Qiao, H.
Osaka J. Math.
51 (2014), 47–66

EULER–MARUYAMA APPROXIMATION FOR SDES WITH JUMPS AND NON-LIPSCHITZ COEFFICIENTS

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(Received May 6, 2011, revised May 1, 2012)

Abstract

In this paper we show that stochastic differential equations with jumps and non-Lipschitz coefficients have (ξ, W, N_p)-pathwise unique strong solutions by the Euler–Maruyama approximation. Moreover, the Euler–Maruyama discretisation has an optimal strong convergence rate.

1. Introduction

Consider the following stochastic differential equation (SDE) with jumps in $\mathbb{R}^d$:

$$Y_t = \xi + \int_0^t b(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s, Y_s)\,\mathrm{d}W_s + \int_0^{t+}\int_{\mathbb{U}_0} f(s, Y_{s-}, u)\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u), \tag{1}$$

where $b\colon \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma\colon \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$, $f\colon \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d$, are Borel measurable functions; $\{W_t,\ t \geqslant 0\}$ is an m-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ and $\{p_t,\ t \geqslant 0\}$ is a stationary Poisson point process of the class (quasi left-continuous) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with values in $\mathbb{U}$ and characteristic measure ν, ξ is an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable square integrable random variable. Here ν is a σ-finite measure defined on a measurable normed space $(\mathbb{U}, \mathcal{U})$ with the norm $\|\cdot\|_{\mathbb{U}}$. Fix $\mathbb{U}_0 \in \mathcal{U}$ with $\nu(\mathbb{U} - \mathbb{U}_0) < \infty$ and $\int_{\mathbb{U}_0} \|u\|_{\mathbb{U}}^2\,\nu(\mathrm{d}u) < \infty$. Let $N_p((0, t], \mathrm{d}u)$ be the counting measure of p_t such that $\mathbb{E}N_p((0, t], A) = t\nu(A)$ for $A \in \mathcal{U}$. Denote

$$\tilde{N}_p((0, t], \mathrm{d}u) := N_p((0, t], \mathrm{d}u) - t\nu(\mathrm{d}u),$$

the compensator of p_t.

In this paper, we study existence and uniqueness of solutions to Equation (1) under non-Lipschitz conditions. Firstly, introduce some concepts. Given W and N_p on a probability space, we recall that a strong solution to Equation (1) is a càdlàg process Y which is adapted to the filtration generated by W and N_p and satisfies Equation (1). A weak solution of Equation (1) is a triple (Y, W, N_p) on a filtered probability space

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2010 Mathematics Subject Classification. 60H10, 60J75, 34A12.
This work is supported by NSF (No. 11001051) of China.

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ such that Y_t is adapted to $\mathcal{F}_t$, W_t and $N_p((0,t], \mathrm{d}u)$ are $\mathcal{F}_t$-Wiener process and $\mathcal{F}_t$-Poisson process, respectively, and (Y, W, N_p) solves Equation (1). We say that (ξ, W, N_p)-pathwise uniqueness holds for Equation (1) if for any filtered probability space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t\geqslant 0}, P')$ carrying ξ', W' and $N_{p'}$ such that the joint distribution of $(\xi', W', N_{p'})$ is the same as that of the given (ξ, W, N_p), Equation (1) with ξ', W' and $N_{p'}$ instead of ξ, W and N_p cannot have more than one weak solution.

Let us recall some recent results. By Euler–Maruyama approximation, Skorokhod [12] proved that Equation (1) has a weak solution. And by smooth approximation of the coefficients, Situ [11, Theorem 175, p. 147] showed that Equation (1) has a weak solution. However, by successive approximation Cao, He and Zhang [2] obtained that Equation (1) has a pathwise unique strong solution. Here we use the Skorokhod weak convergence technique and Lemma 1.1 in [3] to show that Equation (1) has a (ξ, W, N_p)-pathwise unique strong solution. Our approach is very close in spirit to the Yamada–Watanabe theorem. As we know, it seems the first time to show solutions to stochastic differential equations with jumps by the method.

Next assume the following non-Lipschitz conditions for Equation (1): for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$,

$$(\mathbf{H}_b) \qquad |b(t,x) - b(t,y)| \leqslant \lambda(t)|x-y|\kappa_1(|x-y|),$$

where $|\cdot|$ stands for Euclidean norm in $\mathbb{R}^d$,

$$(\mathbf{H}_\sigma) \qquad \|\sigma(t,x) - \sigma(t,y)\|^2 \leqslant \lambda(t)|x-y|^2\kappa_2(|x-y|),$$

where $\|\cdot\|$ is the Hilbert–Schmidt norm from $\mathbb{R}^m$ to $\mathbb{R}^d$,

$$(\mathbf{H}_f^1) \qquad \int_{\mathbb{U}_0} |f(t,x,u) - f(t,y,u)|^{p'} \nu(\mathrm{d}u) \leqslant \lambda(t)|x-y|^{p'}\kappa_3(|x-y|),$$

holds for $p' = 2$ and 4,

$$(\mathbf{H}_f^2) \qquad \lim_{h\to 0} \int_{\mathbb{U}_0} |f(t+h,x,u) - f(t,x,u)|^2 \nu(\mathrm{d}u) = 0,$$

$$(\mathbf{H}_{b,\sigma,f}) \qquad \int_0^T \left(|b(t,0)|^2 + \|\sigma(t,0)\|^2 + \int_{\mathbb{U}_0} |f(t,0,u)|^2 \nu(\mathrm{d}u) \right) \mathrm{d}t < \infty,$$

for any $T > 0$, where $\lambda(t) > 0$ is locally square integrable and κ_i is a positive continuous function, bounded on $[1, \infty)$ and satisfying

$$(2) \qquad \lim_{x\downarrow 0} \frac{\kappa_i(x)}{\log x^{-1}} = \delta_i < \infty, \quad i = 1, 2, 3.$$

REMARK 1.1. In [2] b, σ, f satisfy for $x, y \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$

$$\begin{aligned}&|b(t,x)-b(t,y)|^2+\|\sigma(t,x)-\sigma(t,y)\|^2+\int_{\mathbb{U}_0}|f(t,x,u)-f(t,y,u)|^2\nu(\mathrm{d}u)\\&\leqslant \lambda(t)\gamma\left(\sup_{r\leqslant t}|x_r-y_r|^2\right),\end{aligned}\tag{3}$$

and

$$\int_0^T\left(|b(t,0)|^2+|\sigma(t,0)|^2+\int_{\mathbb{U}_0}|f(t,0,u)|^2\nu(\mathrm{d}u)\right)\mathrm{d}t<\infty,$$

for any $T > 0$, where $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is concave nondecreasing continuous function such that $\gamma(0) = 0$ and $\int_{0+}(1/\gamma(u))\,\mathrm{d}u = \infty$, and $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ stands for the space of all càdlàg mappings from $\mathbb{R}_+$ to $\mathbb{R}^d$.

Comparing our conditions with (3), one will find that in our conditions the modulus of continuity for b in x is different from that for σ, which is convenient to control. In Section 4, we give an example to demonstrate it. Besides, b,σ and f do not depend on all the path but only on the value at t. Therefore, our conditions are more general.

One of our aims is to prove the following result.

Theorem 1.2. *Suppose that* $\dim \mathbb{U} < \infty$ *and the coefficients* b, σ *and* f *satisfy* ($\mathbf{H}_b$), ($\mathbf{H}_\sigma$), ($\mathbf{H}^1_f$), ($\mathbf{H}^2_f$) *and* ($\mathbf{H}_{b,\sigma,f}$). *Then* Equation (1) *has a* (ξ, W, N_p)*-pathwise unique strong solution.*

To prove Theorem 1.2, we construct the Euler–Maruyama approximation Y^n for the solution of Equation (1). In [6], Higham and Kloeden also constructed the Euler–Maruyama approximation of the solution to Equation (1) and showed the Euler–Maruyama discretisation has a strong convergence order of one half. However, they required that b satisfies a one-sided Lipschitz condition and σ and f satisfy global Lipschitz conditions. Our other aim is to prove that under our non-Lipschitz conditions the Euler–Maruyama discretisation still has a strong convergence order of one half.

We shall also consider the more general equation

$$\begin{aligned}X_t=\xi&+\int_0^t b(s,X_s)\,\mathrm{d}s+\int_0^t\sigma(s,X_s)\,\mathrm{d}W_s+\int_0^{t+}\int_{\mathbb{U}_0}f(s,X_{s-},u)\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u)\\&+\int_0^{t+}\int_{\mathbb{U}-\mathbb{U}_0}g(s,X_{s-},u)N_p(\mathrm{d}s\,\mathrm{d}u),\end{aligned}\tag{4}$$

where $g\colon \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d$ is a Borel measurable function.

In Section 3 we show Equation (4) has a (ξ, W, N_p)-pathwise unique strong solution. In Section 4 an example is given to illustrate our result. Section 5 is Appendix to justify some conditions.

Throughout the paper, C with or without indices will denote different positive constants (depending on the indices) whose values are not important.

2. Proof of Theorem 1.2

Before proceeding to our proof, we first prepare a crucial lemma.

Lemma 2.1. *Under* $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H}^1_f)$, *pathwise uniqueness holds for* Equation (1).

Proof. Consider any filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geqslant 0}, \tilde{P})$ carrying $\tilde{\xi}$, $\tilde{W}$ and $N_{\tilde{p}}$ such that the joint distribution of $(\tilde{\xi}, \tilde{W}, N_{\tilde{p}})$ is the same as that of the given (ξ, W, N_p). Assume that Y^1_t, Y^2_t are two weak solutions to Equation (1) with $\tilde{\xi}$, $\tilde{W}$ and $N_{\tilde{p}}$ instead of ξ, W and N_p. Set $Z_t := Y^1_t - Y^2_t$. Then

$$\begin{aligned} Z_t = {}& \int_0^t (b(s, Y^1_s) - b(s, Y^2_s))\,\mathrm{d}s + \int_0^t (\sigma(s, Y^1_s) - \sigma(s, Y^2_s))\,\mathrm{d}\tilde{W}_s \\ &+ \int_0^{t+}\int_{\mathbb{U}_0} (f(s, Y^1_{s-}, u) - f(s, Y^2_{s-}, u))\tilde{N}_{\tilde{p}}(\mathrm{d}s\,\mathrm{d}u). \end{aligned}$$

By Itô's formula we have that

$$|Z_t|^2 =: A^1_t + M^c_t + A^2_t + M^d_t + A^3_t,$$

where

$$\begin{aligned} A^1_t &= 2\sum_i \int_0^t Z^i_s(b^i(s, Y^1_s) - b^i(s, Y^2_s))\,\mathrm{d}s, \\ M^c_t &= 2\sum_{i,j} \int_0^t Z^i_s(\sigma^{ij}(s, Y^1_s) - \sigma^{ij}(s, Y^2_s))\,\mathrm{d}\tilde{W}^j_s, \\ A^2_t &= \int_0^t |\sigma(s, Y^1_s) - \sigma(s, Y^2_s)|^2\,\mathrm{d}s, \\ M^d_t &= \int_0^{t+}\int_{\mathbb{U}_0} (|Z_{s-} + f(s, Y^1_{s-}, u) - f(s, Y^2_{s-}, u)|^2 - |Z_{s-}|^2)\tilde{N}_{\tilde{p}}(\mathrm{d}s\,\mathrm{d}u), \\ A^3_t &= \int_0^{t+}\int_{\mathbb{U}_0} \Bigg(|Z_{s-} + f(s, Y^1_{s-}, u) - f(s, Y^2_{s-}, u)|^2 - |Z_{s-}|^2 \\ &\qquad - 2\sum_i Z^i_{s-}(f^i(s, Y^1_{s-}, u) - f^i(s, Y^2_{s-}, u))\Bigg)\nu(\mathrm{d}u)\,\mathrm{d}s. \end{aligned}$$

For A^1_t, A^2_t and A^3_t, by $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$, $(\mathbf{H}^1_f)$ and Taylor's formula it holds that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]}|A^1_t|\Big)+\mathbb{E}\Big(\sup_{t\in[0,T]}|A^2_t|\Big)+\mathbb{E}\Big(\sup_{t\in[0,T]}|A^3_t|\Big)\\
&\leqslant 2\mathbb{E}\int_0^T|Z_s|\,|b(s,Y^1_s)-b(s,Y^2_s)|\,\mathrm{d}s+\mathbb{E}\int_0^T\|\sigma(s,Y^1_s)-\sigma(s,Y^2_s)\|^2\,\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^T\int_{\mathbb{U}_0}|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^2\nu(\mathrm{d}u)\,\mathrm{d}s\\
&\leqslant 2\mathbb{E}\int_0^T\lambda(s)|Z_s|^2\kappa_1(|Z_s|)\,\mathrm{d}s+\mathbb{E}\int_0^T\lambda(s)|Z_s|^2\kappa_2(|Z_s|)\,\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^T\lambda(s)|Z_s|^2\kappa_3(|Z_s|)\,\mathrm{d}s\\
&\leqslant C\mathbb{E}\int_0^T\lambda(s)\rho_\eta(|Z_s|^2)\,\mathrm{d}s\\
&\leqslant C\int_0^T\lambda(s)\rho_\eta\Big(\mathbb{E}\Big(\sup_{u\in[0,s]}|Z_u|^2\Big)\Big)\,\mathrm{d}s,
\end{aligned}$$

where

$$\rho_\eta(x):=\begin{cases}x\log x^{-1}, & 0<x\leqslant\eta,\\ (\log\eta^{-1}-1)x+\eta, & x>\eta,\end{cases}$$

for $0<\eta<1/\mathrm{e}$.

Using $(\mathbf{H}_\sigma)$, Burkholder's inequality and Young's inequality gives that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[0,T]}|M^c_t|\Big)&\leqslant C\mathbb{E}\Big(\int_0^T|Z_s|^2|\sigma(s,Y^1_s)-\sigma(s,Y^2_s)|^2\,\mathrm{d}s\Big)^{1/2}\\
&\leqslant C\mathbb{E}\Big(\sup_{s\in[0,T]}|Z_s|^2\int_0^T\|\sigma(s,Y^1_s)-\sigma(s,Y^2_s)\|^2\,\mathrm{d}s\Big)^{1/2}\\
&\leqslant \frac14\mathbb{E}\Big(\sup_{s\in[0,T]}|Z_s|^2\Big)+C\mathbb{E}\int_0^T\lambda(s)\rho_\eta(|Z_s|^2)\,\mathrm{d}s\\
&\leqslant \frac14\mathbb{E}\Big(\sup_{s\in[0,T]}|Z_s|^2\Big)+C\int_0^T\lambda(s)\rho_\eta\Big(\mathbb{E}\Big(\sup_{u\in[0,s]}|Z_u|^2\Big)\Big)\,\mathrm{d}s.
\end{aligned}$$

By $(\mathbf{H}^1_f)$, Burkholder's inequality, Young's inequality and the mean value theorem we obtain that

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]}|M_t^d|\right)\\
&\leqslant C\mathbb{E}\left(\int_0^{T+}\int_{\mathbb{U}_0}(|Z_{s-}+f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^2-|Z_{s-}|^2)^2N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right)^{1/2}\\
&\leqslant C\mathbb{E}\left(\int_0^{T+}\int_{\mathbb{U}_0}|Z_{s-}|^2\,|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^2N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right.\\
&\qquad\qquad\left.+\int_0^{T+}\int_{\mathbb{U}_0}|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^4N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right)^{1/2}\\
&\leqslant C\mathbb{E}\left(\int_0^{T+}\int_{\mathbb{U}_0}|Z_{s-}|^2|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^2N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right)^{1/2}\\
&\quad+C\mathbb{E}\left(\int_0^{T+}\int_{\mathbb{U}_0}|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^4N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right)^{1/2}\\
&\leqslant C\mathbb{E}\left(\sup_{s\in[0,T]}|Z_{s-}|^2\int_0^{T+}\int_{\mathbb{U}_0}|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^2N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right)^{1/2}\\
&\quad+C\mathbb{E}\left(\sup_{s\in[0,T]}|Z_{s-}|^2\int_0^{T+}\int_{\mathbb{U}_0}\frac{|f(s,Y^1_{s-},u)-f(s,Y^2_{s-},u)|^4}{|Z_{s-}|^2}I_{\{Z_{s-}\neq 0\}}N_{\tilde p}(\mathrm{d}s\,\mathrm{d}u)\right)^{1/2}\\
&\leqslant \frac{2}{4}\mathbb{E}\left(\sup_{s\in[0,T]}|Z_s|^2\right)+C\int_0^T\lambda(s)\rho_\eta\left(\mathbb{E}\left(\sup_{u\in[0,s]}|Z_u|^2\right)\right)\mathrm{d}s.
\end{aligned}
$$

The above estimates give that

$$
\mathbb{E}\left(\sup_{t\in[0,T]}|Z_t|^2\right)\leqslant C\int_0^T\lambda(s)\rho_\eta\left(\mathbb{E}\left(\sup_{u\in[0,s]}|Z_u|^2\right)\right)\mathrm{d}s.
$$

Applying Lemma 3.6 in [8] yields the conclusion. □

We now give

Proof of Theorem 1.2. We divide the proof into four steps.

STEP 1. We define two processes.

Let

$$
0=t_0^n<t_1^n<t_2^n<\cdots<t_i^n<t_{i+1}^n<\cdots
$$

be a sequence of partitions of $\mathbb{R}_+$ such that for every $T>0$

$$
\sup_{i:\,t^n_{i+1}\leqslant T}(t^n_{i+1}-t^n_i)\to 0
$$

as $n \to \infty$. We define the Euler–Maruyama approximation as the process $\{Y_t^n\}$ satisfying

$$
\begin{aligned}
Y_t^n = \xi &+ \int_0^t b(s, Y^n_{k_n(s)})\,\mathrm{d}s + \int_0^t \sigma(s, Y^n_{k_n(s)})\,\mathrm{d}W_s \\
&+ \int_0^{t+}\int_{\mathbb{U}_0} f(s, Y^n_{k_n(s)}, u)\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u),
\end{aligned}
\tag{5}
$$

where $k_n(s) := t_i^n$ for $s \in [t_i^n, t_{i+1}^n)$.

Set

$$\zeta_t := \int_0^{t+}\int_{\mathbb{U}_0} u\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u) + \int_0^{t+}\int_{\mathbb{U}-\mathbb{U}_0} uN_p(\mathrm{d}s\,\mathrm{d}u).$$

STEP 2. We obtain some sequences.

Now take two subsequences $\{Y^l\}$, $\{Y^m\}$ of the approximation $\{Y^n\}$. To four sequences of processes $\{Y^l\}$, $\{Y^m\}$, $\{W^n;\, W^n = W\}$ and $\{\zeta^n;\, \zeta^n = \zeta\}$ on $(\Omega, \mathcal{F}, P)$, by Appendix and [12, Corollary 2, p. 13] there exist subsequences $l(j)$, $m(j)$, a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, carrying $\mathcal{B}([0,T]) \times \hat{\mathcal{F}}$-measurable sequences $\hat{Y}^{l(j)}$, $\bar{Y}^{m(j)}$, $\hat{W}^j$ and $\hat{\zeta}^j$, where $\mathcal{B}([0,T])$ denotes Borel σ-algebra on $[0,T]$, such that for every positive integer j finite dimensional distributions of $(\hat{Y}^{l(j)}, \bar{Y}^{m(j)}, \hat{W}^j, \hat{\zeta}^j)$ and $(Y^{l(j)}, Y^{m(j)}, W, \zeta)$ coincide, and for any $t \in [0,T]$

$$
\begin{aligned}
&\hat{Y}_t^{l(j)} \to \hat{Y}_t, \quad \hat{W}_t^j \to \hat{W}_t,\\
&\bar{Y}_t^{m(j)} \to \bar{Y}_t, \quad \hat{\zeta}_t^j \to \hat{\zeta}_t,
\end{aligned}
$$

in probability as $j \to \infty$, where $\hat{Y}$, $\bar{Y}$, $\hat{W}$ and $\hat{\zeta}$ are some $\mathcal{B}([0,T]) \times \hat{\mathcal{F}}$-measurable stochastic processes. By [12, Lemma 4, p. 65] and [11, Remark 128, p. 94] there exist Poisson point processes $\{\hat{p}^j\}$ and $\hat{p}$ such that

$$
\begin{aligned}
\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u) &= N_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u) - \nu(\mathrm{d}u)\,\mathrm{d}s,\\
\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u) &= N_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u) - \nu(\mathrm{d}u)\,\mathrm{d}s,\\
\hat{\zeta}_t^j &:= \int_0^{t+}\int_{\mathbb{U}_0} u\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u) + \int_0^{t+}\int_{\mathbb{U}-\mathbb{U}_0} uN_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u),\\
\hat{\zeta}_t &:= \int_0^{t+}\int_{\mathbb{U}_0} u\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u) + \int_0^{t+}\int_{\mathbb{U}-\mathbb{U}_0} uN_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u).
\end{aligned}
$$

Set

$$
\begin{aligned}
\hat{\mathcal{F}}_t^j &:= \sigma(\hat{Y}_s^{l(j)}, \hat{W}_s^j, N_{\hat{p}^j}((0,s], \mathrm{d}u), s \leqslant t),\\
\hat{\mathcal{F}}_t &:= \sigma(\hat{Y}_s, \hat{W}_s, N_{\hat{p}}((0,s], \mathrm{d}u), s \leqslant t).
\end{aligned}
$$

Then for every j, $(\hat{W}_t^j, N_{\hat{p}^j}((0,t], \mathrm{d}u), \hat{\mathcal{F}}_t^j)$ and $(\hat{W}_t, N_{\hat{p}}((0,t], \mathrm{d}u), \hat{\mathcal{F}}_t)$ are Wiener processes and Poisson processes.

STEP 3. We show that $\hat{Y}$ satisfies Equation (1).

$\{\hat{Y}^{l(j)}_t\}$ satisfies the following equation

$$
\begin{aligned}
\hat{Y}^{l(j)}_t = \hat{Y}^{l(j)}_0 &+ \int_0^t b(s, \hat{Y}^{l(j)}_{k_{l(j)}(s)})\, \mathrm{d}s + \int_0^t \sigma(s, \hat{Y}^{l(j)}_{k_{l(j)}(s)})\, \mathrm{d}\hat{W}^j_s \\
&+ \int_0^{t+}\int_{\mathbb{U}_0} f(s, \hat{Y}^{l(j)}_{k_{l(j)}(s)}, u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\, \mathrm{d}u).
\end{aligned}
\tag{6}
$$

By (15),

$$
\hat{\mathbb{E}}\left(\sup_{0\leqslant t\leqslant T}|\hat{Y}^{l(j)}_t|^2\right) = \mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|Y^{l(j)}_t|^2\right) \leqslant C_T.
$$

Because $\hat{Y}^{l(j)}_t$ converges to $\hat{Y}_t$ in probability for every t, we can choose a subsequence which is denoted by $l(j)$ again, such that P-a.s. as $j \to \infty$

$$
\hat{Y}^{l(j)}_t \to \hat{Y}_t, \quad t = r_k,\ k = 1, 2, \dots,
$$

where $\{r_k, k = 1, 2, \dots\} \subset [0, T]$ are rational numbers in $[0, T]$. By Fatou's lemma, it holds that

$$
\begin{aligned}
\hat{\mathbb{E}}\left(\sup_{0\leqslant t\leqslant T}|\hat{Y}_t|^2\right) &= \hat{\mathbb{E}}\left(\sup_k|\hat{Y}_{r_k}|^2\right) \leqslant \hat{\mathbb{E}}\left(\sup_k\left|\lim_{j\to\infty}\hat{Y}^{l(j)}_{r_k}\right|^2\right) \\
&\leqslant \liminf_{j\to\infty}\hat{\mathbb{E}}\left(\sup_k|\hat{Y}^{l(j)}_{r_k}|^2\right) = \liminf_{j\to\infty}\hat{\mathbb{E}}\left(\sup_{0\leqslant t\leqslant T}|\hat{Y}^{l(j)}_t|^2\right) \\
&\leqslant C_T.
\end{aligned}
$$

Thus it follows from Chebyshev's inequality that

$$
\begin{aligned}
\lim_{N\to\infty}\hat{P}\left\{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}| > N\right\} &\leqslant \lim_{N\to\infty}\hat{P}\left\{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_s| > N\right\} = 0, \\
\lim_{N\to\infty}\hat{P}\left\{\sup_{0\leqslant s\leqslant T}|\hat{Y}_{s-}| > N\right\} &\leqslant \lim_{N\to\infty}\hat{P}\left\{\sup_{0\leqslant s\leqslant T}|\hat{Y}_s| > N\right\} = 0.
\end{aligned}
$$

Therefore for any $\varepsilon > 0$ there exists a N such that

$$
\hat{P}\left\{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}| > N\right\} + \hat{P}\left\{\sup_{0\leqslant s\leqslant T}|\hat{Y}_s| > N\right\} < \frac{\varepsilon}{2}.
$$

If we prove for any $\delta > 0$

$$
\hat{P}\left\{1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}_s|\leqslant N}\left|\int_0^t b(s,\hat{Y}^{l(j)}_{k_{l(j)}(s)})\mathrm{d}s - \int_0^t b(s,\hat{Y}_s)\mathrm{d}s\right| > \frac{\delta}{3}\right\} < \frac{\varepsilon}{6},
\tag{7}
$$

$$(8)\quad \hat{P}\left\{1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}_s|\leqslant N}\left|\int_0^t \sigma(s,\hat{Y}^{l(j)}_{k_{l(j)}(s)})\mathrm{d}\hat{W}^j_s-\int_0^t \sigma(s,\hat{Y}_s)\mathrm{d}\hat{W}_s\right|>\frac{\delta}{3}\right\}<\frac{\varepsilon}{6},$$

$$(9)\quad \begin{aligned}\hat{P}\Bigg\{&1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}_s|\leqslant N}\Bigg|\int_0^{t+}\int_{\mathbb{U}_0} f(s,\hat{Y}^{l(j)}_{k_{l(j)}(s)},u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\\&\qquad-\int_0^{t+}\int_{\mathbb{U}_0} f(s,\hat{Y}_{s-},u)\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u)\Bigg|>\frac{\delta}{3}\Bigg\}<\frac{\varepsilon}{6},\end{aligned}$$

then

$$\begin{aligned}&\int_0^t b(s,\,\hat{Y}^{l(j)}_{k_{l(j)}(s)})\,\mathrm{d}s+\int_0^t \sigma(s,\,\hat{Y}^{l(j)}_{k_{l(j)}(s)})\,\mathrm{d}\hat{W}^j_s+\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}^{l(j)}_{k_{l(j)}(s)},\,u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\\&\to\int_0^t b(s,\,\hat{Y}_s)\,\mathrm{d}s+\int_0^t \sigma(s,\,\hat{Y}_s)\,\mathrm{d}\hat{W}_s+\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}_{s-},\,u)\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u)\end{aligned}$$

in probability. Taking the limit in (6) as $j\to\infty$, one sees that $\hat{Y}$ satisfies Equation (1).

Making use of $(\mathbf{H}_b)$ and $(\mathbf{H}_\sigma)$ and the convergence of $\hat{Y}^{l(j)}_t$ and $\hat{W}^j_t$ to $\hat{Y}_t$ and $\hat{W}_t$, respectively, we can, by the same way to the proof of the theorem of Section 3, Chapter 2 in [12], show that (7) and (8) hold.

To (9), we insert a stochastic integral $\int_0^{t+}\int_{\mathbb{U}_0} f(s,\hat{Y}_{s-},u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)$ (adapted to the filtration $(\hat{\mathcal{F}}_t)_{t\geqslant 0}$) and obtain

$$\begin{aligned}&\hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant T}|\hat{Y}_s|\leqslant N}\Bigg|\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}^{l(j)}_{k_{l(j)}(s)},\,u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\\&\qquad\qquad-\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}_{s-},\,u)\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u)\Bigg|>\frac{\delta}{3}\Bigg\}\\&\leqslant\hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\Bigg|\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}^{l(j)}_{k_{l(j)}(s)},\,u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\\&\qquad\qquad-\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}_{s-},\,u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\Bigg|>\frac{\delta}{6}\Bigg\}\\&\quad+\hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\Bigg|\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}_{s-},\,u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\\&\qquad\qquad-\int_0^{t+}\int_{\mathbb{U}_0} f(s,\,\hat{Y}_{s-},\,u)\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u)\Bigg|>\frac{\delta}{6}\Bigg\}\\&=:I_1+I_2.\end{aligned}$$

For I_1, by Chebyshev's inequality and Burkholder's inequality we obtain that

$$I_1\leqslant\frac{36}{\delta^2}\hat{\mathbb{E}}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\int_0^{t+}\int_{\mathbb{U}_0}|(s,\,\hat{Y}^{l(j)}_{k_{l(j)}(s)},\,u)-f(s,\,\hat{Y}_{s-},\,u)|^2\nu(\mathrm{d}u)\,\mathrm{d}s.$$

We claim that for every $s \in [0, t]$

$$1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\int_{\mathbb{U}_0}|(s, \hat{Y}^{l(j)}_{k_{l(j)}(s)}, u) - f(s, \hat{Y}_{s-}, u)|^2\nu(\mathrm{d}u) \to 0$$

in probability as $j \to \infty$. In fact, for any $\eta > 0$

$$\begin{aligned}
&\hat{P}\left\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\int_{\mathbb{U}_0}|(s, \hat{Y}^{l(j)}_{k_{l(j)}(s)}, u) - f(s, \hat{Y}_{s-}, u)|^2\nu(\mathrm{d}u) > \eta\right\}\\
&\leqslant \hat{P}\Big\{1_{|\hat{Y}^{l(j)}_{k_{l(j)}(s)}-\hat{Y}_{s-}|\leqslant h}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\\
&\qquad\cdot\int_{\mathbb{U}_0}|(s, \hat{Y}^{l(j)}_{k_{l(j)}(s)}, u) - f(s, \hat{Y}_{s-}, u)|^2\nu(\mathrm{d}u) > \eta\Big\}\\
&\quad + \hat{P}\{|\hat{Y}^{l(j)}_{k_{l(j)}(s)} - \hat{Y}_{s-}| > h\}.
\end{aligned}$$

Take a small enough $h > 0$ and then by $(\mathbf{H}^1_f)$ the claim is justified. Thus by dominated convergence theorem it holds that $I_1 < \varepsilon/12$.

To I_2, we calculate. For any $\varrho > 0$,

$$\begin{aligned}
I_2 \leqslant\;& \hat{P}\left\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\left|\int_0^{t+}\int_{\mathbb{U}_0\cap\{0<\|u\|_{\mathbb{U}}<\varrho\}} f(s,\hat{Y}_{s-},u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\right| > \frac{\delta}{18}\right\}\\
&+\hat{P}\left\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\left|\int_0^{t+}\int_{\mathbb{U}_0\cap\{0<\|u\|_{\mathbb{U}}<\varrho\}} f(s,\hat{Y}_{s-},u)\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u)\right| > \frac{\delta}{18}\right\}\\
&+\hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\Bigg|\int_0^{t+}\int_{\mathbb{U}_0\cap\{\|u\|_{\mathbb{U}}\geqslant\varrho\}} f(s,\hat{Y}_{s-},u)\tilde{N}_{\hat{p}^j}(\mathrm{d}s\,\mathrm{d}u)\\
&\qquad\qquad -\int_0^{t+}\int_{\mathbb{U}_0\cap\{\|u\|_{\mathbb{U}}\geqslant\varrho\}} f(s,\hat{Y}_{s-},u)\tilde{N}_{\hat{p}}(\mathrm{d}s\,\mathrm{d}u)\Bigg| > \frac{\delta}{18}\Bigg\}\\
=:\;& I_{21}+I_{22}+I_{23}.
\end{aligned}$$

And by Chebyshev's inequality and Burkholder's inequality

$$\begin{aligned}
&I_{21} + I_{22}\\
&\leqslant 2\cdot\left(\frac{18}{\delta}\right)^2\hat{\mathbb{E}}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\int_0^{t+}\int_{\mathbb{U}_0\cap\{0<\|u\|_{\mathbb{U}}<\varrho\}}|f(s, \hat{Y}_{s-}, u)|^2\nu(\mathrm{d}u)\,\mathrm{d}s.
\end{aligned}$$

Because $\hat{\mathbb{E}}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\int_0^{t+}\int_{\mathbb{U}_0\cap\{0<\|u\|_{\mathbb{U}}<\varrho\}}|f(s, \hat{Y}_{s-}, u)|^2\nu(\mathrm{d}u)\,\mathrm{d}s < \infty$, by Fubini's theorem

$$\begin{aligned}
&\hat{\mathbb{E}}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}\int_0^{t+}\int_{\mathbb{U}_0\cap\{0<\|u\|_{\mathbb{U}}<\varrho\}}|f(s, \hat{Y}_{s-}, u)|^2\nu(\mathrm{d}u)\,\mathrm{d}s\\
&= \int_{\mathbb{U}_0\cap\{0<\|u\|_{\mathbb{U}}<\varrho\}}\left(\hat{\mathbb{E}}\int_0^{t+}1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N}|f(s, \hat{Y}_{s-}, u)|^2\,\mathrm{d}s\right)\nu(\mathrm{d}u).
\end{aligned}$$

Noting $\{0 < \|u\|_{\mathbb{U}} < \varrho\} \downarrow \emptyset$ for $\varrho \downarrow 0$, by absolute continuity of the Lebesgue integral one can take a small enough $\varrho > 0$ such that $I_{21} + I_{22} < \varepsilon/24$.

Finally, we treat I_{23}. Consider the partition sequence on $[0,t]$: $0 = t_0^{\tilde{n}} < t_1^{\tilde{n}} < \cdots < t_{\tilde{n}}^{\tilde{n}} = t$, such that $\lim_{\tilde{n}\to\infty} \max_k (t_{k+1}^{\tilde{n}} - t_k^{\tilde{n}}) = 0$.

$$
\begin{aligned}
I_{23} \leqslant\ & \hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N} 1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N} \\
&\qquad \cdot \bigg| \int_0^{t+} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(s, \hat{Y}_{s-}, u) \tilde{N}_{\hat{p}^j}(\mathrm{d}s\, \mathrm{d}u) \\
&\qquad\quad - \sum_{k=0}^{\tilde{n}-1} \int_{t_k^{\tilde{n}}}^{t_{k+1}^{\tilde{n}}} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) \tilde{N}_{\hat{p}^j}(\mathrm{d}s\, \mathrm{d}u) \bigg| > \frac{\delta}{54} \Bigg\} \\
&+ \hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N} 1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N} \\
&\qquad \cdot \bigg| \sum_{k=0}^{\tilde{n}-1} \int_{t_k^{\tilde{n}}}^{t_{k+1}^{\tilde{n}}} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) \tilde{N}_{\hat{p}^j}(\mathrm{d}s\, \mathrm{d}u) \\
&\qquad\quad - \sum_{k=0}^{\tilde{n}-1} \int_{t_k^{\tilde{n}}}^{t_{k+1}^{\tilde{n}}} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) \tilde{N}_{\hat{p}}(\mathrm{d}s\, \mathrm{d}u) \bigg| > \frac{\delta}{54} \Bigg\} \\
&+ \hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N} 1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N} \\
&\qquad \cdot \bigg| \sum_{k=0}^{\tilde{n}-1} \int_{t_k^{\tilde{n}}}^{t_{k+1}^{\tilde{n}}} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) \tilde{N}_{\hat{p}}(\mathrm{d}s\, \mathrm{d}u) \\
&\qquad\quad - \int_0^{t+} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(s, \hat{Y}_{s-}, u) \tilde{N}_{\hat{p}}(\mathrm{d}s\, \mathrm{d}u) \bigg| > \frac{\delta}{54} \Bigg\} \\
\leqslant\ & 2 \cdot \left(\frac{54}{\delta}\right)^2 \hat{\mathbb{E}} 1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N} 1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N} \\
&\cdot \int_0^{t+} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} \bigg| f(s, \hat{Y}_{s-}, u) - \sum_{k=0}^{\tilde{n}-1} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) 1_{[t_k^{\tilde{n}}, t_{k+1}^{\tilde{n}})}(s) \bigg|^2 \nu(\mathrm{d}u)\, \mathrm{d}s \\
&+ \hat{P}\Bigg\{1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}^{l(j)}_{k_{l(j)}(s)}|\leqslant N} 1_{\sup_{0\leqslant s\leqslant t}|\hat{Y}_{s-}|\leqslant N} \\
&\qquad \cdot \sum_{k=0}^{\tilde{n}-1} \bigg| \int_{t_k^{\tilde{n}}}^{t_{k+1}^{\tilde{n}}} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) N_{\hat{p}^j}(\mathrm{d}s\, \mathrm{d}u) \\
&\qquad\quad - \int_{t_k^{\tilde{n}}}^{t_{k+1}^{\tilde{n}}} \int_{\mathbb{U}_0 \cap \{\|u\|_{\mathbb{U}} \geqslant \varrho\}} f(k_{\tilde{n}}(s), \hat{Y}_{k_{\tilde{n}}(s)}, u) N_{\hat{p}}(\mathrm{d}s\, \mathrm{d}u) \bigg| > \frac{\delta}{54} \Bigg\}.
\end{aligned}
$$

From $(\mathbf{H}_f^1)$, $(\mathbf{H}_f^2)$ and [12, Lemma 4, p. 65] it follows $I_{23} < \varepsilon/24$.

STEP 4. We show that Equation (1) has a (ξ, W, N_p)-pathwise unique strong solution.

In the same way as Step 3 we can prove that $\bar{Y}$ satisfies Equation (1). Since the initial values in both cases are the same ($\hat{Y}_0^{l(j)} = \bar{Y}_0^{m(j)}$ because $Y_0^{l(j)} = Y_0^{m(j)} = \xi$) and the joint distribution of the initial value, $\hat{W}$ and $N_{\hat{p}}$ coincides with distribution of ξ, W and N_p, by Lemma 2.1 we conclude that $\hat{Y}_t = \bar{Y}_t$ for all t (a.s.). Hence, by applying Lemma 1.1 in [3] we obtain that Y_t^n converges in probability to Y_t in $(\Omega, \mathcal{F}, P)$. By the same way as Step 3 it holds that Y satisfies Equation (1).

3. The convergence rate for the Euler–Maruyama approximation

In the section we consider the convergence rate for the Euler–Maruyama approximation $\{Y_t^n\}$ defined in (5), that is, for a fixed timestep Δt and $t_i^n = i\Delta t$,

$$
\begin{aligned}
Y_t^n = \xi &+ \int_0^t b(s, Y_{k_n(s)}^n)\, \mathrm{d}s + \int_0^t \sigma(s, Y_{k_n(s)}^n)\, \mathrm{d}W_s \\
&+ \int_0^{t+} \int_{\mathbb{U}_0} f(s, Y_{k_n(s)}^n, u) \tilde{N}_p(\mathrm{d}s\, \mathrm{d}u),
\end{aligned}
$$

where $k_n(s) = t_i^n$ for $s \in [t_i^n, t_{i+1}^n)$.

Theorem 3.1. *Suppose $\mathbb{U} = \mathbb{R}^d$ and b, σ and f satisfy those conditions in* Theorem 1.2. *Moreover, b, σ are independent of t and $f(t, x, u) = f_0(x)u$, where $f_0(x)$ is a real function in x. Then there exists a $T_0 > 0$ such that*

$$
\mathbb{E}\left(\sup_{0 \leqslant t \leqslant T_0} |Y_t^n - Y_t|^2 \right) = O(\Delta t),
$$

where $O(\Delta t)$ means that $O(\Delta t)/\Delta t$ is bounded.

Proof. Set $H_t := Y_t^n - Y_t$ and then H_t satisfies the following equation

$$
\begin{aligned}
H_t = &\int_0^t (b(Y_{k_n(s)}^n) - b(Y_s))\, \mathrm{d}s + \int_0^t \left(\sigma(Y_{k_n(s)}^n) - \sigma(Y_s)\right) \mathrm{d}W_s \\
&+ \int_0^{t+} \int_{\mathbb{U}_0} (f_0(Y_{k_n(s)-}^n) - f_0(Y_{s-}))u \tilde{N}_p(\mathrm{d}s\, \mathrm{d}u).
\end{aligned}
$$

By Itô's formula we obtain that

$$
|H_t|^2 =: J_1 + J_2 + J_3 + J_4 + J_5,
$$

where

$$J_1 = 2\sum_i \int_0^t H_s^i(b^i(Y^n_{k_n(s)}) - b^i(Y_s))\,\mathrm{d}s,$$

$$J_2 = 2\sum_{i,j} \int_0^t H_s^i(\sigma^{ij}(Y^n_{k_n(s)}) - \sigma^{ij}(Y_s))\,\mathrm{d}W_s^j,$$

$$J_3 = \int_0^t \|\sigma(Y^n_{k_n(s)}) - \sigma(Y_s)\|^2\,\mathrm{d}s,$$

$$J_4 = \int_0^{t+}\int_{\mathbb{U}_0} (|H_{s-} + f_0(Y^n_{k_n(s)-})u - f_0(Y_{s-})u|^2 - |H_{s-}|^2)\,\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u),$$

$$J_5 = \int_0^{t+}\int_{\mathbb{U}_0} \bigg(|H_{s-} + f_0(Y^n_{k_n(s)-})u - f_0(Y_{s-})u|^2 - |H_{s-}|^2 - 2\sum_i H^i_{s-}(f_0(Y^n_{k_n(s)-})u^i - f_0(Y_{s-})u^i)\bigg)\,\nu(\mathrm{d}u)\,\mathrm{d}s.$$

For $T > 0$ and J_1, J_3 and J_5, by the same technique as that of dealing with A_t^1, A_t^2 and A_t^3 in Lemma 2.1, one can get

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|J_1|\Big) + \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|J_3|\Big) + \mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|J_5|\Big)\\
&\leqslant C\mathbb{E}\int_0^T \rho_\eta(|H_s|^2)\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^T |H_s|^2\,\mathrm{d}s + 2\mathbb{E}\int_0^T \rho_\eta^2(|Y^n_{k_n(s)} - Y^n_s|)\,\mathrm{d}s\\
&\quad + C\mathbb{E}\int_0^T \rho_\eta(|Y^n_{k_n(s)} - Y^n_s|^2)\,\mathrm{d}s\\
&\leqslant C\int_0^T \rho_\eta\Big(\mathbb{E}\Big(\sup_{0\leqslant r\leqslant s}|H_r|^2\Big)\Big)\,\mathrm{d}s + \frac{1}{2}\int_0^T \mathbb{E}\Big(\sup_{0\leqslant r\leqslant s}|H_r|^2\Big)\,\mathrm{d}s\\
&\quad + 2\int_0^T \rho_\eta^2\bigg(\Big(\mathbb{E}\Big(\sup_{0\leqslant r\leqslant s}|Y^n_{k_n(r)} - Y^n_r|^2\Big)\Big)^{1/2}\bigg)\,\mathrm{d}s\\
&\quad + C\int_0^T \rho_\eta\Big(\mathbb{E}\Big(\sup_{0\leqslant r\leqslant s}|Y^n_{k_n(r)} - Y^n_r|^2\Big)\Big)\,\mathrm{d}s,
\end{aligned}\tag{10}$$

where Jensen's inequality is used in the last inequality.

For J_2, by the same means as that of dealing with M_t^1 in Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\leqslant t\leqslant T}|J_2|\Big) &\leqslant \frac{1}{4}\mathbb{E}\Big(\sup_{0\leqslant s\leqslant T}|H_s|^2\Big) + C\int_0^T \rho_\eta\Big(\mathbb{E}\Big(\sup_{0\leqslant r\leqslant s}|H_r|^2\Big)\Big)\,\mathrm{d}s\\
&\quad + C\int_0^T \rho_\eta\Big(\mathbb{E}\Big(\sup_{0\leqslant r\leqslant s}|Y^n_{k_n(r)} - Y^n_r|^2\Big)\Big)\,\mathrm{d}s.
\end{aligned}\tag{11}$$

For J_4, by the similar method to that of dealing with M_t^2 in Lemma 2.1, one can obtain

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|J_4|\right) \leqslant \frac{2}{4}\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|H_s|^2\right) + C\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|Y^n_{k_n(s)} - Y^n_s|^2\right) + C\int_0^T \rho_\eta\left(\mathbb{E}\left(\sup_{0\leqslant r\leqslant s}|H_r|^2\right)\right)\mathrm{d}s. \tag{12}$$

Next, for $t_i^n \leqslant s < t_{i+1}^n$, it follows from (5) that

$$\begin{aligned} Y_s^n &= Y^n_{t_i^n} + \int_{t_i^n}^s b(Y^n_{k_n(r)})\,\mathrm{d}r + \int_{t_i^n}^s \sigma(Y^n_{k_n(r)})\,\mathrm{d}W_r + \int_{t_i^n}^{s+}\int_{\mathbb{U}_0} f_0(Y^n_{k_n(r)-})u\,\tilde{N}_p(\mathrm{d}r\,\mathrm{d}u)\\ &= Y^n_{t_i^n} + b(Y^n_{t_i^n})(s-t_i^n) + \sigma(Y^n_{t_i^n})(W_s - W_{t_i^n}) + f_0(Y^n_{t_i^n})\int_{t_i^n}^{s+}\int_{\mathbb{U}_0} u\,\tilde{N}_p(\mathrm{d}r\,\mathrm{d}u). \end{aligned}$$

(15) and Burkholder's inequality admit us to get

$$\begin{aligned} \mathbb{E}\left(\sup_{t_i^n\leqslant s<t_{i+1}^n}|Y^n_{k_n(s)} - Y^n_s|^2\right) &\leqslant 3C|t^n_{i+1} - t^n_i|^2 + 3C\mathbb{E}\left(\sup_{t_i^n\leqslant s<t_{i+1}^n}|W_s - W_{t_i^n}|^2\right)\\ &\quad + 3C\mathbb{E}\left(\sup_{t_i^n\leqslant s<t_{i+1}^n}\left|\int_{t_i^n}^{s+}\int_{\mathbb{U}_0} u\,\tilde{N}_p(\mathrm{d}r\mathrm{d}u)\right|^2\right)\\ &\leqslant C\Delta t, \end{aligned} \tag{13}$$

where the last constant C is independent of Δt.

Combining (10), (11), (12) and (13), we have

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|H_t|^2\right) &\leqslant C\int_0^T \rho_\eta\left(\mathbb{E}\left(\sup_{0\leqslant r\leqslant s}|H_r|^2\right)\right)\mathrm{d}s + \frac{1}{2}\int_0^T \mathbb{E}\left(\sup_{0\leqslant r\leqslant s}|H_r|^2\right)\mathrm{d}s\\ &\quad + 2\int_0^T \rho_\eta^2\left(\left(\mathbb{E}\left(\sup_{0\leqslant r\leqslant s}|Y^n_{k_n(r)} - Y^n_r|^2\right)\right)^{1/2}\right)\mathrm{d}s\\ &\quad + C\int_0^T \rho_\eta\left(\mathbb{E}\left(\sup_{0\leqslant r\leqslant s}|Y^n_{k_n(r)} - Y^n_r|^2\right)\right)\mathrm{d}s\\ &\quad + C\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|Y^n_{k_n(s)} - Y^n_s|^2\right)\\ &\leqslant C\int_0^T \rho_\eta\left(\mathbb{E}\left(\sup_{0\leqslant r\leqslant s}|H_r|^2\right)\right)\mathrm{d}s + 2T\rho_\eta^2(C(\Delta t)^{1/2})\\ &\quad + CT\rho_\eta(C(\Delta t)^2) + C\Delta t. \end{aligned}$$

By Lemma 2.1 in [13] it holds that

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|H_t|^2\right)\leqslant A^{\exp\{-CT\}},$$

where $A = 2T\rho_\eta^2(C(\Delta t)^{1/2}) + CT\rho_\eta(C(\Delta t)^2) + C\Delta t$. Thus, there exists a $T_0 > 0$ such that

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T_0}|Y_t^n - Y_t|^2\right) = O(\Delta t).$$

The proof is completed. □

4. Existence of a (ξ, W, N_p)-pathwise unique strong solution for Equation (4)

Theorem 4.1. *Suppose* $\dim \mathbb{U} < \infty$. *Under* $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$, $(\mathbf{H}_f^1)$, $(\mathbf{H}_f^2)$ *and* $(\mathbf{H}_{b,\sigma,f})$, *then* Equation (4) *has a* (ξ, W, N_p)*-pathwise unique strong solution.*

Proof. let D_p be the domain of p_t and $D = \{s \in D_p : p_s \in \mathbb{U} - \mathbb{U}_0\}$. Since $\nu(\mathbb{U} - \mathbb{U}_0) < \infty$, D is a discrete set in $(0, \infty)$ a.s. Set $\sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots$ be the enumeration of all elements in D. It is easy to see that σ_n is a stopping time for each n and $\lim_{n\to\infty}\sigma_n = +\infty$ a.s. (We disregard the trivial case of $\nu(\mathbb{U} - \mathbb{U}_0) = 0$.)

Set

$$X_t^1 = \begin{cases} Y_t, & t \in [0, \sigma_1), \\ Y_{\sigma_1-} + g(\sigma_1, Y_{\sigma_1-}, p_{\sigma_1}), & t = \sigma_1. \end{cases}$$

The process $\{X_t^1, t \in [0, \sigma_1]\}$ is clearly the unique solution of Equation (4) in the time interval $[0, \sigma_1]$. Next, set $\tilde{\mathcal{F}}_t = \mathcal{F}_{t+\sigma_1}$, $\tilde{X}_0 = X_{\sigma_1}^1$, $\tilde{W}_t = W_{t+\sigma_1} - W_{\sigma_1}$, $\tilde{p}_t = p_{t+\sigma_1}$ and $\tilde{\sigma}_1 = \sigma_2 - \sigma_1$. We can determine the process $\tilde{X}_t^2$ on $[0, \tilde{\sigma}_1]$ with respect to $\tilde{\mathcal{F}}_t$, $\tilde{X}_0$, $\tilde{W}_t$, $\tilde{p}_t$ in the same way as X_t^1. Define X_t by

$$X_t = \begin{cases} X_t^1, & t \in [0, \sigma_1], \\ \tilde{X}_{t-\sigma_1}^2, & t \in [\sigma_1, \sigma_2]. \end{cases}$$

It is easy to see that $\{X_t, t \in [0, \sigma_2]\}$ is the unique solution of Equation (4) in the time interval $[0, \sigma_2]$. Continuing this process, X_t is determined uniquely in the time interval $[0, \sigma_n]$ for every n and hence X_t is determined globally. Thus the proof is complete. □

5. An example

Let

$$
\begin{aligned}
&b(t,x) := \lambda(t)\sum_{k\geqslant 1}\frac{\sin(kx)}{k^2},\\
&\sigma(t,x) := \sqrt{\lambda(t)}(1^{-3/2}\sin x, 2^{-3/2}\sin 2x, \ldots, m^{-3/2}\sin mx),\\
&f(t,x,u) = \sqrt{\lambda(t)}\left(\sum_{k\geqslant 1}\frac{\sin^2(kx)}{k^3}\right)\|u\|_{\mathbb{U}},\\
&\mathbb{U}_0 = \{u\in\mathbb{U},\ \|u\|_{\mathbb{U}}\leqslant 1\},\\
&g(t,x,u) = 0,
\end{aligned}
$$

where $\lambda(t)$ is continuous, bounded on (0, 1] and locally square integrable. Then by Lemma 3.1 and 4.1 in [1] and the Hölder inequality

$$
\begin{aligned}
|b(t,x)-b(t,y)| &\leqslant \lambda(t)\sum_{k\geqslant 1}\frac{|\sin(kx)-\sin(ky)|}{k^2} \leqslant 2\lambda(t)\sum_{k\geqslant 1}\frac{|\sin(k(x-y))/2|}{k^2}\\
&\leqslant C\lambda(t)|x-y|\tilde{\kappa}_1(|x-y|),\\
\|\sigma(t,x)-\sigma(t,y)\|^2 &= \lambda(t)\sum_{k=1}^{m}\frac{|\sin(kx)-\sin(ky)|^2}{k^3}\\
&\leqslant \lambda(t)\sum_{k\geqslant 1}\frac{|\sin(kx)-\sin(ky)|^2}{k^3} \leqslant 4\lambda(t)\sum_{k\geqslant 1}\frac{|\sin(k(x-y))/2|^2}{k^3}\\
&\leqslant C\lambda(t)|x-y|^2\tilde{\kappa}_2(|x-y|),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{U}_0}|f(t,x,u)-f(t,y,u)|^2\nu(\mathrm{d}u)\\
&\leqslant C\lambda(t)\left(\sum_{k\geqslant 1}\left|\frac{\sin^2(kx)-\sin^2(ky)}{k^3}\right|\right)^2\\
&\leqslant C\lambda(t)\left(\sum_{k\geqslant 1}\frac{|\sin(kx)-\sin(ky)|^2}{k^3}\right)\left(\sum_{k\geqslant 1}\frac{|\sin(kx)+\sin(ky)|^2}{k^3}\right)\\
&\leqslant C\lambda(t)\sum_{k\geqslant 1}\frac{|\sin(kx)-\sin(ky)|^2}{k^3}\\
&\leqslant C\lambda(t)|x-y|^2\tilde{\kappa}_2(|x-y|),
\end{aligned}
$$

where

$$\tilde{\kappa}_1(x) := \begin{cases} \log x^{-1}, & 0 < x \leqslant \eta, \\ \log \eta^{-1} - 1 + \dfrac{\eta}{x}, & x > \eta, \end{cases}$$

and

$$\tilde{\kappa}_2(x) := \begin{cases} \log x^{-1}, & 0 < x \leqslant \eta, \\ \dfrac{(((\log \eta^{-1})^{1/2} - (1/2)(\log \eta^{-1})^{-1/2})x + (1/2)(\log \eta^{-1})^{-1/2}\eta)^2}{x^2}, & x > \eta. \end{cases}$$

We take $\kappa_1(x) := C\tilde{\kappa}_1(x)$ and $\kappa_2(x) := C\tilde{\kappa}_2(x)$. It is easily justified that $\kappa_1(x)$ and $\kappa_2(x)$ satisfy (2).

Note that $b(t,x)$ does not satisfy the condition (3) because $\log x^{-1} < (\log x^{-1})^2$ for $0 < x \leqslant \eta$. Thus our result generalizes one in [2] in some sense.

6. Appendix

We show that $\{Y^n\}$, $\{W^n; W^n = W\}$ and $\{\zeta^n; \zeta^n = \zeta\}$ satisfy conditions in [12, Corollary 2, p. 13], i.e.

$$\begin{cases} \text{(i)} \quad \lim\limits_{N\to\infty} \lim\limits_{n\to\infty} \sup\limits_{0\leqslant t\leqslant T} P\{|\eta_t^n| > N\} = 0, \\ \text{(ii)} \quad \forall \delta > 0, \ \lim\limits_{h\to 0} \lim\limits_{n\to\infty} \sup\limits_{|s-t|\leqslant h} P\{|\eta_s^n - \eta_t^n| > \delta\} = 0. \end{cases} \tag{14}$$

Firstly we deal with $\{Y^n\}$. It holds by Burkholder's inequality and $(\mathbf{H}_{b,\sigma,f})$ that

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |Y_t^n|^2\right) &\leqslant 4\mathbb{E}|\xi|^2 + C\int_0^T \left(1 + \mathbb{E}\left(\sup_{0\leqslant s\leqslant t} |Y_{k_n(s)}^n|^2\right)\right) \mathrm{d}t \\ &\leqslant 4\mathbb{E}|\xi|^2 + C\int_0^T \left(1 + \mathbb{E}\left(\sup_{0\leqslant s\leqslant t} |Y_s^n|^2\right)\right) \mathrm{d}t. \end{aligned}$$

Gronwall's inequality gives that

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |Y_t^n|^2\right) \leqslant C_T, \tag{15}$$

where C_T is independent of n. Then applying Chebyshev's inequality yields that

$$\begin{aligned} \lim_{N\to\infty} \lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} P\{|Y_t^n| > N\} &\leqslant \lim_{N\to\infty} \lim_{n\to\infty} P\left\{\sup_{0\leqslant t\leqslant T} |Y_t^n| > N\right\} \\ &\leqslant \lim_{N\to\infty} \lim_{n\to\infty} \frac{1}{N^2}\mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |Y_t^n|^2\right) \\ &\leqslant \lim_{N\to\infty} \frac{1}{N^2} \sup_n \mathbb{E}\left(\sup_{0\leqslant t\leqslant T} |Y_t^n|^2\right). \end{aligned}$$

Assume $t > s$ and then

$$Y_t^n - Y_s^n = \int_s^t b(r, Y_{k_n(r)}^n)\,\mathrm{d}r + \int_s^t \sigma(r, Y_{k_n(r)}^n)\,\mathrm{d}W_r + \int_s^{t+}\int_{\mathbb{U}_0} f(r, Y_{k_n(r)}^n, u)\,\tilde{N}_p(\mathrm{d}r\,\mathrm{d}u).$$

By $(\mathbf{H}_{b,\sigma,f})$, (15) and Burkholder's inequality we obtain that

$$\mathbb{E}|Y_s^{(n)} - Y_t^{(n)}|^2 \leqslant C_T|s-t|,$$

where C_T is independent of n. Then for any $\delta > 0$,

$$\begin{aligned}\lim_{h\to 0}\lim_{n\to\infty}\sup_{|s-t|\leqslant h} P\{|Y_s^n - Y_t^n| > \delta\} &\leqslant \lim_{h\to 0}\lim_{n\to\infty}\sup_{|s-t|\leqslant h}\frac{1}{\delta^2}\mathbb{E}|Y_s^n - Y_t^n|^2\\ &\leqslant \lim_{h\to 0}\sup_n\sup_{|s-t|\leqslant h}\frac{1}{\delta^2}\mathbb{E}|Y_s^n - Y_t^n|^2.\end{aligned}$$

By $\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}|W_t|^2\big) \leqslant 4T$ and $\mathbb{E}(|W_s - W_t|^2) = |s-t|$ we know that $\{W_t,\ t\geqslant 0\}$ satisfies (14).

Because $\nu(\mathbb{U}-\mathbb{U}_0) < \infty$, by Chebyshev's inequality and Burkholder's inequality it holds that

$$\begin{aligned}\lim_{N\to\infty}\sup_{0\leqslant t\leqslant T} P\{\|\zeta_t\|_{\mathbb{U}} > N\} &\leqslant \lim_{N\to\infty} P\left\{\sup_{0\leqslant t\leqslant T}\|\zeta_t\|_{\mathbb{U}} > N\right\}\\ &\leqslant \lim_{N\to\infty} P\left\{\sup_{0\leqslant t\leqslant T}\left\|\int_0^{t+}\int_{\mathbb{U}_0} u\,\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u)\right\|_{\mathbb{U}} > \frac{N}{2}\right\}\\ &\quad + \lim_{N\to\infty} P\left\{\sup_{0\leqslant t\leqslant T}\left\|\int_0^{t+}\int_{\mathbb{U}-\mathbb{U}_0} u\,N_p(\mathrm{d}s\,\mathrm{d}u)\right\|_{\mathbb{U}} > \frac{N}{2}\right\}\\ &\leqslant \lim_{N\to\infty}\frac{4}{N^2}\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\left\|\int_0^{t+}\int_{\mathbb{U}_0} u\,\tilde{N}_p(\mathrm{d}s\,\mathrm{d}u)\right\|_{\mathbb{U}}^2\right)\\ &\quad + \lim_{N\to\infty} P\left\{\sum_{0\leqslant t\leqslant T}\|p(t)\|_{\mathbb{U}} 1_{\mathbb{U}-\mathbb{U}_0}(p(t)) > \frac{N}{2}\right\}\\ &\leqslant \lim_{N\to\infty}\frac{16}{N^2}\int_0^T\int_{\mathbb{U}_0}\|u\|_{\mathbb{U}}^2\,\nu(\mathrm{d}u)\,\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to 0}\sup_{|s-t|\leqslant h} P\{\|\zeta_s-\zeta_t\|_{\mathbb{U}}>\delta\} &\leqslant \lim_{h\to 0}\sup_{|s-t|\leqslant h} P\left\{\left\|\int_s^{t+}\int_{\mathbb{U}_0} u\,\tilde{N}_p(\mathrm{d}r\,\mathrm{d}u)\right\|_{\mathbb{U}}>\frac{\delta}{2}\right\}\\
&\quad+\lim_{h\to 0}\sup_{|s-t|\leqslant h} P\left\{\left\|\int_s^{t+}\int_{\mathbb{U}-\mathbb{U}_0} u\,N_p(\mathrm{d}r\,\mathrm{d}u)\right\|_{\mathbb{U}}>\frac{\delta}{2}\right\}\\
&\leqslant \lim_{h\to 0}\sup_{|s-t|\leqslant h}\frac{4}{\delta^2}\mathbb{E}\left(\left\|\int_s^{t+}\int_{\mathbb{U}_0} u\,\tilde{N}_p(\mathrm{d}r\,\mathrm{d}u)\right\|_{\mathbb{U}}^2\right)\\
&\quad+\lim_{h\to 0}\sup_{|s-t|\leqslant h} P\left\{\sum_{s<r\leqslant t}|p(r)|_{\mathbb{U}}1_{\mathbb{U}-\mathbb{U}_0}(p(r))>\frac{\delta}{2}\right\}\\
&\leqslant \lim_{h\to 0}\sup_{|s-t|\leqslant h}\frac{4}{\delta^2}\int_s^t\int_{\mathbb{U}_0}\|u\|_{\mathbb{U}}^2\,\nu(\mathrm{d}u)\,\mathrm{d}r\\
&\quad+\lim_{h\to 0}\sup_{|s-t|\leqslant h} P\{N_p((s,t]\times(\mathbb{U}-\mathbb{U}_0))\geqslant 1\}\\
&\leqslant \lim_{h\to 0}\sup_{|s-t|\leqslant h}(1-\mathrm{e}^{-|s-t|\nu(\mathbb{U}-\mathbb{U}_0)}).
\end{aligned}$$

From this we obtain that $\{\zeta_t,\ t\geqslant 0\}$ satisfies (14).

ACKNOWLEDGEMENTS. The author would like to thank Professor Xicheng Zhang for his valuable discussions and also wish to thank referees for suggestions and improvements.

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