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EXISTENCE, NONEXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR PARAMETRIC NONLINEAR ELLIPTIC EQUATIONS

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Abstract

We consider a parametric nonlinear elliptic equation driven by the Dirichlet p-Laplacian. We study the existence, nonexistence and multiplicity of positive solutions as the parameter λ varies in $\mathbb{R}_0^+$ and the potential exhibits a p-superlinear growth, without satisfying the usual in such cases Ambrosetti–Rabinowitz condition. We prove a bifurcation-type result when the reaction has $(p-1)$-sublinear terms near zero (problem with concave and convex nonlinearities). We show that a similar bifurcation-type result is also true, if near zero the right hand side is $(p-1)$-linear.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary $\partial\Omega$ and $p > 1$ be a real number. In this paper we study the following nonlinear parametric Dirichlet problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u = f(z, u, \lambda) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The aim of this study is to establish the existence, nonexistence and multiplicity of positive smooth solutions of (P_λ) as the parameter λ varies over $]0, +\infty[$ and when the reaction term $f(z, x, \lambda)$ exhibits a $(p-1)$-superlinear growth as x goes to $+\infty$. However, we do not employ the usual in such cases Ambrosetti–Rabinowitz condition (*AR*-condition for short). Instead, we use a weaker condition which permits a much slower growth for $x \mapsto f(z, x, \lambda)$ near $+\infty$. Our setting incorporates, as a very special case, equations involving the combined effects of concave and convex nonlinearities. Such problems were studied by Ambrosetti, Brezis and Cerami [2] (semilinear equations, i.e. $p = 2$) and by Garcia Azorero, Manfredi and Peral Alonso [7] and Guo and Zhang [12] (nonlinear equations, i.e. $p \neq 2$; in Guo and Zhang [12] it is assumed that $p \geq 2$). In all the aforementioned works, the reaction term has the form

$$f(x, \lambda) = \lambda |x|^{q-2} x + |x|^{r-2} x, \quad \text{for all} \quad x \in \mathbb{R},\ \lambda > 0, \quad \text{with} \quad 1 < q < p < r < p^*$$

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(recall that $p^* = Np/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$).

Recently, Hu and Papageorgiou [14] extended these results by considering reactions of the form

$$f(z, x, \lambda) = \lambda |x|^{q-2}x + f_0(z, x), \quad \text{for all} \quad x \in \mathbb{R},\ \lambda > 0, \quad \text{with} \quad 1 < q < p,$$

$f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ being a Carathéodory function (i.e., $z \mapsto f_0(z, x)$ is measurable for all $x \in \mathbb{R}$ and $x \mapsto f_0(z, x)$ is continuous for a.a. $z \in \Omega$) with subcritical growth in x and which satisfies the *AR*-condition.

We should mention that there are alternative ways to generalize the *AR*-condition and incorporate more general "superlinear" reactions. For more information in this direction, we refer to the works of Li and Yang [17] and Miyagaki and Souto [19].

Other parametric equations driven by the p-Laplacian were also considered by Brock, Itturiaga and Ubilla [4], Guo [11], Hu and Papageorgiou [13] and Takeuchi [22]. However, their hypotheses preclude $(p-1)$-superlinear terms.

We will prove the following bifurcation-type result: there exists $\lambda^* > 0$ s.t. for all $0 < \lambda < \lambda^*$ problem (P_λ) admits at least two positive smooth solutions; for $\lambda = \lambda^*$ there is at least one positive smooth solution; and for $\lambda > \lambda^*$ there is no positive solution. This holds for both problems with $(p-1)$-sublinear reaction near zero (see Theorem 10 below) and problems with $(p-1)$-linear reaction near zero (see Theorem 13 below). Our approach is variational, based on the critical point theory coupled with suitable truncation techniques.

2. Mathematical background

In this section we recall some basic notions and analytical tools which we will use in the sequel. So, let X be a Banach space and X^* its topological dual. By $\langle \cdot\,, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$ be a functional. A point $x_0 \in X$ is called a *critical point* of φ if $\varphi'(x_0) = 0$. A number $c \in \mathbb{R}$ is a *critical value* of φ if there exists a critical point $x_0 \in X$ of φ, s.t. $\varphi(x_0) = c$.

We say that $\varphi \in C^1(X)$ satisfies the *Cerami condition at level* $c \in \mathbb{R}$ (the C_c-condition, for short), if the following holds: *every sequence* $(x_n) \subset X$*, s.t.*

$$\varphi(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty,$$

admits a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, then we say that φ satisfies the *Cerami condition* (C-condition, for short).

Using this compactness-type condition, we can have the following minimax characterization of certain critical values of a C^1 functional. The result is known as the *mountain pass theorem*.

Theorem 1. *If X is a Banach space, $\varphi \in C^1(X)$, $x_0, x_1 \in X$, $0 < \rho < \|x_1 - x_0\|$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \leq \inf_{\|x-x_0\|=\rho} \varphi(x) = \eta_\rho,$$

and φ satisfies the C_c-condition, where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1], X)\colon \gamma(i) = x_i,\ i = 0, 1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ. Moreover, if $c = \eta_\rho$, then there exists a critical point $x \in X$ s.t. $\varphi(x) = c$ and $\|x - x_0\| = \rho$.

In the study of problem (P_λ), we will use the Sobolev space $W = W_0^{1,p}(\Omega)$, endowed with the norm $\|u\| = \|Du\|_p$, whose dual is the space $W^* = W^{-1,p'}(\Omega)$ $(1/p + 1/p' = 1)$. We will also use the space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega})\colon u(z) = 0 \text{ for all } z \in \partial\Omega\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega})\colon u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int}(C_+) = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here $n(z)$ denotes the outward unit normal to $\partial\Omega$ at a point z.

Concerning ordered Banach spaces, in the sequel we will use the following simple fact about them.

Lemma 2. *If X is an ordered Banach space with positive (order) cone C and $x_0 \in \operatorname{int}(C)$, then for every $y \in X$ we can find $t > 0$ s.t. $tx_0 - y \in \operatorname{int}(C)$.*

A nonlinear map $A\colon X \to X^*$ is of *type* $(S)_+$ if, for every sequence $(x_n) \subset X$ s.t.

$$x_n \rightharpoonup x \quad \text{in} \quad X \quad \text{and} \quad \limsup_n \langle A(x_n), x_n - x\rangle \leq 0,$$

we have $x_n \to x$ in X.

Let $A\colon W \to W^*$ be defined by

$$\langle A(u), v\rangle = \int_\Omega |Du|^{p-2} Du \cdot Dv\, dz \quad \text{for all} \quad u, v \in W_0^{1,p}(\Omega). \tag{1}$$

We have the following result (see, for example, Papageorgiou and Kyritsi [20]).

Proposition 3. *The map* $A\colon W \to W^*$ *defined by* (1) *is continuous, strictly monotone* (*hence maximal monotone too*) *and of type* $(S)_+$.

Next, let us recall some basic facts about the spectrum of the negative Dirichlet p-Laplacian. Let $m \in L^\infty(\Omega)_+$, $m \neq 0$ and consider the following nonlinear weighted eigenvalue problem:

$$\begin{cases} -\Delta_p u = \hat{\lambda} m(z)|u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

By an *eigenvalue* of (2) we mean a number $\hat{\lambda}(m) \in \mathbb{R}$ s.t. problem (2) has a nontrivial solution $u \in W$. Nonlinear regularity theory (see, for example, Papageorgiou and Kyritsi [20], pp. 311–312) implies that $u \in C_0^1(\overline{\Omega})$. We know that (2) has a smallest eigenvalue $\hat{\lambda}_1(m) > 0$, which is simple and isolated. Moreover, the following variational characterization is available:

$$\hat{\lambda}_1(m) = \min_{u \in W \setminus \{0\}} \frac{\|Du\|_p^p}{\int_\Omega m(z)|u|^p\, dz}. \tag{3}$$

The minimum in (3) is attained on the one-dimensional eigenspace of $\hat{\lambda}_1(m)$. Note that, if $m, m' \in L^\infty(\Omega)_+ \setminus \{0\}$, $m \neq m'$ and $m \leq m'$, then because of (3) we see that $\hat{\lambda}_1(m) > \hat{\lambda}_1(m')$. If $m = 1$, we simply write $\hat{\lambda}_1$ for $\hat{\lambda}_1(1)$. Let $\hat{u}_1 \in C_0^1(\overline{\Omega})$ be the L^p-normalized eigenfunction corresponding to $\hat{\lambda}_1$. It is clear from (3) that $\hat{u}_1$ does not change sign, and so we may assume $\hat{u}_1 \in C_+$. In fact the nonlinear maximum principle of Vázquez [23] implies that $\hat{u}_1 \in \operatorname{int}(C_+)$. Every eigenfunction u corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$ is necessarily *nodal* (i.e., sign changing).

Finally, in what follows we denote by $|\cdot|_N$ the Lebesgue measure on $\mathbb{R}^N$. For all $x \in \mathbb{R}$, we set

$$x^\pm = \max\{\pm x, 0\}.$$

3. Problems with concave and convex nonlinearities

In this section, we consider problems with reactions which are concave (i.e. $(p-1)$-sublinear) near zero and convex (i.e. $(p-1)$-superlinear) near $+\infty$. More precisely, the hypotheses on $f(z, x, \lambda)$ are the following (by p^* we denote the Sobolev critical exponent, defined as in Introduction):

H $f\colon \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a Carathéodory function s.t. $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$ and all $\lambda > 0$. We set

$$F(z, x, \lambda) = \int_0^x f(z, s, \lambda)\, ds \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in \mathbb{R},\ \lambda > 0$$

and assume:

(i) $f(z, x, \lambda) \leq a(z, \lambda) + c|x|^{r-1}$ for a.a. $z \in \Omega$ and all $x \in \mathbb{R}$, $\lambda > 0$, with $p < r < p^*$ and $a(\,\cdot\,, \lambda) \in L^\infty(\Omega)_+$ s.t. the function $\lambda \mapsto \|a(\,\cdot\,, \lambda)\|_\infty$ is bounded on bounded sets and goes to 0 as $\lambda \to 0^+$, $c > 0$;
(ii) for all $\lambda > 0$

$$\lim_{x\to+\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty \quad \text{uniformly for a.a.} \quad z \in \Omega,$$

and there exist $\tau \in](r - p)\max\{1, N/p\}, p^*[$ and, for all bounded $I \subset \mathbb{R}_0^+$, a real number $\beta_0 > 0$ s.t.

$$\liminf_{x\to+\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{x^\tau} \geq \beta_0 \quad \text{for all} \quad \lambda \in I; \tag{4}$$

(iii) there exist $\delta_0 > 0$, $\mu \in]1, p[$ and $\eta_0 > 0$ s.t.

$$f(z, x, \lambda) \geq \eta_0 x^{\mu-1} \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in [0, \delta_0],\ \lambda > 0;$$

(iv) for a.a. $z \in \Omega$ and all $x \geq 0$ the function $\lambda \mapsto f(z, x, \lambda)$ is increasing, for all $\lambda > \lambda' > 0$, $s > 0$ there exists $\mu_s > 0$ s.t.

$$f(z, x, \lambda) - f(z, x, \lambda') \geq \mu_s \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \geq s$$

and for all compact $K \subset \mathbb{R}_0^+$

$$\lim_{\lambda\to+\infty} f(z, x, \lambda) = +\infty \quad \text{uniformly for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in K;$$

(v) for all $\xi > 0$ and every bounded interval $I \subset \mathbb{R}_0^+$, we can find $\sigma_\xi^I > 0$ s.t. the function $x \mapsto f(z, x, \lambda) + \sigma_\xi^I x^{p-1}$ is nondecreasing on $[0, \xi]$ for a.a. $z \in \Omega$ and all $\lambda \in I$.

REMARK 4. Since we are interested in positive solutions and hypotheses **H** (ii)–(v) concern only the positive semiaxis $\mathbb{R}^+$, by truncating things if necessary, we may (and will) assume that $f(z, x, \lambda) = 0$ for a.a. $z \in \Omega$ and all $x \leq 0$, $\lambda > 0$. Hypothesis **H** (i) imposes a growth condition only from above, since from below the other hypotheses imply that for every $\lambda > 0$ we can find $\xi^* > 0$ s.t. $f(z,x,\lambda) \geq -\xi^*$ for a.a. $z \in \Omega$, all $x \geq 0$. Indeed, from **H** (ii) we see that for $x > 0$ large, say for $x \geq M > 0$, we have $f(z, x, \lambda) \geq 0$ for a.a. $z \in \Omega$. Similarly, hypothesis **H** (iii) implies that $f(z, x, \lambda) \geq 0$ for a.a. $z \in \Omega$, all $x \in [0, \delta_0]$. Finally, for $x \in [\delta_0, M]$ we use **H** (v) and obtain the required bound from below. Hypothesis **H** (ii) classifies problem (P_λ) as *p-superlinear*, since it implies that near ∞ the potential function $x \mapsto F(z, x, \lambda)$ grows faster than x^p. Evidently, this is the case if $x \mapsto f(z, x, \lambda)$ is $(p - 1)$-superlinear near $+\infty$, i.e.

$$\lim_{x\to+\infty} \frac{f(z, x, \lambda)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a.} \quad z \in \Omega \quad \text{and all} \quad \lambda > 0.$$

In the literature, such problems are usually studied using the *AR*-condition. We recall that f satisfies the (unilateral) *AR-condition* uniformly in $\lambda > 0$, if there exist $M > 0$, $\tau > p$ s.t.

$$(5)\qquad 0 < \tau F(z, x, \lambda) \leq f(z, x, \lambda)x \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \geq M,\ \lambda > 0.$$

Integrating (5), we obtain the weaker condition

$$(6)\qquad c_1 x^\tau \leq F(z, x, \lambda) \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \geq M,\ \lambda > 0 \quad (c_1 > 0).$$

Clearly (6) implies the much weaker condition

$$(7)\qquad \lim_{x \to +\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty \quad \text{uniformly for a.a.} \quad z \in \Omega \quad \text{and all} \quad \lambda > 0.$$

Here, instead of the *AR*-condition (5), we employ the more general conditions (7) and (4). Similar assumptions can be found in Costa and Magalhães [5] and Fei [6]. Other ways to relax the *AR*-condition in the study of p-superlinear problems can be found in the papers of Jeanjean [15], Miyagaki and Souto [19] and Schechter and Zou [21]. Finally, note that hypothesis **H** (iii) implies that $x \mapsto F(z, x, \lambda)$ is p-sublinear near zero. Therefore hypotheses **H** correspond to problems with *concave and convex nonlinearities*.

EXAMPLE 5. The following functions $f_i \colon \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ $(i = 1, 2, 3)$ satisfy hypotheses **H**:

$$f_1(x, \lambda) = \lambda x^{q-1} + x^{r-1} \quad (1 < q < p < r < p^*),$$

$$f_2(x, \lambda) = \lambda x^{q-1} + x^{p-1}\left(\ln(1 + x) + \frac{1}{p}\frac{x}{1 + x}\right) \quad (1 < q < p),$$

$$f_3(x, \lambda) = \begin{cases} \lambda x^{q-1} & \text{if} \quad 0 \leq x \leq 1, \\ p\lambda x^{p-1}\left(\ln(x) + \dfrac{1}{p}\right) & \text{if} \quad x > 1, \end{cases} \qquad (1 < q < p).$$

Of course, we set $f_i(x, \lambda) = 0$ for all $x \leq 0$, $\lambda > 0$ and for $i = 1, 2, 3$. Note that $f_1(x, \lambda)$ is the reaction term used by Ambrosetti, Brezis and Cerami [2] (for $p = 2$), by Garcia Azorero, Manfredi and Peral Alonso [7] (for $p > 1$) and by Guo and Zhang [12] (for $p \geq 2$). Functions $f_2(x, \lambda)$ and $f_3(x, \lambda)$ do not satisfy the *AR*-condition. So, our work generalizes significantly those in [7] and [12].

For all $\lambda > 0$ and $u \in W$, we denote

$$(8)\qquad N_f^\lambda(u)(z) = f(z, u(z), \lambda) \quad \text{for a.a.} \quad z \in \Omega.$$

By a (*weak*) *solution* of (P_λ) we mean a function $u \in W$ s.t.

$$A(u) = N_f^\lambda(u) \quad \text{in} \quad W^*,$$

that is,

$$\int_\Omega |Du|^{p-2} Du \cdot Dv \, dz = \int_\Omega f(z, u, \lambda) v \, dz \quad \text{for all} \quad v \in W.$$

We say that u is *positive* if $u(z) > 0$ for a.a. $z \in \Omega$. Set

$$\mathcal{P} = \{\lambda \in \mathbb{R}_0^+ : (P_\lambda) \text{ has a positive solution}\}.$$

The following Propositions illustrate the properties of the set $\mathcal{P}$.

Proposition 6. *If hypotheses* **H** *hold, then* $\mathcal{P} \neq \emptyset$ *and for all* $\lambda \in \mathcal{P}$, $\mu \in]0, \lambda[$ *we have* $\mu \in \mathcal{P}$.

Proof. Let $e \in W \setminus \{0\}$, $e \geq 0$ be the unique solution of the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_p e = 1 & \text{in} \quad \Omega, \\ e = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{9}$$

Nonlinear regularity theory (see [20]) and the nonlinear maximum principle (see Vázquez [23]) imply that $e \in \mathrm{int}(C_+)$.

Claim. *There exists* $\tilde{\lambda} > 0$ *s.t., for all* $\lambda \in]0, \tilde{\lambda}[$, *we can find* $\tilde{\xi} > 0$ *s.t.*

$$\|a(\,\cdot\,, \lambda)\|_\infty + c(\tilde{\xi}\|e\|_\infty)^{r-1} < \tilde{\xi}^{p-1} \quad (c > 0 \text{ as in } \mathbf{H} \text{ (i)}). \tag{10}$$

We argue by contradiction. So, suppose we can find a sequence $(\lambda_n) \subset \mathbb{R}_0^+$ s.t. $\lambda_n \to 0$ and

$$\xi^{p-1} \leq \|a(\,\cdot\,, \lambda_n)\|_\infty + c(\xi\|e\|_\infty)^{r-1} \quad \text{for all} \quad n \in \mathbb{N},\ \xi > 0.$$

Passing to the limit as $n \to \infty$ and using hypothesis **H** (i), we obtain

$$1 \leq c\xi^{r-p}\|e\|_\infty^{r-1} \quad \text{for all} \quad \xi > 0.$$

Since $r > p$, letting $\xi \to 0^+$ we reach a contradiction. This proves the claim.

Now, we fix $\lambda \in]0, \tilde{\lambda}[$. Set $\tilde{u} = \tilde{\xi} e \in \mathrm{int}(C_+)$. We have

$$A(\tilde{u}) = \tilde{\xi}^{p-1} \quad \text{(see (9))},$$

which implies

$$A(\tilde{u}) \geq N_f^\lambda(\tilde{u}) \quad \text{in} \quad W^* \quad \text{(see (10) and } \mathbf{H} \text{ (i))}, \tag{11}$$

therefore $\tilde{u}$ is an upper solution for problem (P_λ). We consider the following truncation of $f(z, x, \lambda)$:

(12)
$$\tilde{f}(z, x, \lambda) = \begin{cases} f(z, x, \lambda) & \text{if} \quad x < \tilde{u}(z), \\ f(z, \tilde{u}(z), \lambda) & \text{if} \quad x \geq \tilde{u}(z), \end{cases} \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},\ \lambda \in]0, \tilde{\lambda}[.$$

Evidently, $(z, x) \mapsto \tilde{f}(z, x, \lambda)$ is a Carathéodory function. We set

$$\tilde{F}(z, x, \lambda) = \int_0^x \tilde{f}(z, s, \lambda)\, ds$$

and consider the functional $\tilde{\varphi}_\lambda \colon W \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \tilde{F}(z, u, \lambda)\, dz \quad \text{for all} \quad u \in W.$$

It is clear from (12) that $\tilde{\varphi}_\lambda \in C^1(W)$ is coercive. Also, exploiting the compact embedding of W into $L^r(\Omega)$ (by the Sobolev embedding theorem), we can easily check that $\tilde{\varphi}_\lambda$ is sequentially weakly l.s.c. Thus, by the Weierstrass theorem, we can find $u_0 \in W$ s.t.

$$\tilde{\varphi}_\lambda(u_0) = \inf_{u \in W} \tilde{\varphi}_\lambda(u) = \tilde{m}_\lambda. \tag{13}$$

Let $\delta_0 > 0$ be as postulated in hypothesis **H** (iii) and let $t \in]0, 1[$ be s.t.

$$0 \leq t\hat{u}_1(z) \leq \min\{\tilde{u}(z), \delta_0\} \quad \text{for all} \quad z \in \overline{\Omega}$$

(recall that $\tilde{u}, \hat{u}_1 \in \mathrm{int}(C_+)$ and use Lemma 2). Then, by virtue of hypothesis **H** (iii), we have

$$F(z, t\hat{u}_1(z), \lambda) \geq \frac{\eta_0}{\mu}(t\hat{u}_1(z))^\mu \quad \text{for a.a.} \quad z \in \Omega. \tag{14}$$

So, we get

$$\begin{aligned} \tilde{\varphi}_\lambda(t\hat{u}_1) &= \frac{t^p}{p}\|D\hat{u}_1\|_p^p - \int_\Omega F(z, t\hat{u}_1, \lambda)\, dz \quad \text{(see (12) and (14))} \\ &\leq t^\mu\left[\frac{t^{p-\mu}}{p}\hat{\lambda}_1 - \frac{\eta_0}{\mu}\|\hat{u}_1\|_\mu^\mu\right] \quad \text{(see (3), (14) and recall } \|\hat{u}_1\|_p = 1). \end{aligned}$$

Since $\mu < p$ (see **H** (iii)), choosing $t \in]0, 1[$ even smaller if necessary, from the inequality above we infer that

$$\tilde{\varphi}_\lambda(t\hat{u}_1) < 0,$$

which in turn implies

$$\tilde{m}_\lambda < 0 = \tilde{\varphi}_\lambda(0).$$

So, by (13) $u_0 \neq 0$.

From (13) we deduce that u_0 is a critical point of $\tilde{\varphi}_\lambda$, that is,

$$(15) \qquad A(u_0) = N^\lambda_{\tilde{f}}(u_0) \quad \text{in} \quad W^* \quad (N^\lambda_{\tilde{f}} \text{ defined as in (8), with } \tilde{f} \text{ instead of } f).$$

On (15) we act with $u_0^- \in W$ and we obtain

$$\|Du_0^-\|_p = 0 \quad \text{(see (12))},$$

i.e. $u_0 \geq 0$ a.e. in Ω.

Also, on (15) we act with $(u_0 - \tilde{u})^+ \in W$. Then,

$$\begin{aligned}\langle A(u_0), (u_0 - \tilde{u})^+\rangle &= \int_\Omega \tilde{f}(z, u_0, \lambda)(u_0 - \tilde{u})^+ \, dz \\ &= \int_\Omega f(z, \tilde{u}, \lambda)(u_0 - \tilde{u})^+ \, dz \quad \text{(see (12))} \\ &\leq \langle A(\tilde{u}), (u_0 - \tilde{u})^+\rangle \quad \text{(see (11))},\end{aligned}$$

that is,

$$\begin{aligned}\langle A(u_0) - A(\tilde{u}), (u_0 - \tilde{u})^+\rangle &= \int_{\{u_0 > \tilde{u}\}} (|Du_0|^{p-2} Du_0 - |D\tilde{u}|^{p-2} D\tilde{u}) \cdot (Du_0 - D\tilde{u}) \, dz \\ &\leq 0.\end{aligned}$$

So we have

$$|\{u_0 > \tilde{u}\}|_N = 0,$$

i.e. $u_0 \leq \tilde{u}$. So (15) becomes

$$A(u_0) = N^\lambda_f(u_0) \quad \text{in} \quad W^*.$$

We have proved that $u_0 \in W \setminus \{0\}$, $0 \leq u_0 \leq \tilde{u}$ and u_0 solves problem (P_λ). As before, nonlinear regularity theory (see [20]) assures that $u_0 \in C_+ \setminus \{0\}$. Set $\xi = \|u_0\|_\infty$, $I =]0, \tilde{\lambda}[$ and find $\tilde{\sigma} = \sigma^I_\xi$ as in hypothesis **H** (v). We have

$$-\Delta_p u_0(z) + \tilde{\sigma} u_0(z)^{p-1} = f(z, u_0(z), \lambda) + \tilde{\sigma} u_0(z)^{p-1} \geq 0 \quad \text{for a.a.} \quad z \in \Omega,$$

so

$$\Delta_p u_0(z) \leq \tilde{\sigma} u_0(z)^{p-1} \quad \text{for a.a.} \quad z \in \Omega,$$

hence $u_0 \in \operatorname{int}(C_+)$ (see [23]). Thus, u_0 is a smooth positive solution of (P_λ), in particular $\lambda \in \mathcal{P}$. Therefore $]0, \tilde{\lambda}[\subseteq \mathcal{P}$, in particular $\mathcal{P} \neq \emptyset$.

Next, let $\lambda \in \mathcal{P}$ and $0 < \mu < \lambda$. We can find a positive solution $u_\lambda \in \mathrm{int}(C_+)$ for problem (P_λ). By hypothesis **H** (iv) we have

$$A(u_\lambda) = N_f^\lambda(u_\lambda) \geq N_f^\mu(u_\lambda) \quad \text{in} \quad W^*, \tag{16}$$

therefore u_λ is an upper solution for problem (P_μ). We truncate $x \mapsto f(z, x, \lambda)$ at $u_\lambda(z)$ and we argue as above. Via the direct method (using this time (16) instead of (11)), we produce a positive solution $u_\mu \in \mathrm{int}(C_+)$ for problem (P_μ), s.t. $0 \leq u_\mu \leq u_\lambda$ in $\overline{\Omega}$. Therefore, $\mu \in \mathcal{P}$. □

Denote

$$\lambda^* = \sup \mathcal{P}.$$

Proposition 7. *If hypotheses **H** hold, then $\lambda^* < +\infty$.*

Proof. Hypotheses **H** (ii), (iii) and (iv) imply that we can find $\bar{\lambda} > 0$ large s.t.

$$f(z, x, \bar{\lambda}) \geq \hat{\lambda}_1 x^{p-1} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \geq 0. \tag{17}$$

To see (17) note that by choosing $\delta_0 > 0$ even smaller if necessary, from **H** (iii) we have

$$f(z, x, \lambda) \geq \hat{\lambda}_1 x^{p-1} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \in [0, \delta_0].$$

Also, from hypothesis **H** (ii) we see that we can find $M > 0$ large enough s.t.

$$f(z, x, \lambda) \geq \hat{\lambda}_1 x^{p-1} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \geq M.$$

Finally, invoking **H** (v), we infer that for all $\lambda > 0$ big, we have

$$f(z, x, \lambda) \geq \hat{\lambda}_1 M^{p-1} \geq \hat{\lambda}_1 x^{p-1} \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all} \quad x \in [\delta_0, M].$$

From these estimates we have (17) for $\lambda > 0$ big.

We will prove that $\lambda^* \leq \bar{\lambda}$, arguing by contradiction. So, let $\lambda > \bar{\lambda}$ and suppose that problem (P_λ) has a nontrivial positive solution $u_\lambda \in W$. As before, we obtain $u_\lambda \in \mathrm{int}(C_+)$. By virtue of Lemma 2, we can find $t > 0$ s.t.

$$t\hat{u}_1(z) \leq u_\lambda(z) \quad \text{for all} \quad z \in \overline{\Omega}.$$

Let $t > 0$ be the largest such positive real number. Let $\xi = \|u_\lambda\|_\infty$, $I = [\bar{\lambda}, \lambda]$ and choose $\bar{\sigma} = \sigma_\xi^I$ as in hypothesis **H** (v). We have

$$\begin{aligned}
&-\Delta_p u_\lambda + \bar{\sigma} u_\lambda^{p-1} \\
&= f(z, u_\lambda, \lambda) + \bar{\sigma} u_\lambda^{p-1} \\
&= f(z, u_\lambda, \bar{\lambda}) + \bar{\sigma} u_\lambda^{p-1} + \theta^*(z) \quad (\text{we set } \theta^*(z) = f(z, u_\lambda, \lambda) - f(z, u_\lambda, \bar{\lambda})) \\
&\geq \hat{\lambda}_1 u_\lambda^{p-1} + \bar{\sigma} u_\lambda^{p-1} + \theta^*(z) \quad (\text{see } (17)) \\
&\geq \hat{\lambda}_1 (t\hat{u}_1)^{p-1} + \bar{\sigma}(t\hat{u}_1)^{p-1} + \theta^*(z) \quad (\text{recall } t\hat{u}_1 \leq u_\lambda) \\
&= -\Delta_p(t\hat{u}_1) + \bar{\sigma}(t\hat{u}_1)^{p-1} + \theta^*(z).
\end{aligned}$$

Since $u_\lambda \in \operatorname{int}(C_+)$, using hypothesis **H** (iv), we see that for every compact $K \subset \Omega$ we can find $\mu_K > 0$ s.t.

$$\theta^*(z) \geq \mu_K \quad \text{for a.a.} \quad z \in K.$$

Then, from Proposition 2.6 of Arcoya and Ruiz [3], we infer that $u_\lambda - t\hat{u}_1 \in \operatorname{int}(C_+)$, which contradicts the maximality of $t > 0$.

This proves that for $\lambda > \bar{\lambda}$ problem (P_λ) has no nontrivial positive solution in W and so $\lambda^* \leq \bar{\lambda}$, in particular $\lambda^* < +\infty$. □

Proposition 8. *If hypotheses **H** hold, then $\lambda^* \in \mathcal{P}$ and so $\mathcal{P} =]0, \lambda^*]$.*

Proof. Let $(\lambda_n) \subset]0, \lambda^*[\subseteq \mathcal{P}$ be an increasing sequence s.t. $\lambda_n \to \lambda^*$. To each λ_n there corresponds a positive smooth solution $u_n = u_{\lambda_n} \in \operatorname{int}(C_+)$ for problem (P_{λ_n}). For all $m > n \geq 1$ we have

$$A(u_m) = N_f^{\lambda_m}(u_m) \geq N_f^{\lambda_n}(u_m) \quad \text{in} \quad W^* \quad (\text{see hypothesis } \mathbf{H} \text{ (iv)}). \tag{18}$$

Truncating $x \mapsto f(z, x, \lambda_n)$ at $u_m(z)$ and reasoning as in the proof of Proposition 6, using the direct method and (18) we obtain a smooth positive solution for (P_{λ_n}) with values in $[0, u_m(z)]$, with negative energy. So, without any loss of generality, we may assume that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all} \quad n \in \mathbb{N}, \tag{19}$$

with

$$\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega F(z, u, \lambda)\, dz \quad \text{for all} \quad \lambda > 0,\ u \in W.$$

Also, we have

$$A(u_n) = N_f^{\lambda_n}(u_n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{20}$$

From (19) we have

(21) $$\|Du_n\|_p^p - \int_\Omega pF(z, u_n, \lambda_n)\, dz < 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Acting on (20) with $u_n \in W$, we obtain

(22) $$\|Du_n\|_p^p - \int_\Omega f(z, u_n, \lambda_n)u_n\, dz = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Subtracting (22) from (21), we get

(23) $$\int_\Omega [f(z, u_n, \lambda_n)u_n - pF(z, u_n, \lambda_n)]\, dz < 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Hypotheses **H** (i), (ii) imply that we can find $\beta_1 \in]0, \beta_0[$ and $c_2 > 0$ s.t.

(24) $$\beta_1 x^\tau - c_2 \le f(z, x, \lambda)x - pF(z, x, \lambda) \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge 0,\ \lambda \in]0, \lambda^*].$$

Using (24) in (23), we see that

(25) $$(u_n) \quad \text{is bounded in} \quad L^\tau(\Omega).$$

Claim. *There exists $u^* \in W$ s.t., up to a subsequence,*

(26) $$u_n \rightharpoonup u^* \text{ in } W \quad \text{and} \quad u_n \to u^* \text{ in } L^r(\Omega) \quad \text{as} \quad n \to \infty.$$

First, suppose that $N \neq p$. From hypothesis **H** (ii) it is clear that we can always assume $\tau \le r < p^*$. So, we can find $t \in [0, 1[$ s.t.

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*} \quad (\text{recall that } p^* = +\infty \text{ if } N < p).$$

From the interpolation inequality (see, for example, Gasiński and Papageorgiou [8], p. 905) we have

$$\|u_n\|_r \le \|u_n\|_\tau^{1-t} \|u_n\|_{p^*}^t \quad \text{for all} \quad n \in \mathbb{N},$$

which (together with (25) and the Sobolev embedding theorem) implies

(27) $$\|u_n\|_r^r \le c_3 \|Du_n\|_p^{tr} \quad \text{for all} \quad n \in \mathbb{N} \quad (c_3 > 0).$$

From hypothesis **H** (i) we have

(28) $$f(z, u_n(z), \lambda_n)u_n(z) \le c_4(1 + |u_n(z)|^r) \quad \text{for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N} \ (c_4 > 0).$$

From (20), we have for all $n \in \mathbb{N}$ and some $c_5, c_6 > 0$

$$\begin{aligned}\|Du_n\|_p^p &= \int_\Omega f(z, u_n, \lambda_n)u_n\, dz \\ &\leq c_5(1 + \|u_n\|_r^r) \quad \text{(see (28))} \\ &\leq c_6(1 + \|Du_n\|_p^{tr}) \quad \text{(see (27))}.\end{aligned}$$

The restriction on τ in hypothesis **H** (ii) implies that $tr < p$. So, from the inequality above we infer that (u_n) is bounded in W and we can find $u^* \in W$ satisfying (26).

If $N = p$, then by the Sobolev theorem W is (compactly) embedded in $L^\eta(\Omega)$ for all $\eta \in [1, +\infty[$ (see, for example, Gasiński and Papageorgiou [8], p. 222) while now $p^* = +\infty$. So, in the above argument, we replace p^* by some $\eta > r$ large enough s.t.

$$tr = \frac{\eta(r - \tau)}{\eta - \tau} < p \quad \text{(see } \mathbf{H} \text{ (ii))}.$$

Then, again we deduce that (u_n) is bounded in W and (26) holds. So, the Claim is proved.

On (20) we act with $u_n - u^* \in W$ and we pass to the limit as $n \to \infty$. We obtain

$$\lim_n \langle A(u_n), u_n - u^*\rangle = 0 \quad \text{(see (26))},$$

which implies

$$u_n \to u^* \quad \text{in} \quad W \quad \text{(see Proposition 3)}. \tag{29}$$

Therefore, if on (20) we pass to the limit as $n \to \infty$ and use (29), then

$$A(u^*) = N_f^{\lambda^*}(u^*),$$

i.e. $u^* \in C_+$ (by nonlinear regularity theory) and it solves (P_{λ^*}).

We need to show that $u^* \neq 0$. We argue by contradiction. So, suppose $u^* = 0$ and consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_p w = \eta_0 (w^+)^{\mu - 1} & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{30}$$

(see **H** (iii)). Since $\mu < p$, the energy functional for (30), defined by

$$\psi(w) = \frac{1}{p}\|Dw\|_p^p - \frac{\eta_0}{\mu}\|w^+\|_\mu^\mu \quad \text{for all} \quad w \in W,$$

is coercive and of course it is sequentially weakly l.s.c. Hence, by the Weierstrass theorem, we can find a minimizer $w \in W$ of ψ. Note that, since $\mu < p$, we have

$$\psi(w) = \inf_{u \in W} \psi(u) < 0 = \psi(0),$$

so $w \in W \setminus \{0\}$. Then

$$A(w) = \eta_0 (w^+)^{\mu-1} \quad \text{in} \quad W^*,$$

which implies $w \in \operatorname{int}(C_+)$ and it solves (30).

From Ladyzhenskaya and Uraltseva [16] (p. 286, see also [20], p. 311) we can find $\hat{M} > 0$ s.t. $\|u\|_\infty \le \hat{M}$ for all $n \ge 1$. Then we can apply Theorem 1 of Lieberman [18] (see also [20], p. 312) and find $\alpha \in]0,1[$ and $c_7 > 0$ s.t.

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_7 \quad \text{for all} \quad n \in \mathbb{N}.$$

Recalling that $C_0^{1,\alpha}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, we may assume that $u_n \to u^* = 0$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ s.t.

$$(31) \qquad 0 \le u_n(z) \le \delta_0 \quad \text{for all} \quad z \in \overline{\Omega} \quad \text{and all} \quad n \ge n_0.$$

Fix $n \ge n_0$ and choose $t_n > 0$ s.t.

$$t_n w(z) \le u_n(z) \quad \text{for all} \quad z \in \overline{\Omega} \quad \text{(recall } u_n \in \operatorname{int}(C_+) \text{ and use Lemma 2).}$$

Let t_n be the biggest such number and suppose that $t_n \in]0,1[$. Set $\xi = \|u_n\|_\infty$, $I =]0, \lambda^*]$ and let $\sigma_n = \sigma_\xi^I$ be as in hypothesis **H** (v). Then

$$\begin{aligned}
&-\Delta_p(t_n w) + \sigma_n (t_n w)^{p-1} \\
&= t_n^{p-1} \eta_0 w^{\mu-1} + \sigma_n (t_n w)^{p-1} \quad \text{(see (30))} \\
&< \eta_0 (t_n w)^{\mu-1} + \sigma_n (t_n w)^{p-1} \quad \text{(recall that } t_n \in]0,1[\text{ and } \mu < p) \\
&\le \eta_0 u_n^{\mu-1} + \sigma_n u_n^{p-1} \quad \text{(since } t_n w \le u_n) \\
&\le f(z, u_n, \lambda_n) + \sigma_n u_n^{p-1} \quad \text{(since } n \ge n_0, \text{ see (31) and hypothesis } \mathbf{H} \text{ (iii))} \\
&= -\Delta_p u_n + \sigma_n u_n^{p-1}.
\end{aligned}$$

Note that if we set

$$h_1(z) = t_n^{p-1} \eta_0 w^{\mu-1} + \sigma_n (t_n w)^{p-1}, \quad h_2(z) = \eta_0 u_n^{\mu-1} + \sigma_n u_n^{p-1},$$

then $h_1, h_2 \in C(\overline{\Omega})$ and

$$h_1(z) < h_2(z) \quad \text{for all} \quad z \in \Omega.$$

Moreover, we have

$$h_2(z) \le f(z, u_n, \lambda_n) + \sigma_n u_n^{p-1} \quad \text{a.e. in} \quad \Omega.$$

Therefore, we can apply Proposition 2.6 of Arcoya and Ruiz [3] (see also Guedda and Veron [10]) and we have

$$u_n - t_n w \in \operatorname{int}(C_+),$$

which contradicts the maximality of t_n. Therefore $t_n \geq 1$ and so we have $w \leq u_n$ for all $n \geq n_0$, hence $w \leq 0$, a contradiction. Thus, $u^* \neq 0$.

As before, by using hypothesis **H** (v) and the nonlinear maximum principle of Vázquez [23], we have $u^* \in \operatorname{int}(C_+)$. So, $\lambda^* \in \mathcal{P}$, i.e., $\mathcal{P} =]0, \lambda^*]$. $\square$

Proposition 9. *If hypotheses **H** hold, then for all $\lambda \in]0, \lambda^*[$ problem (P_λ) has at least two positive smooth solutions $u_0, \hat{u} \in \operatorname{int}(C_+)$ s.t. $u_0 \leq \hat{u}$ in $\overline{\Omega}$ and $u_0 \neq \hat{u}$.*

Proof. From Proposition 8, we know that $\lambda^* \in \mathcal{P}$, i.e., there is a solution $u^* \in \operatorname{int}(C_+)$ for problem (P_{λ^*}). We have

$$A(u^*) = N_f^{\lambda^*}(u^*) \geq N_f^{\lambda}(u^*) \quad \text{in} \quad W^* \quad (\text{see } \mathbf{H} \text{ (iv)}), \tag{32}$$

so u^* is an upper solution of (P_λ) when $\lambda \in]0, \lambda^*[$. In what follows $\lambda \in]0, \lambda^*[$. We truncate $x \mapsto f(z, x, \lambda)$ at $u^*(z)$ and, using the direct method and (32), as in the proof of Proposition 6, we obtain a solution $u_0 \in \operatorname{int}(C_+)$ for problem (P_λ), s.t. $0 \leq u_0(z) \leq u^*(z)$ for all $z \in \overline{\Omega}$. For $\xi = \|u^*\|_\infty$ and $I =]0, \lambda^*]$, let $\hat{\sigma} = \sigma_\xi^I$ be as postulated by hypothesis **H** (v). We have

$$\begin{aligned}
&-\Delta_p u_0 + \hat{\sigma} u_0^{p-1} \\
&= f(z, u_0, \lambda) + \hat{\sigma} u_0^{p-1} \\
&= f(z, u_0, \lambda^*) + \hat{\sigma} u_0^{p-1} + \hat{\theta}(z) \quad (\text{we set } \hat{\theta}(z) = f(z, u_0, \lambda) - f(z, u_0, \lambda^*)) \\
&\leq f(z, u^*, \lambda^*) + \hat{\sigma}(u^*)^{p-1} + \hat{\theta}(z) \quad (\text{see } \mathbf{H} \text{ (v) and recall } u_0 \leq u^*) \\
&= -\Delta_p u^* + \hat{\sigma}(u^*)^{p-1} + \hat{\theta}(z).
\end{aligned}$$

By virtue of hypothesis **H** (iv), for every compact $K \subset \Omega$, we have

$$\operatorname{esssup}_K \hat{\theta} < 0.$$

Invoking Proposition 2.6 of Arcoya and Ruiz [3], we have

$$u^* - u_0 \in \operatorname{int}(C_+). \tag{33}$$

We consider the following truncation of $x \mapsto f(z, x, \lambda)$:

(34)

$$g(z,x,\lambda) = \begin{cases} f(z, u_0(z), \lambda) & \text{if} \quad x \leq u_0(z), \\ f(z, x, \lambda) & \text{if} \quad x > u_0(z), \end{cases} \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R},\ \lambda \in]0, \lambda^*[.$$

This is a Carathéodory function. We set

$$G(z, x, \lambda) = \int_0^x g(z, s, \lambda)\, ds \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in \mathbb{R},\ \lambda \in \,]0, \lambda^*[$$

and consider the C^1 functional $\psi_\lambda \colon W \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G(z, u, \lambda)\, dz \quad \text{for all} \quad u \in W.$$

Claim 1. *ψ_λ satisfies the C-condition.*

Let $(u_n) \in W$ be a sequence s.t.

(35) $$|\psi_\lambda(u_n)| \le c_8 \quad \text{for all} \quad n \in \mathbb{N} \quad (c_8 > 0)$$

and

(36) $$\lim_n (1 + \|u_n\|)\psi_\lambda'(u_n) = 0 \quad \text{in} \quad W^*.$$

From (35) we have

(37) $$\|Du_n\|_p^p - \int_\Omega pG(z, u_n, \lambda)\, dz \le pc_8 \quad \text{for all} \quad n \in \mathbb{N}.$$

From (36) we have

(38) $$\left| A(u_n), v\rangle - \int_\Omega g(z, u_n, \lambda) v\, dz \right| \le \varepsilon_n \frac{\|v\|}{1 + \|u_n\|} \quad \text{for all } v \in W,\ n \in \mathbb{N}\ (\varepsilon_n \to 0^+ \text{ as } n \to \infty).$$

In (38) we choose $v = -u_n^- \in W$. Then,

$$\begin{aligned} \|Du_n^-\|_p^p &\le \varepsilon_n + \int_\Omega f(z, u_0, \lambda)(-u_n^-)\, dz \quad \text{(see (34))} \\ &\le c_9(1 + \|Du_n^-\|_p) \quad \text{for some} \quad c_9 > 0 \quad \text{(see } \mathbf{H} \text{ (i))}, \end{aligned}$$

which implies that (u_n^-) is bounded in W.

Next, in (38) we choose $v = u_n^+ \in W$. Then,

(39) $$-\|Du_n^+\|_p^p + \int_\Omega g(z, u_n^+, \lambda)u_n^+\, dz \le \varepsilon_n \quad \text{for all} \quad n \in \mathbb{N}.$$

We add (37) and (39) and use (34) and the boundedness of (u_n^-) to obtain, for all $n \in \mathbb{N}$,

(40) $$\int_\Omega [f(z, u_n^+, \lambda)u_n^+ - pF(z, u_n^+, \lambda)]\, dz \le c_{10} \quad (c_{10} > 0).$$

From (40), using hypothesis **H** (ii) and the interpolation inequality, as in the proof of Proposition 8, we show that (u_n^+) is bounded in W as well. Thus, (u_n) is bounded in W. So, we may assume that there exists $u \in W$ s.t.

$$u_n \rightharpoonup u \quad \text{in} \quad W \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^r(\Omega) \quad \text{as} \quad n \to \infty,$$

from which, using as before Proposition 3, we show that $u_n \to u$ in W (as in the proof of Proposition 8), hence ψ_λ satisfies the C-condition. This proves Claim 1.

Claim 2. *u_0 is a local minimizer of ψ_λ.*

We can always assume that u_0 is the only nontrivial positive solution of problem (P_λ) in the order interval

$$\mathcal{I} = \{u \in W : 0 \leq u(z) \leq u^*(z) \text{ for a.a. } z \in \Omega\},$$

or otherwise we already have a second nontrivial smooth solution and we are done (see also [9]).

We introduce the following truncation of $x \mapsto g(z, x, \lambda)$:

$$\hat{g}(z, x, \lambda) = \begin{cases} f(z, u_0(z), \lambda) & \text{if} \quad x \leq u_0(z), \\ f(z, x, \lambda) & \text{if} \quad u_0(z) < x < u^*(z), \\ f(z, u^*(z), \lambda) & \text{if} \quad x \geq u^*(z), \end{cases} \tag{41}$$

for a.a. $z \in \Omega$ and all $x \in \mathbb{R}$, $\lambda \in \mathbb{R}_0^+$. This is a Carathéodory function. As usual, we set

$$\hat{G}(z, x, \lambda) = \int_0^x \hat{g}(z, s, \lambda)\, ds \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in \mathbb{R},\ \lambda \in \mathbb{R}_0^+$$

and consider the functional $\hat{\psi}_\lambda \in C^1(W)$ given by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \hat{G}(z, u, \lambda)\, dz \quad \text{for all} \quad u \in W.$$

Evidently $\hat{\psi}_\lambda$ is coercive (see (41)) and is as well sequentially weakly l.s.c. So, we can find $\hat{u}_0 \in W$ s.t.

$$\hat{\psi}_\lambda(\hat{u}_0) = \inf_W \hat{\psi}_\lambda,$$

in particular $\hat{u}_0$ is a critical point of $\hat{\psi}\lambda$, i.e.

$$A(\hat{u}_0) = N_{\hat{g}}^\lambda(\hat{u}_0) \quad \text{in} \quad W^* \quad (N_{\hat{g}}^\lambda \text{ defined as in (8)}). \tag{42}$$

On (42) we act with $(u_0 - \hat{u}_0)^+ \in W$. Then

$$\begin{aligned}\langle A(\hat{u}_0), (u_0 - \hat{u}_0)^+\rangle &= \int_\Omega \hat{g}(z, \hat{u}_0, \lambda)(u_0 - \hat{u}_0)^+ \, dz \\ &= \int_\Omega f(z, u_0, \lambda)(u_0 - \hat{u}_0)^+ \, dz \quad (\text{since } u_0 \leq u^*, \text{ see (41)}) \\ &= \langle A(u_0), (u_0 - \hat{u}_0)^+\rangle,\end{aligned}$$

which implies

$$\begin{aligned}\langle A(u_0) - A(\hat{u}_0), (u_0 - \hat{u}_0)^+\rangle &= \int_{\{u_0 > \hat{u}_0\}} (|Du_0|^{p-2} Du_0 - |D\hat{u}_0|^{p-2} D\hat{u}_0) \cdot (Du_0 - D\hat{u}_0) \, dz \\ &= 0.\end{aligned}$$

So

$$|\{u_0 > \hat{u}_0\}|_N = 0,$$

i.e. $u_0 \leq \hat{u}_0$. Also, acting on (42) with $(\hat{u}_0 - u^*)^+ \in W$, we have

$$\begin{aligned}\langle A(\hat{u}_0), (\hat{u}_0 - u^*)^+\rangle &= \int_\Omega \hat{g}(z, \hat{u}_0, \lambda)(\hat{u}_0 - u^*)^+ \, dz \\ &= \int_\Omega f(z, u^*, \lambda)(\hat{u}_0 - u^*)^+ \, dz \quad (\text{see (41) and recall } u_0 \leq u^*) \\ &\leq \langle A(u^*), (\hat{u}_0 - u^*)^+\rangle \quad (\text{see (32)}),\end{aligned}$$

i.e.

$$\begin{aligned}\langle A(\hat{u}_0) - A(u^*), (\hat{u}_0 - u^*)^+\rangle &= \int_{\{\hat{u}_0 > u^*\}} (|D\hat{u}_0|^{p-2} D\hat{u}_0 - |Du^*|^{p-2} Du^*) \cdot (D\hat{u}_0 - Du^*) \, dz \\ &\leq 0.\end{aligned}$$

So

$$|\{\hat{u}_0 > u^*\}|_N = 0,$$

i.e. $\hat{u}_0 \leq u^*$. Hence, (42) becomes

$$A(\hat{u}_0) = N_f^\lambda(\hat{u}_0) \quad \text{in} \quad W^* \quad (\text{see (41) and (34)})$$

and $\hat{u}_0 \in \operatorname{int}(C_+) \cap \mathcal{I}$ is a solution of problem (P_λ). This implies

$$\hat{u}_0 = u_0 \quad (\text{recall that } u_0 \text{ is the only nontrivial solution of } (P_\lambda) \text{ in } \mathcal{I}).$$

Note that

$$\hat{\psi}_\lambda(u) = \psi_\lambda(u) \quad \text{for all} \quad u \in \mathcal{I}.$$

Recall, also, that $u^* - u_0 \in \text{int}(C_+)$ (see (33)) and $u_0 \in \text{int}(C_+)$. Therefore, $\mathcal{I}$ is a neighborhood of u_0 in the topology of $C_0^1(\overline{\Omega})$, and so u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_λ. By virtue of Theorem 1.2 of Garcia Azorero, Manfredi and Peral Alonso [7], it is also a local W-minimizer of ψ_λ. This proves Claim 2.

We may assume that u_0 is an isolated critical point of ψ_λ (otherwise we have a whole sequence of distinct positive smooth solutions converging to u_0). Therefore we can find $\rho \in]0, 1[$ small enough s.t.

$$\psi_\lambda(u_0) < \inf_{\|u-u_0\|=\rho} \psi_\lambda(u) = \eta_\rho \tag{43}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29).

Clearly hypothesis **H** (ii) implies that

$$\lim_{t\to+\infty} \psi_\lambda(t\hat{u}_1) = -\infty. \tag{44}$$

Then, (43), (44) and Claim 1 permit the use of Theorem 1 (the mountain pass theorem). So, we obtain $\hat{u} \in W$ s.t.

$$\psi_\lambda(u_0) < \eta_\rho \leq \psi_\lambda(\hat{u}) \quad \text{(see (43))} \tag{45}$$

and

$$\psi_\lambda'(\hat{u}) = 0. \tag{46}$$

From (45) we have $\hat{u} \neq u_0$. From (46), we have

$$A(\hat{u}) = N_g^\lambda(\hat{u}) \quad \text{in} \quad W^*. \tag{47}$$

Acting on (47) with $(u_0 - \hat{u})^+ \in W$, as before we show that $u_0 \leq \hat{u}$. Hence (47) becomes

$$A(\hat{u}) = N_f^\lambda(\hat{u}) \quad \text{in} \quad W^* \quad \text{(see (34))},$$

so $\hat{u} \in \text{int}(C_+)$ (nonlinear regularity) is a solution of (P_λ). $\square$

Summarizing the situation, we have the following bifurcation-type result for problem (P_λ).

Theorem 10. *If hypotheses **H** hold, then there exists $\lambda^* \in \mathbb{R}_0^+$ s.t.*
(a) *for every $\lambda \in]0, \lambda^*[$ problem (P_λ) has at least two positive smooth solutions $u_0, \hat{u} \in \text{int}(C_+)$ s.t. $u_0 \leq \hat{u}$ in $\overline{\Omega}$ and $u \neq \hat{u}$;*
(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive smooth solution $u^* \in \text{int}(C_+)$;*
(c) *for every $\lambda > \lambda^*$ problem (P_λ) has no positive solution.*

REMARK 11. If $p = 2$ and $0 < \lambda < \lambda^*$, then the two positive solutions $u_0, \hat{u} \in \operatorname{int}(C_+)$ satisfy

$$\hat{u} - u_0 \in \operatorname{int}(C_+).$$

Indeed, if $\xi = \|\hat{u}\|_\infty$ and $I =]0, \lambda^*]$, then we find $\hat{\sigma} = \sigma_\xi^I$ as in hypothesis **H** (v) and we have

$$\begin{aligned} -\Delta(\hat{u} - u_0) + \hat{\sigma}(\hat{u} - u_0) &= f(z, \hat{u}, \lambda) + \hat{\sigma}\hat{u} - f(z, u_0, \lambda) - \hat{\sigma} u_0 \\ &\geq 0 \quad (\text{see } \mathbf{H} \text{ (v)}), \end{aligned}$$

i.e.

$$\Delta(\hat{u} - u_0) \leq \hat{\sigma}(\hat{u} - u_0) \quad \text{a.e. in} \quad \Omega,$$

which implies

$$\hat{u} - u_0 \in \operatorname{int}(C_+) \quad (\text{see Vázquez [23]}).$$

Finally, note that, if $f(z, \ \cdot\ , \lambda) \in C^1(\mathbb{R})$, then by the mean value theorem **H** (v) is automatically true.

4. Problems with $(p-1)$-linear nonlinearities near zero

In the previous section, we examined problems in which the reaction was concave near the origin (see hypothesis **H** (iii)). Here, we consider equations in which $x \mapsto f(z, x, \lambda)$ exhibits $(p-1)$-linear growth near zero. We show that in this case we can still have a bifurcation-type theorem similar to Theorem 10.

The new hypotheses on the nonlinearity $f(z, x, \lambda)$ are the following.
H′ $f : \Omega \times \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a Carathéodory function s.t. $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$ and all $\lambda > 0$. We set

$$F(z, x, \lambda) = \int_0^x f(z, s, \lambda)\, ds \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in \mathbb{R},\ \lambda > 0.$$

Let hypotheses **H′** (i), (ii), (iv), (v) be as **H** (i), (ii), (iv), (v) and

(iii) for all bounded $I \subset \mathbb{R}_0^+$ there exist $\eta_0 \in L^\infty(\Omega)$, $\eta_0(z) \geq \hat{\lambda}_1$ for a.a. $z \in \Omega$, $\eta_0 \not\equiv \hat{\lambda}_1$, and $\eta_1 > 0$ s.t.

$$\begin{aligned} \eta_0(z) &\leq \liminf_{x \to 0^+} \frac{f(z, x, \lambda)}{x^{p-1}} \leq \limsup_{x \to 0^+} \frac{f(z, x, \lambda)}{x^{p-1}} \\ &\leq \eta_1 \quad \text{uniformly for a.a.} \quad z \in \Omega \quad \text{and all} \quad \lambda \in I. \end{aligned}$$

EXAMPLE 12. Let $\eta > \hat{\lambda}_1$, $1 < q < p < r < p^*$. The following function satisfies hypotheses **H′**:

$$f(x, \lambda) = \begin{cases} \lambda x^{r-1} + \eta x^{p-1} & \text{if } 0 \leq x \leq 1, \\ \lambda x^{q-1} + p\eta x^{p-1}\left(\ln(x) + \dfrac{1}{p}\right) & \text{if } x > 1, \end{cases} \quad \text{for a.a. } z \in \Omega \text{ and all } \lambda \in \mathbb{R}_0^+.$$

Again this $(p-1)$-superlinear function (at ∞) does not satisfy the *AR*-condition.

A careful inspection of the proofs in Section 3 reveals that they remain essentially unchanged. The only two parts which need to be modified are the following:
(A) in the proof of Proposition 6, the part where we show that the minimizer u_0 is nontrivial;
(B) in the proof of Proposition 8, the part where we show that $u^* \neq 0$.
First we deal with (A). By virtue of hypothesis $\mathbf{H}'$ (iii), given $\varepsilon > 0$, we can find $\delta > 0$ s.t.

$$F(z, x, \lambda) \geq \frac{1}{p}(\eta_0(z) - \varepsilon)x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta] \text{ and all } \lambda \in]0, \tilde{\lambda}[. \tag{48}$$

Let $t \in]0, 1[$ be s.t.

$$0 \leq t\hat{u}_1(z) \leq \min\{\delta, \tilde{u}(z)\} \quad \text{for all} \quad z \in \overline{\Omega} \quad \text{(see Lemma 2).} \tag{49}$$

Then,

$$\begin{aligned}\tilde{\varphi}_\lambda(t\hat{u}_1) &= \frac{t^p}{p}\|D\hat{u}_1\|_p^p - \int_\Omega F(z, t\hat{u}_1, \lambda)\,dz \quad \text{(see (12) and (49))}\\ &\leq \frac{t^p}{p}\int_\Omega(\hat{\lambda}_1 - \eta_0(z))\hat{u}_1(z)^p\,dz + \frac{t^p}{p}\varepsilon \quad \text{(see (48), (49) and recall } \|\hat{u}_1\|_p = 1).\end{aligned}$$

Since

$$\int_\Omega(\hat{\lambda}_1 - \eta_0(z))\hat{u}_1(z)^p\,dz < 0,$$

by choosing $\varepsilon > 0$ small enough we see that

$$\tilde{\varphi}_\lambda(u_0) \leq \tilde{\varphi}_\lambda(t\hat{u}_1) < 0 \quad \text{(see (13))},$$

i.e. $u_0 \neq 0$.

Next we deal with (B). Again we argue indirectly. So, suppose that $u^* = 0$. Then, $u_n \to 0$ in W (see (29) and in fact, using Theorem 1 of Lieberman [18], we show that $u_n \to 0$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$ (see the proof of Proposition 8). Therefore we can find $n_0 \in \mathbb{N}$ s.t.

$$0 \leq u_n(z) \leq 1 \quad \text{for all} \quad n \geq n_0 \quad \text{and} \quad z \in \overline{\Omega}.$$

Hypotheses $\mathbf{H}'$ (i), (iii) imply that

$$|f(z, x, \lambda)| \leq c_{11}|xt^{p-1}| \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad x \in [0, 1],\ \lambda \in]0, \lambda^*] \quad (c_{11} > 0),$$

which implies

$$|f(z, u_n(z), \lambda_n)| \leq c_{11}|u_n(z)|^{p-1} \quad \text{for a.a.} \quad z \in \Omega \quad \text{and all} \quad n \geq n_0.$$

So, the sequence

$$\left(\frac{N_f^{\lambda_n}(u_n)}{\|u_n\|^{p-1}}\right)$$

is bounded in $L^{p'}(\Omega)$. Hence, passing if necessary to a subsequence, we may assume that

$$\frac{N_f^{\lambda_n}(u_n)}{\|u_n\|^{p-1}} \rightharpoonup h \quad \text{in} \quad L^{p'}(\Omega) \quad \text{as} \quad n \to \infty. \tag{50}$$

Set $y_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$. Then $|y_n| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \rightharpoonup y \text{ in } W \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \quad \text{as} \quad n \to \infty. \tag{51}$$

Recall that

$$A(y_n) = \frac{N_f^{\lambda_n}(u_n)}{\|u_n\|^{p-1}} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{(see (20))}. \tag{52}$$

Acting on (52) with $y_n - y \in W$, passing to the limit as $n \to \infty$ and using (50) and (51), we obtain

$$\lim_n \langle A(y_n), y_n - y\rangle = 0,$$

hence $y_n \to y$ (see Proposition 3). In particular, we have

$$\|y\| = 1 \quad \text{and} \quad y(z) \geq 0 \quad \text{for a.a.} \quad z \in \Omega. \tag{53}$$

Moreover, using hypothesis $\mathbf{H}'$ (iii) and reasoning as in the proof of Theorem 2.8 of [14] (see also [1], proof of Proposition 31), we show that there exists $m \in L^\infty(\Omega)$ s.t.

$$h(z) = m(z)y(z)^{p-1} \quad \text{and} \quad \eta_0(z) \leq m(z) \leq \eta_1 \quad \text{for a.a.} \quad z \in \Omega. \tag{54}$$

So, if in (52) we pass to the limit as $n \to \infty$ and use (53) and (54), then

$$A(y) = m(z)y^{p-1},$$

i.e., $y \in W$ solves the Dirichlet problem

$$\begin{cases} -\Delta_p y = m(z)y^{p-1} & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{55}$$

But, note that

$$\hat{\lambda}_1(m) < \hat{\lambda}_1(\hat{\lambda}_1) = 1 \quad \text{(see (3) and (54))}.$$

So, from (55) it follows that y must be nodal, contradicting (53). This proves that $u^* \neq 0$.

So, we can state the following bifurcation-type theorem.

Theorem 13. *If hypotheses* $\boldsymbol{H}'$ *hold, then there exists* $\lambda^* \in \mathbb{R}_0^+$ *s.t.*
(a) *for every* $\lambda \in]0,\lambda^*[$ *problem* (P_λ) *has at least two positive smooth solutions* $u_0, \hat{u} \in \mathrm{int}(C_+)$ *s.t.* $u_0 \leq \hat{u}$ *in* $\overline{\Omega}$ *and* $u_0 \neq \hat{u}$;
(b) *for* $\lambda = \lambda^*$ *problem* (P_λ) *has at least one positive smooth solution* $u^* \in \mathrm{int}(C_+)$;
(c) *for every* $\lambda > \lambda^*$ *problem* (P_λ) *has no positive solution.*

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