Doi, S., Nishitani, T. and Ueda, H.
Osaka J. Math.
49 (2012), 1065–1085

NOTE ON LOWER BOUNDS OF ENERGY GROWTH FOR SOLUTIONS TO WAVE EQUATIONS

SHIN-ICHI DOI, TATSUO NISHITANI and HIDEO UEDA

(Received July 20, 2010, revised March 25, 2011)

Abstract

In this note we study lower bounds of energy growth for solutions to wave equations which are *compact in space* perturbations of the wave equation $\partial_t^2 u - \Delta u = 0$. Assuming that there exists a null bicharacteristic $(x(t), \xi(t))$, parametrized by the time t, such that $x(t)$ remains inside a ball and $\xi(t)$ outside a ball for $t \geq 0$ we prove that the solution operator $R(t)$ is bounded from below by constant times $\sqrt{|\xi(t)|/|\xi(0)|}$ in the operator norm. We apply this result to examples constructed by the same idea as in Colombini and Rauch [1] and show that there exist compact in space perturbations which cause $\exp(ct^\alpha)$ growth of the energy for any given $0 \leq \alpha \leq 1$.

1. Introduction

In this note we are interested in lower bounds of energy growth for solutions to

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(t, x)\partial_{x_j} u) = 0 \tag{1.1}$$

where $a_{ij}(t, x) = a_{ji}(t, x)$ are smooth with bounded derivatives of all orders such that

$$\begin{cases} a_{ij}(t, x) = \delta_{ij}, \quad |x| \geq R_1, \\ A^{-2}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \leq A^2|\xi|^2, \quad (t, x) \in \mathbb{R}^{1+n} \end{cases} \tag{1.2}$$

with some $R_1 > 0$, $A > 0$, that is, (1.1) is a *compact in space perturbation* of the wave equation $\partial_t^2 u - \Delta u = 0$.

There are many detailed studies about upper and lower bounds of energy of solutions to wave equations $a_{ij}(t, x) = a_{ij}(x)$ with lower order terms. We refer to [8] for compact manifolds without boundary case and [6] for compact manifolds with boundary case.

In the case that $a(t, x)$ depends only on t, and hence not compact in space perturbation, there are also many results about lower bounds of energy, see for example [3], [10], [2], [9].

2010 Mathematics Subject Classification. Primary 35L20; Secondary 35B40, 35B45.

In compact in space perturbation case, in Colombini and Rauch [1] they have studied an example which would give exponentially growing solutions. Unfortunately the proof there is not complete because there is no null bicharacteristic which is periodic and amplifying at the same time (see Remark below). Nevertheless essentially the same type examples gives not only exponentially but also $\exp t^{\alpha}$ $(0 < \alpha < 1)$ growing solutions. To prove this we first formulate a result, in terms of a null bicharacteristic, which gives a lower bound of energy (Theorem 1.1). Then we apply this result to these examples to get the desired growth of energy (Theorems 1.2 and 1.3).

In what follows we put

$$a(t, x, \xi) = \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i \xi_j.$$

Denote by $\mathcal{H}$ the Hilbert space which is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^n} \sum_{i=1}^{n} |\partial_{x_i} u|^2\, dx = \int_{\mathbb{R}^n} |\nabla u|^2\, dx.$$

Let $\mathcal{R}(t, 0)$ be the solution operator defined by

$$C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \ni \begin{pmatrix} u(0,\ \cdot\,) \\ \partial_t u(0,\ \cdot\,) \end{pmatrix} \mapsto \begin{pmatrix} u(t,\ \cdot\,) \\ \partial_t u(t,\ \cdot\,) \end{pmatrix} \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$$

which extends uniquely to bounded operator in $\mathcal{H} \times L^2$. We first give a simple upper bound on the possible growth of $\|\mathcal{R}(t, 0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$;

Proposition 1.1. *We have*

$$\|\mathcal{R}(t, 0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)} \le C \exp\left(\frac{1}{2} \int_0^t \left[\sup_{x,\xi} \frac{|\partial_t a(\tau, x, \xi)|}{a(\tau, x, \xi)} \right] d\tau \right)$$

with some $C > 0$.

We now investigate lower bounds on $\|\mathcal{R}(t, 0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$. We assume that there is a bicharacteristic $(x(t), \xi(t))$ of $\sqrt{a(t, x, \xi)}$ or $-\sqrt{a(t, x, \xi)}$ with $\xi(t) \neq 0$;

$$\frac{dx}{dt} = \pm \frac{\partial}{\partial \xi} \sqrt{a(t, x, \xi)}, \quad \frac{d\xi}{dt} = \mp \frac{\partial}{\partial x} \sqrt{a(t, x, \xi)} \tag{1.3}$$

such that

$$|x(t)| \le C^*, \quad |\xi(t)| \ge c^* \tag{1.4}$$

with some $C^* > 0$, $c^* > 0$ for $t \ge 0$. Then we have

Theorem 1.1. *Assume that there is a bicharacteristic verifying* (1.4). *Then there is a positive constant* C *such that*

$$\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)} \geq C \exp\left(\frac{1}{4}\int_0^t \frac{\partial_t a}{a}(\tau, x(\tau), \xi(\tau))\,d\tau\right) \geq C A^{-1}\sqrt{\frac{|\xi(t)|}{|\xi(0)|}}.$$

REMARK. It will be observed in the remark in section 4 that if $|x(t)|$ remains in a bounded set for $t \geq 0$ then we have

$$\int_0^t \frac{\partial_t a}{a}(\tau, x(\tau), \xi(\tau))\,d\tau = \log \frac{a(t, x(t), \xi(t))}{a(0, x(0), \xi(0))}$$

and hence

$$\int_0^t \frac{\partial_t a}{a}(\tau, x(\tau), \xi(\tau))\,d\tau \to \infty, \quad t \to \infty$$

is equivalent to $\lim_{t\to\infty}|\xi(t)| = \infty$. In particular, if $\xi(t)$ is periodic in t then Theorem 1.1 gives no information about energy growth.

We now construct examples following Colombini and Rauch [1] to which one can apply Theorem 1.1 to get lower bounds on $\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$. Our construction works in all dimensions $n \geq 2$ though we present only the case $n = 2$ for simplicity. Consider the wave equation

$$\partial_t^2 u - \sum_{i=1}^{2} \partial_{x_i}(a(t,x)\partial_{x_i} u) = 0 \tag{1.5}$$

that is, $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = a(t,x)$ which is smooth with bounded derivatives of all orders and

$$C^{-1} \leq a(t,x) \leq C, \quad (t,x) \in \mathbb{R}^{1+2}, \quad a(t,x) = 1 \quad \text{when} \quad |x| \geq 2 \tag{1.6}$$

with some $C > 0$.

Theorem 1.2. *For any non-negative bounded measurable function* $\delta(t)$ *on* $[0,\infty)$ *and for any* $\epsilon > 0$ *there exists* $a(t,x)$ *satisfying* (1.6) *such that for the associate solution operator* $\mathcal{R}$ *to* (1.5) *we have*

$$C_1 \exp\left(\int_0^t \delta(\tau)\,d\tau\right) \leq \|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)} \leq C_2 \exp\left((2+\epsilon)\int_0^t \delta(\tau)\,d\tau\right)$$

with some $C_i > 0$ *independent of* ϵ.

If we impose some conditions on $\delta(t)$ the upper bound of energy growth in Theorem 1.2 can be improved. Denote by H^1 the usual Sobolev space $H^1(\mathbb{R}^n)$ then

Theorem 1.3. *Let* $\delta(t)$ *be a smooth non-negative bounded function on* $[0, \infty)$ *such that* $\delta'(t) \leq 0$, $\delta''(t) \geq 0$. *Then there exists* $a(t, x)$ *verifying* (1.6) *such that for the associate solution operator* $\mathcal{R}(t, 0)$ *to* (1.5) *we have*

$$C_1 \exp\left(\int_0^t \delta(s)\,ds\right) \leq \|\mathcal{R}(t, 0)\|_{\mathrm{Hom}(H^1 \times L^2; \mathcal{H} \times L^2)} \leq C_2 \exp\left(\int_0^t \delta(s)\,ds\right)$$

with some constants $C_i > 0$.

Let us take $\delta(t) = (1-\kappa)(1+t)^{-\kappa}$, $0 \leq \kappa < 1$. Then Theorem 1.3 shows that there is an $a(t, x)$ satisfying (1.6) such that the solution operator $\mathcal{R}(t, 0)$ verifies

$$C_1 e^{(1+t)^{1-\kappa}} \leq \|\mathcal{R}(t, 0)\|_{\mathrm{Hom}(H^1 \times L^2; \mathcal{H} \times L^2)} \leq C_2 e^{(1+t)^{1-\kappa}}.$$

If we choose $\delta(t) = m(1+t)^{-1}$, $m > 0$ then from Theorem 1.3 one can find an $a(t, x)$ with (1.6) such that the associate $\mathcal{R}(t, 0)$ satisfies

$$C_1(1+t)^m \leq \|\mathcal{R}(t, 0)\|_{\mathrm{Hom}(H^1 \times L^2; \mathcal{H} \times L^2)} \leq C_2(1+t)^m.$$

2. Preliminaries

Let $c(x, y, \xi) \in C^\infty(\mathbb{R}^{3n})$ verify for any $l \in \mathbb{N}$

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma c(x, y, \xi)| \leq C_{\alpha\beta\gamma l} \langle \xi \rangle^{m-|\alpha|} \langle x-y \rangle^{2l} \langle y \rangle^{-l} \langle x \rangle^{-l}, \quad \forall \alpha, \beta, \gamma. \tag{2.1}$$

We define $\mathrm{Op}(c)$ by

$$\mathrm{Op}(c)u(x) = \int e^{i(x-y)\xi} c(x, y, \xi) u(y)\,d\xi\,dy.$$

Let us denote $g = |dx|^2 + \langle \xi \rangle^{-2} |d\xi|^2$ and by $S(w, g)$ the set of all $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ verifying

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha\beta} w(x, \xi) \langle \xi \rangle^{-|\alpha|}, \quad \forall \alpha$$

(see [4]). We assume that a positive function $w(x, \xi)$ is g continuous and σ, g temperate (see [4]). For $a(x, \xi) \in S(w, g)$ we define

$$\mathrm{Op}^t(a)u(x) = \int e^{i(x-y)\xi} a((1-t)x + ty, \xi) u(y)\,d\xi\,dy.$$

Lemma 2.1. *Let c verify* (2.1) *and let* $\mathrm{Op}(c) = \mathrm{Op}^{1/2}(b) = \mathrm{Op}^w(b)$. *Then we have* $b \in S(\langle\xi\rangle^m \langle x\rangle^{-k}, g)$ *for any* $k \in \mathbb{N}$. *If* $b \in S(\langle\xi\rangle^m \langle x\rangle^{m'}, g)$ *with* $m < 0$, $m' < 0$ *then* $\mathrm{Op}^w(b)$ *is compact in* $L^2(\mathbb{R}^n)$.

Proof. Let us write $\mathrm{Op}(c) = \mathrm{Op}^w(b) = B$. Recall that $b(x, \xi)$ is given by

$$b(x,\xi) = \int e^{i(\eta-\xi)\theta} c\left(x + \frac{\theta}{2}, x - \frac{\theta}{2}, \eta\right) d\eta\, d\theta$$

(see for example [7]). We first show that

$$b(x,\xi) \in S(\langle\xi\rangle^m \langle x\rangle^{-k}, g) \tag{2.2}$$

for any $k \in \mathbb{N}$. To see this we consider

$$\begin{aligned}\partial_\xi^\alpha \partial_x^\beta b &= \sum_{\beta'+\beta''=\beta} \frac{\beta!}{\beta'!\,\beta''!} \int e^{i\eta\theta} \partial_\xi^\alpha \partial_x^{\beta'} \partial_y^{\beta''} c\left(x + \frac{\theta}{2}, x - \frac{\theta}{2}, \eta + \xi\right) d\eta\, d\theta \\ &= \sum_{\beta'+\beta''=\beta} \frac{\beta!}{\beta'!\,\beta''!} \int e^{i\eta\theta} \langle D_\eta\rangle^N \langle\theta\rangle^{-N} \langle D_\theta\rangle^M \langle\eta\rangle^{-M} \\ &\qquad\qquad \times \partial_\xi^\alpha \partial_x^{\beta'} \partial_y^{\beta''} c\left(x + \frac{\theta}{2}, x - \frac{\theta}{2}, \eta + \xi\right) d\eta\, d\theta.\end{aligned}$$

Noting that

$$\langle\eta + \xi\rangle^{m-|\alpha|} \le C^{|m|+|\alpha|} \langle\xi\rangle^{m-|\alpha|} \langle\eta\rangle^{|m|+|\alpha|}$$

we see that

$$\begin{aligned}|\partial_\xi^\alpha \partial_x^\beta b| \le C_{\alpha\beta l N} \int & \langle\xi\rangle^{m-|\alpha|} \langle\theta\rangle^{-N+2l} \langle\eta\rangle^{-M+|m|+|\alpha|} \\ & \times \left\langle x - \frac{\theta}{2}\right\rangle^{-l} \left\langle x + \frac{\theta}{2}\right\rangle^{-l} d\eta\, d\theta.\end{aligned}$$

Since $C_l \langle x\rangle^{-l} \ge \langle x - \theta/2\rangle^{-l} \langle x + \theta/2\rangle^{-l}$ we get the desired assertion choosing $M \ge n + 1 + |m| + |\alpha|$, $N \ge n + 1 + 2l$.

We turn to the second assertion. Assume that $b(x,\xi) \in S(\langle\xi\rangle^m \langle x\rangle^{m'}, g)$. Since $B^* B = \mathrm{Op}^w(\bar{b} b)$ and $\bar{b} b \in S(\langle\xi\rangle^{2m} \langle x\rangle^{2m'}, g)$ we see

$$(B^* B)^N = \mathrm{Op}^w(b_N), \quad b_N \in S(\langle\xi\rangle^{2Nm} \langle x\rangle^{2Nm'}, g).$$

We remark that the kernel $K_N(x, y)$ of $\mathrm{Op}^w(b_N)$ is in $L^2(\mathbb{R}^{2n})$ taking N large. Indeed

$$\begin{aligned}|K_N(x,y)| &= \left| \int e^{i(x-y)\xi} \langle x - y\rangle^{-L} \langle D_\xi\rangle^L b_N\left(\frac{x+y}{2}, \xi\right) d\xi \right| \\ &\le C_L \langle x + y\rangle^{2Nm'} \langle x - y\rangle^{-L} \int \langle\xi\rangle^{2Nm}\, d\xi\end{aligned}$$

which proves $K_N(x, y) \in L^2(\mathbb{R}^{2n})$ choosing N, L so that $2Nm' < -n/2$, $L > n/2$, $2Nm < -n$. Thus $(B^*B)^N$ is compact in $L^2(\mathbb{R}^n)$ and hence B is also compact in $L^2(\mathbb{R}^n)$. $\square$

Lemma 2.2. *Let $a_i(x,\xi) \in S(\langle\xi\rangle^{m_i}, g)$ and assume that $\partial_x^\alpha a_i(x,\xi) = 0$ for $|x| \geq R$ if $\alpha \neq 0$. Then we have*

$$a_1 \# a_2 = \sum_{|\alpha+\beta|<N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\,\beta!} a_{1(\beta)}^{(\alpha)} a_{2(\alpha)}^{(\beta)} + r_N, \quad r_N \in S(\langle\xi\rangle^{m_1+m_2-N}\langle x\rangle^{-1}, g)$$

where $a_{(\beta)}^{(\alpha)} = \partial_\xi^\alpha \partial_x^\beta a(x, \xi)$. In particular for $a_1 = a_2 = a \in S(\langle\xi\rangle^m, g)$ we have

$$a \# a = a^2 + r, \quad r \in S(\langle\xi\rangle^{2m-2}\langle x\rangle^{-1}, g).$$

Proof. Recall that one has $\mathrm{Op}^w(a_1)\,\mathrm{Op}^w(a_2) = \mathrm{Op}^w(a_1 \# a_2) = \mathrm{Op}^w(b)$ with

$$b(x, \xi) = 2^{2n} \int e^{2i\tilde{z}\tilde{\eta}-2iz\eta} a_1(x + \tilde{z}, \xi + \eta) a_2(x + z, \xi + \tilde{\eta})\, dz\, d\eta\, d\tilde{z}\, d\tilde{\eta}.$$

Applying the Taylor formula and making integration by parts it suffices to estimate terms such as

$$\begin{aligned} &\int e^{2i\tilde{z}\tilde{\eta}-2iz\eta} \partial_\xi^\alpha \partial_x^\beta a_1(x + \tilde{z}, \xi + \theta_1\eta) \\ &\quad \times \partial_\xi^\beta \partial_x^\alpha a_2(x + z, \xi + \theta_2\tilde{\eta})\, dz\, d\eta\, d\tilde{z}\, d\tilde{\eta} \end{aligned} \tag{2.3}$$

where $|\alpha + \beta| = N$ and $|\theta_i| \leq 1$. Since we have $\langle x\rangle \leq C\langle\tilde{z}\rangle$ if $\beta \neq 0$ and $\langle x\rangle \leq C\langle z\rangle$ if $\alpha \neq 0$ on the support of the integrand then the oscillatory integral (2.3) defines a symbol in $S(\langle\xi\rangle^{m_1+m_2-N}\langle x\rangle^{-1}, g)$. $\square$

3. Reduction

We are concerned with the Cauchy problem

$$\begin{cases} D_t^2 u - \displaystyle\sum_{i,j=1}^n D_{x_i}(a_{ij}(t, x) D_{x_j} u) = 0, \\ u(0, x) = \chi(x) f(x), \quad D_t u(0, x) = \chi(x) g(x) \end{cases} \tag{3.1}$$

where $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$ and $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ on $|x| \leq C^*$. We assume that $a_{ij}(t,x)$ verifies (1.2) and hence $\partial_t^k \partial_x^\alpha a_{ij}(t,x)$ are bounded in $\mathbb{R} \times \{|x| \leq R_1\}$. Let us set

$$h(t, x, \xi) = \sqrt{\sum_{i,j} a_{ij}(t, x)\xi_i\xi_j + \psi(\xi)}$$

where $0 \le \psi(\xi) \in C_0^\infty(\mathbb{R}^n)$ with $\psi(\xi) = 1$ near the origin and $\psi(\xi) = 0$ for $|\xi| \ge c^*$. Since

$$\mathrm{Op}^w\Big(\sum_{i,j} a_{ij}(t,x)\xi_i\xi_j\Big) = \sum_{i,j} D_{x_i} a_{ij}(t,x) D_{x_j} + 4^{-1} \sum \partial_{x_i}\partial_{x_j} a_{ij}(t,x)$$

we have with $H = \mathrm{Op}^w(h)$ that

$$\sum_{i,j} D_{x_i} a_{ij}(t,x) D_{x_j} = H^2 + b^w - \psi(D), \quad b \in C^1(\mathbb{R}; S(\langle x\rangle^{-1}, g))$$

by Lemma 2.2. Writing $b = (b/h) \# h + r'$ with $r' \in S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g)$ one has

$$\sum_{i,j} D_{x_i} a_{ij}(t,x) D_{x_j} u = H(Hu) + B'Hu + R'u - \psi(D)u$$

where $B', R' \in C^1(\mathbb{R}; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$. On the other hand one can write

$$D_t h = \Big(\frac{D_t h}{h}\Big) \# h + b'' \# h + r'', \quad b'', r'' \in C^1(\mathbb{R}; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$$

from Lemma 2.2 and hence

$$D_t(Hu) = H D_t u + \Big(\frac{D_t h}{h}\Big)^w Hu + B''Hu + R''u$$

with $B'', R'' \in C^1(\mathbb{R}; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$. Thus the equation (3.1) can be written with $U = (Hu, D_t u)$ (where u is the solution to (3.1)) as

$$D_t U = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} U + BU + R_1 u + R_2 u, \tag{3.2}$$

with $R_1 \in C^1(\mathbb{R}; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$ where

$$B = \begin{pmatrix} \Big(\frac{D_t h}{h}\Big)^w & 0 \\ 0 & 0 \end{pmatrix} + B_{-1}, \quad B_{-1} \in C^1(\mathbb{R}; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$$

and $R_2 = R_2(\xi)$ vanishes outside a neighborhood of the origin. Fix $T > 0$ and consider the Cauchy problem in the strip $[0, T] \times \mathbb{R}^n$. From the finite propagation speed, choosing $\tilde\chi \in C_0^\infty(\mathbb{R}^n)$ so that

$$\mathrm{supp}_x\, u(t, \cdot) \subset \{x \mid \tilde\chi(x) = 1\}, \quad 0 \le t \le T$$

we have $R_2 u = R_2 \tilde\chi u$ and note that $R_2\tilde\chi \in C^1([0, T]; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$.

Lemma 3.1. *Let $k(t,x,\xi) \in C^1([0,T]; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$ and $K(t) = \mathrm{Op}^w(k)$. Then the mapping*

$$H^1 \times L^2 \ni (f, g) \mapsto K(t)u(t) \in C^0([0, T]; L^2)$$

is compact.

Proof. Let $(f_n, g_n) \in H^1 \times L^2$ be bounded in $H^1 \times L^2$. Then it is clear from the energy inequality for the wave equation (3.1) that

$$\|\langle D\rangle u_n(t)\| + \|D_t u_n\| \le C, \quad 0 \le t \le T$$

with C independent of n. From this we have $\|u_n(t') - u_n(t)\| \le C|t' - t|$. Since $\|K(t') - K(t)\|_{\mathrm{Hom}(L^2)} \le C'|t' - t|$ for t', $t \in [0, T]$ by the assumption it is clear that $\{K(t)u_n(t)\}$ is an equi-continuous sequence. It is also clear that $\{K(t)u_n(t)\}$ is uniformly bounded in $C^0([0, T]; L^2)$. Since for each $t \in [0, T]$, $\{K(t)u_n(t)\}$ contains a convergent (in L^2) subsequence, then Ascoli-Arzela theorem implies that we can take a subsequence $\{K(t)u_{n_k}(t)\}$ which converges in $C^0([0, T]; L^2)$. □

Let us consider the solution $V(t)$ to the Cauchy problem

$$D_t V = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} V + BV, \quad V(0) = U(0) = \begin{pmatrix} H(0)u(0) \\ D_t u(0) \end{pmatrix}. \tag{3.3}$$

Then from the energy inequality for the hyperbolic system (3.3) it follows that with $R = R_1 + R_2$

$$\|U(t) - V(t)\| \le C \int_0^t \|R\tilde{\chi}u(s)\|\, ds, \quad t \in [0, T].$$

Thanks to Lemma 3.1 this proves that the mapping: $H^1 \times L^2 \ni (f, g) \mapsto U(t) - V(t) \in L^2 \times L^2$ is compact.

Let us denote by $\mathcal{R}(t, 0)$ the solution operator;

$$\mathcal{R}(t, 0)\colon (u(0), D_t u(0)) \mapsto (u(t), D_t u(t))$$

of the Cauchy problem

$$\begin{cases} D_t^2 u - \displaystyle\sum_{i,j=1}^n D_{x_i}(a_{ij}(t, x) D_{x_j} u) = 0, \\ u(0, x) = f(x), \quad D_t u(0, x) = g(x) \end{cases}$$

so that $U(t) = \begin{pmatrix} H(t) & 0 \\ 0 & 1 \end{pmatrix} \mathcal{R}(t, 0)\chi \begin{pmatrix} f \\ g \end{pmatrix}$. Then we conclude that

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} H(t) & 0 \\ 0 & 1 \end{pmatrix} \mathcal{R}(t, 0)\chi \begin{pmatrix} f \\ g \end{pmatrix} - V(t)$$

is compact in $\mathrm{Hom}(H^1\times L^2;L^2\times L^2)$. Denoting by $\mathcal{T}(t,0)$ the solution operator to (3.3)

$$\mathcal{T}(t,0)\colon V(0)\mapsto V(t)$$

we see that

$$\begin{pmatrix} H(t) & 0 \\ 0 & 1 \end{pmatrix}\mathcal{R}(t,0)\chi-\mathcal{T}(t,0)\begin{pmatrix} H(0) & 0 \\ 0 & 1 \end{pmatrix}\chi$$

is compact. Let us set

$$H^{-1}(t)=\mathrm{Op}^w(h^{-1}(t,x,\xi))$$

and note that $H^{-1}(t)$ is bounded in $\mathrm{Hom}(L^2;H^1)$ with a bound independent of t. From Lemma 2.2 we see $H^{-1}H=1+r^w$ with $r\in C^1(\mathbb{R};S(\langle\xi\rangle^{-2}\langle x\rangle^{-1},g))$, which is compact in $\mathrm{Hom}(L^2;H^1)$ by Lemma 2.1, and hence we see that

$$\mathcal{R}(t,0)\chi-\begin{pmatrix} H^{-1}(t) & 0 \\ 0 & 1 \end{pmatrix}\mathcal{T}(t,0)\begin{pmatrix} H(0) & 0 \\ 0 & 1 \end{pmatrix}\chi \tag{3.4}$$

is compact in $H^1\times L^2$.

We diagonalize the system (3.3) up to zero-th order term. Let us set

$$T=T_0+T_{-1}=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}+\begin{pmatrix} \left(\dfrac{D_t h}{2h^2}\right)^w & 0 \\ 0 & -\left(\dfrac{D_t h}{2h^2}\right)^w \end{pmatrix}$$

where $D_t h/2h^2\in S(\langle\xi\rangle^{-1}\langle x\rangle^{-1},g)$. Let us put

$$\Lambda=\Lambda_1+\Lambda_0=H\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+\left(\frac{D_t h}{2h}\right)^w\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then, noting that $T_{-1}\in C^1(\mathbb{R};S(\langle\xi\rangle^{-1}\langle x\rangle^{-1},g))$, it is easy to check that

$$D_t(TV)=\Lambda TV+RV,\quad R\in C^1(\mathbb{R};S(\langle\xi\rangle^{-1}\langle x\rangle^{-1},g)). \tag{3.5}$$

Let $W(t)$ be the solution to

$$D_t W=\Lambda W \tag{3.6}$$

with $W(0)=T(0)V(0)$. Then from the energy inequality we have

$$\|T(t)V(t)-W(t)\|\le C\int_0^t\|R(s)V(s)\|\,ds.$$

From the same arguments proving Lemma 3.1 it follows that the operator

$$V(0) \mapsto T(t)V(t) - W(t)$$

is compact in $L^2 \times L^2$. Let us denote by $\mathcal{S}(t,0)$ the solution operator of the system (3.6)

$$\mathcal{S}(t, 0)\colon W(0) \mapsto W(t)$$

then we conclude that

$$T(t)\mathcal{T}(t, 0) - \mathcal{S}(t, 0)T(0)$$

is compact in $L^2 \times L^2$. Since $T_0^{-1}T(t) = I + R$, $R \in C^1(\mathbb{R}; S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g))$ and hence we see that

$$\mathcal{T}(t, 0) - T_0^{-1}\mathcal{S}(t, 0)T(0)$$

is compact. Inserting this into (3.4) we get

Proposition 3.1. *Let* $\chi \in C_0^\infty(\mathbb{R}^n)$. *Then*

$$\mathcal{R}(t, 0)\chi - \begin{pmatrix} H^{-1}(t) & 0 \\ 0 & 1 \end{pmatrix} T_0^{-1}\mathcal{S}(t, 0)T(0)\begin{pmatrix} H(0) & 0 \\ 0 & 1 \end{pmatrix}\chi$$

is compact in $H^1 \times L^2$.

4. Lower bounds (proof of Theorem 1.1)

In this section we essentially follow the arguments in [8]. Recall that the system (3.6) consists of uncoupled two single equations so that $\mathcal{S}(t, 0)$ is diagonal. Let us consider

$$D_t U = \Lambda_1 U \tag{4.1}$$

and denote by $\mathcal{U}(t, s)$ the solution operator

$$\mathcal{U}(t, s)\colon U(s) \mapsto U(t).$$

Note that $\mathcal{U}(t, s)$ is unitary because $H^* = H$. Let us put

$$P(t) = \mathcal{S}(t, 0)\chi(x)\mathcal{U}(0, t) = \operatorname{diag}(P_1(t), P_2(t)).$$

Since $\mathcal{U}(t, s)$ satisfies $D_t\mathcal{U}(0, t) = -\mathcal{U}(0, t)\Lambda_1(t)$ it is easy to see

$$\begin{cases} D_t P_1(t) = [H, P_1] + \left(\dfrac{D_t h}{2h}\right)^w P_1, \quad P_1(0) = \chi(x), \\ D_t P_2(t) = -[H, P_2] + \left(\dfrac{D_t h}{2h}\right)^w P_2, \quad P_2(0) = \chi(x). \end{cases} \tag{4.2}$$

Since the arguments is the same for the second equation, we consider the first equation. Writing $P_1 = P$ it yields

$$\begin{cases} D_t P(t) = [H, P] + \left(\dfrac{D_t h}{2h}\right)^w P, \\ P(0) = \chi(x). \end{cases} \tag{4.3}$$

Following [11] we look for $Q(t) = \mathrm{Op}^w(q)$, $q(t, x, \xi) \in S(1, g)$ solving the equation (4.3). Then q must satisfy

$$\partial_t q = \{h, q\} + \frac{\partial_t h}{2h} q, \quad q(0, x, \xi) = \chi(x). \tag{4.4}$$

Lemma 4.1. *There is a solution* $q(t, x, \xi) \in C^1(\mathbb{R}; S(1, g))$ *to* (4.4) *such that* $q(t, x, \xi)$ *vanishes outside some compact set in* x *and hence*

$$q(t, x, \xi) \in C^1([0, T]; S(\langle x\rangle^{-1}, g))$$

for any $T > 0$.

Proof. Let $(X(t), \Xi(t))$ be a bicharacteristic of $-h(t, x, \xi)$, that is

$$\begin{cases} \dfrac{d}{dt} X(t) = -\dfrac{\partial h}{\partial \xi}(t, X, \Xi), \quad X(s) = x, \\ \dfrac{d}{dt}\Xi(t) = \dfrac{\partial h}{\partial x}(t, X, \Xi), \quad \Xi(s) = \xi. \end{cases} \tag{4.5}$$

Then from the ellipticity of h it is not difficult to check that

$$X(t; x, \xi) \in S(1, g), \quad \Xi(t; x, \xi) \in S(\langle\xi\rangle, g).$$

From

$$\frac{d}{dt} q(t, X(t), \Xi(t)) = \frac{\partial_t h}{2h}(t, X(t), \Xi(t)) q(t, X(t), \Xi(t))$$

we have

$$q(s, x, \xi) = \exp\left(\int_0^s \frac{\partial_t h}{2h}(\tau, X(\tau), \Xi(\tau))\, d\tau\right) \chi(X(0)). \tag{4.6}$$

From this we conclude that $q(t, x, \xi) \in C^1(\mathbb{R}; S(1, g))$. Since $|dX(t)/dt| = |(\partial h/\partial \xi)(t, X, \Xi)| \le C$ and $\chi(x)$ has compact support it is clear that for each t, $q(t, x, \xi)$ vanishes outside some compact set $|x| \le C_t$. This proves the assertion. □

Let us set

$$Q(t) = \mathrm{Op}^w(q(t, x, \xi)).$$

Then by Lemma 2.2 we have

$$D_t Q = [H, Q] + \left(\frac{D_t h}{2h}\right)^w Q + R, \quad R \in C^1(\mathbb{R}, S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g)).$$

We remark that

$$\frac{d}{dt} \log h(t, X(t), \Xi(t)) = \frac{\partial_t h}{h}(t, X(t), \Xi(t)) \tag{4.7}$$

which follows from (4.5). Since $\partial_t h/h = \partial_t a/2a$ for $|\xi| \geq c^*$ and $\chi(x) = 1$ for $|x| \leq C^*$ it follows from (4.6) and (4.7) that

$$\begin{aligned} q(t, X(t), \Xi(t)) &= \exp\left(\int_0^t \frac{\partial_t a}{4a}(\tau, X(\tau), \Xi(\tau))\, d\tau\right) \\ &= \sqrt[4]{\frac{a(t, X(t), \Xi(t))}{a(0, X(0), \Xi(0))}} \geq A^{-1}\sqrt{\frac{|\Xi(t)|}{|\Xi(0)|}} \end{aligned} \tag{4.8}$$

provided $|X(t)| \leq C^*$ and $|\Xi(t)| \geq c^*$ for $t \geq 0$.

REMARK. From (4.7) it follows that

$$\int_0^t \frac{\partial_t a}{a}(\tau, X(\tau), \Xi(\tau))\, d\tau = \log \frac{a(t, X(t), \Xi(t))}{a(0, X(0), \Xi(0))}.$$

With $\mathcal{S}(t, s) = \mathrm{diag}(\mathcal{S}_1(t, s), \mathcal{S}_2(t, s))$, $\mathcal{U}(t, s) = \mathrm{diag}(\mathcal{U}_1(t, s), \mathcal{U}_2(t, s))$ we recall that $P(t) = \mathcal{S}_1(t, 0)\chi(x)\mathcal{U}_1(0, t)$. Following [11] we estimate the difference $P(t) - Q(t)$. Since $(P(t) - Q(t))\mathcal{U}_1(t, 0) = \mathcal{S}_1(t, 0)\chi(x) - Q(t)\mathcal{U}_1(t, 0)$ setting with $f \in L^2$

$$u(t) = \mathcal{S}_1(t, 0)\chi(x) f, \quad v(t) = Q(t)\mathcal{U}_1 f$$

we have

$$D_t(u - v) = (H + (D_t h/2h)^w)(u - v) - R(t)\mathcal{U}_1(t, 0) f. \tag{4.9}$$

From the energy inequality (see for example Theorem 23.1.2 in [4]) for any $T > 0$ there is $C > 0$ such that

$$\|u(t) - v(t)\| \leq C \int_0^t \|R(s)\mathcal{U}_1(s, 0) f\|\, ds, \quad t \in [0, T].$$

Since $\|\langle D\rangle^{-1}\mathcal{U}_1(t,0)f - \langle D\rangle^{-1}\mathcal{U}_1(t',0)f\| \le C|t-t'|\|f\|$, from the proof of Lemma 3.1 it follows that $L^2 \ni f \mapsto R(t)\mathcal{U}_1(t,0)f \in C^0([0,T];L^2)$ is compact. Since $\mathcal{U}_1(t,0)$ is unitary we conclude that $P(t)-Q(t)$ is compact in L^2. Since $\phi(D)Q(t)$ is compact in L^2 if

$$\phi(\xi) \in C_0^\infty(\mathbb{R}^n), \quad \phi(\xi) = 0, \quad |\xi| \ge \frac{c^*}{2} \tag{4.10}$$

and hence $\phi(D)P(t)$ is also compact in L^2.

Assume that there is a bicharacteristic $(x(t),\xi(t))$ of $-h(t,x,\xi)$ satisfying (1.4). Since $h(t,x,\xi) = \sqrt{a(t,x,\xi)}$ for $|\xi| \ge c^*$ then from the homogeneity in ξ it is easy to see that $(x(t),\lambda\xi(t))$, $\lambda \ge 1$ is also a bicharacteristic of $-h(t,x,\xi)$.

Let $t > 0$ be fixed. Since $|\lambda\xi(t)| \ge R$ if $c^*\lambda \ge R$ then we have

$$\begin{aligned}
&\sup_{|\xi|\ge R}\sup_x |q(t,x,\xi)| \ge \sup_{|\xi|\ge R}|q(t,x(t),\xi)| \ge |q(t,x(t),\lambda\xi(t))| \\
&= \exp\left(\int_0^t \frac{\partial_t a}{4a}(\tau,x(\tau),\lambda\xi(\tau))\,d\tau\right) = \exp\left(\int_0^t \frac{\partial_t a}{4a}(\tau,x(\tau),\xi(\tau))\,d\tau\right) \\
&\ge A^{-1}\sqrt{\frac{|\xi(t)|}{|\xi(0)|}}.
\end{aligned}$$

Let us set

$$\exp\left(\int_0^t \frac{\partial_t a}{4a}(\tau,x(\tau),\xi(\tau))\,d\tau\right) = G(t).$$

Noting $\mathrm{Op}^w(q(t,x,\xi)) = \mathrm{Op}^0(q(t,x,\xi)) + K$ where K is compact in L^2 we apply Theorem 3.3 in [5] to conclude

$$\|Q(t)\|_{\mathrm{Hom}(L^2)/\mathcal{K}} \ge G(t)$$

which proves that

$$\|P(t)\|_{\mathrm{Hom}(L^2)/\mathcal{K}} \ge G(t).$$

Recalling that $\mathcal{U}(0,t)$ is unitary we conclude that

$$\|\mathcal{S}(t,0)\chi\|_{\mathrm{Hom}(L^2)/\mathcal{K}} \ge G(t). \tag{4.11}$$

To prove Theorem 1.1 it suffices to show

Proposition 4.1. *Let $\Phi(\xi) = 1-\phi(\xi)$ with $\phi(\xi)=1$ near $\xi=0$ verifying* (4.10). *Then there is a $C>0$ such that*

$$\|\Phi(D)\mathcal{R}(t,0)\chi\|_{\mathrm{Hom}(H^1\times L^2;H^1\times L^2)} \ge CG(t). \tag{4.12}$$

Proof. Let us set

$$M(t) = \begin{pmatrix} H^{-1}(t) & 0 \\ 0 & 1 \end{pmatrix} T_0^{-1}, \quad L = T(0)\begin{pmatrix} H(0) & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$\mathcal{R}(t,0)\chi - M(t)\mathcal{S}(t,0)L\chi$$

is compact in $H^1 \times L^2$. Since

$$[L,\chi] = \begin{pmatrix} S(\langle x\rangle^{-1}, g) & 0 \\ S(\langle x\rangle^{-1}, g) & 0 \end{pmatrix} + S(\langle \xi\rangle^{-1}\langle x\rangle^{-1}, g)$$

and hence compact from $H^1 \times L^2$ to $L^2 \times L^2$ then $M(t)\mathcal{S}(t,0)[L,\chi]$ is compact in $H^1 \times L^2$. Thus we see that $\mathcal{R}(t,0)\chi - M(t)\mathcal{S}(t,0)\chi L$ is compact in $H^1 \times L^2$. Hence one can write

$$\Phi(D)\mathcal{R}(t,0)\chi = \Phi(D)M(t)\mathcal{S}(t,0)\chi L + \tilde{K}$$

where $\tilde{K}$ is compact in $H^1 \times L^2$. Since $[\Phi(D), M(t)] \in S(\langle\xi\rangle^{-2}\langle x\rangle^{-1}, g)$ we get $\Phi(D)\mathcal{R}(t,0)\chi - M(t)\Phi(D)\mathcal{S}(t,0)\chi L$ is compact in $H^1 \times L^2$. Since $\phi(D)\mathcal{S}(t,0)\chi$ is compact in L^2 we conclude that

$$\Phi(D)\mathcal{R}(t,0)\chi = M(t)\mathcal{S}(t,0)\chi L + \hat{K}$$

where $\hat{K}$ is compact in $H^1 \times L^2$. We denote

$$M^{-1}(t) = T_0\begin{pmatrix} H(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} H^{-1}(0) & 0 \\ 0 & 1 \end{pmatrix} T_0^{-1}$$

so that we have

$$M^{-1}(t)M(t) = I + K_1, \quad LL^{-1} = I + K_2, \quad K_i \in S(\langle\xi\rangle^{-1}\langle x\rangle^{-1}, g).$$

Thanks to Lemma 2.1 we see that K_i are compact in L^2. Consider

$$\begin{aligned} &M^{-1}(t)[M(t)\mathcal{S}(t,0)\chi L + \hat{K}]L^{-1} \\ &= (I + K_1)\mathcal{S}(t,0)\chi(I + K_2) + K_3 = \mathcal{S}(t,0)\chi + K_4 \end{aligned}$$

where K_3 and K_4 are compact in L^2. From (4.11) it follows that

$$\begin{aligned} G(t) &\le \|\mathcal{S}(t,0)\chi + K_4\|_{\mathrm{Hom}(L^2\times L^2)} \\ &= \|M^{-1}(t)[M(t)\mathcal{S}(t,0)\chi L + \hat{K}]L^{-1}\|_{\mathrm{Hom}(L^2\times L^2)} \end{aligned}$$

$$\le \|M^{-1}(t)\|_{\mathrm{Hom}(H^1\times L^2;L^2\times L^2)}\|M(t)\mathcal{S}(t,0)\chi L+\hat{K}\|_{\mathrm{Hom}(H^1\times L^2;H^1\times L^2)}$$
$$\times \|L^{-1}\|_{\mathrm{Hom}(L^2\times L^2;H^1\times L^2)}$$
$$\le C\|M(t)\mathcal{S}(t,0)\chi L+\hat{K}\|_{\mathrm{Hom}(H^1\times L^2;H^1\times L^2)}$$

where we note that C is independent of t. This proves (4.12). □

To prove Theorem 1.1 note that $\Phi(\xi)$ vanishes near $\xi=0$ and hence $\|\Phi(D)u\|_{H^1}\le C\|u\|_{\mathcal{H}}$. Thus it follows from (4.12)

$$\|\mathcal{R}(t,0)\chi\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}\ge C'G(t)$$

which proves Theorem 1.1.

REMARK. Let $\alpha(t)$ be any positive function such that $\alpha(t)/G(t)\to 0$ as $t\to\infty$. Then from (4.12) and the uniform boundedness principle it follows that there exists $(f,g)\in H^1\times L^2$ such that

$$\limsup_{t\to\infty}\alpha(t)^{-1}\|\mathcal{R}(t,0)\chi^t(f,g)\|_{H^1\times L^2}=\infty.$$

Note that the initial data $\chi^t(f,g)={}^t(\chi f,\chi g)$ has compact support which is a main difference from Theorem 1 in [10].

5. Proof of Theorems 1.2 and 1.3

In this section we construct examples to which one can apply Theorem 1.1 and we prove Theorems 1.2 and 1.3. We follow the construction given by Colombini and Rauch in [1] and generalize it a little bit. We first check Proposition 1.1.

Let $E(t)$ denote the standard energy;

$$E(t)=\int_{\mathbb{R}^n}\left\{|\partial_t u|^2+\sum_{i,j=1}^n a_{ij}(t,x)\partial_{x_i}u\partial_{x_j}\bar{u}\right\}dx.$$

Then for any initial data in $C_0^\infty(\mathbb{R}^n)$ it is easy to see

$$\frac{d}{dt}E(t)=\int_{\mathbb{R}^n}\sum_{i,j=1}^n\partial_t a_{ij}(t,x)\partial_{x_i}u\partial_{x_j}\bar{u}\,dx$$
$$\le\sup_{x,\xi}\frac{|\partial_t a(t,x,\xi)|}{a(t,x,\xi)}\int_{\mathbb{R}^n}\left\{\sum_{i,j=1}^n a_{ij}(t,x)\partial_{x_i}u\partial_{x_j}\bar{u}\right\}dx$$
$$\le\left[\sup_{x,\xi}\frac{|\partial_t a(t,x,\xi)|}{a(t,x,\xi)}\right]E(t)$$

because

$$\sup_{\zeta\in\mathbb{C}^n}\frac{\sum_{i,j=1}^n \partial_t a_{ij}(t,x)\zeta_i\bar{\zeta}_j}{\sum_{i,j=1}^n a_{ij}(t,x)\zeta_i\bar{\zeta}_j}=\sup_{\xi\in\mathbb{R}^n}\frac{\sum_{i,j=1}^n \partial_t a_{ij}(t,x)\xi_i\xi_j}{\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j}$$

which proves the assertion since

$$A^{-2}\int_{\mathbb{R}^n}|\nabla u|^2\,dx\le\int_{\mathbb{R}^n}\left\{\sum_{i,j=1}^n a_{ij}(t,x)\partial_{x_i}u\partial_{x_j}\bar{u}\right\}dx\le A^2\int_{\mathbb{R}^n}|\nabla u|^2\,dx$$

for all $t\in\mathbb{R}$.

In what follows we take $n=2$ while the same argument works in the general $n\ge 2$. To apply Theorem 1.1 we look for $a(t,x)$ such that the hamiltonian $\mp\sqrt{a(t,x)}|\xi|$ admits a bicharacteristic $(x(t),\xi(t))$ such that $|\xi(t)|$ is away from zero while $|x(t)|$ remains to be bounded when $t\to+\infty$;

$$\begin{cases}\dfrac{d}{dt}x(t)=\mp\dfrac{\partial(\sqrt{a(t,x)}|\xi|)}{\partial\xi},\\[2mm] \dfrac{d}{dt}\xi(t)=\pm\dfrac{\partial(\sqrt{a(t,x)}|\xi|)}{\partial x}.\end{cases}\tag{5.1}$$

Using the standard identification $\mathbb{C}\ni u+iv\mapsto(u,v)\in\mathbb{R}^2$ of $\mathbb{R}^2$ with the complex plane we write

$$x=re^{i\theta},\quad \xi=\rho e^{i\phi}.$$

Let $\delta(t)$ be a smooth function on $\mathbb{R}$ with bounded derivatives of all order. Motivated by [1] we choose $a(t,x)=a(t,r,\theta)$ so that

$$\sqrt{a(t,r,\theta)}=\exp\left(\chi(r)(r-1-2\delta(t)f\left(\theta-t-\frac{\pi}{2}\right)\right)\tag{5.2}$$

where $\chi(r)\in C_0^\infty(\mathbb{R})$, $0\le\chi(r)\le 1$ which is zero near $r=0$ and identically equal to 1 on a small neighborhood of $r=1$. Here $f(t)\in C^\infty(\mathbb{R})$ is 2π periodic verifying

$$f(0)=0,\quad f'(0)=1.\tag{5.3}$$

To simplify notations let us write $h(t,r,\theta)=\sqrt{a(t,r,\theta)}$ then the Hamilton equation (5.1) with the hamiltonian $-\sqrt{a}|\xi|=-h(t,r,\theta)\rho$ yields

$$\begin{cases}\dfrac{d}{dt}r=-h\cos(\theta-\phi),\\[2mm] \dfrac{d}{dt}\theta=\dfrac{h}{r}\sin(\theta-\phi),\\[2mm] \dfrac{d}{dt}\phi=-\dfrac{\partial h}{\partial r}\sin(\phi-\theta)+\dfrac{1}{r}\dfrac{\partial h}{\partial\theta}\cos(\phi-\theta)\end{cases}\tag{5.4}$$

and

$$\frac{d}{dt}\rho = \rho\left[\frac{\partial h}{\partial r}\cos(\phi-\theta) + \frac{1}{r}\frac{\partial h}{\partial \theta}\sin(\phi-\theta)\right]. \tag{5.5}$$

Lemma 5.1. *Let* $a(t, r, \theta)$ *be given by* (5.2). *Then* $\theta(t) = t + \pi/2$, $\phi(t) = t$, $r(t) = 1$ *solve* (5.4). *Moreover we have*

$$\rho(t) = \rho(0)\exp\left(2\int_0^t \delta(s)\, ds\right).$$

Proof. Since $(\partial h/\partial r)(t,1,\theta(t)) = h(t,1,\theta(t))$ the first assertion is clear from (5.3). Note that when $\theta(t) = t + \pi/2$, $\phi(t) = t$, $r(t) = 1$ we have from (5.5) that

$$\frac{d}{dt}\rho = -\rho\frac{\partial h}{\partial \theta}(t, 1, \theta(t)) = 2\delta(t)\rho$$

which proves the assertion. □

We now prove Theorem 1.2. Suppose that $\delta(t)$ is given. Take $\chi_1(t) \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi_1(t) \le 1$ verifying $\int \chi_1(t)dt = 1$, $\chi_1(t) = 1$ for $|t| \le 1/4$ and $\chi_1(t) = 0$ for $|t| \ge 3/4$. Define $\tilde\delta(t)$ by

$$\tilde\delta(t) = \int \chi_1(t-s)\delta(s)\, ds$$

then it is easy to see that

$$\begin{cases} C_1 + \displaystyle\int_0^t \delta(s)\, ds \le \int_0^t \tilde\delta(s)\, ds \le C_1 + \int_0^t \delta(s)\, ds, \\ \displaystyle\int_0^t |\tilde\delta'(s)|\, ds \le C_2\int_0^t \delta(s)\, ds + C_3 \end{cases} \tag{5.6}$$

with some constants C_i independent of $t \ge 0$. For any given $\sigma > 0$ small it is clear that one can find a 2π periodic $f(t)$ verifying (5.3) such that

$$\sup|f'(t)| \le 1, \quad \sup|f(t)| \le \sigma. \tag{5.7}$$

We define $\sqrt{a(t, r, \theta)}$ by (5.2) with this $f(t)$ and $\delta(t) = \tilde\delta(t)$. Choosing $\rho(0) = 1$, for example, from Lemma 5.1 there exists a solution $(x(t), \xi(t))$ to the Hamilton system (5.1) with $a(t, x)$ such that

$$|x(t)| = 1, \quad \forall t \in \mathbb{R}, \quad \frac{|\xi(t)|}{|\xi(0)|} = \exp\left(2\int_0^t \tilde\delta(s)\, ds\right),$$

which clearly verifies (1.4). Applying Theorem 1.1, together with (5.6) we obtain the lower bound of Theorem 1.2.

To get the upper bound we note that

$$\frac{|\partial_t a(t,x)|}{a(t,x)} \le 4|\tilde{\delta}'(t)| \sup|f| + 4\tilde{\delta}(t)\sup|f'| \le 4\sigma|\tilde{\delta}'(t)| + 4\tilde{\delta}(t).$$

Thus we have

$$\int_0^t \left(\sup_x \frac{|\partial_t a(s,x)|}{a(s,x)}\right) ds \le 4\int_0^t \tilde{\delta}(s)\,ds + 4\sigma\int_0^t |\tilde{\delta}'(s)|\,ds. \tag{5.8}$$

On the other hand from Proposition 1.1 it follows that

$$\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)} \le C_1 \exp\left(\frac{1}{2}\int_0^t \left[\sup_x \frac{|\partial_t a(\tau,x)|}{a(\tau,x)}\right] d\tau\right)$$

which, together with (5.6) and (5.8), proves the upper bound taking $\sigma > 0$ small enough.

We turn to the proof of Theorem 1.3. Assume that u solves (1.5) with initial data $(u(0), \partial_t u(0)) = (\phi, \psi)$ where $a(t,x)$ is given by (5.2) with f verifying (5.7). We consider a modified energy;

$$\tilde{E}(t) = E(t) + \beta(t)\mathrm{Re}(\partial_t u, u) + \gamma(t)\|u\|^2$$

where

$$E(t) = \int_{\mathbb{R}^2} (|\partial_t u|^2 + a(t,x)|\nabla u|^2)\,dx, \quad (u,v) = \int_{\mathbb{R}^2} u\bar{v}\,dx, \quad \|u\|^2 = (u,u).$$

Real valued functions $\beta(t)$ and $\gamma(t)$ will be determined later. Noting

$$\frac{d}{dt}E(t) \le 4(-\delta'(t)\sigma + \delta(t))\int_{\mathbb{R}^n} a(t,x)|\nabla u|^2\,dx$$

we put $\alpha(t) = 4(-\delta'(t)\sigma + \delta(t))$. Since $\partial_t^2 u = \sum_{i=1}^2 \partial_{x_i}(a(t,x)\partial_{x_i} u)$ we see

$$\begin{aligned}\frac{d}{dt}\tilde{E} &\le \beta\|\partial_t u\|^2 + (\alpha-\beta)\int_{\mathbb{R}^n} a|\nabla u|^2\,dx + (\beta' + 2\gamma)\,\mathrm{Re}(\partial_t u, u) + \gamma'\|u\|^2 \\ &= \beta\tilde{E} + (\alpha - 2\beta)\int_{\mathbb{R}^n} a|\nabla u|^2\,dx + (\beta' + 2\gamma - \beta^2)\,\mathrm{Re}(\partial_t u, u) + (\gamma' - \beta\gamma)\|u\|^2.\end{aligned}$$

We choose $\beta = 2\delta$ and $2\gamma = \beta^2 - \beta'$. Since $\gamma \ge \beta^2/2$ we have

$$E(t) \le 2\tilde{E}(t). \tag{5.9}$$

Taking (5.9) into account we obtain

$$\frac{d}{dt}\tilde{E} \le (-8\delta'\sigma + 2\delta)\tilde{E} + (6\delta\delta' - \delta'' - 4\delta^3)\|u\|^2$$

where we remark that $\delta \ge 0$, $\delta' \le 0$, $\delta'' \ge 0$ and hence

$$\frac{d}{dt}\tilde{E}(t) \le (-8\delta'(t)\sigma + 2\delta(t))\tilde{E}(t).$$

Thus we obtain

$$\begin{aligned}\tilde{E}(t) &\le \tilde{E}(0)e^{8\sigma\delta(0)} \exp\left(2\int_0^t \delta(s)\,ds\right)\\ &\le C\|(\phi,\psi)\|^2_{H^1\times L^2} \exp\left(2\int_0^t \delta(s)\,ds\right)\end{aligned}$$

for $\tilde{E}(0) \le C'\|(\phi,\psi)\|^2_{H^1\times L^2}$ with some $C' > 0$. Thanks to (5.9) one obtains

$$\|\mathcal{R}(t,0)^t(\phi,\psi)\|_{\mathcal{H}\times L^2} \le C_2\|(\phi,\psi)\|_{H^1\times L^2} \exp\left(\int_0^t \delta(s)\,ds\right). \tag{5.10}$$

On the other hand, from Proposition 4.1 it follows that

$$C_1 \exp\left(\int_0^t \delta(s)\,ds\right) \le \|\mathcal{R}(t,0)\chi\|_{\mathrm{Hom}(H^1\times L^2;\mathcal{H}\times L^2)}$$

which together with (5.10) proves the assertion.

We finally give a little bit more general examples than we took in the proof of Theorems 1.2 and 1.3. Let $\phi(t)$, $\rho(t)$, $r(t) \in C^\infty(\mathbb{R})$ satisfy the followings;

(i) r, ρ are strictly positive on $\mathbb{R}$,

(ii) ϕ', r, ρ'/ρ have bounded derivatives of all orders,

(iii) we have

$$\sup_{t\in\mathbb{R}}\left|\int_0^t re^{i\phi}\,ds\right| < \infty.$$

We put

$$\begin{aligned}\tilde{a}(t,x_1,x_2) = r(t)^2&\{\sin(r(t)^{-2}p_1(t)(x_1 - x_1(t))) + 2\}\\ &\times\{\sin(r(t)^{-2}p_2(t)(x_2 - x_2(t))) + 2\}\end{aligned}$$

where, with the standard identification of $\mathbb{C}$ and $\mathbb{R}^2$,

$$\begin{aligned}x(t) = (x_1(t), x_2(t)) &= -2\int_0^t re^{i\phi}\,ds + x(0),\\ (p_1, p_2) &= \frac{2r}{\rho}(\rho' e^{i\phi} + i\rho\phi' e^{i\phi}).\end{aligned}$$

Then $\{x(t);\ t \in \mathbb{R}\}$ is contained in a compact set by the assumption (iii) and $\tilde{a}(t, x)$ is smooth and $\inf_{\mathbb{R}^3} \tilde{a}(t, x) > 0$.

Lemma 5.2. *$x(t)$ and $\xi(t) = \rho(t)e^{i\phi}$ solve the Hamilton equation with the hamiltonian $-\sqrt{\tilde{a}}|\xi|$.*

Proof. Since $\tilde{a}(t, x(t)) = 4r(t)^2$ the first equation of (5.1) follows easily. To check the second equation of (5.1) it suffices to note $\nabla_x \tilde{a}(t,x(t)) = 2(p_1(t), p_2(t))$ and $d\xi/dt = \rho/(2r)(p_1,\ p_2)$. □

We define $a(t, x)$ by

$$a(t, x) = \chi(x)\tilde{a}(t, x) + (1 - \chi(x))$$

where $\chi(x) \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \chi(x) \leq 1$ which is identically equal to 1 on a small neighborhood of $\{x(t); t \in \mathbb{R}\}$, for which one can apply Theorem 1.1 with $(x(t), \xi(t))$.

If we take $r(t) = 1 + e^{-t}$ $(t \geq 0)$, $\phi(t) = t$ and $x(0) = (1, 3)$ for instance, then we see easily that

$$|x(t) - 2ie^{it}| = \sqrt{2}e^{-t}$$

and hence the orbit $\{x(t);\ t \geq 0\}$ is not closed.

References

[1] F. Colombini and J. Rauch: *Smooth localized parametric resonance for wave equations*, J. Reine Angew. Math. **616** (2008), 1–14.

[2] F. Colombini and S. Spagnolo: *Hyperbolic equations with coefficients rapidly oscillating in time*: *a result of nonstability*, J. Differential Equations **52** (1984), 24–38.

[3] F. Hirosawa: *On the asymptotic behavior of the energy for the wave equations with time depending coefficients*, Math. Ann. **339** (2007), 819–838.

[4] L. Hörmander: The Analysis of Linear Partial Differential Operators, III, Springer, Berlin, 1985.

[5] L. Hörmander: *Pseudo-differential operators and hypoelliptic equations*; in Singular Integrals (Proc. Sympos. Pure Math. **10**, Chicago, Ill., 1966), Amer. Math. Soc., Providence, RI, 1966, 138–183.

[6] H. Koch and D. Tataru: *On the spectrum of hyperbolic semigroups*, Comm. Partial Differential Equations **20** (1995), 901–937.

[7] A. Martinez: An Introduction to Semiclassical and Microlocal Analysis, Universitext, Springer, New York, 2002.

[8] J. Rauch and M. Taylor: *Decay of solutions to nondissipative hyperbolic systems on compact manifolds*, Comm. Pure Appl. Math. **28** (1975), 501–523.

[9] M. Reissig and J. Smith: *L^p-L^q estimate for wave equation with bounded time dependent coefficient*, Hokkaido Math. J. **34** (2005), 541–586.

[10] M. Reissig and K. Yagdjian: *About the influence of oscillations on Strichartz-type decay estimates*, Rend. Sem. Mat. Univ. Politec. Torino **58** (2000), 375–388.

[11] M.E. Taylor: Pseudodifferential Operators, Princeton Univ. Press, Princeton, NJ, 1981.

Shin-ichi Doi
Department of Mathematics
Osaka University
Machikaneyama 1-1, Toyonaka, 560-0043, Osaka
Japan
e-mail: sdoi@math.sci.osaka-u.ac.jp

Tatsuo Nishitani
Department of Mathematics
Osaka University
Machikaneyama 1-1, Toyonaka, 560-0043, Osaka
Japan
e-mail: nishitani@math.sci.osaka-u.ac.jp

Hideo Ueda
Department of Mathematics
Osaka University
Machikaneyama 1-1, Toyonaka, 560-0043, Osaka
Japan
e-mail: hideomathjp@cr.math.sci.osaka-u.ac.jp