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ON THE S^1-FIBRED NILBOTT TOWER

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Abstract

We shall introduce a notion of S^1-*fibred nilBott tower*. It is an iterated S^1-bundle whose top space is called an S^1-fibred nilBott manifold and the S^1-bundle of each stage realizes a *Seifert construction*. The S^1-fibred nilBott tower is a generalization of *real Bott tower* from the viewpoint of fibration. In this note we shall prove that any S^1-fibred nilBott manifold is *diffeomorphic* to an infranilmanifold. According to the group extension of each stage, there are two classes of S^1-fibred nilBott manifolds which is defined as *finite type* or *infinite type*. We discuss their properties.

Contents

1. Introduction

Let M be a closed aspherical manifold which is the top space of an iterated S^1-bundle over a point:

$$M = M_n \to M_{n-1} \to \cdots \to M_1 \to \{\mathrm{pt}\}. \tag{1.1}$$

Suppose X is the universal covering of M and each X_i is the universal covering of M_i and put $\pi_1(M_i) = \pi_i$ $(i = 1, \dots, n-1)$ and $\pi_1(M) = \pi$.

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DEFINITION 1.1. An S^1-*fibred nilBott tower* is a sequence (1.1) which satisfies I, II and III below. The top space M is said to be an S^1-*fibred nilBott manifold* (*of depth* n).
I. Each M_i is a fiber space over M_{i-1} with fiber S^1.
II. For the group extension

$$1 \to \mathbb{Z} \to \pi_i \to \pi_{i-1} \to 1 \tag{1.2}$$

associated to the fiber space I, there is an equivariant principal bundle:

$$\mathbb{R} \to X_i \xrightarrow{p_i} X_{i-1}. \tag{1.3}$$

III. Each π_i normalizes $\mathbb{R}$.

The purpose of this paper is to prove the following results.

Theorem 1.2. *Suppose that M is an S^1-fibred nilBott manifold.*
(i) *If every cocycle of $H^2_\phi(\pi_{i-1},\mathbb{Z})$ which represents a group extension* (1.2) *is* of finite order, *then M is diffeomorphic to a Riemannian flat manifold.*
(ii) *If there exists a cocycle of $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ which represents a group extension* (1.2) *is* of infinite order, *then M is diffeomorphic to an infranilmanifold. In addition, M cannot be diffeomorphic to any Riemannian flat manifold.*

As a consequence, we have the following classification. (See Proposition 4.1 and Proposition 4.2.)

Proposition 1.3. *The* 3*-dimensional S^1-fibred nilBott manifolds of finite type are those of* $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$.

Proposition 1.4. *Any* 3*-dimensional S^1-fibred nilBott manifold of infinite type is either a Heisenberg nilmanifold* $\mathsf{N}/\Delta(k)$ *or an Heisenberg infranilmanifold* $\mathsf{N}/\Gamma(k)$.

Real Bott manifolds consist of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_3$ among these $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$.
(Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [13]. We quote the notations $\mathcal{G}_i$, $\mathcal{B}_i$ there.)

Masuda and Lee [8] have also proved the above results.

By (1.2) of Definition 1.1, a 3-dimensional S^1-fibred nilBott manifold M gives a group extension:

$$1 \to \mathbb{Z} \to \pi_1(M) \to Q \to 1$$

where Q is the fundamental group of a Klein Bottle K or a torus T^2. Then this group extension gives a 2-cocycle in the group cohomology $H^2_\phi(Q, \mathbb{Z})$ with a homomorphism $\phi\colon Q \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$. Conversely we have shown

Theorem 1.5. *Every cocycle of $H^2_\phi(Q,\mathbb{Z})$ can be realized as a diffeomorphism class of an S^1-fibred nilBott manifold.*

2. Seifert construction

We shall explain the Seifert construction briefly. It is a tool to construct a closed aspherical manifold for a given extension. Let

$$1 \to \Delta \to \pi \xrightarrow{\nu} Q \to 1 \tag{2.1}$$

be a group extension and $\phi\colon Q \to \mathrm{Aut}(\Delta)$ a conjugation function defined by a section $s\colon Q \to \pi$ for the projection ν. Define $f\colon Q\times Q \to \Delta$ by $s(\alpha)s(\beta) = f(\alpha,\beta)s(\alpha\beta)$. Then f defines the group π which is the product $\Delta\times Q$ with the group law:

$$(n,\alpha)(m,\beta) = (n\cdot\phi(\alpha)(m)\cdot f(\alpha,\beta), \alpha\beta). \tag{2.2}$$

$(\forall n, m \in \Delta, \ \forall\alpha,\beta\in Q)$ (cf. [10] for example).

Suppose Δ is a torsionfree finitely generated nilpotent group. By the Mal'cev's *unique existence* theorem, there is a simply connected nilpotent Lie group $\mathcal{N}$ containing Δ as a discrete uniform subgroup. (See [12] for example.) Moreover if Q acts smoothly and properly discontinuously on a contractible smooth manifold W such that the quotient space W/Q is compact, then there is a map $\lambda\colon Q\to \mathrm{Map}(W,\mathcal{N})$ whose images consist of smooth maps of W into $\mathcal{N}$ satisfying:

$$f(\alpha,\beta) = (\bar\phi(\alpha)\circ\lambda(\beta)\circ\alpha^{-1})\cdot\lambda(\alpha)\cdot\lambda(\alpha\beta)^{-1} \quad (\alpha,\beta\in Q), \tag{2.3}$$

here $\bar\phi\colon Q\to\mathrm{Aut}(\mathcal{N})$ is the unique extension of ϕ by Mal'cev's *unique existence* property. We simply write $f=\delta^1\lambda$ for (2.3). And an action of π on $\mathcal{N}\times W$ is obtained by

$$(n,\alpha)(x,w) = (n\cdot\bar\phi(\alpha)(x)\cdot\lambda(\alpha)(\alpha w), \alpha w). \tag{2.4}$$

This action $(\pi, \mathcal{N}\times W)$ is said to be a Seifert construction. (See [5] for details.)

In particular, when Q is a finite group F and $W=\{pt\}$ it follows $\mathrm{Map}(W,\mathcal{N}) = \mathcal{N}$ for which there is a smooth map $\chi\colon F\to\mathcal{N}$ satisfying $f=\delta^1\chi$:

$$f(\alpha,\beta) = \bar\phi(\alpha)(\chi(\beta))\cdot\chi(\alpha)\cdot\chi(\alpha\beta)^{-1} \quad (\alpha,\beta\in F). \tag{2.5}$$

Let $\mathrm{E}(\mathcal{N})$ be a semidirect product $\mathcal{N}\rtimes\mathcal{K}$ with $\mathcal{K}$ be a maximal compact subgroup of $\mathrm{Aut}(\mathcal{N})$. And we can define a discrete faithful representation $\rho\colon\pi\to\mathrm{E}(\mathcal{N})$ by

$$\rho((n,\alpha)) = (n\cdot\chi(\alpha), \mu(\chi(\alpha)^{-1})\circ\bar\phi(\alpha)), \tag{2.6}$$

(here μ is a conjugation map). Then the action of π on $\mathcal{N}$ is defined by

$$(n,\alpha)(x) = \rho((n,\alpha))(x) = n\cdot\bar\phi(\alpha)(x)\cdot\chi(\alpha). \tag{2.7}$$

Note that the action $(\pi, \mathcal{N})$ is a Seifert construction and if π is torsionfree $\mathcal{N}/\pi$ is an infranilmanifold (cf. [5] or [10]).

3. S^1-fibred nilBott tower

In this section we shall gives a proof of Theorem 1.2 of Introduction and apply our theorem to torus actions.

3.1. Proof of Theorem 1.2. Suppose that

$$M = M_n \xrightarrow{S^1} M_{n-1} \xrightarrow{S^1} \cdots \xrightarrow{S^1} M_1 \xrightarrow{S^1} \{\mathrm{pt}\} \tag{3.1}$$

is an S^1-fibred nilBott tower. By the definition, there is a group extension of the fiber space;

$$1 \to \mathbb{Z} \to \pi_i \to \pi_{i-1} \to 1 \tag{3.2}$$

for any i. The conjugate by each element of π_i defines a homomorphism $\phi\colon \pi_{i-1} \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$. With this action, $\mathbb{Z}$ is a π_{i-1}-module so that the group cohomology $H^*_\phi(\pi_{i-1}, \mathbb{Z})$ is defined. Then the above group extension (3.2) represents a 2-cocycle in $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ (cf. [10]).

Proof of Theorem 1.2. Given a group extension (3.2), we suppose by induction that there exists a torsionfree finitely generated nilpotent normal subgroup Δ_{i-1} of finite index in π_{i-1} such that the induced extension $\tilde{\Delta}_i$ is a central extension:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} & \longrightarrow & 1.
\end{array} \tag{3.3}$$

It is easy to see that $\tilde{\Delta}_i$ is a torsionfree finitely generated normal nilpotent subgroup of finite index in π_i. Then π_i is a virtually nilpotent subgroup, i.e. $1 \to \tilde{\Delta}_i \to \pi_i \to F_i \to 1$ where $F_i = \pi_i/\tilde{\Delta}_i$ is a finite group. Let $\tilde{N}_i$, N_{i-1} be a nilpotent Lie group containing $\tilde{\Delta}_i$, Δ_{i-1} as a discrete cocompact subgroup respectively. Let $\mathrm{A}(\tilde{N}_i) = \tilde{N}_i \rtimes \mathrm{Aut}(\tilde{N}_i)$ be the affine group. If $\tilde{K}_i$ is a maximal compact subgroup of $\mathrm{Aut}(\tilde{N}_i)$, then the subgroup $\mathrm{E}(\tilde{N}_i) = \tilde{N}_i \rtimes \tilde{K}_i$ is called the euclidean group of $\tilde{N}_i$. Then there exists a faithful homomorphism (see (2.6)):

$$\rho_i : \pi_i \to \mathrm{E}(\tilde{N}_i) \tag{3.4}$$

for which $\rho_i|_{\tilde{\Delta}_i} = \mathrm{id}$ and the quotient $\tilde{N}_i/\rho_i(\pi_i)$ is an infranilmanifold. The explicit formula is given by the following

$$\rho_i((n,\alpha)) = (n\cdot\chi(\alpha),\, \mu(\chi(\alpha)^{-1})\circ\bar{\phi}(\alpha)) \tag{3.5}$$

for $n \in \tilde{\Delta}_i$, $\alpha \in F_i$ where $\chi: F_i \to \tilde{N}_i$, $\bar{\phi}\colon F_i \to \mathrm{Aut}(\tilde{N}_i)$. As $\tilde{\Delta}_i \le \tilde{N}_i$, there is a 1-dimensional vector space R containing $\mathbb{Z}$ as a discrete uniform subgroup which has a central group extension (cf. [12]):

$$1 \to \mathsf{R} \to \tilde{N}_i \to N_{i-1} \to 1$$

where $N_{i-1} = \tilde{N}_i/\mathsf{R}$ is a simply connected nilpotent Lie group. As $\mathbb{Z} \le \mathsf{R}\cap\tilde{\Delta}_i$ is discrete cocompact in R and $\mathsf{R}\cap\tilde{\Delta}_i/\mathbb{Z} \to \tilde{\Delta}_i/\mathbb{Z} \cong \Delta_{i-1}$ is an inclusion, noting that Δ_{i-1} is torsionfree, it follows that $\mathsf{R}\cap\tilde{\Delta}_i = \mathbb{Z}$. We obtain the commutative diagram in which the vertical maps are inclusions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \tilde{N}_i & \longrightarrow & N_{i-1} & \longrightarrow & 1. \end{array} \tag{3.6}$$

On the other hand, (3.4) induces the following group extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \rho_i} & & \downarrow{\scriptstyle \hat{\rho}_i} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \rho_i(\pi_i) & \longrightarrow & \hat{\rho}_i(\pi_{i-1}) & \longrightarrow & 1. \end{array} \tag{3.7}$$

Since $\tilde{\Delta}_i$ and $\tilde{N}_i$ centralizes $\mathbb{Z}$ and R respectively, $\hat{\rho}_i$ is a homomorphism from π_{i-1} into $\mathrm{E}(N_{i-1})$. The explicit formula is given by the following:

$$\hat{\rho}_i((\bar{n},\alpha)) = (\bar{n}\cdot\bar{\chi}(\alpha),\, \mu(\bar{\chi}(\alpha)^{-1})\circ\hat{\phi}(\alpha)) \tag{3.8}$$

for $\bar{n} \in \Delta_{i-1}$, $\alpha \in F_i$ where $\bar{\chi} = p_i\circ\chi: F_i \to N_{i-1}$, $\hat{\phi}\colon F_i \to \mathrm{Aut}(\mathrm{N}_{i-1})$;

$$\hat{\phi}(\alpha)(\bar{x}) = \overline{\bar{\phi}(\alpha)(x)}.$$

Using (1.3) and Mal'cev's unique extension property (compare [12]), it is easy to check that the above $\hat{\phi}\colon F_i \to \mathrm{Aut}(\mathrm{N}_{i-1})$ is a well-defined homomorphism. Thus we obtain an equivariant fibration:

$$(\mathbb{Z},\mathsf{R}) \to (\rho_i(\pi_i),\tilde{N}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}). \tag{3.9}$$

Suppose by induction that (π_{i-1}, X_{i-1}) is equivariantly diffeomorphic to the infranil-action $(\hat{\rho}_i(\pi_{i-1}), N_{i-1})$ as above. We have two Seifert fibrations from (1.3):

$$(\mathbb{Z}, \mathsf{R}) \to (\pi_i, X_i) \xrightarrow{p_i} (\pi_{i-1}, X_{i-1})$$

and (3.9):

$$(\mathbb{Z}, \mathsf{R}) \to (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).$$

As $\rho_i \colon \pi_i \to \rho_i(\pi_i)$ is isomorphic such that $\rho_i|_{\mathbb{Z}} = \mathrm{id}$, the Seifert rigidity implies that (π_i, X_i) is equivariantly diffeomorphic to $(\rho_i(\pi_i), \tilde{N}_i)$. This shows the induction step. If $M = X/\pi$, then (π, X) is equivariantly diffeomorphic to an infranil-action $(\rho(\pi), \tilde{N})$ for which $\rho \colon \pi \to E(\tilde{N})$ is a faithful representation.

We have shown that M is diffeomorphic to an infranilmanifold $\tilde{N}/\rho(\pi)$. According to Cases I, II (stated in Theorem 1.2), we prove that $\tilde{N}$ is isomorphic to a vector space for Case I or $\tilde{N}$ is a nilpotent Lie group but not a vector space for Case II respectively.

CASE I. As every cocycle of $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ representing a group extension (3.2) is finite, the cocycle in $H^2(\Delta_{i-1}, \mathbb{Z})$ for the induced extension of (3.3) that $1 \to \mathbb{Z} \to \tilde{\Delta}_i \to \Delta_{i-1} \to 1$ is also finite. By induction, suppose that Δ_{i-1} is isomorphic to a free abelian group $\mathbb{Z}^{i-1}$. Then the cocycle in $H^2(\mathbb{Z}^{i-1}, \mathbb{Z})$ is zero, so $\tilde{\Delta}_i$ is isomorphic to a free abelian group $\mathbb{Z}^i$. Hence the nilpotent Lie group N_i is isomorphic to the vector space $\mathbb{R}^i$. This shows the induction step. In particular, π_i is isomorphic to a Bieberbach group $\rho_i(\pi_i) \leq \mathrm{E}(\mathbb{R}^i)$. As a consequence X/π is diffeomorphic to a Riemannian flat manifold $\mathbb{R}^n/\rho(\pi)$.

CASE II. Suppose that π_{i-1} is virtually free abelian until $i-1$ and the cocycle $[f] \in H^2_\phi(\pi_{i-1}, \mathbb{Z})$ representing a group extension $1 \to \mathbb{Z} \to \pi_i \to \pi_{i-1} \to 1$ is of infinite order in $H^2_\phi(\pi_{i-1}, \mathbb{Z})$. Note that π_{i-1} contains a torsionfree normal free abelian subgroup $\mathbb{Z}^{i-1}$. As in (3.3), there is a central group extension of $\tilde{\Delta}_i$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow \mathrm{i} & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \mathbb{Z}^{i-1} & \longrightarrow & 1
\end{array} \tag{3.10}$$

where $[\pi_{i-1} : \mathbb{Z}^{i-1}] < \infty$. Recall that there is a transfer homomorphism $\tau \colon H^2(\mathbb{Z}^{i-1}, \mathbb{Z}) \to H^2_\phi(\pi_{i-1}, \mathbb{Z})$ such that $\tau \circ \mathrm{i}^* = [\pi_{i-1} : \mathbb{Z}^{i-1}] \colon H^2_\phi(\pi_{i-1}, \mathbb{Z}) \to H^2_\phi(\pi_{i-1}, \mathbb{Z})$, see [1, (9.5) Proposition p. 82] for example. The restriction $\mathrm{i}^*[f]$ gives the bottom extension sequence of (3.10). If $\mathrm{i}^*[f] = 0 \in H^2(\mathbb{Z}^2, \mathbb{Z})$, then $0 = \tau \circ \mathrm{i}^*[f] = [\pi_{i-1} : \mathbb{Z}^{i-1}][f] \in H^2_\phi(\pi_{i-1}, \mathbb{Z})$. So $\mathrm{i}^*[f] \neq 0$. Therefore $\tilde{\Delta}_i$ (respectively $\tilde{N}_i$) is not abelian (respectively not isomorphic to a vector space). As a consequence, $\tilde{N}$ is a simply connected (non-abelian) nilpotent Lie group. □

In order to study S^1-fibred nilBott manifolds further, we introduce the following definition:

DEFINITION 3.1. If an S^1-fibred nilBott manifold M satisfies Case I (respectively Case II) of Theorem 1.2, then M is said to be an S^1-fibred nilBott manifold of finite type (respectively of infinite type).

Apparently there is no inter between finite type and infinite type. And S^1-fibred nilBott manifolds are of finite type until dimension 2.

REMARK 3.2. Let M be an S^1-fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 1.2). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension

$$1 \to \mathbb{Z}^n \to \rho(\pi) \to H \to 1 \tag{3.11}$$

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and H is the holonomy group of $\rho(\pi)$. We may identify $\rho(\pi)$ with π whenever π is torsionfree.

Proposition 3.3. *Suppose M is an S^1-fibred nilBott manifold of finite type. Then the holonomy group of π is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $\mathrm{O}(n)$ $(0 \leq s \leq n)$.*

Proof. Let M be an S^1-fibred nilBott manifold of finite type. Recall an equivariant fibration:

$$(\mathbb{Z}, \mathsf{R}) \to (\pi_i, \tilde{N}_i) \xrightarrow{p_i} (\pi_{i-1}, N_{i-1}).$$

If f is a cocycle in $H^2_{\phi}(\pi_{i-1}, \mathbb{Z})$ for Case I representing (3.2), then there exists a map $\lambda\colon \pi_{i-1} \to \mathsf{R}$ such that

$$f(\alpha, \beta) = \bar{\phi}(\alpha)(\lambda(\beta)) + \lambda(\alpha) - \lambda(\alpha\beta) \quad (\alpha, \beta \in \pi_{i-1}) \tag{3.12}$$

(see [3]). Moreover let $(n, \alpha) \in \pi_i$ and $(x, w) \in \tilde{N}_i = \mathsf{R} \times N_{i-1}$, then the action of π_i is given by

$$(n, \alpha)(x, w) = (n + \bar{\phi}(\alpha)(x) + \lambda(\alpha), \alpha w) \tag{3.13}$$

$(n \in \mathbb{Z},\ \alpha \in \pi_{i-1})$. See (2.4). As we have shown in Case I of Theorem 1.2, N_{i-1}/π_{i-1} is a Riemannian flat manifold $\mathbb{R}^{i-1}/\pi_{i-1}$, we may assume that

$$\alpha w = b_\alpha + A_\alpha w \quad (w \in \mathbb{R}^{i-1})$$

($b_\alpha \in \mathbb{R}^i$, $A_\alpha \in \mathrm{O}(i-1)$) in the above action of (3.13). Then the above action (3.13) has the formula:

$$(n,\alpha)\begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} n+\lambda(\alpha) \\ b_\alpha \end{pmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & 0 \\ 0 & A_\alpha \end{pmatrix}\right)\begin{bmatrix} x \\ w \end{bmatrix}, \tag{3.14}$$

where $\begin{bmatrix} x \\ w \end{bmatrix} \in \tilde{N}_i = \mathsf{R}\times\mathbb{R}^{i-1} = \mathbb{R}^i$. Suppose inductively that $\{A_\alpha \mid \alpha \in \pi_{i-1}\} \leq (\mathbb{Z}_2)^{i-1}$. Here

$$(\mathbb{Z}_2)^{i-1} = \left\{\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}\right\} \leq \mathrm{O}(i-1). \tag{3.15}$$

Since $\bar{\phi}(\pi_{i-1}) \leq \{\pm 1\}$, the holonomy group H_i of π_i is isomorphic to $(\mathbb{Z}_2)^s$, $(0 \leq s \leq i)$. This proves the induction step. □

3.2. Torus actions on S^1-fibred nilBott manifolds. Given an effective T^k-action on a closed aspherical manifold M, define an orbit map $ev\colon T^k \to M$ by $ev(t) = tx$ ($\exists x \in M$). Then ev induces a homomorphism of the fundamental groups $ev_*\colon \pi_1(T^k) \to \pi_1(M)$ which is known to be injective by Conner and Raymond [3]. But $ev_*\colon H_1(T^k) \to H_1(M)$ is not necessarily injective.

DEFINITION 3.4. When $ev_*\colon H_1(T^k) \to H_1(M)$ is injective, we call that the T^k-action is homologically injective.

Corollary 3.5. *Each S^1-fibred nilBott manifold of finite type M_i admits a homologically injective T^k-action where $k = \operatorname{Rank} H_1(M_i)$. Moreover, the action is maximal, i.e. $k = \operatorname{Rank} C(\pi_i)$.*

Proof. We suppose by induction that there is a *homologically injective* maximal T^{k-1}-action on $M_{i-1} = T^{i-1}/H_{i-1}$ such that $k-1 = \operatorname{Rank} H_1(M_{i-1}) = \operatorname{Rank} C(\pi_{i-1})$ $(k-1>0)$. Since π_i, π_{i-1} are Bieberbach groups, there are two group extensions

$$1 \to \mathbb{Z}^i \to \pi_i \xrightarrow{h_i} H_i \to 1,$$

$$1 \to \mathbb{Z}^{i-1} \to \pi_{i-1} \xrightarrow{h_{i-1}} H_{i-1} \to 1$$

where H_i, H_{i-1} are holonomy groups of π_i, π_{i-1} respectively and $\mathbb{Z}^i = \pi_i \cap \mathbb{R}^i$, $\mathbb{Z}^{i-1} =$

$\pi_{i-1} \cap \mathbb{R}^{i-1}$. We have a following diagram

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^i & \longrightarrow & \mathbb{Z}^{i-1} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle h_{i-1}} & & \\
 & & & & H_i & = & H_{i-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\tag{3.16}
$$

Let $p \colon \mathbb{R}^i = \mathsf{R} \times \mathbb{R}^{i-1} \to T^i = S^1 \times T^{i-1}$ be the canonical projection such that $\operatorname{Ker} p = \mathbb{Z}^i = \pi_i \cap \mathbb{R}^i$. By Proposition 3.3, $H_i = (\mathbb{Z}_2)^s$ for some s $(1 \le s \le i)$. The action $(\pi_i, \mathbb{R}^i)$ induces an isometric action (H_i, T^i) from (3.14). We may represent the action as follows:

$$
\hat{\alpha}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{pmatrix} = \begin{pmatrix} t_{\hat{\alpha}} \cdot \psi(\hat{\alpha})(z_1) \\ {z_2}' \\ \vdots \\ {z_i}' \end{pmatrix}
\tag{3.17}
$$

here $\hat{\alpha} = h_i((n, \alpha)) \in H_i$, $t_{\hat{\alpha}} = p(n + \lambda(\alpha)) \in S^1$, and $\psi : H_i \to \{\pm 1\}$ is defined by

$$
\psi(\hat{\alpha})(z_1) = \begin{cases} z_1 & \text{if} \quad \bar{\phi}(\alpha) = 1, \\ \bar{z}_1 & \text{if} \quad \bar{\phi}(\alpha) = -1. \end{cases}
\tag{3.18}
$$

Note that $(t_{\hat{\alpha}})^2 = p(n+\lambda(\alpha))p(n+\lambda(\alpha)) = p(2n+2\lambda(\alpha))$. By (3.14) if $\bar{\phi}(\alpha) = 1$, then

$$
(n, \alpha)^2 \begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} 2n + 2\lambda(\alpha) \\ b_\alpha + A_\alpha w \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix}.
\tag{3.19}
$$

Since $2n + 2\lambda(\alpha) \in \mathbb{Z}$, $(t_{\hat{\alpha}})^2 = 1$ i.e. $t_{\hat{\alpha}} = \pm 1$.

If $\psi(\hat{\alpha}) = 1$ for all $\hat{\alpha}$, it follows from (3.17) that the left translation of S^1 on $T^i = S^1 \times T^{i-1}$ induces an S^1-action on $M_i = T^i/H_i$ so that T^k-action on $M_i = T^i/H_i$ follows

$$\begin{pmatrix} t \\ t' \end{pmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix} = \begin{bmatrix} t \cdot z_1 \\ t' \cdot \begin{pmatrix} z_2 \\ \vdots \\ z_i \end{pmatrix} \end{bmatrix} \tag{3.20}$$

where $(t, t') \in S^1 \times T^{k-1}$, $[z_1, \dots, z_i] \in M_i = T^i/H_i$. On the other hand, if there is an element $\hat{\alpha}$ of H_i which $\psi(\hat{\alpha})(z) = \bar{z}$, then M_i admits a T^{k-1}-action by the induction hypothesis. The group extension (3.11) gives rise to a group extension:

$$1 \to \mathbb{Z}/[\pi_i, \pi_i] \cap \mathbb{Z} \to \pi_i/[\pi_i, \pi_i] \xrightarrow{\nu_i} \pi_{i-1}/[\pi_{i-1}, \pi_{i-1}] \to 1. \tag{3.21}$$

As in the proof of Proposition 3.3, $[(0, \alpha), (n, 1)] = ((\phi(\alpha) - 1)(n), 1)$. It follows that $[\pi_i, \pi_i] \cap \mathbb{Z} = \{1\}$ or $[\pi_i, \pi_i] \cap \mathbb{Z} = 2\mathbb{Z}$ according to whether $H_i = \operatorname{Ker} \psi$ or not. So (3.21) becomes

$$1 \to \mathbb{Z} \to H_1(M_i) \xrightarrow{\nu_i} H_1(M_{i-1}) \to 1, \tag{3.22}$$

or

$$1 \to \mathbb{Z}_2 \to H_1(M_i) \xrightarrow{\nu_i} H_1(M_{i-1}) \to 1. \tag{3.23}$$

For (3.22), it follows $k = \operatorname{Rank} H_1(M_i)$ for which M_i admits a homologically injective T^k-action as above. For (3.23), $k - 1 = \operatorname{Rank} H_1(M_i)$ and M_i admits a homologically injective T^{k-1}-action by the induction hypothesis.

Now we show the action is maximal. Suppose $\psi(\hat{\alpha}) = 1$ for all $\hat{\alpha}$. Noting that the group extension $1 \to \mathbb{Z} \to \pi_i \xrightarrow{p_i} \pi_{i-1} \to 1$ is a central extension, we obtain a group extension:

$$1 \to \mathbb{Z} \to C(\pi_i) \xrightarrow{p_i} p_i(C(\pi_i)) \to 1.$$

On the other hand, since M_i admits the above T^k-action, $\mathbb{Z}^k \subset C(\pi_i)$. Let $\operatorname{Rank} C(\pi_i) = k + l$, $(l = 0, 1, 2, \dots)$, then $\mathbb{Z}^{k+l-1} \subset p_i(C(\pi_i))$. By the induction hypothesis, $k - 1 = \operatorname{Rank} C(\pi_{i-1}) \geq \operatorname{Rank} p_i(C(\pi_i))$. Therefore $l = 0$ that is $k = \operatorname{Rank} C(\pi_i)$.

Assume that there exists an element $\hat{\alpha} \in H_i$ such that $\psi(\hat{\alpha})(z) = \bar{z}$. It is easy to check that $\mathbb{Z} \cap C(\pi_i) = \{1\}$, i.e. $C(\pi_i) \leq C(\pi_{i-1})$ and since M_i admits T^{k-1}-action, $\mathbb{Z}^{k-1} \leq C(\pi_i)$. By the induction hypothesis, $k - 1 = \operatorname{Rank} C(\pi_i)$. Hence in each case the torus action is maximal. $\square$

4. 3-dimensional S^1-fibred nilBott towers

By the definition of S^1-fibred nilBott manifold M_n, M_2 is either a torus T^2 or a Klein bottle K so that M_2 is a Riemannian flat manifold.

4.1. 3-dimensional S^1-fibred nilBott manifolds of finite type. Any 3-dimensional S^1-fibred nilBott manifold M_3 of finite type is a Riemannian flat manifold. It is known that there are just 10-isomorphism classes $\mathcal{G}_1, \dots, \mathcal{G}_6$, $\mathcal{B}_1, \dots, \mathcal{B}_4$ of 3-dimensional Riemannian flat manifolds. (Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [13].) In particular, for Riemannian flat 3-manifolds corresponding to $\mathcal{B}_2$ and $\mathcal{B}_4$, we have shown that there are two S^1-fibred nilBott towers: $\mathcal{B}_2 \to K \to S^1 \to \{\text{pt}\}$ and $\mathcal{B}_4 \to K \to S^1 \to \{\text{pt}\}$ in [10]. Remark that every real Bott manifold is an S^1-fibred nilBott manifold of finite type and $\mathcal{B}_2$ and $\mathcal{B}_4$ are not real Bott manifolds. And the following proposition has been proved. See [10] for details.

Proposition 4.1. *The* 3*-dimensional S^1-fibred nilBott manifolds of finite type are those of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$.*

4.2. 3-dimensional S^1-fibred nilBott manifolds of infinite type. Any 3-dimensional S^1-fibred nilBott manifold M_3 of infinite type is an infranil-Heisenberg manifold. The 3-dimensional simply connected nilpotent Lie group N_3 is isomorphic to the Heisenberg Lie group N which is the product $\mathsf{R} \times \mathbb{C}$ with group law:

$$(x, z) \cdot (y, w) = (x + y - \operatorname{Im} \bar{z}w, z + w).$$

Then a maximal compact Lie subgroup of Aut(N) is $\mathrm{U}(1) \rtimes \langle \tau \rangle$ which acts on N

$$\begin{aligned} &e^{\mathbf{i}\theta}(x, z) = (x, e^{\mathbf{i}\theta} z), \quad (e^{\mathbf{i}\theta} \in \mathrm{U}(1)), \\ &\tau(x, z) = (-x, \bar{z}). \end{aligned} \tag{4.1}$$

A 3-dimensional compact infranilmanifold is obtained as a quotient N/Γ where Γ is a torsionfree discrete uniform subgroup of $\mathrm{E}(\mathsf{N}) = \mathsf{N} \rtimes (\mathrm{U}(1) \rtimes \langle \tau \rangle)$. See [4].

Let

$$S^1 \to M_3 \to M_2$$

be an S^1-fibred nilBott manifold of infinite type which has a group extension $1 \to \mathbb{Z} \to \pi_3 \to \pi_2 \to 1$. As before this group extension contains a central group extension $1 \to \mathbb{Z} \to \tilde{\Delta}_3 \to \Delta_2 \to 1$. Since $\mathsf{R} \subset \mathsf{N}$ is the center, this induces the commutative diagram of central extensions (cf. (3.16)):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_3 & \longrightarrow & \Delta_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathsf{N} & \longrightarrow & \mathbb{C} & \longrightarrow & 1. \end{array} \tag{4.2}$$

Using this, we obtain an embedding:

(4.3)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \hat{\rho}} & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathrm{E}(\mathsf{N}) & \longrightarrow & \mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) & \longrightarrow & 1. \end{array}$$

Note that $\mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) = \mathbb{R}^2 \rtimes \mathrm{O}(2) = \mathrm{E}(2)$. Since $\mathsf{R} \cap \pi_3 = \mathbb{Z}$ from (4.3), $\hat{\rho}(\pi_2)$ is a Bieberbach group in E(2) so that $\mathbb{R}^2/\hat{\rho}(\pi_2)$ is either T^2 or K.

Define $L : \mathrm{E}(\mathsf{N}) \to \mathrm{U}(1) \rtimes \langle\tau\rangle$ to be the canonical projection.

CASE (i). Suppose $L(\pi_3) = \{1\}$. Then $\hat{\rho}(\pi_2) \leq \mathbb{C}$. So we may assume $\pi_3 = \tilde{\Delta}_3$ from (4.2). For each $k \in \mathbb{Z}$, we introduce the nilpotent group $\Delta(k)$ which is a subgroup of N generated by

$$c = (2k, 0), \quad a = (0, k), \quad b = (0, k\mathbf{i}).$$

Put $\mathsf{Z} = \langle c\rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that

(4.4)
$$[a, b] = c^{-k}.$$

Then $\tilde{\Delta}_3 \leq \mathsf{N}$ is isomorphic to $\Delta(k)$ for some $k \in \mathbb{Z}$. Since R is the center of N, we have a principal bundle

$$S^1 = \mathsf{R}/\mathsf{Z} \to \mathsf{N}/\Delta(k) \to \mathbb{C}/\mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [9] for example.)

CASE (ii). Suppose that the holonomy group of π_3 is nontrivial. Then we note that $L(\pi_3) = \mathbb{Z}_2 \leq \mathrm{U}(1) \rtimes \langle\tau\rangle$, but not in U(1). By (3.16) $L(\pi_3) = L(\pi_2)$, first remark that $L(\pi_2)$ is not contained in U(1). For this, suppose that (b, A) is an element of $\pi_2 \leq \mathbb{R}^2 \rtimes \mathrm{O}(2)$. Then for any $x \in \mathbb{R}^2$, $(b, A)x \neq x$, because the action of π_2 on $\mathbb{R}^2$ is free. Therefore $\det(A - I) = 0$. This implies that if $A \in \mathrm{SO}(2) = \mathrm{U}(1)$, then $A = I$. So $L(\pi_2) = L(\pi_3)$ is not contained in U(1).

Suppose that there exists an element $g \in \pi_3$ such that $L(g) = (e^{\mathbf{i}\theta}, \tau) \in \mathrm{U}(1) \rtimes \langle\tau\rangle$. Noting (4.1), it follows $L(g)^2 = 1$. Then $L(\pi_3) = (\mathrm{U}(1) \cap L(\pi_3)) \cdot \langle L(g)\rangle$. Let $\pi_3' = L^{-1}(\mathrm{U}(1) \cap L(\pi_3)) \leq \pi_3$ which has the commutative diagram:

(4.5)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \xrightarrow{p_3} & \pi_2 & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3' & \longrightarrow & \pi_2' & \longrightarrow & 1. \end{array}$$

Here $\pi_2' = p_3(\pi_3')$. Since π_2' also acts on $\mathbb{R}^2$ freely, it follows $L(\pi_2') = L(\pi_3') = \mathrm{U}(1) \cap L(\pi_3) = \{1\}$. Hence $L(\pi_2) = L(\pi_3) = \mathbb{Z}_2 = \langle L(g)\rangle$. In particular M_2 is the Klein bottle K.

Let $n = (x, 0)$ be a generator of $\mathbb{Z} \leq \mathsf{N}$. Choose $h \in \pi_3$ with $L(h) = 1$ such that the subgroup $\langle p_3(g), p_3(h)\rangle$ is the fundamental group of K. It has a relation $p_3(g)p_3(h)p_3(g)^{-1} = p_3(h)^{-1}$. Then $\langle n, g, h\rangle$ is isomorphic to π_3. In particular, those generators satisfy

$$\begin{aligned} &ghg^{-1} = n^k h^{-1} \quad (\exists k \in \mathbb{Z}),\\ &gng^{-1} = L(g)n = \tau n = n^{-1}, \quad ahnh^{-1} = L(h)n = n. \end{aligned} \tag{4.6}$$

On the other hand, fix a non-zero integer k. Let $\Gamma(k)$ be a subgroup of $\mathrm{E}(\mathsf{N})$ generated by

$$n = ((k, 0), I), \quad \alpha = \left(\left(0, \frac{k}{2}\right), \tau\right), \quad \beta = ((0, k\mathbf{i}), I), \tag{4.7}$$

where $(a, x) \in \mathsf{N} = \mathsf{R} \times \mathbb{C} \leq \mathrm{E}(\mathsf{N})$.

Note that $\alpha^2 = ((0, k), I)$. Then it is easily checked that

$$\alpha\beta\alpha^{-1} = n^k \beta^{-1}, \quad \alpha n \alpha^{-1} = n^{-1}, \quad \beta n \beta^{-1} = n, \tag{4.8}$$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathrm{E}(\mathsf{N}) & \longrightarrow & \mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) & \longrightarrow & 1\\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \langle n\rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle\hat{\alpha}, \hat{\beta}\rangle & \longrightarrow & 1. \end{array} \tag{4.9}$$

Then the subgroup generated by $\hat{\alpha}^2$, $\hat{\beta}$ is isomorphic to the subgroup of translations of $\mathbb{R}^2$; $t_1 = \begin{bmatrix} k \\ 0 \end{bmatrix}$, $t_2 = \begin{bmatrix} 0 \\ k \end{bmatrix}$. Let $T^2 = \mathbb{R}^2/\langle t_1, t_2\rangle$. Then it is easy to see that the quotient $\gamma = [\hat{\alpha}]$ of order 2 acts on T^2 as

$$\gamma(z_1, z_2) = (-z_1, \bar{z}_2). \tag{4.10}$$

As a consequence, $\mathbb{R}^2/\langle\hat{\alpha}, \hat{\beta}\rangle = T^2/\langle\gamma\rangle$ turns out to be K. So $M_3 = \mathsf{N}/\Gamma(k)$ is an S^1-fibred nilBott manifold:

$$S^1 \to \mathsf{N}/\Gamma(k) \to K$$

where $S^1 = \mathsf{R}/\langle n\rangle$ is the fiber (but not an action).

Compared (4.6) with $\Gamma(k)$, π_3 is isomorphic to $\Gamma(k)$ with the following commutative arrows of isomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \langle n\rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle\hat{\alpha}, \hat{\beta}\rangle & \longrightarrow & 1. \end{array} \tag{4.11}$$

As both (π_3, X_3) and $(\Gamma(k), \mathsf{N})$ are Seifert actions, the isomorphism of (4.11) implies that they are equivariantly diffeomorphic, i.e. $M_3 = X_3/\pi_3 \cong \mathsf{N}/\Gamma(k)$. This shows the following.

Proposition 4.2. *A 3-dimensional S^1-fibred nilBott manifold M_3 of infinite type is either a Heisenberg nilmanifold $\mathsf{N}/\Delta(k)$ or a Heisenberg infranilmanifold $\mathsf{N}/\Gamma(k)$.*

5. Realization

5.1. Realization of S^1-fibration over a Klein bottle K. Let Q be a fundamental group of a Klein Bottle K, then Q has a presentation:

$$\{g, h \mid ghg^{-1} = h^{-1}\}. \tag{5.1}$$

A group extension $1 \to \mathbb{Z} \to \pi \to Q \to 1$ for any 3-dimensional S^1-fibred nilBott manifold over K represents a 2-cocycle in $H^2_\phi(Q, \mathbb{Z})$ for some representation ϕ. Conversely, given any representation $\phi\colon Q \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, we shall prove that any element of $H^2_\phi(Q, \mathbb{Z})$ can be realized as an S^1-fibred nilBott manifold.

We must consider following cases of a representation ϕ:

CASE 1. $\phi(g) = 1$, $\phi(h) = 1$.

CASE 2. $\phi(g) = 1$, $\phi(h) = -1$.

CASE 3. $\phi(g) = -1$, $\phi(h) = 1$.

CASE 4. $\phi(g) = -1$, $\phi(h) = -1$.

Suppose ϕ_i $(i = 1, 2, 3, 4)$ is the representation ϕ for Case i. Any element of $H^2_{\phi_i}(Q, \mathbb{Z})$ gives rise to a group extension

$$1 \to \mathbb{Z} \to \pi \xrightarrow{p} Q \to 1.$$

Then π is generated by $\tilde{g}$, $\tilde{h}$, n such that $\langle n\rangle = \mathbb{Z}$ and $p(\tilde{g}) = g$, $p(\tilde{h}) = h$. There exists $k \in \mathbb{Z}$ which satisfies

$$\tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{5.2}$$

Put $\pi = {}_i\pi(k)$ for each $k \in \mathbb{Z}$ and $[f_k]$ denotes the 2-cocycle of $H^2_{\phi_i}(Q, \mathbb{Z})$ representing ${}_i\pi(k)$. Note that $[f_0] = 0$.

CASE 1: Since ϕ_1 is trivial, $H^2_{\phi_1}(Q, \mathbb{Z}) = H^2(Q, \mathbb{Z}) \approx H^2(K, \mathbb{Z}) \approx \mathbb{Z}_2$, and the group ${}_1\pi(k)$ satisfies the following presentation:

$$\tilde{g}n\tilde{g}^{-1} = n, \quad \tilde{h}n\tilde{h}^{-1} = n, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{5.3}$$

Lemma 5.1. *The groups ${}_1\pi(0)$, ${}_1\pi(1)$ are isomorphic to $\mathcal{B}_1$, $\mathcal{B}_2$ respectively.*

Proof. First we shall discuss ${}_1\pi(0)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(0)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = n$ and $t_3 = \tilde{h}$. Remark that a group generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(0)$. Using the relation (5.3),

$$\begin{aligned}\varepsilon^2 &= t_1,\\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g}\tilde{h}\tilde{g}^{-1} = \tilde{h}^{-1} = t_2^{-1},\\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g} n \tilde{g}^{-1} = n = t_3.\end{aligned}$$

Compared these relations with those of $\mathcal{B}_1$, ${}_1\pi(0)$ is isomorphic to $\mathcal{B}_1$ (due to the Wolf's notation [13]).

Second, we shall discuss ${}_1\pi(1)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(1)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{g}^{-2} n$ and $t_3 = \tilde{h}$. A group generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(1)$. By using the relation (5.3),

$$\begin{aligned}\varepsilon^2 &= t_1,\\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g}\tilde{g}^{-2} n \tilde{g}^{-1} = \tilde{g}^{-1} n \tilde{g}^{-1} = \tilde{g}^{-2} n = t_1,\\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g} h \tilde{g}^{-1} = \tilde{g}^2 \tilde{g}^{-2} n \tilde{h}^{-1} = t_1 t_2 t_3^{-1}.\end{aligned}$$

This implies that ${}_1\pi(1)$ is isomorphic to $\mathcal{B}_2$. (See [13].) □

For arbitrary $k \in \mathbb{Z}$, we have the following.

Proposition 5.2. *The group extension ${}_1\pi(k)$ is isomorphic to $\mathcal{B}_1$, or $\mathcal{B}_2$ in accordance with k is even or odd.*

Proof. Take $[f_1] \in H^2_{\phi_1}(Q, \mathbb{Z}) \approx \mathbb{Z}_2$ by Lemma 5.1, then

$$\begin{aligned} n = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h} &= (0, g)(0, h)(-f_1(g^{-1}, g), g^{-1})(0, h)\\ &= f_1(g, h) - f_1(g^{-1}, g) + f_1(gh, g^{-1}) + f_1(h^{-1}, h), \end{aligned} \tag{5.4}$$

and so

$$n^k = kf_1(g, h) - kf_1(g^{-1}, g) + kf_1(gh, g^{-1}) + kf_1(h^{-1}, h). \tag{5.5}$$

Since $[kf_1] \in H^2_{\phi_1}(Q,\mathbb{Z})$, we can construct a group H_k which is represented by (kf_1, ϕ_1). Then H_k is generated by the elements n and $g' = (0, g)$, $h' = (0, h)$ satisfying that

$$(n, \alpha)(m, \beta) = (n + \phi_1(\alpha)(m) + kf_1(\alpha, \beta), \alpha\beta) \quad (\forall n, m \in \mathbb{Z},\ \forall \alpha, \beta \in Q).$$

It follows

$$\begin{aligned} g'h'g'^{-1}h' &= (0, g)(0, h)(-kf_1(g^{-1}, g), g^{-1})(0, h) \\ &= kf_1(g, h) - kf_1(g^{-1}, g) + kf_1(gh, g^{-1}) + kf_1(h^{-1}, h) \\ &= n^k \quad \text{(from (5.5))}. \end{aligned}$$

Thus we obtain $g'h'g'^{-1} = n^k h'^{-1}$. In view of (5.2), a correspondence $g' \mapsto \tilde{g}$, $h' \mapsto \tilde{h}$ gives an isomorphism Ψ of the group extensions:

(5.6)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_k & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_1\pi(k) & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

If we recall that $[f_k]$ (resp. $[k \cdot f_1]$) represents ${}_1\pi(k)$ (resp. H_k), then it follows $[f_k] = k \cdot [f_1]$. Noting that $[f_1]$ is a two torsion element, the result follows. □

CASE 2: Let $\phi_2(g) = 1$, $\phi_2(h) = -1$, then ${}_2\pi(k)$ has the following presentation.

(5.7)
$$\tilde{g}n\tilde{g}^{-1} = n, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1},$$

for some $k \in \mathbb{Z}$.

Proposition 5.3. *The groups ${}_2\pi(0)$, ${}_2\pi(1)$ are isomorphic to $\mathcal{B}_3$, $\mathcal{B}_4$ respectively.*

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_2\pi(0)$ be as before. Put $\alpha = \tilde{h}\tilde{g}$, $\varepsilon = \tilde{h}^{-1}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}^{-2}$ and $t_3 = n$. Note that the group generated by α, ε, t_1, t_2, t_3 coincides with ${}_2\pi(0)$. Using the relation (5.7),

$$\begin{aligned} \tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}\tilde{h}^{-1}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1, \\ \varepsilon^2 &= t_2, \\ \varepsilon\alpha\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h} = \tilde{h}^{-1}\tilde{g} = t_2\alpha, \\ \alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1}, \\ alt_3\alpha^{-1} &= \tilde{h}\tilde{g}n\tilde{g}^{-1}\tilde{h}^{-1} = n^{-1} = t_3^{-1}, \\ \varepsilon t_1\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{g}^2\tilde{h} = \tilde{h}^{-1}\tilde{g}\tilde{h}^{-1}\tilde{g} = \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1, \\ \varepsilon t_3\varepsilon^{-1} &= \tilde{h}^{-1}n\tilde{h} = n^{-1} = t_3^{-1}. \end{aligned}$$

Since these relations correspond to those of $\mathcal{B}_3$ (cf. [13]), ${}_2\pi(0)$ is isomorphic to $\mathcal{B}_3$.

Let $\tilde{g}, \tilde{h}, n \in {}_2\pi(1)$ be as above. Put $\alpha = \tilde{h}\tilde{g}$, $\varepsilon = n^{-1}\tilde{h}^{-1}$, $t_1 = n^{-1}\tilde{g}^2$, $t_2 = \tilde{h}^{-2}$, and $t_3 = n^{-1}$. Using the relation (5.7), we obtain the following presentation:

$$\begin{aligned}
\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}n\tilde{h}^{-1}\tilde{g}\tilde{g} = n^{-1}\tilde{g}^2 = t_1,\\
\varepsilon^2 &= t_2,\\
\varepsilon\alpha\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h}n = \tilde{h}^{-1}\tilde{g}n = t_2t_3\alpha,\\
\alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1},\\
\alpha t_3\alpha^{-1} &= \tilde{h}\tilde{g}n^{-1}\tilde{g}^{-1}\tilde{h}^{-1} = n = t_3^{-1},\\
\varepsilon t_1\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{g}^2\tilde{h}n = n^{-1}\tilde{g}^2 = t_1,\\
\varepsilon t_3\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{h}n = n = t_3^{-1}.
\end{aligned}$$

This implies that ${}_2\pi(1)$ is isomorphic to $\mathcal{B}_4$. (See [13]). □

Proposition 5.4. *$H^2_{\phi_2}(Q, \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$.*

Proof. We first show that $H^2_{\phi_2}(Q,\mathbb{Z})$ is a 2-torsion group. Let Q' be the subgroup of Q generated by $g, h^2 \in Q$ satisfying that $gh^2g^{-1} = (ghg^{-1})^2 = h^{-2}$. We have a commutative diagram:

(5.8)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_2\pi(k) & \xrightarrow{p} & Q & \longrightarrow & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \xrightarrow{p} & Q' & \longrightarrow & 1
\end{array}$$

where π' is the subgroup of ${}_2\pi(k)$ generated by $n, \tilde{g}, \tilde{h}^2$. Note that

$$\tilde{g}\tilde{h}^2\tilde{g}^{-1} = n^k\tilde{h}^{-1}n^k\tilde{h}^{-1} = \tilde{h}^{-2}.$$

Since the subgroup $\langle\tilde{g}, \tilde{h}^2\rangle$ of π' maps isomorphically onto Q' and a restriction $\phi_2|Q' = \mathrm{id}$, it follows $\pi' = \mathbb{Z}\times Q'$. This shows that the restriction homomorphism $\iota^*\colon H^2_{\phi_2}(Q,\mathbb{Z}) \to H^2(Q', \mathbb{Z})$ is the zero map, equivalently $\iota^*[f_k] = 0$. Using the transfer homomorphism $\tau\colon H^2(Q', \mathbb{Z}) \to H^2_{\phi_2}(Q, \mathbb{Z})$ and by the property $\tau\circ\iota^*([f]) = [Q : Q'][f] = 2[f]$ $(\forall [f] \in H^2_{\phi_2}(Q, \mathbb{Z}))$, we obtain $2[f] = 0$.

Let $[f_k]$ be a 2-cocycle of ${}_2\pi(k)$. Similarly as in the proof of Proposition 5.2 we obtain

(5.9)
$$[f_k] = k\cdot[f_1].$$

As a consequence, $H^2_{\phi_2}(Q, \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$. □

The following is obvious using Proposition 5.3 and Proposition 5.4.

Corollary 5.5. *The group extension* ${}_2\pi(k)$ *is isomorphic to* $\mathcal{B}_3$ *or* $\mathcal{B}_4$ *in accordance with* k *is even or odd.*

CASE 3: The group ${}_3\pi(k)$ has the following presentation for some $k \in \mathbb{Z}$;

$$\tilde{g}n\tilde{g}^{-1} = n^{-1}, \quad \tilde{h}n\tilde{h}^{-1} = n, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{5.10}$$

Lemma 5.6. *The groups* ${}_3\pi(0)$, ${}_3\pi(k)$ *are isomorphic to* $\mathcal{G}_2$, $\Gamma(k)$ *respectively.* (*cf.* (4.7).)

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_3\pi(0)$ be as before. Put $\alpha = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}$ and $t_3 = n$. Note that the group generated by α, t_1, t_2, t_3 coincides with ${}_3\pi(0)$. By using the relation (5.10), it is easy to check that:

$$\alpha^2 = t_1,$$
$$\alpha t_2 \alpha^{-1} = t_2^{-1},$$
$$\alpha t_3 \alpha^{-1} = t_3^{-1}.$$

And so ${}_3\pi(0)$ is isomorphic to $\mathcal{G}_2$. (See [13].)

Suppose $\tilde{g}, \tilde{h}, n \in {}_3\pi(k)$ $(k \neq 0)$. By the relations (4.6) and (5.10), ${}_3\pi(k)$ is isomorphic to $\Gamma(k)$ (cf. (4.7)). □

Proposition 5.7. $H^2_{\phi_3}(G, \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}$.

Proof. From Theorem 1.2 and Lemma 5.6, $\Gamma(k)$ represents the torsionfree element $[f_k]$ in $H^2_{\phi_3}(G, \mathbb{Z})$. Moreover as in the proof of Proposition 5.2, we can show that $[f_k] = k \cdot [f_1]$. Therefore $H^2_{\phi_3}(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. □

CASE 4. The group ${}_4\pi(k)$ has the following presentation.

$$\tilde{g}n\tilde{g}^{-1} = n^{-1}, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{5.11}$$

Put $\alpha = \tilde{g}\tilde{h}$. It is easy to check that

$$\alpha n \alpha^{-1} = n, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \alpha\tilde{h}\alpha = n^k\tilde{h}^{-1}. \tag{5.12}$$

In view of (5.7), this implies that ${}_4\pi(k)$ is isomorphic to ${}_2\pi(k)$.

We have shown that any element of $H^2_{\phi_i}(Q, \mathbb{Z})$ can be realized an S^1-fibred nilBott manifold M_3, and obtain the following table:

		Case 1	Case 2 and 4	Case 3
	$H^2_\phi(Q,\mathbb{Z})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$\pi_1(M_3)$	$[f]=0$	$\mathcal{B}_1$	$\mathcal{B}_3$	$\mathcal{G}_2$
	$[f]\neq 0$: torsion	$\mathcal{B}_2$	$\mathcal{B}_4$	—
	$[f]$: torsionfree	—	—	$\Gamma(k)$

5.2. Realization of S^1-fibration over T^2. Let $\mathbb{Z}^2$ be the fundamental group of a torus T^2 generated by α, β. Given a representation $\phi\colon \mathbb{Z}^2 \to \mathbb{Z} = \{\pm 1\}$, we shall show that any element of $H^2_\phi(\mathbb{Z}^2,\mathbb{Z})$ can be realized as an S^1-fibred nilBott manifold.

We must consider following cases of a representation ϕ:

CASE 5. $\phi(\alpha)=1$, $\phi(\beta)=1$.

CASE 6. $\phi(\alpha)=1$, $\phi(\beta)=-1$.

CASE 7. $\phi(\alpha)=-1$, $\phi(\beta)=-1$.

Suppose ϕ_i $(i=5,6,7)$ is the representation ϕ for Case i. Any element of $H^2_{\phi_i}(\mathbb{Z}^2,\mathbb{Z})$ gives rise to a group extension

$$1 \to \mathbb{Z} \to \pi \xrightarrow{p} \mathbb{Z}^2 \to 1.$$

Then π is generated by $\tilde{\alpha}, \tilde{\beta}, m$ such that $\langle m\rangle = \mathbb{Z}$ and $p(\tilde{\alpha})=\alpha$, $p(\tilde{\beta})=\beta$. There exists $k\in\mathbb{Z}$ which satisfies

$$\tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}. \tag{5.13}$$

Put $\pi = {}_i\pi(k)$ for each $k\in\mathbb{Z}$ and $[f_k]$ denotes the 2-cocycle of $H^2_{\phi_i}(\mathbb{Z}^2,\mathbb{Z})$ representing ${}_i\pi(k)$. Note that $[f_0]=0$.

CASE 5: The group ${}_5\pi(k)$ has the following presentation.

$$\tilde{\alpha}m\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}, \tag{5.14}$$

for some $k\in\mathbb{Z}$. Compared these relations with (4.4),

Proposition 5.8. *The groups ${}_5\pi(0)$, ${}_5\pi(k)$ are isomorphic to $\pi_1(T^3)$, $\pi_1(\Delta(-k))$ respectively.*

Similarly as in the proof of Proposition 5.7, we obtain

Proposition 5.9. *$H^2_{\phi_5}(\mathbb{Z}^2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.*

CASE 6: The group ${}_6\pi(k)$ has the following presentation.

$$\tilde{\alpha}m\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m^{-1}, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}, \tag{5.15}$$

for some $k\in\mathbb{Z}$.

Proposition 5.10. *The groups* ${}_6\pi(0)$, ${}_6\pi(1)$ *are isomorphic to* $\mathcal{B}_1$, $\mathcal{B}_2$ *respectively.*

Proof. First let $k=0$. Put $m=\tilde{h}$, $\tilde{\alpha}=n$, $\tilde{\beta}=\tilde{g}$, then we can check easily that ${}_6\pi(0)$ is isomorphic to ${}_1\pi(0)$. So ${}_6\pi(0)$ is isomorphic to $\mathcal{B}_1$.

Suppose $k=1$. Put $m=n$, $\tilde{\alpha}=\tilde{g}$, $m^{-1}\tilde{\beta}=\tilde{h}$, then it is easy to check that ${}_6\pi(1)$ is isomorphic to $\mathcal{B}_2$. □

Moreover similarly as in the proof of Proposition 5.4, we obtain

Proposition 5.11. $H^2_{\phi_6}(\mathbb{Z}^2,\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}_2$.

CASE 7: The group ${}_7\pi(k)$ has the following presentation.

$$\tilde{\alpha}m^{-1}\tilde{\alpha}^{-1}=m,\quad \tilde{\beta}m\tilde{\beta}^{-1}=m^{-1},\quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1}=m^k\tilde{\beta}, \tag{5.16}$$

for some $k\in\mathbb{Z}$. Then it is easy to check that ${}_7\pi(k)$ is isomorphic to ${}_6\pi(k)$ if we put $g=\tilde{\alpha}\tilde{\beta}$.

We have shown that any element of $H^2_{\phi}(\mathbb{Z}^2,\mathbb{Z})$ can be realized an S^1-fibred nilBott manifold M_3, and we obtain the following table:

		Case 5	Case 6 and 7
	$H^2_{\phi}(\mathbb{Z}^2,\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}_2$
$\pi_1(M_3)$	$[f]=0$	$\mathcal{G}_1$	$\mathcal{B}_1$
	$[f]\neq 0$: torsion	—	$\mathcal{B}_2$
	$[f]$: torsionfree	$\Delta(k)$	—

6. Halperin–Carlsson conjecture

Theorem 6.1 (Halperin–Carlsson conjecture [11])**.** *Let* T^s *be an arbitrary effective action on an* m*-dimensional* S^1*-fibred nilBott manifold* M *of finite type. Then*

$${}_sC_j\le b_j\quad (= \textit{the } j\textit{-th Betti number of } M). \tag{6.1}$$

In particular $2^s\le\sum_{j=0}^m \operatorname{Rank} H_j(M)$.

Proof. By Corollary 3.5, M admits a homologically injective T^k-action where $k=\operatorname{Rank}C(\pi)$ where $\pi=\pi_1(M)$. Then we have shown in [6] that any homologically injective T^k-actions on any closed aspherical manifold satisfies that

$${}_kC_j\le b_j\quad (= \text{the } j\text{-th Betti number of } M).$$

It follows from the result of Conner–Raymond [3] that there is an injective homomorphism $1\to\mathbb{Z}^s\to C(\pi)$. This shows that $s\le k$ so we obtain

$${}_sC_j\le b_j. \tag{6.2}$$

□

REMARK 6.2. This result is obtained when M_i is a real Bott manifold by Masuda, Choi and Oum [2].

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