Wiemeler, M.
Osaka J. Math.
50 (2013), 153–160

QUASITORIC MANIFOLDS HOMEOMORPHIC TO HOMOGENEOUS SPACES

MICHAEL WIEMELER

(Received February 4, 2011, revised May 30, 2011)

Abstract

We present some classification results for quasitoric manifolds M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which admit an action of a compact connected Lie-group G such that $\dim M/G \leq 1$. In contrast to Kuroki's work [7, 6] we do not require that the action of G extends the torus action on M.

1. Introduction

Quasitoric manifolds are certain $2n$-dimensional manifolds on which an n-dimensional torus acts such that the orbit space of this action may be identified with a simple convex polytope. They were first introduced by Davis and Januszkiewicz [2] in 1991.

In [7, 6] Kuroki studied quasitoric manifolds M which admit an extension of the torus action to an action of some compact connected Lie-group G such that $\dim M/G \leq 1$. Here we drop the condition that the G-action extends the torus action in the case where the first Pontrjagin-class of M is equal to the negative of a sum of squares of elements of $H^2(M)$. In this note all cohomology groups are taken with coefficients in $\mathbb{Q}$. We have the following two results.

Theorem 1.1. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which is homeomorphic (or diffeomorphic) to a homogeneous space G/H with G a compact connected Lie-group. Then M is homeomorphic (diffeomorphic) to $\prod S^2$. In particular, all Pontrjagin-classes of M vanish.*

Theorem 1.2. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Assume that the compact connected Lie-group G acts smoothly and almost effectively on M such that $\dim M/G = 1$. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2-bundle over a product of two-spheres.*

2000 Mathematics Subject Classification. 57S15, 57S25.

Part of the research was supported by SNF Grants Nos. 200021-117701 and 200020-126795.

The proofs of these theorems are based on Hauschild's study [4] of spaces of q-type. A space of q-type is defined to be a topological space X satisfying the following cohomological properties:

- The cohomology ring $H^*(X)$ is generated as a $\mathbb{Q}$-algebra by elements of degree two, i.e. $H^*(X) = \mathbb{Q}[x_1, \ldots, x_n]/I_0$ and $\deg x_i = 2$.
- The defining ideal I_0 contains a definite quadratic form Q.

The note is organised as follows. In Section 2 we show that a quasitoric manifold M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ is of q-type. In Section 3 we prove Theorem 1.1. In Section 4 we recall some properties of cohomogeneity one manifolds. In Section 5 we prove Theorem 1.2.

The results presented in this note form part of the outcome of my Ph.D. thesis [10] written under the supervision of Prof. Anand Dessai at the University of Fribourg. I would like to thank Anand Dessai for helpful discussions.

2. Quasitoric manifolds with $p_1(M) = -\sum a_i^2$

In this section we study quasitoric manifolds M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. To do so we first introduce some notations from [4] and [5, Chapter VII]. For a topological space X we define the topological degree of symmetry of X as

$$N_t(X) = \max\{\dim G;\ G \text{ compact Lie-group, } G \text{ acts effectively on } X\}.$$

Similarly one defines the semi-simple degree of symmetry of X as

$$N_t^{ss}(X) = \max\{\dim G;\ G \text{ compact semi-simple Lie-group, } G \text{ acts effectively on } X\}$$

and the torus-degree of symmetry as

$$T_t(X) = \max\{\dim T;\ T \text{ torus, } T \text{ acts effectively on } X\}.$$

In the above definitions we assume that all groups act continuously.

Another important invariant of a topological space X used in [4] is the so called embedding dimension of its rational cohomology ring. For a local $\mathbb{Q}$-algebra A, we denote by $\operatorname{edim} A$ the embedding dimension of A. By definition, we have $\operatorname{edim} A = \dim_{\mathbb{Q}} \mathfrak{m}_A/\mathfrak{m}_A^2$, where $\mathfrak{m}_A$ is the maximal ideal of A. In case that $A = \bigoplus_{i\geq 0} A^i$ is a positively graded local $\mathbb{Q}$-algebra, $\mathfrak{m}_A$ is the augmentation ideal $A_+ = \bigoplus_{i>0} A^i$. If furthermore A is generated by its degree two part, then $\mathfrak{m}_A^2 = \bigoplus_{i>2} A^i$. Therefore for a quasitoric manifold M over the polytope P we have $\operatorname{edim} H^*(M) = \dim_{\mathbb{Q}} H^2(M) = m - n$ where m is the number of facets of P and n is its dimension.

Lemma 2.1. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Then M is a manifold of q-type.*

Proof. The discussion at the beginning of Section 3 of [8] together with Corollary 6.8 of [2, p. 448] shows that there are a basis $u_{n+1}, \dots, u_m$ of $H^2(M)$ and $\lambda_{i,j} \in \mathbb{Z}$ such that

$$p_1(M) = \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left(\sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2.$$

Therefore

$$\begin{aligned} 0 &= \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left(\sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2 + \sum_i a_i^2 \\ &= \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left(\sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2 + \sum_j \left(\sum_{i=n+1}^{m} \mu_{i,j} u_i \right)^2 \end{aligned}$$

with some $\mu_{i,j} \in \mathbb{Q}$ follows.

Because

$$\sum_{i=n+1}^{m} X_i^2 + \sum_{j=1}^{n} \left(\sum_{i=n+1}^{m} \lambda_{i,j} X_i \right)^2 + \sum_j \left(\sum_{i=n+1}^{m} \mu_{i,j} X_i \right)^2$$

is a positive definite bilinear form the statement follows. □

Proposition 2.2. *Let M be a quasitoric manifold of q-type over the n-dimensional polytope P. Then we have for the number m of facets of P:*

$$m \geq 2n$$

Proof. By Theorem 3.2 of [4, p. 563], we have

$$n \leq T_t(M) \leq \operatorname{edim} H^*(M) = m - n.$$

Therefore we have $2n \leq m$. □

REMARK 2.3. The inequality in the above proposition is sharp, because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $p_1(M) = 0$.

By Theorem 5.13 of [4, p. 573], we have for a manifold M of q-type that $N_t^{ss} \leq \dim M + \operatorname{edim} M$. Hence, for a quasitoric manifold M, we get:

Proposition 2.4. *Let M as in* Proposition 2.2. *Then we have*

$$N_t^{ss}(M) \leq 2n + m - n = n + m.$$

REMARK 2.5. The inequality in the above proposition is sharp because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $SU(2) \times \cdots \times SU(2)$ acts on M and has dimension $3n$.

3. Quasitoric manifolds which are also homogeneous spaces

In this section we prove Theorem 1.1. Recall from Lemma 2.1 that a quasitoric manifold M with first Pontrjagin-class equal to the negative of the sum of squares of elements of $H^2(M)$ is a manifold of q-type.

Let M be a quasitoric manifold over the polytope P which is also a homogeneous space and is of q-type.

Let G be a compact connected Lie-group and $H \subset G$ a closed subgroup such that M is homeomorphic or diffeomorphic to G/H. Because $\chi(M) > 0$ and M is simply connected, we have $\operatorname{rank} G = \operatorname{rank} H$ and H is connected. Therefore we may assume that G is semi-simple and simply connected.

Let T be a maximal torus of G. Then $(G/H)^T$ is non-empty. By Theorem 5.9 of [4, p. 572], the isotropy group G_x of a point $x \in (G/H)^T$ is a maximal torus of G. Hence, H is a maximal torus of G.

Now it follows from Theorem 3.3 of [4, p. 563] that

$$T_t(G/H) = \operatorname{rank} G.$$

Because M is quasitoric, we have $n \leq T_t(G/H)$. Combining these inequations, we get

$$\dim G - \dim H = \dim M = 2n \leq 2 \operatorname{rank} G.$$

This equation implies that $\dim G \leq 3 \operatorname{rank} G$.

For a simple simply connected Lie-group G' we have $\dim G' \geq 3 \operatorname{rank} G'$ and $\dim G' = 3 \operatorname{rank} G'$ if and only if $G' = SU(2)$. Therefore we have $G = \prod SU(2)$ and $M = \prod SU(2)/T^1 = \prod S^2$. This proves Theorem 1.1.

4. Cohomogeneity one manifolds

Here we discuss some facts about closed cohomogeneity one Riemannian G-manifolds M with orbit space a compact interval $[-1, 1]$. We follow [3, p. 39-44] in this discussion.

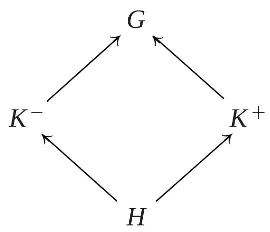
We fix a normal geodesic $c\colon [-1, 1] \to M$ perpendicular to all orbits. We denote by H the principal isotropy group $G_{c(0)}$, which is equal to the isotropy group $G_{c(t)}$ for $t \in \,]-1, 1[$, and by $K^{\pm}$ the isotropy groups of $c(\pm 1)$.

Then M is the union of tabular neighbourhoods of the non-principal orbits $Gc(\pm 1)$ glued along their boundary, i.e., by the slice theorem we have

$$M = G \times_{K^-} D_- \cup G \times_{K^+} D_+, \tag{4.1}$$

where $D_\pm$ are discs. Furthermore $K^\pm/H = \partial D_\pm = S_\pm$ are spheres.

Note that M may be reconstructed from the following diagram of groups.



The construction of such a group diagram from a cohomogeneity one manifold may be reversed. Namely, if such a group diagram with $K^\pm/H = S_\pm$ spheres is given, then one may construct a cohomogeneity one G-manifold from it. We also write these diagrams as $H \subset K^-, K^+ \subset G$.

Now we give a criterion for two group diagrams yielding up to G-equivariant diffeomorphism the same manifold M.

Lemma 4.1 ([3, p. 44]). *The group diagrams $H \subset K^-$, $K_1^+ \subset G$ and $H \subset K^-$, $K_2^+ \subset G$ yield the same cohomogeneity one manifold up to equivariant diffeomorphism if there is an $a \in N_G(H)^0$ with $K_1^+ = aK_2^+a^{-1}$.*

5. Quasitoric manifolds with cohomogeneity one actions

In this section we study quasitoric manifolds M which admit a smooth action of a compact connected Lie-group G which has an orbit of codimension one. As before we do not assume that the G-action on M extends the torus action. We have the following lemma:

Lemma 5.1. *Let M be a quasitoric manifold of dimension $2n$ which is of q-type. Assume that the compact connected Lie-group G acts almost effectively and smoothly on M such that $\dim M/G = 1$. Then we have*:

(1) *The singular orbits are given by G/T where T is a maximal torus of G.*
(2) *The Euler-characteristic of M is $2\,\#W(G)$.*
(3) *The principal orbit type is given by G/S, where $S \subset T$ is a subgroup of codimension one.*
(4) *The center Z of G has dimension at most one.*
(5) $\dim G/T = 2n - 2$.

Proof. At first note that M/G is an interval $[-1, 1]$ and not a circle because M is simply connected. We start with proving (1). Let T be a maximal torus of G. By passing to a finite covering group of G we may assume $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus. Let $x \in M^T$. Then the isotropy group G_x has maximal rank in G. Therefore G_x splits as $G'_x \times Z'$.

By Theorem 5.9 of [4, p. 572], G'_x is a maximal torus of G'. Therefore we have $G_x = T$.

Because $\dim G - \dim T$ is even, x is contained in a singular orbit. In particular we have

$$\chi(M) = \chi(M^T) = \chi(G/K^+) + \chi(G/K^-), \tag{5.1}$$

where $G/K^{\pm}$ are the singular orbits. Furthermore we may assume that G/K^+ contains a T-fixed point. This implies

$$\chi(G/K^+) = \chi(G/T) = \#W(G) = \#W(G'). \tag{5.2}$$

Now assume that all T-fixed points are contained in the singular orbit G/K^+. Then we have $(G/K^-)^T = \emptyset$. This implies

$$\chi(M) = \chi(G/K^+) = \#W(G').$$

Now Theorem 5.11 of [4, p. 573] implies that M is the homogeneous space $G'/G' \cap T = G/T$. This contradicts our assumption that $\dim M/G = 1$.

Therefore both singular orbits contain T-fixed points. This implies that they are of type G/T. This proves (1). (2) follows from (5.1) and (5.2).

Now we prove (3) and (5). Let $S \subset T$ be a minimal isotropy group. Then T/S is a sphere of dimension $\operatorname{codim}(G/T, M) - 1$. Therefore S is a subgroup of codimension one in T and $\operatorname{codim}(G/T, M) = 2$.

If the center of G has dimension greater than one, then $\dim Z' \cap S \geq 1$. That means that the action is not almost effective. Therefore (4) holds. □

By Lemma 5.1, we have with the notation of the previous section that $K^{\pm}$ are maximal tori of G containing $H = S$. In the following we will write $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus.

Because $K^{\pm}$ are maximal tori of the identity component $Z_G(S)^0$ of the centraliser of S, there is some $a \in Z_G(S)^0$ such that $K^- = aK^+a^{-1}$. By Lemma 4.1, we may assume that $K^+ = K^- = T$. Now from Theorem 4.1 of [9, p. 198] it follows that M is a fiber bundle over G/T with fiber the cohomogeneity one manifold with group diagram $S \subset T$, $T \subset T$. Therefore it is a S^2-bundle over G/T.

Lemma 5.2. *Let M and G as in the previous lemma. Then we have*

$$T_t(M) \leq \operatorname{rank} G' + 1.$$

Proof. At first we recall the rational cohomology of G/T. By [1, p. 67], we have

$$H^*(G/T) \cong H^*(BT)/I$$

where I is the ideal generated by the elements of positive degree which are invariant under the action of the Weyl-group of G. Therefore it follows that

$$\dim_{\mathbb{Q}} H^{\mathrm{odd}}(G/T) = 0 \quad \text{and} \quad \dim_{\mathbb{Q}} H^2(G/T) = \operatorname{rank} G'.$$

Therefore the Serre spectral sequence for the fibration $S^2 \to M \to G/T$ degenerates. Hence, we have

$$H^*(M) = H^*(G/T) \otimes H^*(S^2)$$

as $H^*(G/T)$-modules. In particular, we have

$$\dim_{\mathbb{Q}} H^2(M) = \dim_{\mathbb{Q}} H^2(G/T) + \dim_{\mathbb{Q}} H^2(S^2) = \operatorname{rank} G' + 1.$$

Therefore

$$T_t(M) \leq \operatorname{edim} H^*(M) = \dim_{\mathbb{Q}} H^2(M) = \operatorname{rank} G' + 1$$

follows. □

Theorem 5.3. *Let M and G as in the previous lemmas. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2-bundle over a product of two-spheres.*

Proof. Because M is quasitoric we have $n \leq T_t(M)$. By Lemma 5.1 we have

$$\dim G' - \operatorname{rank} G' = \dim G/T = 2n - 2.$$

Now Lemma 5.2 implies

$$\dim G' = 2n - 2 + \operatorname{rank} G' \leq 3 \operatorname{rank} G'.$$

Therefore $\prod SU(2)$ is a finite covering group of G'. This implies the statement about the finite covering group of G.

It follows that $G/T = \prod S^2$. Therefore M is a S^2-bundle over $\prod S^2$. □

Now Theorem 1.2 follows from Theorem 5.3 and Lemma 2.1.

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The University of Manchester
Oxford Road, Manchester M13 9PL
U.K.

Current address:
Institut für Algebra und Geometrie
Karlsruher Institut für Technologie
Kaiserstrasse 89–93
D-76133 Karlsruhe
Germany
e-mail: michael.wiemeler@kit.edu