P_2$.

The cone Q_1 contains two different families of planes Π_λ and Θ_η that pass through the point P_1. The second quadric Q_2 cuts out a conic C_λ on Π_λ, which also passes through P_1. Hence F contains a one-dimensional family of conics meeting the double line Λ. This contradicts Lemma 2.3. □

In Theorem 2.6 below we deal with the following setting.

SETUP 2.5. As before, we consider the following two cases:
(i) For $g = 9$, we let $W = \mathbb{P}^3$ and $\Gamma \subseteq \mathbb{P}^3$ be a smooth, linearly nondegenerate, nonhyperelliptic curve of degree 7 and genus 3.
(ii) For $g = 10$, we let $W = Q \subseteq \mathbb{P}^4$ be a smooth quadric and Γ be a smooth, linearly nondegenerate curve of degree 7 and genus 2 on Q.
In both cases, we let $F = F(\Gamma)$ denote the corresponding surface from Lemma 2.1.

In the case of a curve Γ lying on a smooth surface F, the following result can be found in [18]. In the present more general form, the result was announced without proof in [22, Theorems 4.3.3 and 4.3.7]. Besides, we can quote an explanation in [22, 4.3.9 (ii)] as to why the assumption in Setup 2.5 (i) that the curve Γ is non-hyperelliptic is important. The details of the proof can be found in an unpublished thesis [34] (in Russian). For the reader's convenience we reproduce them below; see also [3].

Theorem 2.6. *In the notation as in* Setup 2.5 *there exists a Sarkisov link*

(2.6.2)
$$\begin{tikzcd}
& \tilde{D} \arrow[dl] \arrow[rr, hook] & & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\sigma_0"] \arrow[rr, dashed, "\chi"] & & \hat{X} \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi"] & & \hat{F} \arrow[ll, hook'] \arrow[dr] & \\
\Gamma \arrow[rr, hook] & & W \arrow[rrrr, dashed, bend right, "\psi"'] & & X_0 & & X & & L, \arrow[ll, hook']
\end{tikzcd}$$

where σ is the blowup of Γ, σ_0 and φ_0 are the anticanonical maps onto $X_0 \subseteq \mathbb{P}^{g-1}$, χ is a flop, $X = X_{2g-2}$ is a smooth Fano threefold of genus g with $\mathrm{Pic}(X) = \mathbb{Z} \cdot (-K_X)$ *anticanonically embedded in $\mathbb{P}^{g+1}$, and φ is the blowup of a line L on X. The exceptional divisor $\hat{F}$ of φ is a proper transform of the surface $F = F(\Gamma) \subseteq W$. The exceptional divisor $\tilde{D}$ of σ is a proper transform of a divisor $D \in |-(12-g)K_X-(25-2g)L|$. The map ψ^{-1} is the double projection with center L, that is, a map given by the linear system $|A - 2L|$ on X, where $A \sim -K_X$ is a hyperplane section of X.*

Conversely, if X is any Fano threefold of genus $g = 9$ or 10 and $L \subseteq X$ is a line, then the double projection with center L defines the above diagram.

In the proof of Theorem 2.6 we use auxiliary results Lemma 2.7, Corollary 2.8, Lemma 2.9 and Corollary 2.10. Let us introduce the following data. We let $\sigma\colon \tilde{X} \to W$ be the blowup of Γ, $\tilde{D}$ be the exceptional divisor, and let $H^* = \sigma^* H$, where H is the positive generator of $\mathrm{Pic}(W) \simeq \mathbb{Z}$. We have (see e.g. [22, Lemma 2.2.14])

$$(H^*)^3 = g - 8, \quad (H^*)^2 \cdot \tilde{D} = 0, \quad H^* \cdot \tilde{D}^2 = -H \cdot \Gamma = -7, \tag{2.6.3}$$

and

$$\tilde{D}^3 = -\deg \mathscr{N}_{\Gamma/W} = \begin{cases} -23 & \text{if} \quad g = 10, \\ -32 & \text{if} \quad g = 9. \end{cases}$$

Letting $\tilde{F} \subseteq \tilde{X}$ be the proper transform of F we get $\tilde{F} \sim (12-g)H^* - \tilde{D}$. The divisor classes $-K_{\tilde{X}} \sim (13-g)H^* - \tilde{D}$ and $\tilde{F}$ form a basis of $\mathrm{Pic}(\tilde{X}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and satisfy the relations

$$-K_{\tilde{X}}^3 = 2g-6 > 0, \quad (-K_{\tilde{X}})^2 \cdot \tilde{F} = 3, \quad -K_{\tilde{X}} \cdot \tilde{F}^2 = -2, \quad \text{and} \quad \tilde{F}^3 = g - 13. \tag{2.6.4}$$

Lemma 2.7. *The divisor $-K_{\tilde{X}}$ is nef and big.*

Proof. First we show that $-K_{\tilde{X}}$ is nef. Consider the case $g = 10$; the proof in the case $g = 9$ is similar. From the exact sequence

$$0 \to \mathscr{I}_{\Gamma}(3) \to \mathscr{O}_W(3) \to \mathscr{O}_{\Gamma}(3) \to 0$$

we obtain by the Riemann–Roch theorem

$$\dim H^0(\mathscr{I}_{\Gamma}(3)) \geq \dim H^0(\mathscr{O}_W(3)) - \dim H^0(\mathscr{O}_{\Gamma}(3)) = 10.$$

The members of the linear system $|-K_{\tilde{X}}|$ are proper transforms of the members of the linear system $|-K_W| = |\mathscr{O}_W(3)|$ passing through Γ. Hence

$$\dim|-K_{\tilde{X}}| \geq 9. \tag{2.7.5}$$

Notice that an irreducible element $G \in |-K_W| = |\mathscr{O}_W(3)|$ cannot be singular along Γ, since otherwise we obtain $G \cdot F \cdot H = 3 \cdot 2 \cdot 2 \geq 2 \cdot 7$, a contradiction. Applying Lemma 2.1 it is easily seen that the only reducible members $\tilde{G} \in |-K_{\tilde{X}}|$ are those of the form $\tilde{G} = \tilde{F} + H^*$. Hence such divisors form a linear subsystem in $|-K_{\tilde{X}}|$ of codimension ≥ 5.

Assume to the contrary that there exists an irreducible curve $\tilde{C}$ on $\tilde{X}$ with $-K_{\tilde{X}} \cdot \tilde{C} < 0$, and let $C = \sigma(\tilde{C}) \subseteq W$. Since $g(\Gamma) = 2$, the curve Γ does not admit any 4-secant line. Indeed, otherwise the projection from this line would send Γ isomorphically to a plane cubic, which is impossible. Since

$$\#(C \cap \Gamma) = \tilde{C} \cdot \tilde{D} > 3H^* \cdot \tilde{C} = 3 \deg C \geq 3,$$

the curve C cannot be a line. If C is contained in a plane $\Pi \subseteq \mathbb{P}^4$ then by the same argument

$$\#(\Pi \cap \Gamma) \geq \#(C \cap \Gamma) > 3 \deg C \geq 6.$$

Since $\deg \Gamma = 7$ and Γ is linearly non-degenerate, we get a contradiction. Thus C is not contained in a plane and so $\deg C \geq 3$. Assume that C is contained in some hyperplane $\Theta \subseteq \mathbb{P}^4$. Then as above

$$\#(\Theta \cap \Gamma) \geq \#(C \cap \Gamma) > 3 \deg C \geq 9,$$

which again leads to a contradiction because $\deg \Gamma = 7$. Therefore C is linearly non-degenerate and $\deg C \geq 4$.

On the other hand, F contains a line, say Υ. Let $\tilde{\Upsilon} \subseteq \tilde{X}$ be its proper transform. We have $-K_{\tilde{X}} \cdot \tilde{\Upsilon} \leq 3 = -K_W \cdot \Upsilon$. Therefore, fixing four general points on $\tilde{\Upsilon}$, a member $\tilde{M} \in |-K_{\tilde{X}}|$ passing through these points is forced to contain $\tilde{\Upsilon}$. The family of all such members has codimension at most 4, while degenerate ones vary in a family of codimension at least five, as we observed before. Hence there exists an irreducible divisor $\tilde{M} \in |-K_{\tilde{X}}|$ containing $\tilde{\Upsilon}$. By our assumption $\tilde{M} \cdot \tilde{C} < 0$, and then also $\tilde{F} \cdot \tilde{C} = \tilde{M} \cdot \tilde{C} - H^* \cdot \tilde{C} < 0$. Thus the intersection $\tilde{M} \cap \tilde{F}$ contains $\tilde{C} \cup \tilde{\Upsilon}$ and so by (2.6.3) we have

$$\begin{aligned} \deg(C + \Upsilon) &= (\tilde{C} + \tilde{\Upsilon}) \cdot H^* \\ &\leq \tilde{M} \cdot \tilde{F} \cdot H^* = -K_{\tilde{X}} \cdot \tilde{F} \cdot H^* = (3H^* - \tilde{D}) \cdot (2H^* - \tilde{D}) \cdot H^* = 5. \end{aligned}$$

It follows that $\deg C = 4$, so $C \subseteq \mathbb{P}^4$ is a rational normal quartic curve. An easy computation gives $\dim H^0(\mathscr{I}_C(2)) = 6$. Picking two distinct points on Γ let us consider the family of quadrics from $H^0(\mathscr{I}_C(2))$ passing through these points. It has dimension four. Such a quadric cuts Γ in at least $13 + 2$ points, hence contains it. It follows that

$$\dim H^0(\mathscr{I}_{C \cup \Gamma}(2)) \geq 6 - 2 = 4.$$

The latter contradicts Lemma 2.1 (b). Therefore in the case $g = 10$ the divisor $-K_{\tilde{X}}$ is nef. Since $-K_{\tilde{X}}^3 = 2g - 6 > 0$, the divisor $-K_{\tilde{X}}$ is big. $\square$

By the base point free theorem we deduce the following result.

Corollary 2.8. *For some $n > 0$ the linear system $|-nK_{\tilde{X}}|$ defines a birational morphism $\sigma_0 \colon \tilde{X} \to X_0 \subseteq \mathbb{P}^N$ whose image is a Fano threefold with at worst Gorenstein canonical singularities. Moreover $-K_{\tilde{X}} = \sigma_0^*(-K_{X_0})$.*

Lemma 2.9. *The morphism σ_0 is small, i.e. it does not contract any divisor.*

Proof. Assume that σ_0 contracts a prime divisor $\Xi \sim -\alpha K_{\tilde{X}} - \beta \tilde{F}$. Then by (2.6.4)

$$0 = \Xi \cdot (-K_{\tilde{X}})^2 = (2g - 6)\alpha - 3\beta.$$

This yields $\beta = (2g/3 - 2)\alpha$. Since $\Xi \neq \tilde{F}$ and $-K_{\tilde{X}}$ is nef by Lemma 2.7, we have

$$0 \leq -K_{\tilde{X}} \cdot \Xi \cdot \tilde{F} = 3\alpha + 2\beta = \alpha\left(\frac{4g}{3} - 1\right).$$

Hence $\alpha > 0$. Furthermore,

$$\Xi \sim \alpha\left(\frac{2g^2}{3} - 11g + 37\right)H^* + \alpha\left(\frac{2g}{3} - 3\right)\tilde{D}.$$

Since $\sigma_* \Xi$ is effective we must have $2g^2/3 - 11g + 37 \geq 0$, a contradiction. $\square$

The next corollary is standard.

Corollary 2.10. *In the notation as above, X_0 has at worst isolated compound Du Val singularities.*

Following the techniques outlined in [22, Section 4.1] we can now prove Theorem 2.6.

Proof of Theorem 2.6. If $-K_{\tilde{X}}$ is ample then the map σ_0 is an isomorphism. In this case we let $\hat{X} = \tilde{X} = X_0$ and χ be the identity map. Otherwise by [29] the contraction $\sigma_0 \colon \tilde{X} \to X_0$ can be completed to a flop triangle as in diagram (2.6.2). Here φ_0 is another small resolution of X_0. Let $\hat{C} \subseteq \hat{X}$ and $\tilde{C} \subseteq \tilde{X}$ be the flopped and the flopping curves, respectively. Then χ induces an isomorphism $\tilde{X} \setminus \tilde{C} \simeq \hat{X} \setminus \hat{C}$.

In both cases the divisor $-K_{\hat{X}} = \varphi^*(-K_{X_0})$ is nef and big. Furthermore,

$$-K_{\hat{X}}^3 = -K_{\tilde{X}}^3 = 2g - 6, \quad (-K_{\hat{X}})^2 \cdot \hat{F} = (-K_{\tilde{X}})^2 \cdot \tilde{F} = 3,$$

and

$$-K_{\hat{X}} \cdot \hat{F}^2 = -K_{\tilde{X}} \cdot \tilde{F}^2 = -2.$$

Since $\operatorname{Pic}(\hat{X}) \simeq \operatorname{Pic}(\tilde{X})$ is of rank 2 the Mori cone $\operatorname{NE}(\hat{X})$ is generated by two extremal rays. One of them has the form $\mathbb{R}_+[T]$, where T is a curve in the fiber of σ (of φ_0, respectively) if χ is an isomorphism (is not an isomorphism, respectively). Let $R \subseteq \operatorname{NE}(\hat{X})$ be the second extremal ray. Since $-K_{\hat{X}}$ is nef and big, R is K-negative. By [31] there exists a contraction $\varphi \colon \hat{X} \to X$ of R.

Since $-K_{\tilde{X}} - \tilde{F} = \sigma^*\mathscr{O}(1)$ is nef we have $(-K_{\tilde{X}} - \tilde{F}) \cdot \tilde{C} > 0$. Therefore $\tilde{F} \cdot \tilde{C} < 0$ and $\hat{F} \cdot \hat{C} > 0$. Since $-K_{\hat{X}} \cdot \hat{F}^2 = -2 < 0$, the divisor $\hat{F}$ is not nef. Hence $\hat{F} \cdot R < 0$ that is, the ray R is not nef. By the classification of extremal rays [31], φ is a birational divisorial contraction. Furthermore, the φ-exceptional divisor coincides with $\hat{F}$. If $\varphi \colon \hat{X} \to X$ contracts $\hat{F}$ to a point, then by [31]

$$(-K_{\hat{X}})^2 \cdot \hat{F} = 4,\ 2 \text{ or } 1.$$

On the other hand, $(-K_{\hat{X}})^2 \cdot \hat{F} = 3$, a contradiction. Hence $\varphi \colon \hat{X} \to X$ contracts $\hat{F}$ to a curve Z. In this case both X and Z are smooth and φ is the blowup of X with center Z [31]. Moreover, X is a Fano threefold of Fano index $r = 1, 2, 3$ or 4. The group $\operatorname{Pic} \hat{X}$ is generated by $\hat{F}$ and

$$-\frac{1}{r}\varphi^* K_X = \frac{1}{r}(-K_{\hat{X}} + \hat{F}).$$

Therefore, the subgroup generated by $\tilde{F}$ and $-K_{\tilde{X}}$ has index r in $\operatorname{Pic}\tilde{X} \simeq \operatorname{Pic}\hat{X}$. This implies that $r = 1$. We have

$$\begin{aligned} -K_X^3 &= -K_{\hat{X}} \cdot (-K_{\hat{X}} + \hat{F})^2 \\ &= -K_{\hat{X}}^3 + 2\hat{F} \cdot (-K_{\hat{X}})^2 - K_{\hat{X}} \cdot F'^2 = 2g - 6 + 6 - 2 = 2g - 2, \end{aligned}$$

i.e. X is a Fano threefold of genus g. Furthermore,

$$\deg Z = -K_X \cdot Z = -K_{\hat{X}} \cdot (-K_{\hat{X}} + \hat{F}) \cdot \hat{F} = 3 - 2 = 1,$$

i.e. $Z \subseteq X$ is a line. Now an easy computation shows that $\hat{F}^3 \neq \tilde{F}^3$, so χ is not an isomorphism.

As for the last statement of Theorem 2.6, the existence of a diagram (2.6.2) follows from [19]. Here Γ is (as a scheme) the base locus of the linear subsystem $\sigma_*|-K_{\tilde{X}}| \subseteq |\mathscr{O}_W(13-g)|$. It remains to show that in the case $g = 9$ the curve Γ is not hyperelliptic. Assume the converse. It was shown already that Γ does not admit a 5-secant line. On the other hand, by [10, Chapter 2, Section 5] Γ admits a 4-secant line, say, N. The projection from N defines a linear system of degree 3 and dimension ≥ 1 on Γ. Hence the curve Γ is hyperelliptic and trigonal. However, this is impossible, since otherwise the linear systems g_2^1 and g_3^1 on Γ define a birational morphism $\Gamma \to \mathbb{P}^1 \times \mathbb{P}^1$ whose image is a divisor of bidegree (2, 3). This contradicts the assumption that $g(\Gamma) = 3$. Now the proof of Theorem 2.6 is completed. □

Corollary 2.11. *In the notation as above we have $X \setminus D \simeq W \setminus F$.*

In the next proposition we describe the flopped and the flopping curves in (2.6.2).

Proposition 2.12. *In the notation as above we let $\tilde{C} \subseteq \tilde{X}$ and $\hat{C} \subseteq \hat{X}$ be the flopping and the flopped curve, respectively. Then the following hold.*
(1) *Any irreducible component $\hat{C}_i \subseteq \hat{X}$ either is a proper transform of a line $L_i \neq L$ on X meeting L, or* (*in the case where L is of type* (β)) *is the negative section Σ of the ruled surface $\hat{F} \simeq \mathbb{F}_3$.*
(2) *The curve $\hat{C}$ is a disjoint union of the $\hat{C}_i$'s.*
(3) *For any $\hat{C}_i$ we have*

$$\mathscr{N}_{\hat{C}_i/\hat{X}} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$$

or

$$\mathscr{N}_{\hat{C}_i/\hat{X}} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2).$$

It follows that χ coincides with the Reid's pagoda [38] *near each $\hat{C}_i$.*

(4) *The curve* $\tilde{C}$ *in* $\tilde{X}$ *is a disjoint union of the* $\tilde{C}_i$*'s, where each* $\tilde{C}_i$ *is the proper transform of a* $(13-g)$*-secant line of* Γ.

Proof. Recall that $\hat{C}$ and $\tilde{C}$ are exceptional loci of φ_0 and σ_0, respectively. The assertion (1) is proven in [18, Proposition 3, (iv)] and [22, Proposition 4.3.1], while (2) and (3) in [6, Proposition 4] and [6, Corollary 12, Theorem 13], respectively. By (2) the curve $\hat{C}$ is a disjoint union of the $\hat{C}_i$'s. Since the construction of a flop is local and symmetric (i.e. flopping an irreducible curve we get an irreducible one), the flopped curve $\tilde{C}$ in $\tilde{X}$ must be in turn a disjoint union of its irreducible components. Finally $(-K_{\hat{X}}-\hat{F})\cdot\hat{C}_i=-1$. Therefore $1=(-K_{\tilde{X}}-\tilde{F})\cdot\tilde{C}_i=\sigma^*\mathscr{O}_W(1)\cdot\tilde{C}_i$. So $\sigma(\tilde{C}_i)$ is a line. Since $-K_{\tilde{X}}\cdot\tilde{C}_i=0$, this line must be $(13-g)$-secant. □

In the next theorem we provide a criterion as to when the surface F as in Theorem 2.6 is normal.

Theorem 2.13. *In the notation of* Theorems 1.1 *and* 2.6, *the surface* F *is normal if and only if* L *is a line of type* (α) *on* X.

Proof. We use the notation of Proposition 2.12. Assume that L is of type (β), and let $\tilde{C}_0$ denote the flopped curve on $\tilde{X}$ which corresponds to the negative section Σ of the ruled surface $\hat{F}\simeq\mathbb{F}_3$. By Remark 5.13 in [38], $\tilde{F}$ is not normal along $\tilde{C}_0$. Since the Picard number $\rho(\tilde{X})=2$ and $\tilde{C}_0$ is contracted by σ_0, it cannot be contracted by σ. Since $\tilde{C}_0$ is a smooth rational curve, σ is an isomorphism at a general point of $\tilde{C}_0$. So F is also non-normal along $\sigma(\tilde{C}_0)$.

Assume to the contrary that L is of type (α), while F is non-normal. Then F is singular along a line Λ. Clearly $\Lambda\neq\Gamma$, so $\tilde{F}$ is also non-normal and singular along $\sigma^{-1}(\Lambda)$. The map χ is an isomorphism near a general ruling $\hat{f}\subseteq\hat{F}\simeq\mathbb{F}_1$. Letting $\tilde{f}=\chi^{-1}(\hat{f})$, the surface $\tilde{F}$ is smooth along $\tilde{f}$ and $\sigma_0(\tilde{f})=\varphi_0(\hat{f})$ is a line on $\sigma_0(\tilde{F})=\varphi_0(\hat{F})\simeq\mathbb{F}_1$. Let $l\subseteq F$ be a general line on a non-normal scroll F and $\tilde{l}$ be its proper transform on $\tilde{F}$. An easy computation shows that $\sigma_0(\tilde{l})$ is again a line on $\sigma_0(\tilde{F})=\varphi_0(\hat{F})\simeq\mathbb{F}_1$. Thus we may suppose that $\tilde{l}=\tilde{f}$. On the other hand, $\tilde{l}\cap\operatorname{Sing}(\tilde{F})\neq\emptyset$, a contradiction. □

3. Construction of a cylinder

In this section we prove Theorem 0.1. Recall that under its assumptions $X=X_{2g-2}$ is a Fano threefold in $\mathbb{P}^{g+1}$ of genus $g=9$ or 10 with $\operatorname{Pic}(X)=\mathbb{Z}\cdot(-K_X)$ and with a non-smooth Fano scheme $\tau(X)$. Such a threefold X exists indeed by virtue of Theorem 2.6. The following result fixes the first assertion of Theorem 0.1.

Theorem 3.1. *Under the assumptions as above the variety* X *contains a cylinder.*

Proof. Assuming that the scheme $\tau(X)$ is not smooth at a point $[L] \in \tau(X)$, it suffices to construct a cylinder in $W \setminus F$ (see Corollary 2.11).

By Theorem 1.1 (4) L is a line of type (β) on X. According to Theorem 2.13 the surface F is non-normal. So by Proposition 2.2 $\Lambda = \mathrm{Sing}(F)$ is a double line on F. Consider the following diagram:

$$\begin{tikzcd} & \bar{W} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ W \arrow[rr, dashed, "\xi"] & & \mathbb{P}^{g-8} \end{tikzcd}$$

where ξ is the projection from Λ, p is the blowup of Λ, and $q = \xi \circ p$. We show below that q is a $\mathbb{P}^{11-g}$-bundle over $\mathbb{P}^{g-8}$. Let $\bar{E} \subseteq \bar{W}$ be the exceptional divisor and $\bar{F} \subseteq \bar{W}$ be the proper transform of F.

In the case $g = 10$ the fibers of ξ are intersections of our smooth quadric $W \subseteq \mathbb{P}^4$ (see Setup 2.5) with planes in $\mathbb{P}^4$ containing Λ. Therefore q is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, whose fibers are the proper transforms of lines in $W \subseteq \mathbb{P}^4$ meeting Λ. The morphism $q\colon \bar{W} \to \mathbb{P}^2$ is given by the linear system $|p^*\mathscr{O}_W(1) - \bar{E}|$. Since $\bar{F} \sim 2p^*\mathscr{O}_W(1) - 2\bar{E}$, the image $q(\bar{F}) = \xi(F)$ is a conic on $\mathbb{P}^2$. Since $\mathscr{N}_{\Lambda/W} \simeq \mathscr{O}_\Lambda \oplus \mathscr{O}_\Lambda(1)$, the $\mathbb{P}^1$-bundle $\bar{E} \to \Lambda$ is that of the Hirzebruch surface $\mathbb{F}_1 \to \mathbb{P}^1$. Furthermore, its negative section $\bar{\Sigma}$ is a fiber of q. It follows that the open set $W \setminus F \simeq \bar{W} \setminus (\bar{F} \cup \bar{E})$ is an $\mathbb{A}^1$-bundle over $\mathbb{P}^2 \setminus q(\bar{F} \cup \bar{\Sigma})$. By [24, Theorem 2] or [25, Theorem], this bundle is trivial over a Zariski open subset $Z \subseteq \mathbb{P}^2 \setminus q(\bar{F} \cup \bar{\Sigma})$. This gives a cylinder contained in $W \setminus F$ and also a cylinder on X.

In the case $g = 9$ the fibers of ξ are planes in $W = \mathbb{P}^3$. The intersection of such a plane with the cubic surface F consists of the double line Λ and a residual line l. Therefore q is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, and $\bar{F} \cup \bar{E}$ intersects each fiber along a pair of lines.

More precisely, we have $\bar{E} \cong \mathbb{F}_0$ and $\bar{F} \cong \mathbb{F}_1$ (see Proposition 2.2 (2a)). Furthermore, $q|_{\bar{E}}$ and $q|_{\bar{F}}$, respectively, yield $\mathbb{P}^1$-bundles with rulings being lines in the fibers of q. A simple computation shows that $\bar{F}|_{\bar{E}} \sim 2\bar{\Sigma} + \bar{l}$, where $\bar{\Sigma}$ (resp. $\bar{l}$) is a section (a ruling, respectively) of the trivial $\mathbb{P}^1$-bundle $\bar{E} \to \Lambda$. Notice that $\bar{\Sigma}$ is a line in a fiber of q and $\bar{l}$ is a section of q. The finite map $p|_{\bar{F}}\colon \bar{F} \to F$ yields a normalization of F. For the curve $\bar{F}|_{\bar{E}}$ there are the following two possibilities:

(i) $\bar{F}|_{\bar{E}} = \Delta_1$, where $\Delta_1 \in |2\bar{\Sigma} + \bar{l}|$ is irreducible, or

(ii) $\bar{F}|_{\bar{E}} = \bar{\Sigma} + \Delta_0$, where $\Delta_0 \in |\bar{\Sigma} + \bar{l}|$ is a (1, 1)-divisor.

We claim that $W \setminus F \simeq \bar{W} \setminus (\bar{F} \cup \bar{E})$ contains a cylinder. In what follows we deal with case (ii) only; case (i) can be treated in a similar fashion. There exists exactly one fiber of q, say $\bar{\Pi}_\infty$, such that $\bar{E}$, $\bar{F}$ and $\bar{\Pi}_\infty$ meet along a common line $\bar{\Sigma}$. Blowing up $\bar{W}^\circ := \bar{W} \setminus \bar{\Pi}_\infty$ along the irreducible curve $\bar{E} \cap \bar{F} \cap \bar{W}^\circ$ we obtain an $\mathbb{F}_1$-bundle $\hat{\pi}\colon \hat{W}^\circ \to \mathbb{A}^1$ together with the proper transforms $\hat{F}^\circ$ and $\hat{E}^\circ$ on $\hat{W}^\circ$ of $\bar{F}$ and $\bar{E}$, respectively. The exceptional divisor $\hat{E}_1^\circ$ is ruled over $\mathbb{A}^1$ with rulings being the (-1)-curves in the fibers isomorphic to $\mathbb{F}_1$. There is a natural $\mathbb{P}^1$-bundle structure $\rho\colon \hat{W}^\circ \to \hat{E}_1^\circ$ which defines in each fiber of ρ the ruling $\mathbb{F}_1 \to \mathbb{P}^1$. The map ρ sends $\hat{E}^\circ$ and $\hat{F}^\circ$ to the intersections $\hat{E}^\circ \cap \hat{E}_1^\circ$ and $\hat{F}^\circ \cap \hat{E}_1^\circ$, respectively. The complement $\hat{W}^\circ \setminus (\hat{E}_1^\circ \cup \hat{E}^\circ \cup \hat{F}^\circ) \simeq \bar{W} \setminus (\bar{E} \cup \bar{F} \cup \bar{\Pi}_\infty) \simeq W \setminus (F \cup \Pi_\infty)$ is again a $\mathbb{P}^1$-bundle over $\hat{E}_1^\circ \setminus (\hat{E}^\circ \cup \hat{F}^\circ)$, where $\Pi_\infty := p_*(\bar{\Pi}_\infty)$. This bundle is trivial over a Zariski open subset $Z \subseteq \hat{E}_1^\circ$ and admits a tautological section defined by $\hat{E}_1^\circ \hookrightarrow \hat{W}^\circ$. After trivialization the map $\rho\colon \rho^{-1}(Z) \to Z$ becomes the first projection $Z \times \mathbb{P}^1 \to Z$. The second projection of the tautological section defines a morphism $f\colon Z \to \mathbb{P}^1$. The automorphism $t \longmapsto (t - f(z))^{-1}$ of $Z \times \mathbb{P}^1$ sends this section to the constant section 'at infinity'. The $\mathbb{A}^1$-bundle $\rho\colon \hat{W}^\circ \setminus \hat{E}_1^\circ \to \hat{E}_1^\circ$ being trivial over Z it defines a cylinder $\rho^{-1}(Z) \setminus \hat{E}_1^\circ \simeq Z \times \mathbb{A}^1$, as required. □

Proof of Theorem 0.1. The first assertion of Theorem 0.1 is a consequence of Theorem 3.1. Let us show the second one. Recall that the automorphism group of a Fano threefold of genus $g = 9$ or 10 with $\mathrm{Pic}(X) = (-K_X) \cdot \mathbb{Z}$ is finite [36].

Fix a moduli space $\mathscr{M}_g$ of the Fano threefolds of genus $g = 9$ or 10 with $\mathrm{Pic}(X) = (-K_X) \cdot \mathbb{Z}$. It can be defined using GIT, and is unique up to a birational equivalence. Let $\mathscr{M}\mathscr{L}_g$ be the moduli space of pairs (X, L), where X is a Fano threefold as above and L is a line on X. Consider a natural projection $\pi\colon \mathscr{M}\mathscr{L}_g \to \mathscr{M}_g$ whose fiber over the point $[X] \in \mathscr{M}_g$ which corresponds to X is isomorphic to $\tau(X)$ up to a finite cover, since the automorphism group $\mathrm{Aut}(X)$ is finite, as we mentioned before. By Theorem 1.1 (3) we have $\dim \mathscr{M}_g = \dim \mathscr{M}\mathscr{L}_g - 1$. By Theorem 2.6 $\mathscr{M}\mathscr{L}_g$ is isomorphic to the moduli space of embedded curves $\Gamma \subseteq W$ of degree 7 and genus $g(\Gamma) = 12 - g$.

Let further $\mathscr{M}'_g \subseteq \mathscr{M}_g$ be the closed subvariety formed by all Fano threefolds X whose Fano scheme $\tau(X)$ is non-smooth, and let $\mathscr{M}\mathscr{L}'_g \subseteq \mathscr{M}\mathscr{L}_g$ be the subvariety formed by all pairs (X, L) such that L is of type (β). Then $\mathscr{M}'_g = \pi(\mathscr{M}\mathscr{L}'_g)$. Since such a Fano threefold X contains at most a finite number of (β)-lines (see Remark 1.2) we have $\dim \mathscr{M}'_g = \dim \mathscr{M}\mathscr{L}'_g$. Now the second assertion of Theorem 0.1 is immediate in view of the following claim. □

Claim 3.2. *Let $\mathscr{H}_g$ be the Hilbert scheme parameterizing the curves Γ on W of degree 7 and arithmetic genus $p_a(\Gamma) = 12 - g$. Then $\dim \mathscr{H}_g = 91 - 7g$. If the surface $F = F(\Gamma)$ is smooth along Γ, then $\mathscr{H}_g$ is smooth at the corresponding point. Furthermore, the subscheme $\mathscr{H}'_g$ of $\mathscr{H}_g$ parameterizing the curves Γ with $F(\Gamma)$ non-normal has codimension 2.*

Proof. Assuming that $F(\Gamma)$ is smooth along Γ, we consider the following exact sequence of vector bundles over Γ:

$$0 \to \mathscr{N}_{\Gamma/F} \to \mathscr{N}_{\Gamma/W} \to \mathscr{N}_{F/W}|_\Gamma \to 0. \tag{3.2.6}$$

Taking into account the relations

$$\deg \mathscr{N}_{\Gamma/F} = 2g(\Gamma) - 2 + \deg \Gamma$$

and

$$\deg \mathscr{N}_{F/W}|_{\Gamma} = \Gamma \cdot F$$

we obtain by (3.2.6) that $H^1(\mathscr{N}_{\Gamma/W}) = 0$ and $\dim H^0(\mathscr{N}_{\Gamma/W}) = 91 - 7g$. Now the first two assertions follow by the standard facts of the deformation theory.

The proof of the last assertion is just a parameter count. By Proposition 2.2 the dimension of the family of curves Γ with a non-normal surface $F(\Gamma)$ equals 13 and 11 in cases (a), (b) and (b′), respectively, while the family of all non-normal surfaces F is of codimension $15 - g$. $\square$

REMARK 3.3. Note that the family of all Fano threefolds X with Picard number 1 of genus 9 or 10 with a non-smooth Hilbert scheme of lines $\tau(X)$ is irreducible, since $\mathscr{M}\mathscr{L}'_g \simeq \mathscr{H}'_g$ is.

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