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A NOTE ON KNOTS WITH H(2)-UNKNOTTING NUMBER ONE

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Abstract

We give an obstruction to unknotting a knot by adding a twisted band, derived from Heegaard Floer homology.

1. Introduction

Many unknotting operations have been defined and studied in knot theory. For example, as well-known, (a), (b) (cf. [8, 10]) and (c) in Fig. 1 are three types of unknotting operations. Especially, (c) was introduced by Hoste, Nakanishi and Taniyama [4], which they called H(n)-move. Here n is the number of arcs inside the circle. Note that an H(n)-move is required to preserve the component number of the diagram. The H(n)-*unknotting number* of a knot is the minimal number of H(n)-moves needed to change the knot into the unknot. In this note, we focus on the special case when n equals two. Given two knots K and K', when K' is obtained from K by applying an H(2)-move, we also alternatively say that K' is obtained from K by adding a twisted band, as shown in Fig. 2. Following [4], we denote the H(2)-unknotting number of a knot K by $u_2(K)$. In this note, we give a necessary condition for a knot K to have $u_2(K) = 1$, by using a method introduced by Ozsváth and Szabó [15].

The question whether a given knot has H(2)-unknotting number one should be traced back to Riley. He made the conjecture that the figure-eight knot could never be unknotted by adding a twisted band. Lickorish confirmed this conjecture in [7]. Here we give a brief review of his method. Given a knot K, let $\Sigma(K)$ denote the double-branched cover of S^3 along K and let $\lambda\colon H_1(\Sigma(K), \mathbb{Z}) \times H_1(\Sigma(K), \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ be the linking form of $\Sigma(K)$. Lickorish proved that if the knot K can be unknotted by adding a twisted band, then $H_1(\Sigma(K), \mathbb{Z})$ is cyclic and it has a generator g such that $\lambda(g, g) = \pm 1/\det(K)$, where $\det(K)$ is the determinant of K. For the figure-eight knot 4_1, the linking form has the form $\lambda(g, g) = 2/5$ for some generator $g \in H_1(\Sigma(4_1)) \cong \mathbb{Z}/5\mathbb{Z}$. If there is another generator $g' = xg$ such that $\lambda(g', g') = \pm 1/5$, we have $2x^2 \equiv \pm 1$ (mod 5), while there is no such an integer x satisfying the condition. Therefore Riley's conjecture holds.

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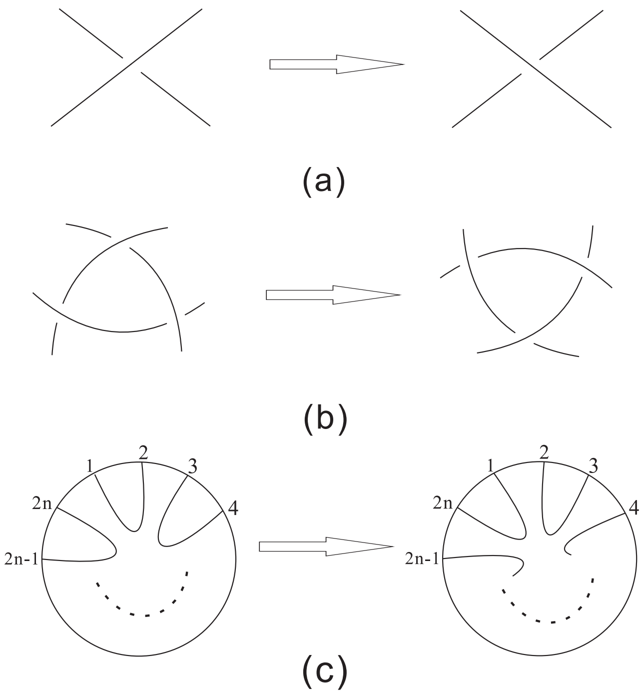

Fig. 1. Some unknotting operations.

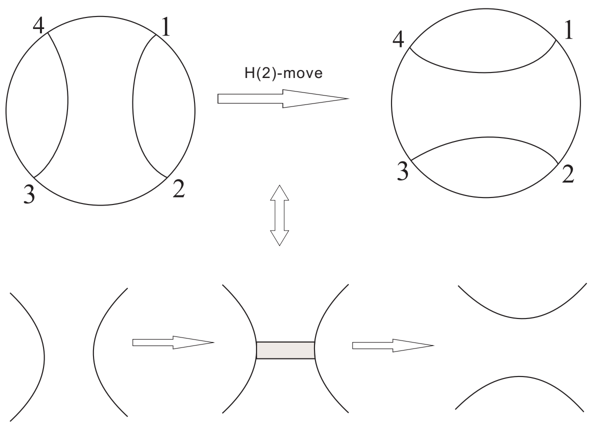

Fig. 2. Adding a twisted band to a knot diagram.

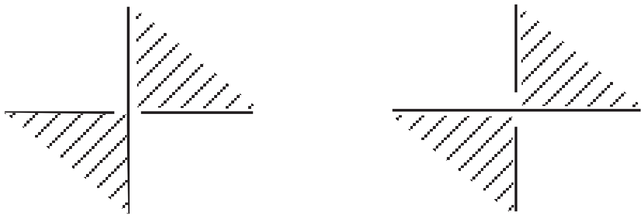

Fig. 3. The sign convention of a crossing.

Now we turn to the description of our result. Consider a negative-definite symmetric $n \times n$ matrix Q over $\mathbb{Z}$, and suppose $|\det(Q)|$ is p. Then define a group

$$G_Q := \mathbb{Z}^n/\mathrm{Im}(Q).$$

A *characteristic vector* for Q is an element in

$$\begin{aligned}\mathrm{char}(Q) &= \{\xi = (\xi_1, \xi_2, \ldots, \xi_n)^t \in \mathbb{Z}^n \mid \xi^t v \equiv v^t Q v \pmod 2 \text{ for any } v \in \mathbb{Z}^n\}\\ &= \{\xi \in \mathbb{Z}^n \mid \xi_i \equiv Q_{ii} \pmod 2 \text{ for } 1 \le i \le n\}.\end{aligned}$$

Suppose p is odd, and consider the map (cf. [12, 15])

$$M_Q \colon G_Q \to \mathbb{Q}$$

defined by

$$M_Q(\alpha) = \max\left\{ \frac{\xi^t Q^{-1} \xi + n}{4} \;\middle|\; \xi \in \mathrm{char}(Q),\ [\xi] = \alpha \in G_Q \right\}.$$

Now we recall the definition of Goeritz matrix. Given a knot diagram, color this diagram in checkerboard fashion such that the unbounded region has black color. Let $f_0, f_1, \ldots, f_k$ denote the black regions and f_0 correspond to the unbounded one. Define the sign of a crossing as in Fig. 3. Then the Goeritz matrix A is the $k \times k$ symmetric matrix defined as follows,

$$q_{ij} = \begin{cases}\text{the signed count of crossings adjacent to } f_i & \text{if } i = j,\\ \text{minus the signed count of crossings joining } f_i \text{ and } f_j & \text{if } i \neq j\end{cases}$$

for $i, j = 1, 2, \ldots, k$.

Our result about H(2)-unknotting number is as follows:

Theorem 1.1. *Let K be an alternating knot with $|\det K| = p$, and let A be the negative-definite Goeritz matrix corresponding to a reduced alternating diagram of K or its mirror image. Since K is a knot, we see that p is an odd number. Suppose G_A is the group presented by A. If $u_2(K) = 1$, then there is an isomorphism $\phi\colon \mathbb{Z}/p\mathbb{Z} \to G_A$ and a sign $\epsilon \in \{+1, -1\}$ with the properties that for all $i \in \mathbb{Z}/p\mathbb{Z}$:*

$$I_{\phi,\epsilon}(i) := \epsilon \cdot M_A(\phi(i)) + \frac{1}{4}\left(\frac{1}{p}\left(\frac{p + (-1)^i p}{2} - i\right)^2 - 1\right) = 0 \pmod 2,$$

and

$$I_{\phi,\epsilon}(i) \geq 0.$$

Here we abuse i to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \ldots, p-1\}$.

If one is familiar with the work in [15], the proof is immediate. We will give the proof in Section 2.

The H(2)-unknotting number of a knot is an interesting knot invariant. It is closely related to the 3-dimensional and 4-dimensional crosscap numbers of a knot. It can be defined in some different viewpoints, as indicated by Taniyama and Yasuhara [17]. Many researches concerning it can be found in [18, 6, 1] and other papers.

In order to check that Theorem 1.1 works better in some cases than the existing criteria, we post the knot $P(13, 4, 11)$ as an example. We determine that it has H(2)-unknotting number 2, which cannot seem to be detected by the other methods that the author knows.

Corollary 1.2. *The pretzel knot $P(13, 4, 11)$ has* H(2)*-unknotting number* 2.

2. Proofs

2.1. Preliminaries. Almost all the ingredients contained in this subsection can be found in [15], or an earlier paper [13]. But for intactness, we include them here. If X is an oriented 3- or 4-manifold, the second cohomology $H^2(X, \mathbb{Z})$ acts on the set of spin^c-structures $\mathrm{Spin}^c(X)$ freely and transitively. Each spin^c-structure $s \in \mathrm{Spin}^c(X)$ has the first Chern class $c_1(s) \in H^2(X, \mathbb{Z})$, and the relation to the action is $c_1(s + h) = c_1(s) + 2h$ for any $h \in H^2(X, \mathbb{Z})$.

Let Y be an oriented rational homology 3-sphere and s be a spin^c-structure over Y. Then there is Heegaard Floer homology associated with the pair (Y, s). In this note, we use Heegaard Floer homology with coefficients in the field $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. There are several versions of this homology. One version is $HF^+(Y, s)$, which is a $\mathbb{Q}$-graded module over the polynomial algebra $\mathbb{F}[U]$. That is

$$HF^+(Y,s) = \bigoplus_{i\in\mathbb{Q}} HF_i^+(Y,s),$$

where multiplication by U lowers the grading by two. In each grading $i \in \mathbb{Q}$, $HF_i^+(Y,s)$ is a finite-dimensional $\mathbb{F}$-vector space. A simpler version is $HF^\infty(Y)$, and it satisfies $HF^\infty(Y,s) = \mathbb{F}[U,U^{-1}]$ for each $s \in \mathrm{Spin}^c(Y)$ [14, Theorem 10.1]. It is also $\mathbb{Q}$-graded and multiplication by U lowers its grading by two.

For any spin^c-structure s, there is a natural $\mathbb{F}[U]$-equivariant map

$$\pi\colon HF^\infty(Y,s) \to HF^+(Y,s),$$

which preserves the $\mathbb{Q}$-grading. We use π_i to denote the restriction of π on the grading i. Then π_i is zero for all sufficiently negative gradings and an isomorphism in all sufficiently positive gradings. Ozsváth and Szabó defined an invariant $d(Y,s)$ from the map π, which is called the *correction term* of the pair (Y,s). Precisely, we have

$$d(Y,s) := \min\{i \in \mathbb{Q} \mid \pi_i \text{ is non-zero}\}.$$

The correction terms for Y and $-Y$, where "$-$" means the reversion of orientation, are related by the formula

$$d(-Y,s) = -d(Y,s)$$

under the natural identification $\mathrm{Spin}^c(Y) \cong \mathrm{Spin}^c(-Y)$.

The map π behaves naturally under cobordisms. Let Y_1 and Y_2 be two oriented rational homology 3-spheres. We say a smooth connected oriented 4-manifold X is a cobordism from Y_1 to Y_2 if the boundary of X is given by $\partial X = -Y_1 \cup Y_2$. Suppose X is a cobordism from Y_1 to Y_2 and t is a spin^c-structure of X. Then there is a homomorphism

$$F^o_{X,t}\colon HF^o(Y_1,s_1) \to HF^o(Y_2,s_2),$$

where HF^o denotes any version of Heegaard Floer homology and s_i is the restriction of t to Y_i for $i = 1, 2$ (we simply express it as $s_i = t|_{Y_i}$). The map π and the map $F^o_{X,t}$ fit into the following commutative diagram:

$$\begin{array}{ccc} HF^\infty(Y_1,s_1) & \xrightarrow{F^\infty_{X,t}} & HF^\infty(Y_2,s_2) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ HF^+(Y_1,s_1) & \xrightarrow{F^+_{X,t}} & HF^+(Y_2,s_2). \end{array}$$

If X is a negative-definite cobordism, the proof of Theorem 9.1 in [13] (also mentioned in the proof of [13, Proposition 9.9]) tells us that $F^{\infty}_{X,t}$ is an isomorphism.

Suppose that Y is an oriented rational homology 3-sphere, that X is a negative-definite simply connected 4-manifold with $\partial X = Y$ and that $t \in \mathrm{Spin}^c(X)$. Then it is shown in [13] that

$$d(Y, t|_Y) \geq \frac{c_1^2(t) + b_2(X)}{4}, \tag{1}$$

$$d(Y, t|_Y) = \frac{c_1^2(t) + b_2(X)}{4} \pmod 2. \tag{2}$$

Here (1) follows directly from [13, Theorem 9.6], while (2) is not clearly written. For readers' convenience, we explain it here. Consider X minus a point as a cobordism W from S^3 to Y. Then we have the following commutative diagram

$$\begin{array}{ccc} HF^{\infty}(S^3, t|_{S^3}) & \xrightarrow{F^{\infty}_{W,t}} & HF^{\infty}(Y, t|_Y) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ HF^{+}(S^3, t|_{S^3}) & \xrightarrow{F^{+}_{W,t}} & HF^{+}(Y, t|_Y), \end{array}$$

and $F^{\infty}_{W,t}$ is an isomorphism. There is an element $\xi \in HF^{\infty}(Y, t|_Y)$ with the property that its $\mathbb{Q}$-grading $\mathrm{gr}(\xi)$ is $d(Y, t|_Y)$. Suppose the preimage of ξ in $HF^{\infty}(S^3, t|_{S^3})$ is η. Then we have

$$d(Y, t|_Y) - \mathrm{gr}(\eta) = \mathrm{gr}(\xi) - \mathrm{gr}(\eta) = \frac{c_1^2(t) - 2\chi(W) - 3\sigma(W)}{4} = \frac{c_1^2(t) + b_2(X)}{4}.$$

The first equality follows from our choice of ξ, the second one follows from Equation (4) in [13], and the last one holds because of the fact that $2\chi(W) + 3\sigma(W) + b_2(X) = 0$. Precisely we have

$$\begin{aligned} &2\chi(W) + 3\sigma(W) + b_2(X) \\ &= 2(b_0(W) - b_1(W) + b_2(W) - b_3(W) + b_4(W)) - 3b_2(W) + b_2(W) \\ &= 2(b_0(W) - b_1(W) - b_3(W) + b_4(W)) \\ &= 2(b_0(W) - 2b_1(W) - 1 + b_4(W)) = 0. \end{aligned}$$

Here $b_i(W)$ denotes the i-th Betti number of W. The first equality comes from our assumption that X is negative-definite. The third equality follows from the fact that $b_3(W) = b_1(W) + 1$, obtained from the relation $H_3(W) \cong H_3(W, S^3 \cup Y) \oplus \mathbb{Z}$, Poincaré duality and the universal coefficient theorem. The last equality comes from the facts that $b_0(W) = 1$ and $b_4(W) = 0$, and our assumption that X is simply connected. For the 3-sphere S^3, as an $\mathbb{F}$-vector space, we know that ([14, Theorem 10.1])

$$HF^\infty(S^3, t|_{S^3}) = \bigoplus_{i=-\infty}^{\infty} \mathbb{F}_{(2i)},$$

where $\mathbb{F}_{(j)}$ denotes the summand supported on grading j. Therefore we see that $\mathrm{gr}(\eta) = 0 \pmod 2$. Now (2) follows.

Remember that $d(S^3, t|_{S^3}) = 0$ and that $HF^\infty(S^3, t|_{S^3}) = \mathbb{F}[U, U^{-1}]$, and therefore we obtain $\mathrm{gr}(\eta) = 0 \pmod 2$. Now (2) follows obviously.

Suppose further for simplicity that X is simply-connected and that the order of $H^2(Y, \mathbb{Z})$ is odd. Then there exists a group structure on the space $\mathrm{Spin}^c(Y)$ by identifying $s \in \mathrm{Spin}^c(Y)$ with $c_1(s) \in H^2(Y, \mathbb{Z})$. In the following, we denote the correction term $d(Y, s)$ by $d(Y, c_1(s))$ if necessary. Let r denote the second Betti number of X. Then we have the following exact sequence:

$$0 \to H_2(X) = \mathbb{Z}^r \xrightarrow{\tau} H^2(X) = \mathbb{Z}^r \xrightarrow{j^*} H^2(Y) \to H_1(X) = 0.$$

Fix a basis for $H_2(X)$ and let B be the matrix of the intersection form of X. Then B is a symmetric negative-definite $r \times r$ integer matrix with $|\det B| = |H^2(Y, \mathbb{Z})|$. A spin^c-structure $s \in \mathrm{Spin}^c(Y)$ is the restriction of a spin^c-structure $t \in \mathrm{Spin}^c(X)$ on Y if and only if $j^*(c_1(t)) = c_1(s)$.

In fact, the map τ under the given basis of $H_2(X)$ is presented by the matrix B. We define φ as the map $\mathrm{Coker}(\tau) = G_B \xrightarrow{j_1^*} H^2(Y)$, where j_1^* is the map induced from j^* on the cokernel of τ. It is obvious from the exact sequence that φ is an isomorphism. Under φ the set of characteristic vectors $\mathrm{char}(B)$ is equal to the set $\{c_1(t) \mid t \in \mathrm{Spin}^c(X)\} \subset H^2(X, \mathbb{Z})$. If we suppose the first Chern class $c_1(t)$ corresponds to the characteristic vector ξ, then $c_1^2(t)$ is equal to $\xi^t B^{-1} \xi$.

Under these identifications, (1) and (2) can be written as follows. For any $s \in \mathrm{Spin}^c(Y)$ and any $\xi \in \mathrm{char}(B)$ with $c_1(s) = \varphi([\xi])$, there are

$$d(Y, c_1(s)) \geq \frac{\xi^t B^{-1} \xi + r}{4}$$

and

$$d(Y, c_1(s)) = \frac{\xi^t B^{-1} \xi + r}{4} \pmod 2.$$

This is equivalent to say under the isomorphism $\varphi \colon G_B \to H^2(Y, \mathbb{Z})$ the following hold for any $\alpha \in G_B$:

$$\begin{aligned} &d(Y, \varphi(\alpha)) \geq M_B(\alpha), \\ &d(Y, \varphi(\alpha)) = M_B(\alpha) \pmod 2. \end{aligned} \tag{3}$$

2.2. Proof of Theorem 1.1. When K is an alternating knot in S^3, the correction terms for $\Sigma(K)$ have an extremely easy combinatorial description as follows.

Theorem 2.1 (Ozsváth–Szabó [15, 16])**.** *If K is an alternating knot and A denotes a Goeritz matrix associated to a reduced alternating projection of K, and G_A is the group presented by A, then there is an isomorphism $\psi\colon H^2(\Sigma(K), \mathbb{Z}) \to G_A$, with the property that*

$$d(\Sigma(K), \beta) = M_A(\psi(\beta))$$

for all $\beta \in H^2(\Sigma(K), \mathbb{Z})$.

For knots with H(2)-unknotting number one, we have the following lemma.

Lemma 2.2 (Montesinos's trick [9])**.** *If the* H(2)*-unknotting number of a knot K is one, then $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$. Here $p = |\mathrm{det}(K)|$ and $S^3_{-p}(C)$ denotes the $-p$-surgery of S^3 along the knot C.*

Proof of Theorem 1.1. If the H(2)-unknotting number of K is one, then by Lemma 2.2 $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$ and $p = |\mathrm{det}(K)|$. Therefore $\epsilon \cdot \Sigma(K) = S^3_{-p}(C)$ bounds a 4-manifold X, which is obtained by attaching a 2-handle to the 4-ball along C with framing $-p$. The intersection form of X is $B = (-p)$. In this case $G_B = \mathbb{Z}/p\mathbb{Z}$, and X is a simply-connected negative-definite 4-manifold.

By (3), there exists a group isomorphism $\varphi\colon G_B = \mathbb{Z}/p\mathbb{Z} \to H^2(S^3_{-p}(C), \mathbb{Z})$ with

$$d(S^3_{-p}(C), \varphi(i)) = d(\epsilon \cdot \Sigma(K), \varphi(i)) = \epsilon \cdot d(\Sigma(K), \varphi(i)) \geq M_B(i)$$

and

$$\epsilon \cdot d(\Sigma(K), \varphi(i)) \equiv M_B(i) \pmod 2$$

Theorem 2.1 implies that for the map $\phi = \psi \circ \varphi\colon \mathbb{Z}/p\mathbb{Z} \to G_A$ (here we automatically identify $H^2(S^3_{-p}(C), \mathbb{Z})$ with $H^2(\Sigma(K), \mathbb{Z})$) we have

$$\epsilon \cdot M_A(\phi(i)) \geq M_B(i)$$

and

$$\epsilon \cdot M_A(\phi(i)) \equiv M_B(i) \pmod 2$$

for all $i \in \mathbb{Z}/p\mathbb{Z}$. In the following calculation, we abuse i to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \ldots, p-1\}$. By definition we see that

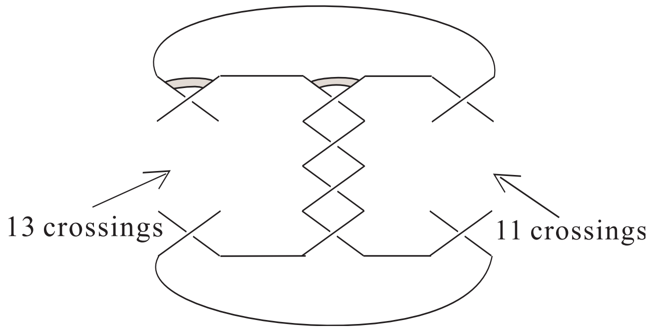

Fig. 4. The pretzel knot $P(13, 4, 11)$.

for any $i \in \mathbb{Z}/p\mathbb{Z}$,

$$\begin{aligned} M_B(i) &= \max\left\{ \frac{u^t B^{-1} u + 1}{4} \;\middle|\; u \text{ is odd, } [u] = i \right\} \\ &= \max\left\{ \frac{-u^2 + p}{4p} \;\middle|\; u \text{ is odd, } [u] = i \right\} \\ &= \begin{cases} \dfrac{-(p-i)^2 + p}{4p} & \text{if } i \text{ is even,} \\ \dfrac{-(i)^2 + p}{4p} & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Writing these two cases in one form we have $M_B(i) = -(1/4)((1/p)((p + (-1)^i p)/2 - i)^2 - 1)$. This completes the proof of Theorem 1.1. □

2.3. An example: proof of Corollary 1.2. The pretzel knot $K = P(13, 4, 11)$ is an alternating knot as shown in Fig. 4. A negative-definite Goeritz matrix associated with the mirror image of this diagram is

$$A = \begin{pmatrix} -17 & 4 \\ 4 & -15 \end{pmatrix},$$

and the determinant is $\det(A) = \det(K) = 239$. Suppose G_A is the group presented by A. In fact, the group G_A is isomorphic to $\mathbb{Z}/239\mathbb{Z}$. In the following calculation, we take the vector $(0, 1)^t$ as a generator of G_A. The inverse of the matrix A is

$$A^{-1} = \frac{1}{239}\begin{pmatrix} -15 & -4 \\ -4 & -17 \end{pmatrix}.$$

Then by definition for any $0 \le r \le 238$ it holds that

$$\begin{aligned} &M_A((0, r)^t) \\ &= \max\left\{ \frac{(u, v)^t A^{-1}(u, v) + 2}{4} \;\middle|\; (u, v)^t \in \mathrm{char}(A),\ [(u, v)^t] = (0, r)^t \in G_A \right\} \\ &= \max\left\{ \frac{478 - (15u^2 + 8uv + 17v^2)}{956} \;\middle|\; u \text{ and } v \text{ are odd},\ [(u, v)^t] = (0, r)^t \in G_A \right\}. \end{aligned}$$

From this expression, we see that in order to obtain the maximum we only need to focus on those representatives $(u, v)^t$ satisfying $|u| \le 17$ and $|v| \le 15$.

By calculation, it is easy to see that for any isomorphism $\phi\colon \mathbb{Z}/239\mathbb{Z} \to \mathbb{Z}/239\mathbb{Z}$ there is

$$I_{\phi,\epsilon}(0) = \epsilon \cdot M_A(\phi(0)) + \frac{119}{2} = \epsilon \cdot M_A((0, 0)^t) + \frac{119}{2} = \frac{\epsilon \cdot (-11) + 119}{2}.$$

The vector which realizes the value of $M_A((0,0)^t)$ is $(u, v)^t = (13, 11)^t$ or $(-13, -11)^t$.

We assume that $u_2(K) = 1$. Then by Theorem 1.1 the value $I_{\phi,\epsilon}(0)$ has to be an even number, and therefore $\epsilon = 1$. Next by calculation we have $I_{\phi,1}(1) = M_A(\phi(1)) - 119/478$. Since 239 is a prime number, any ϕ_j = "multiplication by j" is an automorphism of $\mathbb{Z}/239\mathbb{Z}$. To guarantee that $I_{\phi_j,1}(1)$ is an even number, the isomorphism ϕ_j has to be either ϕ_{15} or ϕ_{224}. By calculation, we see that

$$I_{\phi_{15},1}(1) = M_A((0, 15)^t) - \frac{119}{478} = -4.$$

The vector which realizes the value of $M_A((0, 15)^t)$ is $(u, v)^t = (-9, -11)^t$. Same calculation tells us that $I_{\phi_{224},1}(1) = -4$ as well, which is realized by the vector $(u, v)^t = (9, 11)^t$. Now we see -4 is a negative number, which conflicts with the necessary condition stated in Theorem 1.1. Therefore the H(2)-unknotting number of $P(13, 4, 11)$ has to be at least two. On the other hand, the knot $P(13, 4, 11)$ can be changed into the unknot by adding two twisted bands as shown in Fig. 4. Hence the H(2)-unknotting number of $P(13, 4, 11)$ is two. This completes the proof of Corollary 1.2.

2.4. Comparisons with other criterions. There have been many criterions and properties which can be used to bound the H(2)-unknotting number of a knot. We want to apply them to the knot $P(13, 4, 11)$ and compare the results with Corollary 1.2.

The first one is Lickorish's obstruction that we recalled in the beginning. It does not work for the pretzel knot $K = P(13, 4, 11)$ because of the following reason. It is known that the Goeritz matrix A is a presentation matrix of $H_1(\Sigma(K), \mathbb{Z})$, and A^{-1} represents the linking form λ. It is not hard to see that $H_1(\Sigma(K))$ is cyclic of order 239, and that the generator $g = (0, 1)^t$ satisfies $\lambda(g, g) = -17/239$. Then we see $\lambda(15g, 15g) = (225 \times (-17))/239 = -3825/239 = -1/239$ over $\mathbb{Q}/\mathbb{Z}$. Since 239 is a prime number, the vector $g' = (0, 15)^t$ can work as a generator of $H_1(\Sigma(K), \mathbb{Z})$.

There are two invariants of knots which are closely related to H(2)-unknotting number. Given a knot $K \subset S^3$, the crosscap number of K [2] is defined as follows:

$$\gamma(K) = \min\{\beta_1(F) \mid F \text{ is a non-orientable connected surface in } S^3 \text{ and } \partial F = K\},$$

where $\beta_1(F)$ denotes the rank of the first homology group of F. The 4-dimensional crosscap number of K [11], which we denote $\gamma^*(K)$ here, is by name defined as follows:

$$\gamma^*(K) = \min\left\{\beta_1(F) \,\middle|\, \begin{array}{l} F \text{ is a non-orientable connected smooth surface in } B^4 \text{ and} \\ \partial F = K \subset \partial B^4 = S^3 \end{array}\right\}.$$

Their relation with H(2)-unknotting number is as follows.

Lemma 2.3. *Given a knot $K \subset S^3$, we have $\gamma^*(K) \leq u_2(K) \leq \gamma(K)$.*

Proof. The knot K can be reconstructed from the unknot by adding $u_2(K)$ twisted bands successively. Let D be a disk bounded by the unknot and $b_1, b_2, \ldots, b_{u_2(K)}$ be the bands added to the boundary of D. Then $F := D \cup \bigcup_{i=1}^{u_2(K)} b_i$ is a non-orientable surface in B^4 with $\partial F = K$. We have $\gamma^*(K) \leq \beta_1(F) = u_2(K)$. The second inequality is proved as follows. Suppose S is a non-orientable surface in S^3 which realizes the crosscap number of K. Namely we have $\beta_1(S) = \gamma(K)$ and $\partial S = K$. Then there are $\gamma(K)$ disjoint essential arcs in S, say $\tau_1, \tau_2, \ldots, \tau_{\gamma(K)}$, such that $S - \tau_i$ has one boundary component for $i = 1, 2, \ldots, \gamma(K)$ and $S - \bigcup_{i=1}^{\gamma(K)} \tau_i$ is a disk. If we add twisted bands to K along τ_i for $i = 1, 2, \ldots, \gamma(K)$, the resulting knot is the unknot. Therefore we have $u_2(K) \leq \gamma(K)$. □

Ichihara and Mizushima [5] calculated the crosscap numbers of pretzel knots. According to their calculation, the crosscap number of $P(13, 4, 11)$ is two. Gilmer and Livingston [3] studied the 4-dimensional crosscap number of a knot by using Heegaard Floer homology. Their method and our result in this note are both in spirit derived from Theorem 9.6 in [13]. The author does not know whether their method can verify that the 4-dimensional crosscap number of $P(13, 4, 11)$ is 2 or not.

Yasuhara [18], and Kanenobu and Miyazawa [6] introduced some methods for detecting the H(2)-unknotting number of a knot, but simple calculation shows that their methods cannot be applied to the knot $P(13, 4, 11)$. Taniyama and Yasuhara[17] established the equivalence between H(2)-unknotting number and other two invariants of knots, but there seems no obvious way to apply their relation to the calculation of H(2)-unknotting number.

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