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ON THE CLASSIFICATION OF HOMOGENEOUS 2-SPHERES IN COMPLEX GRASSMANNIANS

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Abstract

In this paper we discuss a classification problem of homogeneous 2-spheres in the complex Grassmann manifold $G(k+1, n+1)$ by theory of unitary representations of the 3-dimensional special unitary group $SU(2)$. First we observe that if an immersion $x\colon S^2 \to G(k+1, n+1)$ is homogeneous, then its image $x(S^2)$ is a 2-dimensional $\rho(SU(2))$-orbit in $G(k+1, n+1)$, where $\rho\colon SU(2) \to U(n+1)$ is a unitary representation of $SU(2)$. Then we give a classification theorem of homogeneous 2-spheres in $G(k+1, n+1)$. As an application we describe explicitly all homogeneous 2-spheres in $G(2, 4)$. Also we mention about an example of non-homogeneous holomorphic 2-sphere with constant curvature in $G(2, 4)$.

1. Introduction

It is one of fundamental problems in differential geometry to study the rigidity and homogeneity of special surfaces and submanifolds in a given Riemannian manifold. The most important and interesting case is that the ambient manifold is a space form. For example, minimal surfaces with constant (Gaussian) curvature in real space forms have been classified completely ([3], [4], [11]), and minimal 2-spheres with constant curvature in the complex projective space $\mathbb{C}P^n$ also have been classified completely ([1], [2]). We wish to study 2-spheres with constant curvature immersed in the complex Grassmannians which is a generalization of the complex projective space $\mathbb{C}P^n$.

The complex Grassmann manifold $G(k+1,n+1)$ is the set of all $(k+1)$-dimensional complex vector subspaces in $\mathbb{C}^{n+1}$, which is isomorphic to a Hermitian symmetric space $U(n+1)/(U(k+1)\times U(n-k))$. We equip $G(k+1,n+1)$ with a canonical Kähler metric which is $U(n+1)$-invariant and has Einstein constant $2(n+1)$. Particularly, $G(1,n+1)$ is the complex projective space $\mathbb{C}P^n$, which has constant holomorphic sectional curvature 4. However, the geometric structure of $G(k+1, n+1)$ is much more complicated when $k \geq 1$. For example, when $k \geq 1$, $G(k+1, n+1)$ does not have constant holomorphic sectional curvature, and the rigidity of holomorphic curves in $G(k+1, n+1)$ fails ([7]). For this reason, it is hard to generalize some perfect results of submanifolds in $\mathbb{C}P^n$

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to the ones of submanifolds in a general complex Grassmannians. However, when the integers k, n are small, there are some results about minimal 2-spheres in $G(k+1, n+1)$. Minimal 2-spheres with constant curvature in $G(2, 4)$ were determined by Z.-Q. Li and Z.-H. Yu ([14]), and holomorphic 2-spheres with constant curvature in $G(2, 5)$ were also investigated by X.-X. Jiao and J.-G. Peng ([10]).

In [1], S. Bando and Y. Ohnita proved that all minimal 2-spheres with constant curvature in $\mathbb{C}P^n$ are homogeneous, and later, H.-Z. Li, C.-P. Wang and F.-E. Wu classified homogeneous 2-spheres in $\mathbb{C}P^n$ in [13] by the method of harmonic sequence. The purpose of this paper is to classify homogeneous 2-spheres in complex Grassmannians (see Theorem 4.1), which is a generalization of [13]. We should point out that our original idea derives from [1] and the observations of the main result in [13]. They inspire us to consider this problem from the angle of the unitary representation theory of $SU(2)$, which is quite different from the method used in [13]. As its applications we give an explicit description of all homogeneous 2-spheres in $G(2, 4)$ and their differential geometric quantities (see Theorems 5.1 and 5.2).

It is worthy to notice that minimal 2-spheres with constant curvature in spheres ([8], [9]) and complex projective spaces ([1]) are always homogeneous, but it is not true for the case when the ambient space is a general complex Grassmannians. To show this phenomenon, we prove the non-homogeneity for an example of holomorphic 2-sphere with constant curvature in $G(2, 4)$ given in [14] (see Theorem 5.3). Therefore, this phenomenon also reflects the complexity of the geometric structure of general complex Grassmannians.

Our paper is organized as follows. In Section 2, we recall some basic facts of unitary representations of $SU(2)$. In Section 3, we prove that if an immersion $x : S^2 \to G(k+1, n+1)$ is homogeneous, then its image $x(S^2)$ is a 2-dimensional $\rho(SU(2))$-orbit in $G(k+1, n+1)$, where $\rho : SU(2) \to U(n+1)$ is a unitary representation of $SU(2)$. In Section 4, we give a classification theorem of homogeneous 2-spheres in $G(k+1, n+1)$ which generalizes main theorem of H.-Z. Li, C.-P. Wang and F.-E. Wu in the case of $\mathbb{C}P^n$. In Section 5, we describe explicitly all homogeneous 2-spheres in $G(2, 4)$.

2. Preliminaries

In this section, we will agree on the same notations in [1], and begin with recalling some basic facts of irreducible unitary representations of the 3-dimensional special unitary group $SU(2)$, which is defined by

$$SU(2) = \left\{ g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix};\ a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\},$$

and its Lie algebra $\mathfrak{su}(2)$ is given by

$$\mathfrak{su}(2) = \left\{ X = \begin{pmatrix} \sqrt{-1}x & -\bar{y} \\ y & -\sqrt{-1}x \end{pmatrix};\ x \in \mathbb{R},\ y \in \mathbb{C} \right\},$$

with a natural basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, which is given by

$$\varepsilon_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Set $T = \{\exp(\theta\varepsilon_1);\ \theta \in \mathbb{R}\}$, then we have the homeomorphism $S^2 \simeq SU(2)/T$. The tangent space of the point $[T] \in S^2$ can be naturally identified with the subspace $\mathrm{span}_{\mathbb{R}}\{\varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$. We define a complex structure on S^2 such that $\varepsilon_2 - \sqrt{-1}\varepsilon_3$ is a vector of $(1, 0)$-type.

For each nonnegative integer n, let V_n be the representation space of $SU(2)$, which is an $(n+1)$-dimensional complex vector space of all complex homogeneous polynomials of degree n in two variables z_0 and z_1. The standard irreducible representation ρ_n of $SU(2)$ on V_n is defined by

$$\rho_n(g) f(z_0, z_1) := f(az_0 + bz_1, -\bar{b}z_0 + \bar{a}z_1), \tag{2.1}$$

where $g \in SU(2)$ and $f \in V_n$. If we view the elements of V_n as polynomial functions of $S^3 = \{(z_0, z_1) \in \mathbb{C}^2;\ |z_0|^2 + |z_1|^2 = 1\}$, we can define a $SU(2)$-invariant Hermitian inner product $(\ ,\ )$ on V_n as follows

$$(f, h) := \frac{(n+1)!}{2\pi^2} \int_{S^3} f \cdot \bar{h}\, dv,$$

where $h \in V_n$ and dv is the volume element of S^3. It is easy to check that $\{u_k^{(n)}\}$ defined by

$$u_k^{(n)} := \frac{1}{\sqrt{k!\,(n-k)!}} z_0^{n-k} z_1^k, \quad 0 \le k \le n,$$

is a unitary basis of V_n. Since $\rho_n(g)u_k^{(n)} \in V_n$, we can write

$$\rho_n(g)u_k^{(n)} = \sum_{l=0}^{n} \lambda_k^l(a, b) u_l^{(n)},$$

where $\{\lambda_k^l(a, b)\}$ are polynomials of degree n in $\{a, \bar{a}, b, \bar{b}\}$. By (2.1), we have

$$\lambda_k^l(a, b) = \sqrt{\frac{l!\,(n-l)!}{k!\,(n-k)!}} \sum_{p+q=n-l} \binom{n-k}{p} \binom{k}{q} a^p (\bar{a})^{k-q} b^{n-k-p} (-\bar{b})^q. \tag{2.2}$$

Let $\mathbb{C}^{n+1}$ be the complex number space of dimension $(n+1)$, and $\{E_i\}_{i=0}^n$ be the standard basis of $\mathbb{C}^{n+1}$. With respect to the unitary basis $\{u_k^{(n)}\}$ of V_n, there is a natural isomorphism between V_n and $\mathbb{C}^{n+1}$. Under such isomorphism, each linear endomorphism

$\rho_n(g)$ (for $g \in SU(2)$) can be represented by a matrix $(\lambda_k^l(a,b))$, which is still denoted by $\rho_n(g)$. It is easy to see $\rho_n(g) \in U(n+1)$, thus we have a Lie group homomorphism

$$\begin{aligned}\rho_n \colon SU(2) &\to U(n+1),\\ g &\mapsto \rho_n(g) = (\lambda_k^l(a,b)).\end{aligned}$$

The representation ρ_n of $SU(2)$ induces an action of $\mathfrak{su}(2)$ on V_n which can be described as follows

$$\begin{aligned}\rho_{n*}(X)(u_k^{(n)}) &= \frac{d}{dt}\Big|_{t=0}\rho(\exp tX)(u_k^{(n)})\\ &= -\sqrt{k(n-k+1)}\bar{y}u_{k-1}^{(n)} + (n-2k)\sqrt{-1}xu_k^{(n)}\\ &\quad + \sqrt{(n-k)(k+1)}yu_{k+1}^{(n)},\end{aligned} \tag{2.3}$$

for $0 \le k \le n$ and $X \in \mathfrak{su}(2)$. Using the matrix notation, we get a Lie algebra homomorphism $\rho_{n*}\colon \mathfrak{su}(2) \to \mathfrak{u}(n+1)$, $X \mapsto \rho_{n*}(X)$, which is the differential of the homomorphism ρ_n. From (2.3), in terms of matrix form, $\rho_{n*}(X)$ can be written as

$$\rho_{n*}(X) = \begin{pmatrix} n\sqrt{-1}x & \sqrt{n}y & & & \\ -\sqrt{n}\bar{y} & (n-2)\sqrt{-1}x & \sqrt{2(n-1)}y & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & -\sqrt{n}\bar{y} & -n\sqrt{-1}x \end{pmatrix}. \tag{2.4}$$

3. Homogeneous 2-spheres in complex Grassmannians

In this section, we shall reduce the classification problem of homogeneous 2-spheres in complex Grassmannians to an algebraic problem on unitary representations of $SU(2)$ whose classification theory is classical and well-known.

An immersion $x\colon S^2 \to G(k+1,n+1)$ is said to be *homogeneous*, if for any two points $p, q \in S^2$ there exists an isometry $\tilde{\sigma}$ of S^2 and a holomorphic isometry σ of $G(k+1,n+1)$ such that $\tilde{\sigma}(p) = q$ and the following diagram communicates

$$\begin{array}{ccc} S^2 & \xrightarrow{x} & G(k+1,n+1) \\ {\scriptstyle\tilde{\sigma}}\downarrow & & \downarrow{\scriptstyle\sigma} \\ S^2 & \xrightarrow{x} & G(k+1,n+1), \end{array} \tag{3.1}$$

i.e., $x \circ \tilde{\sigma} = \sigma \circ x$. We can identify $\tilde{\sigma}$ (resp. σ) with an element of $SO(3)$ (resp. $U(n+1)$). All such σ form a subgroup G of $U(n+1)$ and G acts transitively on $x(S^2)$. It's known that such 2-spheres in $G(k+1,n+1)$ have constant curvature, but they are non-minimal in general. Standard examples of homogeneous 2-spheres in $\mathbb{C}P^n$ are so-called *Veronese sequence* which are completely determined by S. Bando and Y. Ohnita ([1]).

As mentioned above, an immersion $x\colon S^2 \to G(k+1, n+1)$ is said to be homogeneous if the group $G = I(G(k+1, n+1), x(S^2))$ acts transitively on $x(S^2)$, or equivalently, $x(S^2)$ is a 2-dimensional G-orbit in $G(k+1, n+1)$. Since S^2 is compact, we know that G is also a compact Lie subgroup of $U(n+1)$.

Lemma 3.1. *The group $\hat{G} = \{\tilde{\sigma} \in SO(3);\ x \circ \tilde{\sigma} = x\}$ consists the unit element I_3 only.*

Proof. Let $\tilde{\sigma} \in \hat{G}$, then 1 is an eigenvalue of $\tilde{\sigma}$, so the action of $\tilde{\sigma}$ on S^2 has a fixed point p. The differential map of x and $\tilde{\sigma}$ at p satisfy

$$x_{*p} \circ \tilde{\sigma}_{*p} = x_{*p}.$$

Since x_{*p} is injective, it follows that $\tilde{\sigma}_{*p}$ is an identity map. Thus, we obtain $\tilde{\sigma} = I_3$, which completes the proof. $\square$

By this lemma and the commutation diagram (3.1), we obtain a natural Lie group homomorphism $\lambda\colon G \to SO(3)$, $\sigma \mapsto \tilde{\sigma}$. Up to now, we can prove our following principle result.

Theorem 3.2. *If $x\colon S^2 \to G(k+1, n+1)$ is a homogeneous immersion, then there exists a unitary representation $\rho\colon SU(2) \to U(n+1)$ such that $x(S^2)$ is a 2-dimensional $\rho(SU(2))$-orbit in $G(k+1, n+1)$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and $\lambda_*\colon \mathfrak{g} \to \mathfrak{o}(3)$ be the Lie algebra homomorphism induced by λ. Set $H = \ker\lambda$ and $\mathfrak{h} = \ker\lambda_*$, then H is a closed normal Lie subgroup of G whose Lie algebra is $\mathfrak{h}$. Obviously, $\mathfrak{h}$ is an idea of $\mathfrak{g}$. Since G is compact, one can equip $\mathfrak{g}$ with an Ad_G-invariant inner product $\langle\ ,\ \rangle$. The orthogonal complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Ad_G-invariant inner product is denoted by $\mathfrak{h}^\perp$. Then $\mathfrak{h}^\perp$ is a subalgebra of $\mathfrak{g}$ and also an ideal of $\mathfrak{g}$. So there exists a unique connected Lie subgroup K of G with its Lie algebra $\mathfrak{h}^\perp$. By the fundamental homomorphism theorem of Lie algebra, we obtain the following Lie algebra isomorphisms:

$$\lambda_*(\mathfrak{h}^\perp) = \lambda_*(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{h}^\perp.$$

Since $x(S^2)$ is a 2-dimensional G-orbit and H acts on $x(S^2)$ keeping every point fixed, we know that $x(S^2)$ is also a 2-dimensional K-orbit. Therefore, we obtain the relationship $\dim\lambda_*(\mathfrak{h}^\perp) = \dim\mathfrak{h}^\perp = \dim K \geq 2$. One can conclude that $\lambda_*(\mathfrak{h}^\perp) = \mathfrak{o}(3)$ by the well known fact that there is no 2-dimensional subalgebra of $\mathfrak{o}(3)$, and hence $\mathfrak{h}^\perp \cong \mathfrak{o}(3)$. Then we get a covering homomorphism $\lambda|_K\colon K \to SO(3)$, which implies that K is isomorphic to $SU(2)$ or $SO(3)$. If $K \cong SU(2)$, then $\rho\colon SU(2) \xrightarrow{\cong} K \xhookrightarrow{i} U(n+1)$ defines a unitary representation of $SU(2)$. If $K \cong SO(3)$, let ψ be the adjoint representation of $SU(2)$, then the map

$$\rho\colon SU(2) \xrightarrow{\psi} SO(3) \xrightarrow{\cong} K \xhookrightarrow{i} U(n+1)$$

also defines a unitary representation of $SU(2)$. This completes the proof. □

Two homogeneous immersions $x\colon S^2 \to G(k+1,n+1)$ and $x'\colon S^2 \to G(k+1,n+1)$ are said to be *equivalent* or *congruent*, if there exists an element $A \in U(n+1)$ such that $x = A \circ x'$. According to Theorem 3.2, the classification of equivalent classes of homogeneous 2-spheres in $G(k+1, n+1)$ is reduced to the following two problems:
(I) *Classifying the equivalence classes of unitary representations of* $SU(2)$;
(II) *Determining all* $\rho(SU(2))$*-orbits in* $G(k+1,n+1)$ *of dimension* 2, *here* $\rho : SU(2) \to U(n+1)$ *is a unitary representation of* $SU(2)$.

The problem (I) is classical and well-known in the representation theory of compact Lie groups, namely, the set $\{(\rho_n, V_n),\ n = 0, 1, 2, \ldots\}$ forms all inequivalent irreducible unitary representations of $SU(2)$ and every unitary representation ρ of $SU(2)$ can be expressed as direct sum of irreducible ones. To solve the problem (II), we can prove the following theorem:

Theorem 3.3. *An orbit M of $\rho(SU(2))$ on $G(k+1, n+1)$ is a 2-dimensional sphere immersed in $G(k+1, n+1)$ if and only if M goes through the point $W \in G(k+1, n+1)$ which is a $(k+1)$-dimensional vector subspace invariant by T.*

Proof. Suppose that $M = W \cdot \rho(SU(2))$ is a 2-dimensional orbit for some $W \in G(k+1, n+1)$, and H is the isotropy subgroup of $SU(2)$ at the point W. It is easy to see that H is a 1-dimensional closed Lie subgroup of $SU(2)$ and its Lie algebra $\mathfrak{h}$ is a 1-dimensional subalgebra of $\mathfrak{su}(2)$. Thus there is an element X of $\mathfrak{su}(2)$ such that $\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{X\}$. According to some basic theory of linear algebra, there exist $g \in SU(2)$ and a nonzero real number x such that $g^{-1}Xg = x\varepsilon_1$. Hence $g^{-1}\mathfrak{h}g = \mathrm{span}_{\mathbb{R}}\{\varepsilon_1\}$ and it follows that $g^{-1}H^o g = T$, where H^o is the connected component of H which contains the unit. Taking the point $W' = W \cdot \rho(g)$, then the isotropy group at W' is $g^{-1}Hg$ which contains T, i.e., W' is a $(k+1)$-dimensional vector subspace invariant by the action of T. Thus, we prove the sufficiency of our theorem, and the necessity follows from the fact that $\mathfrak{su}(2)$ has no 2-dimensional subalgebra. □

4. Classification theorem of homogeneous 2-spheres in $G(k+1, n+1)$

In this section, we will give a classification theorem of homogeneous 2-spheres in $G(k+1, n+1)$.

In virtue of Theorem 3.3, the problem (II) was reduced to determining of all $(k+1)$-dimensional vector subspaces invariant by T. Let $\rho\colon SU(2) \to U(n+1)$ be a unitary representation of $SU(2)$ and $\rho|_T\colon T \to U(n+1)$ be the restriction of ρ from $SU(2)$ to T. Since T is a torus group, we just need to determine all 1-dimensional vector subspaces invariant by T.

If ρ is irreducible, i.e., $\rho = \rho_n$ for some nonnegative integer n. By (2.2), it is easy to see that

$$\begin{aligned} &\rho_n|_T : T \to U(n+1), \\ &\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} \mapsto \mathrm{diag}\big\{e^{\sqrt{-1}n\theta}, e^{\sqrt{-1}(n-2)\theta}, \dots, e^{-\sqrt{-1}n\theta}\big\}. \end{aligned} \tag{4.1}$$

Hence, $\{E_i,\ 0 \le i \le n\}$ are eigenvectors of $\rho_n(T)$ which belong to eigenvalues

$$e^{\sqrt{-1}n\theta}, \dots, e^{\sqrt{-1}(n-2i)\theta}, \dots, e^{-\sqrt{-1}n\theta},$$

respectively. Thus, $W_i = \mathrm{span}_{\mathbb{C}}\{E_i\} := [E_i]$, $0 \le i \le n$, are all 1-dimensional vector subspaces invariant by T. Define the map $\{\phi_i^{(n)}\}$ by

$$\begin{aligned} \phi_i^{(n)}\colon S^2 = SU(2)/T &\to \mathbb{C}P^n, \\ gT &\mapsto W_i \cdot \rho_n(g) = [f_i^{(n)}], \end{aligned}$$

which are $SU(2)$-equivariant immersions of S^2 into $\mathbb{C}P^n$, here $f_i^{(n)} = (\lambda_i^0, \lambda_i^1, \dots, \lambda_i^n)$. The sequence $\{\phi_0^{(n)}, \phi_1^{(n)}, \dots, \phi_n^{(n)}\}$ is well-known as *Veronese sequence* in $\mathbb{C}P^n$ ([1], [2]). The Gaussian curvature K and the Kähler angle θ of $\phi_i^{(n)}$ are

$$K = \frac{4}{n+2i(n-i)}, \quad \cos\theta = \frac{n-2i}{n+2i(n-i)},$$

respectively.

If ρ is reducible, then $\rho = \rho_{n_1} \oplus \cdots \oplus \rho_{n_r}$ and $\mathbb{C}^{n+1} = \mathbb{C}^{n_1+1} \oplus \cdots \oplus \mathbb{C}^{n_r+1}$ with $n = n_1 + \cdots + n_r + r - 1$, i.e.,

$$\begin{aligned} \rho\colon SU(2) &\to U(n+1), \\ g &\mapsto \rho(g) = \mathrm{diag}\{\rho_{n_1}(g), \rho_{n_2}(g), \dots, \rho_{n_r}(g)\}. \end{aligned}$$

Set $E_{j_\alpha}^{n_\alpha} := E_i$, where $i = n_1 + \cdots + n_{\alpha-1} + j_\alpha + \alpha - 1$ and $0 \le j_\alpha \le n_\alpha$. It follows from (4.1) that a 1-dimensional vector subspace invariant by T must be spanned by a complex vector v with the following form

$$v = c_1 E_{j_1}^{n_1} + \cdots + c_r E_{j_r}^{n_r}, \quad c_\alpha \in \mathbb{C},\ 1 \le \alpha \le r, \tag{4.2}$$

where $\{n_\alpha,\ j_\alpha \mid \alpha = 1, \dots, r\}$ are nonnegative integers satisfying

$$n_1 - 2j_1 = n_2 - 2j_2 = \cdots = n_r - 2j_r. \tag{4.3}$$

In general, a $(k+1)$-dimensional vector subspace invariant by T can be spanned by $k+1$ complex vectors $\{v_i \mid i = 1, \dots, k+1\}$, where each v_i has the form (4.2) and they satisfy $(v_i, v_j) = \delta_{ij}$ with respect to the standard Hermitian inner product $(\ ,\ )$ on $\mathbb{C}^{n+1}$.

Combining Theorems 3.2 and 3.3 together with the above arguments, we obtain the following classification.

Theorem 4.1. *Let $x\colon S^2 \to G(k+1, n+1)$ be a homogeneous 2-sphere in $G(k+1, n+1)$. Then there exist nonnegative integers $\{n_\alpha, j_{1,\alpha} \mid \alpha = 1, \dots, r\}, \dots, \{n_\alpha, j_{k+1,\alpha} \mid \alpha = 1, \dots, r\}$ satisfying*

$$\begin{aligned} &n_1 + \cdots + n_r + r = n+1, \\ &0 \le j_{1,\alpha}, \dots, j_{k+1,\alpha} \le n_\alpha \quad (\alpha = 1, \dots, r), \\ &n_1 - 2j_{1,1} = n_2 - 2j_{1,2} = \cdots = n_r - 2j_{1,r}, \\ &\cdots \\ &n_1 - 2j_{k+1,1} = n_2 - 2j_{k+1,2} = \cdots = n_r - 2j_{k+1,r} \end{aligned}$$

and complex constants $\{c_{i,\alpha} \mid i = 1, \dots, k+1,\ \alpha = 1, \dots, r\}$ satisfying

$$\sum_{\alpha=1}^{r} c_{i,\alpha}\overline{c_{h,\alpha}}\delta_{j_{i,\alpha} j_{h,\alpha}} = \delta_{ih}$$

such that $x = A \circ f$, where $A \in U(n+1)$ and f is defined by

$$f\colon S^2 = SU(2)/T \to G(k+1, n+1),$$

$$gT \mapsto \begin{bmatrix} c_{1,1} f^{(n_1)}_{j_{1,1}} & c_{1,2} f^{(n_2)}_{j_{1,2}} & \cdots & c_{1,r} f^{(n_r)}_{j_{1,r}} \\ c_{2,1} f^{(n_1)}_{j_{2,1}} & c_{2,2} f^{(n_2)}_{j_{2,2}} & \cdots & c_{2,r} f^{(n_r)}_{j_{2,r}} \\ \vdots & \vdots & \cdots & \vdots \\ c_{k+1,1} f^{(n_1)}_{j_{k+1,1}} & c_{k+1,2} f^{(n_2)}_{j_{k+1,2}} & \cdots & c_{k+1,r} f^{(n_r)}_{j_{k+1,r}} \end{bmatrix}.$$

REMARK. Theorem 4.1 is a generalization of main theorem of H.-Z. Li, C.-P. Wang and F.-E. Wu in the case of $\mathbb{C}P^n$ ([13]).

However in order to classify completely it is necessary to determine all $\{n_\alpha, j_{1,\alpha} \mid \alpha = 1, \dots, r\}, \dots, \{n_\alpha, j_{k+1,\alpha} \mid \alpha = 1, \dots, r\}$ and $\{c_{i,\alpha} \mid i = 1, \dots, k+1,\ \alpha = 1, \dots, r\}$ satisfying the above conditions. In next section, we will do it completely in the case of $G(2,4)$. In the case of more general complex Grassmannians one would need more efforts to do them certainly.

5. Explicit description of homogeneous 2-spheres in $G(2,4)$

In this section, we will describe explicitly all homogeneous 2-spheres in $G(2,4)$, from which, one can find a family of non-minimal homogeneous 2-spheres in $G(2,4)$. To do this well, we should consider the following four cases respectively.

CASE I. If $\rho = \rho_3$, then $\rho\colon SU(2) \to U(4)$ and $\rho_*\colon \mathfrak{su}(2) \to \mathfrak{u}(4)$ can be given by (2.2) and (2.4) explicitly as follows

$$\rho(g) = \begin{pmatrix} a^3 & \sqrt{3}a^2 b & \sqrt{3}ab^2 & b^3 \\ -\sqrt{3}a^2\bar{b} & a(|a|^2 - 2|b|^2) & b(2|a|^2 - |b|^2) & \sqrt{3}\bar{a}b^2 \\ \sqrt{3}a\bar{b}^2 & \bar{b}(|b|^2 - 2|a|^2) & \bar{a}(|a|^2 - 2|b|^2) & \sqrt{3}\bar{a}^2 b \\ -\bar{b}^3 & \sqrt{3}\bar{a}\bar{b}^2 & -\sqrt{3}\bar{a}^2\bar{b} & \bar{a}^3 \end{pmatrix}, \tag{5.1}$$

$$\rho_*(X) = \begin{pmatrix} 3\sqrt{-1}x & \sqrt{3}y & 0 & 0 \\ -\sqrt{3}\bar{y} & \sqrt{-1}x & 2y & 0 \\ 0 & -2\bar{y} & -\sqrt{-1}x & \sqrt{3}y \\ 0 & 0 & -\sqrt{3}\bar{y} & -3\sqrt{-1}x \end{pmatrix}. \tag{5.2}$$

By the arguments in Section 4, we know that $[E_k]$, $0 \le k \le 3$ are all 1-dimensional vector subspaces invariant by T. Then $\mathrm{span}_{\mathbb{C}}\{E_k, E_l\}$, $0 \le k < l \le 3$ are all 2-dimensional vector subspaces invariant by T, so we get the following six homogeneous 2-spheres.

(I1) The base point $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a^3 & \sqrt{3}a^2 b & \sqrt{3}ab^2 & b^3 \\ -\sqrt{3}a^2\bar{b} & a(|a|^2 - 2|b|^2) & b(2|a|^2 - |b|^2) & \sqrt{3}\bar{a}b^2 \end{bmatrix}.$$

(I1$'$) The base point $W = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} \sqrt{3}a\bar{b}^2 & \bar{b}(|b|^2 - 2|a|^2) & \bar{a}(|a|^2 - 2|b|^2) & \sqrt{3}\bar{a}^2 b \\ -\bar{b}^3 & \sqrt{3}\bar{a}\bar{b}^2 & -\sqrt{3}\bar{a}^2\bar{b} & \bar{a}^3 \end{bmatrix}.$$

(I2) The base point $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a^3 & \sqrt{3}a^2 b & \sqrt{3}ab^2 & b^3 \\ \sqrt{3}a\bar{b}^2 & \bar{b}(|b|^2 - 2|a|^2) & \bar{a}(|a|^2 - 2|b|^2) & \sqrt{3}\bar{a}^2 b \end{bmatrix}.$$

(I2$'$) The base point $W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} -\sqrt{3}a^2\bar{b} & a(|a|^2 - 2|b|^2) & b(2|a|^2 - |b|^2) & \sqrt{3}\bar{a}b^2 \\ -\bar{b}^3 & \sqrt{3}\bar{a}\bar{b}^2 & -\sqrt{3}\bar{a}^2\bar{b} & \bar{a}^3 \end{bmatrix}.$$

(I3) The base point $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$f: S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a^3 & \sqrt{3}a^2 b & \sqrt{3}ab^2 & b^3 \\ -\bar{b}^3 & \sqrt{3}\bar{a}\bar{b}^2 & -\sqrt{3}\bar{a}^2\bar{b} & \bar{a}^3 \end{bmatrix}.$$

(I3$'$) The base point $W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$f: S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} -\sqrt{3}a^2\bar{b} & a(|a|^2 - 2|b|^2) & b(2|a|^2 - |b|^2) & \sqrt{3}\bar{a}b^2 \\ \sqrt{3}a\bar{b}^2 & \bar{b}(|b|^2 - 2|a|^2) & \bar{a}(|a|^2 - 2|b|^2) & \sqrt{3}\bar{a}^2 b \end{bmatrix}.$$

It is clear that the cases (Ii) and (Ii') ($i = 1, 2, 3$) are Hermitian orthogonal with respect to the standard Hermitian inner product of $\mathbb{C}^4$.

CASE II. If $\rho = \rho_1 \oplus \rho_0 \oplus \rho_0$, then $\rho: SU(2) \to U(4)$ and $\rho_*: \mathfrak{su}(2) \to \mathfrak{u}(4)$ can be written explicitly as follows

$$\rho(g) = \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.3}$$

$$\rho_*(X) = \begin{pmatrix} \sqrt{-1}x & y & 0 & 0 \\ -\bar{y} & -\sqrt{-1}x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.4}$$

by (2.2) and (2.4). Then, the restriction representation $\rho|_T: T \to U(n+1)$ is given by

$$\operatorname{diag}\{e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\} \mapsto \operatorname{diag}\{e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}, 1, 1\}.$$

So, we get two inequivalent homogeneous 2-spheres up to $U(4)$-equivalent.

(II1) The base point $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$f: S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(II1$'$) The base point $W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$f: S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We know that (II1) and (II1$'$) are also Hermitian orthogonal with respect to the standard Hermitian inner product of $\mathbb{C}^4$.

CASE III. If $\rho = \rho_2 \oplus \rho_0$, similarly, $\rho\colon SU(2) \to U(4)$ and $\rho_*\colon \mathfrak{su}(2) \to \mathfrak{u}(4)$ can be written as follows

$$\rho(g) = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 & 0 \\ -\sqrt{2}a\bar{b} & |a|^2-|b|^2 & \sqrt{2}\bar{a}b & 0 \\ \bar{b}^2 & -\sqrt{2}\bar{a}\bar{b} & \bar{a}^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.5}$$

$$\rho_*(X) = \begin{pmatrix} 2\sqrt{-1}x & \sqrt{2}y & 0 & 0 \\ -\sqrt{2}\bar{y} & 0 & \sqrt{2}y & 0 \\ 0 & -\sqrt{2}\bar{y} & -2\sqrt{-1}x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.6}$$

by (2.2) and (2.4). The restriction representation $\rho|_T\colon T \to U(n+1)$ is given by

$$\operatorname{diag}\{e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\} \mapsto \operatorname{diag}\{e^{2\sqrt{-1}\theta}, 1, e^{-2\sqrt{-1}\theta}, 1\}.$$

Hence, all 1-dimensional vector subspaces invariant by T are $[E_0]$, $[E_2]$ and $[v]$, where $v = c_1E_1 + c_2E_3$ with $|c_1|^2 + |c_2|^2 = 1$. Then up to $U(4)$-equivalent, we have two isolate homogeneous 2-spheres and two 1-parameter families of homogeneous 2-spheres.

(III1) The base point $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a^2 & \sqrt{2}ab & b^2 & 0 \\ \bar{b}^2 & -\sqrt{2}\bar{a}\bar{b} & \bar{a}^2 & 0 \end{bmatrix}.$$

(III1$'$) The base point $W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} -\sqrt{2}a\bar{b} & |a|^2-|b|^2 & \sqrt{2}\bar{a}b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(III2) The base point $W_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & 0 & \sin t \end{bmatrix}$, $t \in [0, \pi/2]$, then

$$f_t\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a^2 & \sqrt{2}ab & b^2 & 0 \\ -\sqrt{2}a\bar{b}\cos t & (|a|^2-|b|^2)\cos t & \sqrt{2}\bar{a}b\cos t & \sin t \end{bmatrix}.$$

(III2$'$) The base point $W_t = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -\sin t & 0 & \cos t \end{bmatrix}$, $t \in [0, \pi/2]$, then

$$\tilde{f}_t\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} \bar{b}^2 & -\sqrt{2}\bar{a}\bar{b} & \bar{a}^2 & 0 \\ \sqrt{2}a\bar{b}\sin t & (|b|^2-|a|^2)\sin t & -\sqrt{2}\bar{a}b\sin t & \cos t \end{bmatrix}.$$

It is easy to check that (IIIi) and (IIIi') ($i = 1, 2$) are also Hermitian orthogonal with respect to the standard Hermitian inner product of $\mathbb{C}^4$.

CASE IV. If $\rho = \rho_1 \oplus \rho_1$, then $\rho\colon SU(2) \to U(4)$ and $\rho_*\colon \mathfrak{su}(2) \to \mathfrak{u}(4)$ can be written as follows

$$\rho(g) = \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix}, \tag{5.7}$$

$$\rho_*(X) = \begin{pmatrix} \sqrt{-1}x & y & 0 & 0 \\ -\bar{y} & -\sqrt{-1}x & 0 & 0 \\ 0 & 0 & \sqrt{-1}x & y \\ 0 & 0 & -\bar{y} & -\sqrt{-1}x \end{pmatrix}, \tag{5.8}$$

by (2.2) and (2.4). The restriction representation $\rho|_T\colon T \to U(n+1)$ is given by

$$\mathrm{diag}\{e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\} \mapsto \mathrm{diag}\{e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}, e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\}.$$

Hence, the 1-dimensional vector subspaces invariant by T are $[v_1]$ and $[v_2]$, where $v_1 = c_1E_0 + c_2E_2$ and $v_2 = d_1E_1 + d_2E_3$ with $|c_1|^2 + |c_2|^2 = |d_1|^2 + |d_2|^2 = 1$. Then up to $U(4)$-equivalent, we have two isolate homogeneous 2-spheres and a family of homogeneous 2-spheres.

(IV1) The base point $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \end{bmatrix}.$$

(IV1$'$) The base point $W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then

$$f\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & -\bar{b} & \bar{a} \end{bmatrix}.$$

And also, (IV1) and (IV1$'$) are Hermitian orthogonal with respect to the standard Hermitian inner product of $\mathbb{C}^4$.

(IV2) The base point $W = \begin{bmatrix} c_1 & 0 & c_2 & 0 \\ 0 & d_1 & 0 & d_2 \end{bmatrix}$, with $|c_1|^2 + |c_2|^2 = |d_1|^2 + |d_2|^2 = 1$ and $\mu := c_1d_2 - c_2d_1 \neq 0$, then

$$f(c_1, c_2, d_1, d_2)\colon S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} c_1a & c_1b & c_2a & c_2b \\ -d_1\bar{b} & d_1\bar{a} & -d_2\bar{b} & d_2\bar{a} \end{bmatrix}.$$

Thus, we have completely classified homogeneous 2-spheres in $G(2,4)$.

Next, we will give some geometrical descriptions of these homogeneous 2-spheres in $G(2,4)$. We only compute the geometric quantities of the case (I1). For other cases, we omit the details of calculations and just list the results in Table 1.

Let $\Omega = (\Omega_{A\bar{B}})$, $0 \le A, B \le 3$ be the $\mathfrak{u}(4)$-valued right-invariant Maurer–Cartan form of $U(4)$. The Maurer–Cartan structure equations of $U(4)$ are

$$d\Omega_{A\bar{B}} = \Omega_{A\bar{C}} \wedge \Omega_{C\bar{B}}. \tag{5.9}$$

Then the canonical Kähler metric of $G(2, 4)$ and its Kähler form can be written as

$$ds^2 = \sum_{\alpha,i} \Omega_{\alpha\bar{i}} \cdot \bar{\Omega}_{\alpha\bar{i}},$$
$$\Theta = \frac{\sqrt{-1}}{2} \sum_{\alpha,i} \Omega_{\alpha\bar{i}} \wedge \bar{\Omega}_{\alpha\bar{i}},$$

where the range of the indices are $\alpha = 0, 1$ and $i = 2, 3$ respectively.

We choose a unitary frame field $e = (e_0, e_1, e_2, e_3)$ along f, where $e_A = E_A \cdot \rho(g)$, $A = 0, 1, 2, 3$. It is easily see from (5.2) that the pull back of Maurer–Cartan form can be written as

$$e^*\Omega = \begin{pmatrix} \omega_{0\bar{0}} & \frac{\sqrt{3}}{2}\phi & 0 & 0 \\ -\frac{\sqrt{3}}{2}\bar{\phi} & \omega_{1\bar{1}} & \phi & 0 \\ 0 & -\bar{\phi} & \omega_{2\bar{2}} & \frac{\sqrt{3}}{2}\phi \\ 0 & 0 & -\frac{\sqrt{3}}{2}\bar{\phi} & \omega_{3\bar{3}} \end{pmatrix} \tag{5.10}$$

with $\omega_{0\bar{0}} + \omega_{3\bar{3}} = 0$, $\omega_{1\bar{1}} + \omega_{2\bar{2}} = 0$ and $\omega_{0\bar{0}} = 3\omega_{1\bar{1}}$, where ϕ is a complex-valued $(1, 0)$ form of S^2, which defined up to a factor of absolute value 1, and the induced metric is $f^*\, ds^2 = \phi\bar{\phi}$.

If we write

$$f^*\Omega_{A\bar{B}} = \omega_{A\bar{B}} = a_{A\bar{B}}\phi + b_{A\bar{B}}\bar{\phi}, \tag{5.11}$$

and

$$A = (a_{\alpha\bar{i}}), \quad B = (b_{\alpha\bar{i}}).$$

It follows from (5.10) that

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So the Kähler angle (defined in [5]) of f is $\cos\theta = tr(AA^* - BB^*) = 1$.

The structure equations of S^2 with respect to the induced metric can be written as

$$d\phi = -\rho \wedge \phi, \tag{5.12}$$

$$d\rho = \frac{K}{2}\phi \wedge \bar{\phi}, \tag{5.13}$$

where ρ is the complex connection form and K is the Gaussian curvature of S^2, with respect to the induced metric f^*ds^2. Using the Maurer–Cartan structure equations (5.9) we obtain $d\phi = d\omega_{1\bar{2}} = -(\omega_{2\bar{2}} - \omega_{1\bar{1}}) \wedge \phi$. It gives $\rho = \omega_{2\bar{2}} - \omega_{1\bar{1}}$ by (5.12). Making use of (5.9) again, we get $d\rho = d(\omega_{2\bar{2}} - \omega_{1\bar{1}}) = (1/2)\phi \wedge \bar{\phi}$, which implies $K = 1$ by (5.13).

Taking the exterior derivative of (5.11) and using (5.9), we get the covariant differential of $a_{\alpha\bar{i}}$ and $b_{\alpha\bar{i}}$ (defined in [6]) given as follows

$$Da_{\alpha\bar{i}} = p_{\alpha\bar{i}}\phi + q_{\alpha\bar{i}}\bar{\phi},$$
$$Db_{\alpha\bar{i}} = q_{\alpha\bar{i}}\phi + r_{\alpha\bar{i}}\bar{\phi},$$

where

$$(p_{\alpha\bar{i}}) = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad (q_{\alpha\bar{i}}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (r_{\alpha\bar{i}}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The second identities $q_{\alpha\bar{i}} = 0$ imply that f is a minimal immersion ([6]). By the identity (1′.15) in [15], the square length of the second fundamental form is

$$S = 4\sum_{\alpha,i}(|q_{\alpha\bar{i}}|^2 + |r_{\alpha\bar{i}}|^2) = 6. \tag{5.14}$$

Through some similar straightforward computations, we get the following theorem.

Theorem 5.1. *The differential geometric quantities of homogeneous* 2*-spheres in* $G(2,4)$ *are given in* Table 1, *where* $t \in [0, \pi/2]$ *and* $\mu = c_1d_2 - c_2d_1 \neq 0$, *and* K *is* (*induced*) *Gaussian curvature,* θ *is the Kähler angle and* S *is the square length of the second fundamental form.*

REMARK 1. In the case (IV2), when $|\mu| = 1$, $f(c_1, c_2, d_1, d_2)$ are totally geodesic with $K = 2$. They are all $U(4)$-equivalent to

$$f : S^2 \to G(2,4), \quad gT \mapsto \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & -\bar{b} & \bar{a} \end{bmatrix}. \tag{IV2$'$}$$

The others in the case (IV2) are non-minimal. The one given in (III2) (resp. (III2$'$)) with $t = \pi/4$ is $U(4)$-equivalent to the one given in (IV1) (resp. (IV1$'$)) ([7]).

Table 1.

Case	Minimality	K	$\cos\theta$	S
(I1)	Yes	1	1	6
(I1$^{'}$)	Yes	1	-1	6
(I2)	Yes	$2/5$	$1/5$	0
(I2$^{'}$)	Yes	$2/5$	$-1/5$	0
(I3)	Yes	$2/3$	0	$8/3$
(I3$^{'}$)	Yes	$2/3$	0	$8/3$
(II1)	Yes	4	1	0
(II1$^{'}$)	Yes	4	-1	0
(III1)	Yes	1	0	0
(III1$^{'}$)	Yes	1	0	0
(III2)	Yes	2	1	$4\cos^2 2t$
(III2$^{'}$)	Yes	2	-1	$4\cos^2 2t$
(IV1)	Yes	2	1	0
(IV1$^{'}$)	Yes	2	-1	0
(IV2)		$2/\lvert\mu\rvert^2$	0	$4(1-\lvert\mu\rvert^2)/\lvert\mu\rvert^2$

REMARK 2. Since every closed totally geodesic submanifold of a homogeneous Riemannian manifold is homogeneous ([12]), Theorem 5.1 also contains a complete classification of totally geodesic 2-spheres in $G(2, 4)$.

By Table 1 and the above remarks, we obtain

Theorem 5.2. *Up to $U(4)$-equivalent, the one given in* (I2), (I2$^{'}$), (II1), (II1$^{'}$), (III1), (III1$^{'}$), (IV1), (IV1$^{'}$) *and the one given in* (IV2$^{'}$) *are all totally geodesic* 2*-spheres in $G(2, 4)$.*

REMARK 3. The 1-parameter family of homogeneous holomorphic 2-spheres in (III1) was first discovered by Q.-S. Chi and Y.-B. Zheng in [7].

REMARK 4. There are some differences between our classification and the classification of minimal 2-spheres with constant curvature in $G(2, 4)$ by Z.-Q. Li and Z.-H. Yu ([14]). The case (IV2) in our classification is not contained in theirs, and there is a holomorphic (thus minimal) 2-sphere with constant curvature $K = 4/3$ in $G(2, 4)$ given in [14] which is not contained in ours.

To conclude this section, we want to prove that the holomorphic 2-sphere with $K = 4/3$ in $G(2, 4)$ mentioned in Remark 4 is *not homogeneous*.

Set $S^2 = \mathbb{C} \cup \{\infty\}$ and z is the local coordinate of S^2, and define the map

$$\varphi\colon S^2 \to G(2,4), \quad z \mapsto \begin{bmatrix} 1 & 0 & \sqrt{3}z^2 & 0 \\ 0 & 1 & \sqrt{\frac{8}{3}}z & \sqrt{\frac{1}{3}}z \end{bmatrix}.$$

Theorem 5.3. *The holomorphic embedding φ is not homogeneous.*

Proof. We choose a unitary frame field $e = (e_1, e_2, e_3, e_4)$ along φ as follows:

$$\begin{aligned}
e_1 &= \frac{1}{\lambda_1}(1, 0, \sqrt{3}z^2, 0),\\
e_2 &= \frac{1}{\lambda_2}\left(-2\sqrt{2}\bar{z}|z|^2, 1+3|z|^4, \sqrt{\frac{8}{3}}z, \sqrt{\frac{1}{3}}z(1+3|z|^4)\right),\\
e_3 &= \frac{1}{\lambda_3}\left(0, -\sqrt{\frac{1}{3}}\bar{z}, 0, 1\right),\\
e_4 &= \frac{1}{\lambda_4}\left(-\sqrt{3}\bar{z}^2\left(1+\frac{1}{3}|z|^2\right), -\sqrt{\frac{8}{3}}\bar{z}, 1+\frac{1}{3}|z|^2, -\frac{2\sqrt{2}}{3}|z|^2\right),
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= \sqrt{1+3|z|^4},\\
\lambda_2 &= \sqrt{1+3|z|^2+6|z|^4+10|z|^6+9|z|^8+3|z|^{10}},\\
\lambda_3 &= \sqrt{1+\frac{1}{3}|z|^2},\\
\lambda_4 &= \sqrt{1+\frac{10}{3}|z|^2+4|z|^4+2|z|^6+\frac{1}{3}|z|^8}.
\end{aligned}$$

By direct computation, we get

$$\begin{aligned}
\omega_{1\bar{3}} &= (de_1, e_3) = 0,\\
\omega_{1\bar{4}} &= (de_1, e_4) = \frac{2\sqrt{3}z(1+(1/3)|z|^2)}{\lambda_1\lambda_4}\,dz,\\
\omega_{2\bar{3}} &= (de_2, e_3) = \frac{\sqrt{3}(1+3|z|^4)}{3\lambda_2\lambda_3}\,dz,\\
\omega_{2\bar{4}} &= (de_2, e_4) = -\frac{2\sqrt{6}(2|z|^6+3|z|^4-1)}{3\lambda_2\lambda_4}\,dz.
\end{aligned}$$

Thus, the metric induced by φ is

$$\varphi^* ds^2 = \omega_{1\bar{3}}\bar{\omega}_{1\bar{3}} + \omega_{1\bar{4}}\bar{\omega}_{1\bar{4}} + \omega_{2\bar{3}}\bar{\omega}_{2\bar{3}} + \omega_{2\bar{4}}\bar{\omega}_{2\bar{4}} = \frac{3}{(1+|z|^2)^2}\, dz\, d\bar{z},$$

which implies the (induced) Gaussian curvature $K = 4/3$. Set $\phi = \sqrt{3}/(1+|z|^2)\, dz$, then we have

$$A = \begin{pmatrix} 0 & \dfrac{2z(1+(1/3)|z|^2)(1+|z|^2)}{\lambda_1\lambda_4} \\ \dfrac{(1+3|z|^4)(1+|z|^2)}{3\lambda_2\lambda_3} & -\dfrac{2\sqrt{2}(2|z|^6+3|z|^4-1)(1+|z|^2)}{3\lambda_2\lambda_4} \end{pmatrix} \tag{5.15}$$

by (5.11).

Up to now, we have two ways to show that φ is not homogeneous. The first one is that $z = 0$ is an isolate zero point of $\det A$ and $\operatorname{rank} A\,|_{z=0} = 1$ by (5.15), which implies that φ is not homogeneous. The second one is that the square length of the second fundamental form of φ is given by

$$S = \frac{16(3+2|z|^2+3|z|^4)}{9(1+|z|^2)^2},$$

by the first identity of (5.14), which is not a constant, and hence φ is not homogeneous. $\square$

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References

[1] S. Bando and Y. Ohnita: *Minimal 2-spheres with constant curvature in* $\mathrm{P}_n(\mathbf{C})$, J. Math. Soc. Japan **39** (1987), no. 3, 477–487.

[2] J. Bolton, G.R. Jensen, M. Rigoli and L.M. Woodward: *On conformal minimal immersions of* S^2 *into* $\mathbf{C}\mathrm{P}^n$, Math. Ann. **279** (1988), 599–620.

[3] R.L. Bryant: *Minimal surfaces of constant curvature in* S^n, Trans. Amer. Math. Soc. **290** (1985), 259–271.

[4] E. Calabi: *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geometry **1** (1967), 111–125.

[5] S.S. Chern and J.G. Wolfson: *Minimal surfaces by moving frames*, Amer. J. Math. **105** (1983), 59–83.

[6] S.S. Chern and J.G. Wolfson: *Harmonic maps of the two-sphere into a complex Grassmann manifold*, II, Ann. of Math. (2) **125** (1987), 301–335.

[7] Q.-S. Chi and Y. Zheng: *Rigidity of pseudo-holomorphic curves of constant curvature in Grassmann manifolds*, Trans. Amer. Math. Soc. **313** (1989), 393–406.

[8] M.P. do Carmo and N.R. Wallach: *Representations of compact groups and minimal immersions into spheres*, J. Differential Geometry **4** (1970), 91–104.

[9] M.P. do Carmo and N.R. Wallach: *Minimal immersions of spheres into spheres*, Ann. of Math. (2) **93** (1971), 43–62.

[10] X. Jiao and J. Peng: *Classification of holomorphic spheres of constant curvature in complex Grassmann manifold* $\mathbf{G}_{2,5}$, Differential Geom. Appl. **20** (2004), 267–277.

[11] K. Kenmotsu: *On minimal immersions of* R^2 *into* S^N, J. Math. Soc. Japan **28** (1976), 182–191.

[12] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry, II, Interscience Publishers John Wiley & Sons, Inc., New York, 1969.

[13] H. Li, C. Wang and F. Wu: *The classification of homogeneous* 2*-spheres in* $\mathbb{C}P^n$, Asian J. Math. **5** (2001), 93–108.

[14] Z.-Q. Li and Z.-H. Yu: *Constant curved minimal* 2*-spheres in* $G(2,4)$, Manuscripta Math. **100** (1999), 305–316.

[15] K. Ogiue: *Differential geometry of Käehler submanifolds*, Advances in Math. **13** (1974), 73–114.

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