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A NEW DISTINGUISHED FORM FOR 3-BRAIDS

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Abstract

We show that every 3-strand braid has a representative word of a given form, and furthermore, this form allows us, in most cases, to deduce positivity (or negativity) in the σ-ordering of B_3. The σ-ordering of B_n was introduced by Patrick Dehornoy in the late 1990's, however, other (equivalent) orderings were discovered soon after by Fenn, Greene, Rolfsen, et al.

1. Introduction

The braid groups, denoted B_n, were introduced in 1925 by Emil Artin [1] and can be defined for each $n > 1$ as the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ with relations:
(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| > 1$, and
(2) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ if $|i - j| = 1$.

In this paper, we use this generator and relation description of B_n, but we will also view B_n as a mapping class group of the space $D_n = D^2 - \{p_1, p_2, \dots, p_n\}$, the unit disk with the set of distinguished points (called *punctures*) removed. As a mapping class, the generator σ_i exchanges punctures p_i and p_{i+1} by a counterclockwise half-twist. The n-strand braid

$$\Delta_n = (\sigma_1\sigma_2 \cdots \sigma_{n-1})(\sigma_1\sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)(\sigma_1)$$

is a half-twist of all n strands, and as a mapping class Δ_n^2 is a full Dehn twist about the boundary of D_n which generates the infinite cyclic center of B_n. We will denote $\Delta_3 = \sigma_1\sigma_2\sigma_1$ as simply Δ throughout this paper. Finally, we note that, here, we compose braids on the left.

A solution to the word and conjugacy problems for B_n was discovered in 1968 by Garside in [6] and expanded upon by many others over the years. However, it still remains an interesting endeavor to find distinguished representative words for elements of the braid groups and monoids. The goal of this paper is to give a new distinguished representative word for all 3-braids which can be obtained from a simple algorithm. The geometric nature of this approach and the use of the mapping class group point of view of B_3 makes this approach novel and is an improvement upon previous normal forms which use brute force methods. This new standard form is particularly useful in

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deciding for a 3-braid β, whether $\beta > 1$ is true in the σ-ordering of B_3. Until now, there has been no standard form for braids which claims to determine positivity in the σ-ordering. This is a new and practical feature of the standard form given here for the cases where the integer m in the standard form is greater than or equal to 3.

We give a brief description of each section of this paper. In Section 2, we give a proof of the main theorem by using the mapping class group definition of B_3 and using a Euclidean-like algorithm. In Section 3, we recall the σ-ordering of the braid groups, first defined by Patrick Dehornoy, and show how the distinguished form for a 3-braid can in most cases identify it as positive or negative in this ordering. Finally, in Section 4, we outline future directions in the continuation of this work. We start with the statement of the main theorem.

Theorem 1.1. *Every 3-braid β admits a representative word of the form*

$$\mu = (\sigma_2\sigma_1)^m \omega \tau^r, \tag{1.1}$$

where $\tau = \sigma_1^{-1}\sigma_2\sigma_1$ and ω is a word in only σ_1 and σ_2^{-1}.

2. Proof of the main result

As a matter of convenience, we arrange the punctures in a triangular fashion. Let e be a straight edge between p_1 and p_3, and let α_i be a properly embedded arc which separates D_3 into two components: a component which contains only the puncture p_i and its complement. We also require that $\alpha_1, \alpha_2, \alpha_3$ be pairwise disjoint (see Fig. 1).

Let β be an element of B_3, and isotop $\beta(e)$ so that it intersects the set $\{\alpha_1, \alpha_2, \alpha_3\}$ transversally and minimally. The intersection of $\beta(e)$ and each once-punctured disk region is some number (possibly zero) of parallel arcs enclosing the puncture and possibly an arc which ends at the puncture. The complement of the three once-punctured disk regions is a hexagonal region bounded by the arcs α_i and three subarcs of ∂D_3. The intersection of $\beta(e)$ with this region consists of a disjoint union of embedded edges, each of which connects two of the arcs α_i. Thus, there are three types of these edges; however, all three could not occur in the same diagram, for this would give rise to a closed loop. The braid $\sigma_2\sigma_1$ rotates the diagram by an angle of $2\pi/3$, therefore by applying $(\sigma_2\sigma_1)^m$ for $m = 0, 1$ or 2 to $\beta(e)$, we can assume there is no edge connecting α_1 and α_3 (see Fig. 2).

Let a_i be the geometric intersection number between the arcs $e' = (\sigma_1\sigma_2)^m\beta(e)$ and α_i. Note that since there is no edge between α_1 and α_3, we have $a_2 = a_1 + a_3$. First, there are a few special cases to consider. If $a_3 = 0$, then it must be the case that $a_1 = a_2 = 1$ to avoid a closed loop. Thus, in this case, e' is a straight arc between p_1 and p_2. We apply σ_2, and the resulting braid $\sigma_2(\sigma_1\sigma_2)^m\beta$ fixes e. Similarly, if $a_1 = 0$, then e' is a straight arc between p_2 and p_3. We apply σ_1^{-1} and the resulting braid $\sigma_1^{-1}(\sigma_1\sigma_2)^m\beta$ fixes e. If $a_1 = a_3$, then we must have $a_1 = a_3 = 1$ to avoid a closed

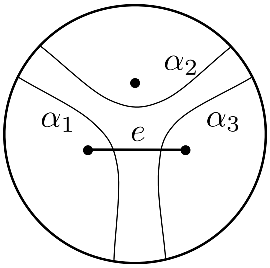

Fig. 1.

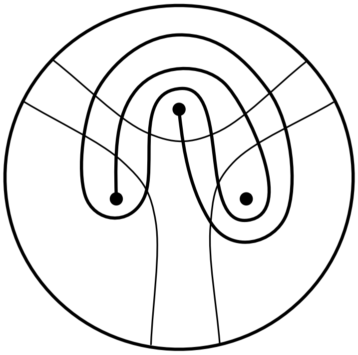

Fig. 2.

loop. In this case e' is a "U"-shaped arc with endpoints p_1 and p_3. We may apply σ_2^2 or σ_1^{-2}, and in either case, the resulting braid $\sigma_2^2(\sigma_1\sigma_2)^m\beta$ or $\sigma_1^{-2}(\sigma_1\sigma_2)^m\beta$ fixes e.

Now suppose that $0 < a_1 < a_3$, and consider $\sigma_2(e')$. This arc has fewer intersections with the set $\{\alpha_1, \alpha_2, \alpha_3\}$ than e'. In particular, if a'_i denotes the geometric intersection numbers of $\sigma_2(e')$ and α_i, then $a'_1 = a_1$, $a'_2 = a_3$, and $a'_3 = a_3 - a_1$. Similarly, if $0 < a_3 < a_1$, we apply σ_1^{-1} and find that the intersection numbers of $\sigma_1^{-1}(e')$ and $\alpha_1, \alpha_2, \alpha_3$ are $a'_1 = a_1 - a_3$, $a'_2 = a_1$, and $a'_3 = a_3$, respectively. We repeat the algorithm of applying σ_2 whenever $a_1 < a_3$ and applying σ_1^{-1} whenever $a_3 < a_1$ a finite number of times until we eventually unravel the arc e' so that the resulting arc is isotopic to e.

Thus, in each case we have generated a word ω^{-1} only involving negative powers of σ_1 and positive powers of σ_2 for which the braid $\omega^{-1}(\sigma_2\sigma_1)^m\beta$ fixes e. However, any braid that fixes e must be a power of $\tau = \sigma_1^{-1}\sigma_2\sigma_1$, the half Dehn twist about e, times a power of the central element $\Delta^2 = (\sigma_2\sigma_1)^3$. Thus,

$$\omega^{-1}(\sigma_2\sigma_1)^m\beta = \Delta^{2k}\tau^r$$

yielding

$$\beta = (\sigma_2\sigma_1)^{-m}\omega\Delta^{2k}\tau^r$$
$$= (\sigma_2\sigma_1)^{3k-m}\omega\tau^r,$$

where ω is a word involving only positive powers of σ_1 and negative powers of σ_2. This is the desired form, so the result is shown.

3. An application to the σ-ordering of B_3

The connection between the braid groups and orderable groups, though long overdue, was not widely recognized when announced by Patrick Dehornoy in 1992 [3] because the methods used were largely unfamiliar to topologists. Several years later, a topological proof of the orderability of B_n was discovered by Fenn, Rolfsen, Wiest, et al. [5]. This ordering used the description of B_n as a mapping class group of an n-punctured disk, yet surprisingly it leads to the same ordering given by Dehornoy. In the years since these discoveries, several other approaches, including ideas from hyperbolic geometry and lamination theory, have been used to show orderability of B_n. Here we describe Dehornoy's ordering of B_n known as the *σ-ordering* starting with a few definitions.

A braid word w is said to be *σ-positive* (resp. σ-negative) if, among the letters $\sigma_i^{\pm 1}$ that occur in w, the one with lowest index occurs positively only (resp. negatively only). For example, the word $w = \sigma_2\sigma_3^{-1}\sigma_2\sigma_3$ is σ-positive since σ_2 appears in w, but σ_2^{-1} does not. By contrast the word $\sigma_1\sigma_2\sigma_1^{-1}$ is neither σ-positive nor σ-negative since both σ_1 and σ_1^{-1} appear in w. For β, β' in B_n, we say that $\beta <_n \beta'$ is true if $\beta^{-1}\beta'$ admits an n-strand representative word that is σ-positive.

For instance, if $\beta = \sigma_1$ and $\beta' = \sigma_2\sigma_1$ in B_3, then $\beta^{-1}\beta'$ admits the word $\sigma_1^{-1}\sigma_2\sigma_1$ which is neither σ-positive nor σ-negative. However, using the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in B_3 we see that this braid also admits the word $\sigma_2\sigma_1\sigma_2^{-1}$ which is σ-positive. Hence, $\beta <_3 \beta'$. The relation $<_n$ is a total left-invariant ordering of B_n for $2 \le n \le \infty$, and we shall refer to this ordering as the *σ-ordering of B_n*. The proof is not included here but can be found in [4], among other sources. The essential properties of the relation $<_n$ needed for the proof are summed up in the following two statements.

Property A (Acyclicity)**.** A braid that admits at least one σ-positive representative word is nontrivial.

Property C (Comparison)**.** Every braid in B_n admits an n-strand representative word that is σ-positive, σ-negative, or empty.

From the definition above, it can be seen that $1 <_n \beta$ if and only if β admits an n-strand representative word that is σ-positive. For a braid β in B_3, we use a representative word in the form given by Theorem 1.1 to deduce σ-positivity. Since we will be referring to 3-strand braids for the remainder of this paper, we simply write $<$ for $<_3$.

First, we make a simple observation about σ-positivity in B_3. Since there are only the letters $\sigma_1^{\pm 1}$ and $\sigma_2^{\pm 1}$ to consider, a 3-strand braid word w is σ-positive if and only if σ_1 appears in w, but σ_1^{-1} does not; or if w is a positive power of σ_2. We can immediately conclude the following.

Proposition 3.1. *Let $\beta \in B_3$ be nontrivial, and choose a representative word for β of the form given in* Equation (1.1). *If $m \geq 0$ and $r \geq 0$, then $1 < \beta$.*

Proof. Let $\beta = (\sigma_2\sigma_1)^m \omega \tau^r$, where ω is a word in only σ_1 and σ_2^{-1}. We simply observe that $\tau = \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$. Therefore, if $m \geq 0$ and $r \geq 0$, only positive powers of σ_1 appear in the word. This proves the result. □

We can also conclude that if $m \geq 3$, the word is σ-positive no matter the value of r due to the fact that the infinite cyclic center of B_3 is generated by $(\sigma_2\sigma_1)^3$.

Proposition 3.2. *Let $\beta \in B_3$ be nontrivial, and choose a representative word for β of the form given in* Equation (1.1). *If $m \geq 3$, then $1 < \beta$.*

Proof. If we have the word $\mu = (\sigma_2\sigma_1)^m \omega \sigma_1^{-1}\sigma_2^r\sigma_1$, then there is only one occurrence of the letter σ_1^{-1}. We can freely reduce this letter by commuting ω and $(\sigma_2\sigma_1)^3$. This proves the result. □

For further cases, we use a special case of a combinatorial method for comparing braid words called *handle reduction* introduced by Dehornoy in [2]. Handle reduction provides an algorithm for finding a σ-positive or σ-negative representative word for a nonempty braid word (which always exists by Property C). By definition, if a nonempty braid word w is neither σ-positive nor σ-negative, then the letter σ_i of lowest index must appear both positively and negatively in w. Therefore, w contains a subword either of the form $\sigma_i \nu \sigma_i^{-1}$ or $\sigma_i^{-1}\nu\sigma_i$ where all letters of ν are $\sigma_k^{\pm 1}$ for $k > i$. A subword of this type is called a *σ_i-handle*, and handle reduction is a process of replacing a σ_i-handle in w with an equivalent word in which the first and last letters $\sigma_i^{\pm 1}$ have been deleted. This process is iterated until no σ_i-handle is left in the word, resulting in an equivalent word that is either σ-positive, σ-negative, or empty.

So, specifically, here is how handle reduction works. A σ_i-handle $\sigma_i^e \nu \sigma_i^{-e}$ is said to be *permitted* if the word ν includes no σ_{i+1}-handle. If v is a permitted σ_i-handle, we define the *reduct* of v to be the word obtained from ν by replacing each letter $\sigma_{i+1}^{\pm 1}$ with $\sigma_{i+1}^{-e}\sigma_i^{\pm 1}\sigma_{i+1}^e$. We say that w' is obtained from w by *handle reduction* (or w *reduces* to w') if w' is obtained by replacing a subword of w that is a permitted handle with its reduct. It should be noted that handle reduction extends free reduction since reducing the σ_i-handle $\sigma_i^e\sigma_i^{-e}$ amounts to deleting it. By a result of Dehornoy, handle reduction converges.

Sparing further exposition on the general case, we move directly to the $n = 3$ case. Here, the only nontrivial σ_i-handles are σ_1-handles $\sigma_1^e\sigma_2^d\sigma_1^{-e}$ which we simply refer to as "handles". It is also immediately apparent that all handles are permitted. Therefore, if each $\sigma_2^{\pm1}$ in a handle is replaced by $\sigma_2^{-e}\sigma_1^{\pm1}\sigma_2^e$, after free reductions are performed, we see that handle reduction in this case simply replaces a subword of the form $\sigma_1^e\sigma_2^t\sigma_1^{-e}$ with its reduct $\sigma_2^{-e}\sigma_1^t\sigma_2^e$.

We establish an algorithm for performing handle reductions. First, at each step we choose to reduce the *leftmost* handle in w. Here is what is meant by leftmost. If w is a word of length l, then a (p, q)*-subword* of w is the word obtained by deleting all letters before position p and after position q, for $1 \le p \le q \le l$. A handle v is said to be *leftmost* in w if there exist p, q such that v is the (p, q)-subword of w, and there is no p', q' with $q' < q$ such that the (p', q')-subword of w is a handle. For example, in the 3-strand braid word $w = \sigma_2\sigma_1\sigma_2^{-3}\sigma_1^{-2}\sigma_2^2\sigma_1$, the (2, 6)-subword $v = \sigma_1\sigma_2^{-3}\sigma_1^{-1}$ is the leftmost handle in w. Secondly, we perform all possible free reductions rather than waiting for trivial handle $\sigma_1^e\sigma_1^{-e}$ or $\sigma_2^e\sigma_2^{-e}$ to be leftmost. This algorithm is referred to as *left handle reduction*, and we use it in proving the following.

Proposition 3.3. *Let $\beta \in B_3$ be nontrivial, and choose a representative word for β of the form given in* Equation (1.1). *If $m \le -4$, then $\beta < 1$.*

Proof. For simplicity's sake, we replace σ_1 with the letter a and σ_2 with the letter b. The inverse of each generator will be the corresponding capital letter. Using this notation, $\mu = (AB)^m\omega Ab^r a$, where $m \ge 4$ and ω is a word in letters a and B only. Thus, we show μ is equivalent to a word in which A appears, but a does not appear.

First, we consider the case that a does not appear in the subword ω. Then, ω is either empty or a power of B. In this case, only appearance of the letter a is as the last letter of μ. Since $(AB)^3$ is in the center of B_3, we can annihilate the letter a by commuting the prefix $(AB)^3$ with the rest of the word. The resulting word has the letter A but no a, hence it is σ-negative.

Now, suppose the letter a does appear in ω. So, $\omega = a^{s_k}B^{s_{k-1}}\cdots a^{s_3}B^{s_2}a^{s_1}$, where $s_i \ge 0$ and some power of a does appear. Consider the leftmost a in ω; call it a_1. The letter a_1 is to the right of the prefix $(AB)^m$, so it is clear that the leftmost handle in μ is of the form $AB^{t_1}a_1$, where $t_1 \ge 1$. We replace $AB^{t_1}a_1$ with its reduct $bA^{t_1}B$. Notice that this replacement deletes a_1 and adds no more a's. Furthermore, we have shaved one AB from our prefix, so the reduced word now has at least three AB's in its prefix.

If there are any remaining a's in ω, consider the leftmost, and call it a_2. Note that a_2 must be to the right of the reduct $bA^{t_1}B$ from the previous step. Therefore, again our leftmost handle is of the form $AB^{t_2}a_2$, where $t_2 \ge 1$. We replace this handle with its reduct $bA^{t_2}B$, again, deleting a_2 while adding no more a's. We keep this repeating this process until there are no more occurrences of the letter a in ω.

Now, the only appearance of a is as the last letter of μ' (the reduction of μ). But since we have $(AB)^3$ in our prefix, we can commute this subword with the rest of the word to annihilate this a. The result is a word in which A appears but a does not appear. Thus, in each case $\beta < 1$. □

4. Conclusions

In this paper, we show that every 3-braid admits a distinguished word of the form $(\sigma_2\sigma_1)^m\omega\sigma_1^{-1}\sigma_2^r\sigma_1$ where ω is a word in σ_1 and σ_2^{-1} only. For example, it can be verified that σ_2 is equivalent to the word $(\sigma_2\sigma_1)^2\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_1$. Even though it may be less practical to use this form for braids like σ_2, the value for m sheds light on the positivity or negativity of the braid in the σ-ordering of B_3. Our results leave remaining cases $-3 \leq m < 3$ open, but initial investigations into these cases suggests that the integer r plays a greater role when $|m|$ is small.

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