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On the foundation of balayage theory

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Introduction

In this paper, we intend to show a new construction of the balayage (sweeping out) of measures which is one of the central themes in the potential theory. Our main tool used here is the noted theorem of Krein-Milman in the theory of general linear topological spaces (see [2], [7], etc). Thus, we begin with some detailed considerations about a linear normed space $H(D)$ and its dual $(H(D))^*$, especially some compact convex subset $\overline{M}_0^+(\overline{D})$ in $(H(D))^*$ generated by the collection of positive measures of norm 1 distributed in the closure of considered open set D (§**1**). The next paragraph (§ **2**) is devoted to the general construction of balayage for open sets, but the same method is also well applicable to the case of closed sets, which is identical with the notion of so-called extremisation owing to M. Brelot (§ **3**).

Now, from a historical point of view, the balayage theory founded by H. Poincaré has been recently reconstructed by means of projection method in the theory of Hilbert space; the most important work of such a kind is appeared in H. Cartan [4], and some interesting works of H. Cartan-J. Deny and of J. Deny follow it. However, in our present work, it seems very interesting that we can find some notable connection between the extreme points of $\overline{M}_0^+(\overline{D})$ (or of $\overline{M}_0^+(F)$) and regular (or resp. stable) boundary-points (§ **4**), and as applications of this fact, we shall offer an elementary criterion in order that a boundary-point be regular or stable (Theorem 17).

§ **5** is devoted to the representation theory and application to Dirichlet's problem in the ordinary or extended form. To obtain the solution, we employ the Banach space method here; thus, we are standing in some different position from the others.

The same theme of this paper has appeared incompletely in the last half of my previous note [10], and the present one is the precision and correction of that. However, we leave the general notion of superharmonicity to be defined as in the first half of [10].

Finally, the author wishes to express his gratitude to Professor Dr. M. Inoue for his kind and precious guidance throughout the development of this work.

§ 1. Preliminary Theorems

1.1. *Preliminary notions and notations.* Let E be a locally compact Hausdorff space; we assume now that for any open set $D \subset E$ there corresponds the family $\mathfrak{L}^+(D)$ of functions defined in D (*called superharmonic in* D) such that:

i) $\mathfrak{L}^+(D)$ forms a positive cône; that is, $\alpha f+\beta g$ belongs to $\mathfrak{L}^+(D)$ together with f and g for any positive numbers α and β,

ii) every function of $\mathfrak{L}^+(D)$ is lower semi-continuous in D,

iii) if $D_1 \subset D_2 (\subset E)$, then $\mathfrak{L}^+(D_2) \subset \mathfrak{L}^+(D_1)$,

and there exists a linear operator Δ_D from $\mathfrak{L}^+(D)$ into positive Radon measures distributed in D which satisfies:

Δ_1) Δ_D is positively linear, *i.e.* $\Delta_D(\alpha f+\beta g)=\alpha\Delta_D(f)+\beta\Delta_D(g)$ for $f, g \in \mathfrak{L}^+(D)$ and $\alpha, \beta \geqq 0$,

Δ_2) if $D_1 \subset D_2$, $\Delta_{D_1}(f)$ coincides with the restriction of $\Delta_{D_2}(f)$ in D_1 for every $f \in \mathfrak{L}^+(D_2)$.[1)]

If $f \in \mathfrak{L}^+(D)$ and simultaneously $-f \in \mathfrak{L}^+(D)$, then f is said to be ***harmonic*** in D. We shall abbreviate Δ_E to Δ and call it generalized Laplacian. If E is n-dimensional Euclidean space R^n for $n \geqq 2$, $\mathfrak{L}^+(D)$ may be adopted as the collection of superharmonic functions in D of the usual sense (e.g. of T. Radó [11]), in which $\Delta(f)$ is defined in such a manner that its restriction in each compact domain $B \subset E$ is the vague limit of the sequence of $\mu_j = -\sum_{i=1}^n \frac{\partial^2 f_j}{\partial x_i^2} dx$, where dx means the n-dimensional Lebesgue measure and $f_j \in \mathfrak{L}^+(B) \cap C^2$, $f_i \nearrow f$ on B.[2)] $\Delta_D(f)$ is naturally the restriction of $\Delta(f)$ in D.

We shall assume moreover that for every positive Radon measure μ distributed in E there cerresponds the potential function $\phi(\mu)$ which satisfies;

1) $\phi(\mu)$ is the function identically infinite or otherwise $\phi(\mu) \in \mathfrak{L}^+(E)$ for which $\Delta\phi(\mu)=\mu$ and it is harmonic outside of the *support of* μ.[3)]

2) Fubini's formula; $\int \phi(\mu)d\nu = \int \phi(\nu)d\mu$ for another positive measure ν in E unless these integrals are meaningless.

3) Modulus (maximal) principle; if $f \geqq 0$ is superharmonic in E and $\phi(\mu)$ has the following properties that i) $\int \phi(\mu)d\mu$ (the *energy of* μ) is finite and ii) $\phi(\mu) \leqq f$ on a ***kernel of*** μ,[3)] then this inequality ii) takes place in the whole E.

4) If K is compact in E, we have a measure λ distributed in $E-K$, whose

1) About these fact, refer to my previous paper [10] § **2,** p. 59~60.

2) C^2 designates the class of functions having continuous partial derivatives up to the order 2. About the assertion, see [10], *ibid.*, **4. 6,** p. 68~69, and N. Bourbaki [1], the article on the localisation of measures, p. 67~69. (About the notation $\nearrow$, refer to the footnote 9).

3) We call such X that $\int_{E-X} |d\mu| = 0$ a ***kernel of*** μ, distinguishing from the *support* which is the intersection of all closed kernels of μ; the support is always uniquely determined.

potential $\phi(\lambda)$ is equal to 1 at least on K and $0 \leqq \phi(\lambda) \leqq 1$ in E. For two distinct points $x, y \in K$, there exists such μ distributed in $E-K$ that $\phi(\mu)(x) \neq \phi(\mu)(y)$.

5) Let $f \geqq 0$ be a continuous function with compact support $K \subset E$; for given $\varepsilon > 0$ and a neighborhood U of K, there exists a Radon measure μ of composed type such that $|f(x) - \phi(\mu)| < \varepsilon$, $\phi(\mu)$ is continuous in E and vanishes in $E-U$.

As an example of such E, we can take primarily the n-dimensional Euclidean space $R^n (n \geqq 3)$ with Newtonian potential (cited as Example *a*)); denoting the Euclidean distance by $r(x, y)$,

$$\phi(\mu)(x) = N_n \int r^{2-n}(x, y) d\mu(y), \qquad N_n = \frac{\Gamma(n/2)}{2(n-1)\pi^{n/2}}.$$

Another example of E is the open unit circle $|z| < 1$ in the complex number plane Z^2 with logarithmic potential (cited as Example *b*)), *i.e.*

$$\phi(\mu)(x) = \iint_{|y|<1} \log \left| \frac{1-\bar{x}y}{x-y} \right| d\mu(y).$$

Next, we enumerate some linear topological spaces of measures and functions whicth shall be made use of successively in the later discussions. For a given measurable X in E, we define:

$C_0(X)$ = space of all continuous functions with compact support in X,

$C(X)$ (or $C_u(X)$) = space of all bounded (resp. uniformly) continuous functions defines in X,

$L_\infty(X)$ = space of all bounded functions defined in X, vanishing at the infinity; that is, each $f \in L_\infty(X)$ is characterized as such a function that for any given $\varepsilon > 0$ there exists a compact $F_\varepsilon \subset X$ outside of which $|f(x)| < \varepsilon$,

$C_\infty(X) = L_\infty(X) \cap C(X)$,

$\mathfrak{M}^+(X)$ = collection of all positive Radon measure defined in X,

$\mathfrak{M}(X)$ = space of all Radon measure defined in X; in other words, it is just the linear envelope of $\mathfrak{M}^+(X)$ over the real field.

Assume always $\phi(\mu) \in L_\infty(E)$ for any μ with compact support.

The first three spaces $C_0(X)$, $C(X)$ and $C_u(X)$ form Banach spaces with respect to the uniform norm (simultaneously, they form Banach algebras, which indicates some significance in regard to the functional representation). If X is compact, these three coincide with each other.

As the topology of $\mathfrak{M}(X)$, we adopt as is customarily done the so-called *vague topology*, that is the topology of simple convergence in $C_0(X)$, then $\mathfrak{M}(X)$ is the toplogical dual of $C_0(X)$. Next Lemma shall be useful in our future work.

LEMMA 1. *If $K \subset E$ is compact, the collection $\mathfrak{M}_0^+(K)$ of such $\mu \in \mathfrak{M}^+(K)$ that $\|\mu\|=1$ is vaguely compact (abbrev. v-compact), and if $F \subset E$ is closed, the collection $\mathfrak{M}_1^+(F)$ of such $\mu \in \mathfrak{M}^+(F)$ that $\|\mu\| \leqq 1$ is also v-compact.*[♮]

To see the first half, it is sufficient to remark that the unit function, $1(x)=1$ on K, is contained in $C(K)$; the latter half shall be refered to N. Bourbaki [1][4], taking notice of the fact that $\mathfrak{M}^+(E)$ is complete for the uniform structure deduced from the vague topology.

1. 2. *Linear normed space $H(D)$ and its dual.* In this section, we shall be occupied to study the linear normed space $H(D)$ defined below and its dual $(H(D))^*$, especially the unit sphere Ξ^* of $(H(D))^*$. These studies contribute us simultaneously to construct the balayage, to criticize the regular boundary-points, and to give a new method for Dirichlet problem: we shall start with the

DEFINITION 1. *Let D be a given open set in E with the compact closure $\bar{D}$ and boundary ∂D. $H(D)$ denotes a linear normed space consisting of the restrictions in $\bar{D}$ of all bounded potentials $\phi(\mu)$ for $\mu \in \mathfrak{M}(E-D)$, in which the norm is defined by*

$$(1.1) \qquad \|f\|_D = \sup_{x \in \bar{D}} |f(x)|, \quad f \in H(D).$$

$H_0(D)$ denotes a linear subspace of $H(D)$ consisting only of those which are continuous in $\bar{D}$, i.e. $H_0(D)=H(D) \cap C(\bar{D})$.

We use sometimes the same letter $\phi(\mu) \in B(E)$ with $\mu \in \mathfrak{M}(E-D)$ for its restriction in $\bar{D}$, that is, an element in $H(D)$ so far as no confusion would occur, where $B(E)$ denotes the space of all bounded potentials in E.

Every function of $H(D)$ is obviously harmonic in D.

DEFINITION 2. *$(H(D))^*$ denotes the dual space of $H(D)$, in which we shall always take the weak topology as functionals, that is, topology of simple convergence in $H(D)$.*

The weak topology thus defined is called *w^*-topology* of $(H(D))^*$, for which every element of $H(D)$ acts as a continuous function on $(H(D))^*$. The unit sphere Ξ^* of $(H(D))^*$ is w^*-compact (S. Kakutani's theorem), which is an easy consequence of the fact that a product of compact spaces is compact.

Now, denoting by $\mathfrak{M}_0^+(\bar{D})$ the collection of measures $\in \mathfrak{M}^+(\bar{D})$ with norm 1, we define for each $\mu \in \mathfrak{M}_0^+(\bar{D})$ a linear functional $\mu^\wedge$ on $H(D)$ in the following manner;

$$(1.2) \qquad \mu^\wedge(f) = \int f d\mu, \quad f \in H(D).$$

♮) $\|\mu\|$ denotes the norm of measure μ, i.e. $\int |d\mu|$.

4) Prop. 7, Corollary 2 to Prop. 8, § 2, Chap. III, and Prop. 6, § 3, Chap. III.

Owing to the property 4) of potential we see that $H(D)$ has the unit function $1(x)(1(x)=1$ for all $x\in\bar{D})$; the collection $M_0^+(\bar{D})$ of such functionals $\mu^\frown$ defined by (1, 2) for $\mu\in\mathfrak{M}_0^+(\bar{D})$ forms a convex subset in the unit sphere Ξ^* of $(H(D))^*$, hence its w^*-closure $\bar{M}_0^+(\bar{D})$ is also convex and w^*-compact, that is, regularly convex in the sense of Krein-Smulian. Thus, the Krein-Milman's extreme points theorem is applicable to $\bar{M}_0^+(\bar{D})$; $\bar{M}_0^+(\bar{D})$ possesses sufficiently many extreme points whose closed convex hull coincides with $\bar{M}_0^+(\bar{D})$ itself.[5] Denoting the set of all extreme points of $\bar{M}_0^+(\bar{D})$ by $Ext.\ \bar{M}_0^+(\bar{D})$, we can translate the above result into the following expression:

THEOREM 1. *For any* $\mu\in\mathfrak{M}_0^+(D)$, $f\in H(\bar{D})$ *and* $\varepsilon>0$, *we can select a finite number of* $\mu_i^\sim\in Ext.\ \bar{M}_0^+(\bar{D})$ *such that*

$$\left|\int f d\mu-\sum\alpha_i\mu_i^\sim(f)\right|<\varepsilon, \tag{1.3}$$

where $\sum\alpha_i=1$ *and* $\alpha_i>0$.

Remark: We denote by $\mu^\sim$ a general element of $\bar{M}_0^+(\bar{D})$, otherwise a limit element of $M_0^+(\bar{D})$ which is not defined primitively by (1.2), distinguishing from any other element $\mu^\frown$ just contained in $M_0^+(\bar{D})$ itself.

An argument quite analogous to the one we discussed above shows that for every measure $\mu\in\mathfrak{M}_0^+(\bar{D})$ (1.2) defines also a linear functional μ^* of norm 1 on $H_0(D)$, *i.e.* $\mu^*\in(H_0(D))^*$, and the collection $M_0^*(\bar{D})$ of such μ^* is compact and convex with respect to the w^*-topology in $(H_0(D))^*$, where $(H_0(D))^*$ means the dual space of $H_0(D)$. It is easily seen that the w^*-topology of $(H_0(D))^*$ is compatible with the topology reduced from that of $(H(D))^*$, considering $(H_0(D))^*$ as a residue space of $(H(D))^*$.

It remains us to prove the compactness of $M_0^*(\bar{D})$, but it follows easily from the fact that $H_0(D)$ is a subspace of $C(D)$ and the application $\mu\to\mu^*$ from $\mathfrak{M}_0^+(\bar{D})$ to $M_0^*(\bar{D})$ is uniformly continuous.

We see next that, according to the restriction from $(H(D))^*$ into $(H_0(D))^*$, to each element $\mu^\sim$ of $\bar{M}_0^+(\bar{D})$ a certain $\mu^*\in M_0^*(D)$ corresponds uniquely (we shall call such μ^* the projection of $\mu^\sim$ in $M_0^*(\bar{D})$), and also to each μ^* there exist by definition some corresponding measures of $\mathfrak{M}_0^+(D)$; among those, we can find at least one measure $\mu\in\mathfrak{M}_0^+(\bar{D})$ which is a vague limit of some subsequence of $\{\mu_\lambda\}$, where $\mu_\lambda^\frown$ converges to $\mu^\sim$ in $(H(D))^*$. Such μ is called a *basic measure* of $\mu^\sim$. But the correspondence $\mu^\sim\to\mu$ is not unique, since so is $\mu^*\to\mu$, and also the inverse correspondence $\mu\to\mu^\sim$ is naturally multi-valued.

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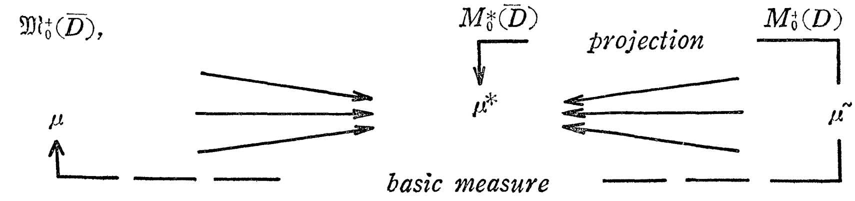
5) Finite convex combinations of extreme points, $\sum\alpha_i\mu_i^\sim$ with $\alpha_i>0$ and $\sum\alpha_i=1$, $\mu_i^\sim$ being extreme in $\bar{M}_0^+(\bar{D})$, are dense in $\bar{M}_0^+(\bar{D})$ with respect to the w^*-topology.



We can however prove that if $\mu^\sim$ is an extreme point of $\bar{M}_0^+(\bar{D})$, μ is uniquely corresponding to $\mu^\sim$ (Theorem 2 below); before going to this, we shall prepare a very important

LEMMA 2. *$\bar{M}_0^+(\bar{D})$ is characterized as such a collection of $\mu^\sim \in (H(D))^*$ that*

i) *$\mu^\sim$ is positive, that is, if $f \in H(D)$ is $\geqq 0$ in $\bar{D}$, we have $\mu^\sim(f) \geqq 0$,*

ii) $\|\mu^\sim\| = 1$.

Proof. Since every $\mu^\sim \in \bar{M}_0^+(\bar{D})$ satisfies evidently the conditions i) and ii) above, it is sufficient to prove that, if $\mu^\sim \in \Xi^*$ has the property i), $\mu^\sim$ belongs necessarily to $\bar{M}_0^+(\bar{D})$. To see this, suppose now it were not so, then since $\bar{M}_0^+(\bar{D})$ is regularly convex, there exists an element $f \in H(D)$ for which

$$\sup_{\nu^\sim \in \bar{M}_0^+(\bar{D})} \nu^\sim(f) = h < \mu^\sim(f).$$

Putting $f_0 = f - h$, we have $\nu^\sim(f_0) = \nu^\sim(f) - h \leqq 0$ for every $\nu^\sim \in \bar{M}_0^+(\bar{D})$ (hence for $\nu \in \mathfrak{M}_0^+(\bar{D})$), so that $f_0(x) \leqq 0$ for all $x \in \bar{D}$; on the other hand, $\mu^\sim(f_0) \geqq \mu^\sim(f) - h > 0$ (since $\mu^\sim \in \Xi^*$ implies $\mu^\sim(1) \leqq 1$ and hence $\mu^\sim(h) = h\mu^\sim(1) \leqq h$), which contradicts with the condition ii). Thus, Lemma 2 is completely proved.

In passing, we shall make a slight remark that in above Lemma the condition i) may be well replaced by that $\mu^\sim(1) = 1$.

THEOREM 2. *If $\mu^\sim \in Ext.\ \bar{M}_0^+(\bar{D})$, then its basic measure μ is uniquely determined and is equal to a point measure of total mass $+1$ (Dirac measure) placed on a certain point of $\bar{D}$.*

Proof. 1°) Let $\mu_\lambda^\sim \to \mu^\sim$ in $(H(D))^*$ and $\mu_\lambda \to \mu$ vaguely;[6)] suppose now that μ is not a point measure and the support K_μ of μ contains at least two mutually distinct points x_1 and x_2. Then, owing to the property 4) of potential, there exists certainly such a measure $\nu \in \mathfrak{M}^+(E-K)$ for a certain compact K containing $\bar{D}$ in its interior that $\phi(\nu)(x_1) \neq \phi(\nu)(x_2)$. Since $\phi(\nu)$ is continuous in $\bar{D}$, we can take such neighborhoods $V(x_i)$ of $x_i (i=1, 2)$ that

$$\sup |\phi(\nu)(x_1') - \phi(\nu)(x_2')| > \varepsilon, \qquad x_i' \in V(x_i), \tag{1.4}$$

for a sufficiently small ε(e. g. $\varepsilon < \frac{1}{2} |\phi(\nu)(x_1) - \phi(\nu)(x_2)|$); here, it is evident that $V(x_1) \cap V(x_2)$ is void.

6) Assume that $\mu_\tau^\sim \to \mu^\sim$ and a sub-sequence $\{\mu_\lambda\}$ of $\{\mu_\tau\}$ converges to μ (by definition); then $\{\mu_\lambda^\sim\}$ converges to $\mu^\sim$ also. A similar argument shows 6)bis essentially.

2°) Denoting the restrictions of μ_λ in $V(x_i)$ by $(\mu_\lambda)_i$ for $i=1,2$, we can select sub-sequences $\{(\mu_{\lambda'})_1^\frown\}$ of $\{(\mu_\lambda)_1^\frown\}$ and $\{(\mu_{\lambda'})_2^\frown\}$ of $\{(\mu_\lambda)_2^\frown\}$ simultaneously,[6) bis] such that $(\mu_{\lambda'})_i^\frown$ converges to a certain $\mu_i^\sim$ in Ξ^* for each i. By hypothesis, basic measures μ_1 of $\mu_1^\sim$ and μ_2 of $\mu_2^\sim$ are not null; since $\hat{\mu_{\lambda'}}-((\mu_{\lambda'})_1^\frown+(\mu_{\lambda'})_2^\frown)$ is positive for every λ', the limit $\mu^\sim-(\mu_1^\sim+\mu_2^\sim)$ is also positive on $H(D)$. Setting $\mathring{\mu}_1^\sim=\mu_1^\sim/\alpha_1$, $\mathring{\mu}_2^\sim=\mu_2^\sim/\alpha_2$ for $\alpha_i=\mu_i^\sim(1)(i=1,2)$, and $\mathring{\mu}_3^\sim=(\mu^\sim-(\mu_1^\sim+\mu_2^\sim))/\alpha_3$ for $\alpha_3=1-(\alpha_1+\alpha_2)$ (but if $\mu^\sim-(\mu_1^\sim+\mu_2^\sim)=0$ and hence $\alpha_3=0$, we should put $\mathring{\mu}_3^\sim=0$), we have $\mathring{\mu}_1^\sim$, $\mathring{\mu}_2^\sim$ and $\mathring{\mu}_3^\sim\in\overline{M}_0^+(\overline{D})$, and

$$(1.5)\qquad \mu^\sim=\alpha_1\mathring{\mu}_1^\sim+\alpha_2\mathring{\mu}_2^\sim+\alpha_3\mathring{\mu}_3^\sim,\qquad \textstyle\sum_{i=1}^3\alpha_i=1.$$

By (1.4) above, we have $\mathring{\mu}_1^\sim(\phi(\nu))=\int\phi(\nu)d(\mu_1/\alpha_1)\neq\int\phi(\nu)d(\mu_2/\alpha_2)=\mathring{\mu}_2^\sim(\phi(\nu))$, so that $\mathring{\mu}_1^\sim\neq\mathring{\mu}_2^\sim$. Thus, $\overset{*}{\mu}{}^\sim=(\alpha_1/(1-\alpha_3))\mathring{\mu}_1^\sim+(\alpha_2/(1-\alpha_3))\mathring{\mu}_2^\sim$ is an inner point of the segment combining $\mathring{\mu}_1^\sim$ and $\mathring{\mu}_2^\sim$, therefore refering to (1.5) $\mu^\sim$ could not be extreme, which is contradictory to the assumption.

3°) Suppose next $\mu^\sim$ has at least two basic measures μ_1 and μ_2; the result just obtained above shows that both μ_1 and μ_2 are point measures and, as is seen in 1°), there exists such $f\in H_0(D)$ that $f(x_1)\neq f(x_2)$ for the supporting points x_i of $\mu_i(i=1,2)$, from which it follows that $\mu^\sim(f)=\int fd\mu_1=f(x_1)\neq f(x_2)=\int fd\mu_2=\mu^\sim(f)$; this is absurd. Thus, the first half of Theorem is proved, which completes our proof.

Let Γ be the set $\subset\overline{D}$ such that for each $x\in\Gamma$ there exists at least one element of $Ext.\ \overline{M}_0^+(D)$ having ε_x as its basic measure,[7)] while Γ_0 a subset of Γ consisting of all such x that $\hat{\varepsilon}_x\in Ext.\ \overline{M}_0^+(\overline{D})$. Every point contained in Γ_0 is called *regular* (it should be noticed that $x\in\Gamma$ does not mean $\hat{\varepsilon}_x\in Ext.\ \overline{M}_0^+(\overline{D})$ unless x is regular; for instance, in the noted example given by H. Lebesgue the original point $0\in\Gamma$ but not $\in\Gamma_0$).

COROLLARY. *If $x\in\Gamma-\Gamma_0$, it must be $x\in\partial D$.*

Proof. Let ε_x be a basic measure of $\mu^\sim\in Ext.\ M_0^+(\overline{D})$ and $\hat{\mu_\lambda}\to\mu^\sim$ (in $(H(D))^*$) and $\mu_\lambda\to\mu$ vaguely; suppose now x to be an inner point of D and take a neighborhood $U(x)$ of x such that $\overline{U}(x)\subset D$. By assumption, we see that $(\mu_\lambda)_{E-\overline{U(x)}}\to 0$ as $\lambda\to+\infty$, that is, $\int fd(\mu_\lambda)_{E-\overline{U(x)}}\leqq\|f\|_\infty\int d(\mu_\lambda)_{E-\overline{U(x)}}\to 0$. Since every $f\in H(D)$ is continuous in $\overline{U(x)}$, we have $f(x)=\lim_\lambda\int_{\overline{U(x)}}d\mu_\lambda=\lim_\lambda\int fd(\mu_\lambda)_{\overline{U(x)}}=\mu^\sim(f)$, so that $\hat{\varepsilon}_x=\mu^\sim$, contradicting with the fact $x\in\Gamma-\Gamma_0$.

Remark: Throughout this paragraph, it would be rather pertinent to use directed systems than sequences, but still in later discussions we shall be content with the latter so far as circumstances permit.

7) ε_x denotes the point measure of total mass $+1$ placed on x.

§ 2. Construction of Balayage.

2.1. *General construction of balayage.* The method we actually use here for constructing the balayage of measures seems peculiar in such a point that its principle is essentially based upon the linear topological space theory. Especially, the Krein-Milman's theorem plays again an important rôle concerning about the discussion of regular boundary-points.

We have seen earlier (Thr. 1) that for any $\mu\in\mathfrak{M}^+(\bar{D})$ there exists a collection of finite linear convex aggregates of extreme points verifying (1.3) in accordance with given $f\in H(D)$ and $\varepsilon>0$; varying f and ε, such collections constitute a base of filter $\mathfrak{F}_\mu$ in $\bar{M}_0^+(\bar{D})$. Since $\bar{M}_0^+(\bar{D})$ is w^*-compact, an ultrafilter (maximal filter) $\mathfrak{F}_\mu^\circ$ containing $\mathfrak{F}_\mu$ converges to an element $\mu_\Gamma^\sim$ in $M_0^+(D)$, for which it holds

$$\mu^\frown(f)=\mu_\Gamma^\sim(f)\quad\text{for all}\quad f\in H(D); \tag{2.1}$$

this implies directly that $\mu^\frown=\mu_\Gamma^\sim\in\bar{M}_0^+(D)$, so that $\mu_\Gamma^\sim$ coincides with $\mu_\Gamma^\frown$ for a certain $\mu_\Gamma\in M_0^+(D)$, (it may be that μ_Γ is equal to μ itself).

On the other hand, let $\{D_p\}$ be a family, countable or not, of open sets in $\bar{D}$ with respect to the relative topology induced in $\bar{D}$, such that $D_p\supset\bar{\Gamma}_*$ for all p and $\Pi_p\bar{D}_p=\bar{\Gamma}_*$,†) where $\Gamma_*=\Gamma\cup\partial D$. Denote next by $M_0^+(\bar{D}_p)$ the collection of $\mu^\frown$ such that $\mu\in\mathfrak{M}_0^+(D_p)$, then the w^*-closure $\bar{M}_0^+(\bar{D}_p)$ of $M_0^+(\bar{D}_p)$ is evidently convex and w^*-compact. We see easily $Ext.\,\bar{M}_0^+(\bar{D})\subset\bar{M}_0^+(\bar{D}_p)$ and therefore all linear convex aggregates of extreme points (a fortiori, those which appear in (1.3)) are contained in $\bar{M}_0^+(\bar{D}_p)$ for all p, so that in $\Pi_p\bar{M}_0^+(\bar{D}_p)$. Finally, $\mu_\Gamma^\frown$ is considered as contained $\Pi_p\bar{M}_0^+(\bar{D}_p)\cap M_0^+(\bar{D})$, which implies that μ_Γ is necessarily distributed in $\bar{\Gamma}_*$.

THEOREM 3. *Such a μ_Γ satisfies*

$$\int f du=\int d\mu_\Gamma\quad \textit{for all}\quad f\in H(D). \tag{2.1$'$}$$

We shall call such μ_Γ as verifies (2.1) or (2.1)$'$ and is distributed in $\bar{\Gamma}_*$ *a balayaged measure* of μ, which is not necessarily unique since it depends upon the selection of ultrafilter $\mathfrak{F}_\mu^\circ$ which contains $\mathfrak{F}_\mu$.

The version of this Theorem appropriates to potentials takes the following

THEOREM 3.$^{\text{bis}}$ *$\phi(\mu)=\phi(\mu_\Gamma)$ eveywhere in $E-\bar{D}$ and excepting a set of capacity 0 on ∂D. If μ_Γ is distributed on ∂D and if $\phi(\mu_\Gamma)$ is bounded, then*

$$\phi(\mu)\geqq\phi(\mu_\Gamma)\ \textit{everywhere in}\ E. \tag{2.2}$$

Here, a set X is called "*of capacity* 0" if X admits no positive measure ν such as $\phi(\nu)$ would be bounded.

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†) E completely regular, since it is locally compact.

Proof of Theorem 3^{bis}. The first part is somewhat trivial; in fact, for any $x\in E-D$, $\phi(\varepsilon_x)$ is bounded on D, *i.e.* $\phi(\varepsilon_x)\in H(D)$, therefore $\phi(\mu)(x)=\int\phi(\varepsilon_x)d\mu=\int\phi(\varepsilon_x)d\mu_\Gamma=\phi(\mu_\Gamma)(x)$. The second part is verified as follows; suppose first that $\phi(\mu)>\phi(\mu_\Gamma)$ on a set $X\subset\partial D$ not of capacity 0, then there exists a positive measure ν on X such that $\phi(\nu)\in H(D)$, so that $\int\phi(\nu)d\mu=\int\phi(\mu)d\nu>\int\phi(\mu_\Gamma)d\nu=\int\phi(\nu)\mu_\Gamma$, contradicting with $(1.6)^{\text{bis}}$. This shows that $\phi(\mu)\leqq\phi(\mu_\Gamma)$ on ∂D excepting a set of capacity 0. Suppose next that $\phi(\mu)<\phi(\mu_\Gamma)$ on a set $Y\subset\partial D$ not of capacity 0, then an analogous arguments to above leads us also to a contradiction, which guarantees the assertion. The last half is more briefly obtained; that is, from the result just obtained above, we have $\phi(\mu)=\phi(\mu_\Gamma)$ on a kernel of μ_Γ, from which we conclude that $\phi(\mu)\geqq\phi(\mu_\Gamma)$ everywhere in E owing to the maximal principle of potentials (see **1.1**). Thus, Theorem 3^{bis} is completely proved.

As is noted before, a balayaged measure is in general not unique, but we have two important cases where it is uniquely determined (as it is or under some restrictions); we state the matters in the following form:

THEOREM 4. i) *If* $x\in\Gamma_0$, $(\varepsilon_x)_\Gamma$ *is uniqely determined and equal to* ε_x *itself.* ii) *If* μ_Γ *and* μ'_Γ *are balayaged measures of the same* $\mu\in\mathfrak{M}_0^+(\bar{D})$, *both of which are distributed in* ∂D *and have bounded potentials* $\phi(\mu_\Gamma)$ *and* $\phi(\mu'_\Gamma)$, *then it holds that* $\mu_\Gamma=\mu'_\Gamma$.

Proof. i) $(\varepsilon_x)\hat{}_\Gamma=\hat{\varepsilon}_x\in Ext.\ \overline{M}_0^+(\bar{D})$ by hypothesis, so that $(\varepsilon_x)_\Gamma$ is a point measure by Theorem 2, *i.e.* $(\varepsilon_x)_\Gamma=\varepsilon_y$ for a certain $y\in D$, therefore $\hat{\varepsilon}_x=\hat{\varepsilon}_y$ or equivalently $f(x)=f(y)$ for all $f\in H(D)$. This implies $x=y$ owing to the property 4) of potential, **1.1**.

ii) By Theorem 3^{bis}, we have $\phi(\mu_\Gamma)=\phi(\mu'_\Gamma)$ on ∂D excepting a set of capacity 0, and so on kernels of both μ_Γ and μ'_Γ, from which it follows simultaneously $\phi(\mu_\Gamma)\geqq\phi(\mu'_\Gamma)$ and $\phi(\mu'_\Gamma)\geqq\phi(\mu_\Gamma)$ eveywhere in E by means of the maximal principle, therefore $\phi(\mu_\Gamma)=\phi(\mu'_\Gamma)$ everywhere in E. This means $\mu_\Gamma=\mu'_\Gamma$.[8]

2.2. *The case in which* $\Gamma\subset\partial D$. More abundant results will be obtained under some restricted situation: we assume first that

*) *Γ is contained in ∂D.*

In the case of Newtonian potential *a*) in $R^n(n\geqq3)$ or of logarithmic potential *b*) in the open unit circle in Z^2, cited in **1.1**, the condition *) is evidently fulfiled. In fact, let x be an inner point of D, Σ_x a sphere with center x such that $\overline{\Sigma}_x\subset D$, and λ_x the spherical measure of total mass $+1$ uniformly distributed on the surface of Σ_x. Now, if $x\in\Gamma$, then it must be that $x\in\Gamma_0$ (since $x\in$ interior of D) by Corollary to Theorem 2, that is $\hat{\varepsilon}_x\in Ext.\ M_0^+(D)$; every $f\in H(D)$ being harmonic in

8) In fact, we have $\int\phi(\nu)d\mu_\Gamma=\int\phi(\mu_\Gamma)d\nu=\int\phi(\mu'_\Gamma)d\nu=\int\phi(\nu)d\mu'_\Gamma$ for all bounded potential $\phi(\nu)$; refering to the property 5) of potential, we conclude that $\mu_\Gamma=\mu'_\Gamma$.

D in the ordinary sense, we have

$$f(x)=\int f d\lambda_x \quad \text{for all} \quad f \in H(D).$$

Take a point $z \in E-\overline{D}$ and draw such a sphere Σ_z with center z as intersects with Σ_x, for which neither $\Sigma_x^1=\Sigma_x \cap \Sigma_z$ nor $\Sigma_x^2=\Sigma_x-\Sigma_z$ is of capacity 0. Denoting the restrictions of λ_x in Σ_x^i by λ_x^i $(i=1,2)$, we see easily

$$\varepsilon_x^{\frown}=\lambda_x^{1\frown}+\lambda_x^{2\frown},$$

in which $\lambda_x^{1\frown}(\phi(\varepsilon_z))>\lambda_x^{2\frown}(\phi(\varepsilon_z))$, *i.e.* $\lambda_x^{1\frown} \neq {}_x^{2\frown}$. This implies that $\varepsilon_x^{\frown}$ is an inner point of $\overline{M}_0^+(\overline{D})$, contradicting with the assumption. Thus, *) is proved.

Now, under this assumption *) Theorem 3^{bis} and the last half of Theorem 4 are resumed as in the following

THEOREM 5. *If $\phi(\mu_\Gamma)$ is bounded, μ_Γ is uniquely determined and $\phi(\mu) \geqq \phi(\mu_\Gamma)$ everywhere in E. If $\phi(\mu)$ is bounded and μ is distributed in ∂D, then $\mu=\mu_\Gamma$.*

From this Theorem, we have directly:

COROLLARY. *If $\phi(\mu_\Gamma)$ is bounded, $(\mu_\Gamma)_\Gamma=\mu_\Gamma$.*

2.3. *Extension of balayage and proper balayage.* Hereafter, we proceed in adopting the assumption *) and, to develop the theory more finely, the further conditions for potentials in addition to the five ones 1)$\sim$5) in **1.1,** that:

6) For every potential $\phi(\mu)$, $\mu \in \mathfrak{M}^+(E)$, there exists a sequence of continuous potentials $\{\phi(\lambda_i)\}$, $\lambda_i \in \mathfrak{M}^+(E)$, such that $\phi(\lambda_i) \nearrow \phi(\mu)$.[9]

7) Conversely, if $\{\phi(\lambda_i)\}$, $\lambda_i \in \mathfrak{M}^+(E)$, is any increasing sequence of potentials such that $\phi(\lambda_i) \leqq \phi(\mu)$ for some $\mu \in \mathfrak{M}^+(E)$, then $\lim\limits_i \phi(\lambda_i)$ defines a potential of a certain positive measure ν, *i.e.* $\phi(\lambda_i) \nearrow \phi(\nu) (\leqq \phi(\mu))$.

These conditions 6)$\sim$7) are well verified in Examples *a*) and *b*) cited in **1.1,** see e.g. H. Cartan [4] and also Appendix I at the end of this paper.

Now, at first, we shall restrict ourselves within the case where D is regularly open, *i.e.* $D=\mathrm{int}\overline{D}$.[10] Then, we see at once $\partial D=\partial \overline{D}$ and that if $\phi(\mu)$, $\mu \in \mathfrak{M}_0^+(\overline{D})$, is bounded, so is $\phi(\mu_\Gamma)$ also. Indeed, since the potential function of a positive measure is lower semi-continuous (by the superharmonicity), we have $\phi(\mu_\Gamma)(x) \leqq \varliminf\limits_{y \to x} \phi(\mu_\Gamma)(y)=\varliminf\limits_{y \to x} \phi(\mu)(y) \leqq K$ for every $x \in \partial D$ and $y \in E-\overline{D}$, so that $\phi(\mu_\Gamma)$ is bounded on a kernel of μ_Γ and hence everywhere in E. Thus, we see in this case $\phi(\mu_\Gamma)$ is uniquely determined and $\phi(\mu_\Gamma) \leqq \phi(\mu)$ for every μ with bounded $\phi(\mu)$.

Next, we shall define belayage for measures in $\mathfrak{M}^+(\overline{D})$ and $\mathfrak{M}(\overline{D})$. For every $\mu \in \mathfrak{M}^+(\overline{D})$ with bounded $\phi(\mu)$, we put

$$\mu_\Gamma=\alpha \mathring{\mu}_\Gamma, \quad \text{for } \alpha=\|\mu\| \text{ and } \mathring{\mu}=\mu/\alpha, \tag{2.3}$$

9) The symbol $g_i \nearrow f$ indicates that $g_i \leqq g_{i+1} \leqq \cdots \leqq f$ and $\lim\limits_i g_i=f$ as a pointwise limit.

10) We denote hereafter by $\mathrm{int}X$ the interior of X.

in which it is clear that $\mathring{\mu}\in\mathfrak{M}_0^+(\bar{D})$, and μ_Γ is uniquely determined since so is $\mathring{\mu}_\Gamma$ (see the above argument). Let $\mu\in\mathfrak{M}(\bar{D})$, assuming $\mu=\mu_1-\mu_2$ for $\mu_i\in\mathfrak{M}^+(\bar{D})$ with bounded $\phi(\mu_i)$ $(i=1,2)$; then we define

$$(2.3)' \qquad \mu_\Gamma=(\mu_1)_\Gamma-(\mu_2)_\Gamma .$$

For thus extended balayaged measures, we have the linearity; $(\alpha\mu+\beta\nu)_\Gamma=\alpha\mu_\Gamma+\beta\nu_\Gamma$ for any real α and β so far as these are uniquely determined.

Next, let $B^+(E)$ be a convex cône in $B(E)$ consisting of all such $\phi(\nu)\in B(E)$ that $\nu\in\mathfrak{M}^+(E)$. For any $\phi(\nu)\in B^+(E)$, put

$$(2.4) \qquad \nu_\Gamma=(\nu_{\bar{D}})_\Gamma+\nu_{E-\bar{D}},$$

or

$$(2.4)' \qquad \phi(\nu)_\Gamma=\phi((\nu_D)_\Gamma)+\phi(\nu_{E-D}),$$

where ν_X indicates the restrsition of ν in X (compact or open). Clearly, $\phi(\nu)_\Gamma$ $(=\phi(\nu_\Gamma))$ is in $B^+(E)$ together with $\phi(\nu)$ and moreover $\phi(\nu)_\Gamma$ belongs to $H(D)$. According to the above argument about μ_Γ, we establish the fundamental relations for such f_Γ, $f\in B^+(E)$, as follows;

THEOREM 6. i) *$f\geqq f_\Gamma$ everywhere in E,* ii) *$f=f_\Gamma$ in $E-D$ and on ∂D excepting a set of capacity* 0, *and* iii) *if $f\in H(D)$, f coincides with f_Γ.*

We shall now consider the case of general (relatively compact) D: To do it, we need to make another assumption, which is proved to be valid in Examples a), b), cited in **1.1** (about the proof, see H. Cartan [3], p. 88) and plays an important rôle in the balayage theory of H. Cartan himself [4], such that;

**) if $\mu_\lambda\to\mu_0$ vaguely for μ_λ, $\mu_0\in\mathfrak{M}^+(E)$ with their energies uniformly bounded, then we have $\int\phi(\mu)d\mu_\lambda\longrightarrow\int\phi(\mu)d\mu_0$ for any $\mu\in\mathfrak{M}^+(E)$ with finite energy.

For given D, take a sequence of such regularly open sets $D_j(D_j=int\ \bar{D})$ as $\bar{D}_j\subset D_{j+1}$ and $\cup_j D_j=D$, and denote for any $\mu\in\mathfrak{M}^+(D)$ with bounded $\phi(\mu)$ the j-th balayage of μ with respect to D_j in the sense of (2.4) by μ_j^0 and the vague limit of $\{\mu_j^0\}$ by μ_Γ. Then we see immediately that such obtained μ_Γ is necessarily distributed in ∂D and satisfies

$$\phi(\mu)\geqq\phi(\mu_j^0)\geqq\phi(\mu_{j+1}^0)\geqq\phi(\mu_\Gamma)$$

in virtue of the lower semi-continuity of application $\mu\to\phi(\mu)$.[11] On the other hand,

11) The application $\mu\in\mathfrak{M}^+(E)\to\phi(\mu)$ is lower semi-continuous with respect to the vague topology; $\varliminf_{\mu\to\nu}\phi(\mu)\geqq\phi(\nu)$. This comes from the fact that every potential is lower semi-continuous and represented as in an integral form;

$$\phi(\mu)=\int\Phi(x,y)d\mu(y),\ \Phi(x,y)=\phi(\varepsilon_x)(y)=\phi(\varepsilon_y)(x).$$

Refer to Theorem 3 in [10].

owing to the above assumption **), we have $\int f\,d\mu = f\,d\mu_j$ (for each j) $= \int f\,d\mu_\Gamma$ for every $f \in H(D)$, so that μ_Γ is a balayaged measure of μ (if D is regularly open, thus defined μ_Γ is identical with the preceeding one). As $\phi(\mu_\Gamma)$ is bouned ($\phi(\mu_\Gamma) \leqq \phi(\mu)$), such μ_Γ is uniquely determined.

Using such μ_Γ, we can define also the balayage for any measure of $\mathfrak{M}^+(\bar{D})$ and hence of $\mathfrak{M}(\bar{D})$ under the bounded condition as analogically as in (2.3) and (2.3)$'$. Also for any $f \in B^+(E)$, an analogue as (2.4) and (2.4)$'$ is quite valid.

Thus, we are now ready to define the proper balayage, which is achieved in the following manner; we begin with

LEMMA 3. *Let f and g be in $B^+(E)$, then $f \geqq g$ implies that $f_\Gamma \geqq g_\Gamma$.*

In fact, we have directly $f_\Gamma = f \geqq g = g_\Gamma$ in $E - \bar{D}$ and on ∂D excepting a set of capacity 0. Assume now $g = \phi(\nu)$, then $f_\Gamma \geqq \phi(\nu)_\Gamma = \phi(\nu_\Gamma)$ on a kernel of ν_Γ and hence in virtue of the maximal principle $f_\Gamma \geqq g_\Gamma$ in E, from which follows Lemma 3.

For a given $\mu \in \mathfrak{M}^+(\bar{D})$, put

$$(2.5) \qquad L_\mu(f) = \int f_\Gamma d\mu \quad \text{for all} \quad f \in B^+(E),$$[12]

then L_μ is linearly prolonged to a linear positive functional on $B(E)$; the positivity may be assured in such a way that if $f - g \geqq 0$ for $f, g \in B^+(E)$, then it yields that $f_\Gamma - g_\Gamma \geqq 0$ (by Lemma 3) and so $L_\mu(f-g) \geqq 0$. Refering to the property 5) of potential, **1.1**, such L_μ defines a uniquely determined positive Radon measure μ_Γ° in E (we owe this fact to a Proposition of N. Bourbaki).[13]

Such μ° is necessarily distributed in ∂D; in fact, let first f_0 be in $C_0(D)$ with compact support $K_0 \subset D$, then by the property 5) of potential, **1.1**, for any neighborhood U of K_0 such that $U \subset D$ and positive number ε we can choose $g \in B(E)$ vanishing at the outside of U and verifying $|f_0 - g| < \varepsilon$ in E, for which we have $L_\mu(g) = \int g_\Gamma d\mu = 0$ (let $g = \phi(\nu_1) - \phi(\nu_2)$, $\nu_i \in \mathfrak{M}^+(\bar{D})$ for $i = 1, 2$, then it vanishes at the outside of U, so that $\hat{\nu_1} = \hat{\nu_2}$ and hence $(\nu_1)_\Gamma = (\nu_2)_\Gamma$), therefore $|L_\mu(f_0)| < \varepsilon$ and, ε being arbitratry, $L_\mu(f_0) = 0$. By the same reasoning, we have $L_\mu(f_*) = 0$ for all $f_* \in C_0(E - \bar{D})$.

LEMMA 4. *Let $g_i \in B^+(E)$ for $i = 1, 2, \cdots$, and assume that $g_i \nearrow f \in B^+(E)$, then $(g_i)_\Gamma \nearrow f_\Gamma$.*

Proof. Accorcing to the sequence $\{g_i\}$, $\{(g_i)_\Gamma\}$ is also increasing by Lemma 3, so that owing to the assumption 7), **2.3**, $h = \lim_i (g_i)_\Gamma$ is also in $B^+(E)$. For every $\nu \in \mathfrak{M}^+(E - D)$ with $\phi(\nu) \in B^+(E)$ (more precisely, $\in H(D)$), we have

12) Owing to the above argument, such functional L_μ is uniquely determined.

13) N. Bourbaki [1], Prop. 2, §**2**, Chap. III.

$$\int f_\Gamma d\nu=\int f d\nu=\lim_i \int g_i d\nu=\lim_i \int (g_i)_\Gamma d\nu$$

$=\int h\,d\nu$, so that $f_\Gamma=h$ in $E-D$ excepting a set of capacity 0 and hence on a kernel of f_Γ; by the maximal principle, we conclude $f_\Gamma \leqq h$ everywhere in E, but $h \leqq f_\Gamma$ is evident, which proves Lemma 4 completely.

Now, for every $f \in B^+(E)$ there exists in $B^+(E)$ a sequence $\{g_i\}$, each g_i being continuous, such that $g_i \nearrow f$, as quoted in the assumption 6), **2. 3**. Using the above Lemma, we see

$$L_\mu(f)=\int f_\Gamma d\mu=\lim_i \int (g_i)_\Gamma d\mu=\lim_i L_\mu(g_i)$$
$$=\lim_i \int g_i d\mu_\Gamma^\circ=\int f\,d\mu_\Gamma^\circ;$$

thus, we can formulate

$$L_\mu(f)=\int f\,d\mu_\Gamma^\circ \qquad \text{for all } f \in B^+(E), \tag{2.6}$$

and moreover (by Theorem 6, iii))

$$\int f\,d\mu=\int f\,d\mu_\Gamma^\circ \qquad \text{for all } f \in H(D). \tag{2.7}$$

THEOREM 7. *Such μ_Γ° is one of balayaged measures of $\mu \in \mathfrak{M}^+(D)$. Therefore, if μ_Γ is unique, it must be $\mu_\Gamma^\circ=\mu_\Gamma$.*

It is sufficient to prove the Theorem in the case where $\mu \in \mathfrak{M}_0^+(\bar{D})$; indeed, (2. 7) shows $\mu^\frown=\mu_\Gamma^{\circ\frown}$ in $M_0^+(D)$, so that μ_Γ° fulfils (2. 3) as a matter of course; as is noted before, μ_Γ° is distributed in $\bar{\Gamma}_*$ (which is just equal to ∂D in the present case). Thus, these two facts guarantee the assertion.

We call thus obtained μ_Γ° *properly balayaged measure* of μ, but hereafter if we say merely the *balayaged measure* (with definite article), we shall always mean such μ_Γ°, while each μ_Γ is distinguished by calling a *general balayaged measure* if necessary. The operation $\mu \to \mu_\Gamma^\circ$ is called *balayage.*

The balayaged measure has the following properties;

THEOREM 8. *Let μ_Γ° be the balayaged measure of $\mu \in \mathfrak{M}^+(\bar{D})$, then we have*

α) *$\phi(\mu)=\phi(\mu_\Gamma^\circ)$ in $E-\bar{D}$ and on ∂D except a set of capacity* 0,

β) *$\phi(\mu) \geqq \phi(\mu_\Gamma^\circ)$ everywhere in E.*

Proof. α) is clear from Theorem 3bis and Theorem 7. β) is proved as follows; according to the assumption for each $x \in E$ there exists a sequence of continuous (and hence bounded) potentials $\phi(\lambda_i)$, $\lambda_i \in \mathfrak{M}^+(E)$, such that $\phi(\lambda_i) \nearrow \phi(\varepsilon_x)$, so that

$$\begin{aligned}\phi(\mu_\Gamma^\circ)(x)=\int\phi(\varepsilon_x)d\mu_\Gamma^\circ&=\lim_i\int\phi(\lambda_i)d\mu_\Gamma^\circ\\&=\lim_i\int(\phi(\lambda_i))_\Gamma d\mu\\&\leqq\lim_i\int\phi(\lambda_i)d\mu\\&=\int\phi(\varepsilon_x)d\mu=\phi(\mu)(x)\,,\end{aligned}$$

which proves β).

Finally, we observe a characterization property of μ_Γ°, which is answered as follows:

THEOREM 9. *μ_Γ° is characterized as a measure of $\mathfrak{M}^+(\partial D)$ whose potential $\phi(\mu_\Gamma^\circ)$ is the minimum among all of $\phi(\nu)$, $\nu\in\mathfrak{M}^+(\partial D)$, which fulfil the condition α) in Theorem 8. If $\phi(\mu)$ is bounded, $\mu_\Gamma^\circ(=\mu_\Gamma)$ is also characterized as a measure of $\mathfrak{M}^+(E-D)$, whose potential is the maximum among all other $\phi(\nu)$ for $\nu\in\mathfrak{M}^+(E-D)$ such that $\phi(\nu)\leqq\phi(\mu)$.*

Proof. With the same notations in the proof of Theorem 9, we see that for each i

$$\begin{aligned}\int\phi(\lambda_i)d\mu_\Gamma^\circ=\int(\phi(\lambda_i))_\Gamma d\mu&=\int\phi(\mu)d(\lambda_i)_\Gamma\\&=\int\phi(\nu)d(\lambda_i)_\Gamma\quad(\text{by }\alpha))\\&=\int(\phi(\lambda_i))_\Gamma d\nu\leqq\int\phi(\lambda_i)d\nu,\end{aligned}$$

from which $\phi(\mu_\Gamma^\circ)(x)\leqq\phi(\nu)(x)$; since μ_Γ° itself satisfies the condition α), the first half of the Theorem is proved. The last half is easily obtained by a simple fact that $\phi(\nu)\leqq\phi(\mu)=\phi(\mu_\Gamma^\circ)$ in $E-\overline{D}$ and on ∂D excepting a set of capacity 0 and so on a kernel of ν.

2.4. *The case of non relatively compact D.* In this section, we shall investigate the balayage for an open D such that $E-\overline{D}$ is non-void and ∂D is compact. But, D itself is assumed not to be relatively compact. Then, we stand in some different situation from the preceding section. Indeed, $\mathfrak{M}_0^+(\overline{D})$ is not vaguely compact (as is easily seen, the measure null is adherent to $\mathfrak{M}_0^+(\overline{D})$ if D extends to the infinite; roughly speaking, if $x_i\in D$ runs to the infinite as $i\longrightarrow+\infty$, then ε_x converges to 0 vaguely), and $H(D)$, leaving its definition to Definition 1, contains no constant function. These two are the most notable differences. Such being the case, it would be convenient to make the one point compactification of D.[14] Let D_∞ be such compactification of D, denoting the additional point by θ, and put $D_\infty=D\cup\theta$. On the other hand, we adjoint the identity 1 to $H(D)$ and

14) See, e.g. L. H. Loomis [8], § 2, Chap. I.

denote by $H(D_\infty)$ a normed linear space generated from $H(D)$ and the unit function 1 ($1(x)=1$ on $\bar{D}$). It is easily seen that every function of $H(D_\infty)$ is well prolonged up to θ, verifying $\theta^\frown(1)=1(\theta)=1$ and

(2.8) $$\theta^\frown(f)=f(\theta)=0 \quad \text{for all} \quad f\in H(D),$$

since we have $H(D)\subset L_\infty(\bar{D})$.

Now, $\mathfrak{M}^+(\bar{D}_\infty)$ is generated, taking vague limits, by $\mathfrak{M}^+(\bar{D})$ and the point measure ε_θ (of total mass $+1$) placed on θ, and $\mathfrak{M}_0^+(\bar{D}_\infty)$ is vaguely compact and convex. The collection $M_0^+(\bar{D}_\infty)$, of such bounded linear functionals $\mu^\frown$, $\mu\in\mathfrak{M}_0^+(\bar{D}_\infty)$, as is defined by

(1.2)′ $$\mu^\frown(f)=\int_{\bar{D}_\infty} f\,d\mu,\ f\in H(D_\infty),$$

is convex, hence the w^*-closure $\bar{M}_0^+(\bar{D}_\infty)$ in $(H(D_\infty))^*$ is w^*-compact and convex, for which the Krein-Milman's theorem is also applicable.

In order to have the analogical argument as before, we need to assume:

#) For arbitrary two points x and $y\in\bar{D}_\infty$ distinct each other, there exists such a $\mu\in\mathfrak{M}^+(\bar{E}-D)$ that $\phi(\mu)(x)\neq\phi(\mu)(y)$.[15]

Then, replacing E, $\mathfrak{M}_0^+(\bar{D})$, $H(D)$ and $\bar{M}_0^+(\bar{D})$ by $E_\infty=E\cup\theta$, $\mathfrak{M}_0^+(\bar{D}_\infty)$, $H(D_\infty)$, and $\bar{M}_0^+(\bar{D}_\infty)$ respectively, we can see easily that the whole theory contained in **1.2** is well revised completely, and consequently we get also general balayaged measures μ_∞ of $\mu\in\mathfrak{M}_0^+(\bar{D})$ in the sense of **2.1**, assuming $\Gamma\subset\theta\cup\partial D$. Let μ_Γ be the restriction of one of these μ_∞ in ∂D (μ_∞ may be distributed in $\theta\cup\partial D$ in general); for such a μ_Γ, we see that Theorems $3^{\text{bis}}\sim 5$ are all true, since $f\in H(D)$ vanishes at θ and hence

(2.9) $$\int f d\mu=\int f d\mu_\Gamma\left(=\int_{\bar{D}_\infty} f d\mu_\infty\right) \text{ for all } f\in H(D).$$

This admits us to construct the properly balayaged measure μ_Γ° of μ, distributed in ∂D, in the quite same manner as in **2.2**$\sim$**2.3.** Theorems 7$\sim$8 are then completely valid.

However we have

(2.10) $$\int d\mu=\int_{\bar{D}_\infty} d\mu_\infty\geqq\int_{\bar{D}_\infty} d\mu_\Gamma=\int d\mu_\Gamma^\circ,$$

and if μ is distributed in D, we have exactly

(2.10)′ $$\int d\mu>\int d\mu_\Gamma^\circ.$$

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15) This assumption #) is well held in $R^n(n\geqq 3)$ with Newtomian potential (Example *a*)); the locus of equidistant points from x and y forms a hypersurface in R^n, whose intersection with $E-\bar{D}$ is at least of Lebesgue measure null, while $E-\bar{D}$ itself is not so.

In fact, let λ be a measure distributed in a certain compact $K \supset \partial D$, whose intersection with the support of μ is void, such that $\phi(\lambda)=1$ on K and $0 \leqq \phi(\lambda) \leqq 1$ in $E-K$; we have then $h=1-\phi(\lambda_\Gamma^\circ) \in H(D_\infty)$ and $\phi(\lambda_\Gamma^\circ) \in H(D)$, so that $\int (d\mu - d\mu_\Gamma^\circ) = \int h(d\mu - d\mu_\Gamma^\circ) > 0$ since $\int \phi(\lambda_\Gamma^\circ) d\mu = \int \phi(\lambda_\Gamma^\circ) d\mu_\Gamma^\circ$ by (2.9)and $\int h d\mu > \int h d\mu_\Gamma = 0$.

Such phenomenon does not occur in the case of a relatively compact D, for which we have always $\int d\mu = \int d\mu_\Gamma^\circ$, since $H(D)$ contains the unit function.

§ 3. The Case of Compact Sets: Extremisation.

3.1. *Balayage in the case of compact sets.* Let K be a compact set in E. By analogy to Definition 1, we define $H(K)$ as a normed linear space consisting of the restrictions in K of all such $\phi(\nu) \in B(E)$ that ν is distributed in $E-K$, with respect to the norm

$$\|f\|_K = \sup_{x \in K} |f(x)| . \tag{3.1}$$

If the interior of K, *intK*, is a non-trivial (open) set, we see easily that $H(K)$ is a linear subspace of $H(\bar{D})$ for $D = intK$. An arguement exactly analogous to that we used in the preceding sections allows us to define $\bar{M}_0^+(K)$ and *Ext.* $\bar{M}_0^+(K)$ again, for which by replacing the letter $\bar{D}$ by K Theorems 1, 2 and Lemma 2 remain valid. However, in order to avoid any confusion, we shall use the notations ∇, ∇_0, and μ_∇ instead of Γ, Γ_0 and μ_Γ respectively. Then, Theorem 3 is also valid and stated as follows:

THEOREM 10. *Let μ be in $\mathfrak{M}^+(K)$ and μ_∇ a balayaged measure of μ; we have then*

$$\int f\, d\mu = \int f\, d\mu_\nabla \quad \textit{for all} \quad f \in H(K), \tag{3.2}$$

and moreover

$$\phi(\mu) = \phi(\mu_\nabla) \quad \textit{outside of } K. \tag{3.3}$$

It should be noticed that in the present case Theorem 3^{bis} is not valid and even if $\phi(\mu_\nabla)$ is bounded; μ_∇ is not uniquely determined in general, since $\phi(\mu_\nabla)$ does not belong to $H(K)$.

We prepare a lemma for later use:

LEMMA 5. *Assume intK not to be void; if μ has the support contained in intK, then $\phi(\mu) \geqq \phi(\mu_\nabla)$ everywhere in E.*

In fact, $\phi(\mu)$ is continuous in $E-intK$ and so bounded in ∂K; for every $x \in \partial K$ it holds that

$$\phi(\mu)(x)=\lim_{\substack{y\to x\\ y\in E-K}}\phi(\mu)(y)=\lim_{y\to x}\phi(\mu_\Gamma)(y)\geqq\phi(\mu_\Gamma)(x).$$

Thus, $\phi(\mu)\geqq\phi(\mu_\Gamma)$ on a kernel of μ_Γ and $\int\phi(\mu_\Gamma)d\mu_\Gamma$ must be finite so that by the maximal principle for potentials it follows the assertion.

Take now a sequence of relatively compact open sets $\{D_j\}$ such that $\overline{D}_{j+1}\subset D_j$ and $\Pi D_j=K$. For a given $\mu\in\mathfrak{M}_0^+(K)$, considering $\mu\in\mathfrak{M}_0^+(\overline{D}_j)$, let us denote a balayaged measure of μ in $\overline{D}_j$ by μ_j for each j. Since $\mathfrak{M}_0^+(\overline{D}_1)$ is vaguely compact, the sequence $\{\mu_j\}$ has such a sub-sequence $\{\mu_{jk}\}$ that μ_{jk} converges to a certain measure μ_Γ° which is necessarily distributed in ∂K. An analogue of the proof of Theorem 8 for μ_Γ shows that μ_Γ° is also a balayaged measure of the present sense.

Let μ and ν be arbitrary in $\mathfrak{M}_0^+(K)$; as $\phi(\mu)\geqq\phi(\mu_j)$ by Lemma 5, we see

$$\begin{aligned}\int\phi(\mu_\Gamma^\circ)d\nu&\leqq\underline{\lim_j}\int\phi(\mu_j)d\nu\\&=\underline{\lim_j}\int\phi(\mu_j)d\nu_\Gamma^\circ\ \text{(by (3.3))}\leqq\int\phi(\mu)d\nu_\Gamma^\circ.\end{aligned}$$

Replacing μ and ν mutually, we get

$$\int\phi(\mu_\Gamma^\circ)d\nu=\int\phi(\mu)d\nu_\Gamma^\circ. \tag{3.4}$$

On the other hand, one sees easily $\phi(\mu_\Gamma^\circ)\leqq\underline{\lim_j}\,\phi(\mu_j)\leqq\phi(\mu)$ everywhere in E. Summarizing these, we have the following.

THEOREM 11. *μ_Γ° is the balayaged measure, uniquely determined, of $\mu\in\mathfrak{M}_0^+(K)$ (or more generally, of $\mu\in\mathfrak{M}^+(K)$), whose potential $\phi(\mu_\Gamma^\circ)$ has the following properties;*

$$\begin{aligned}&\phi(\mu_\Gamma^\circ)=\phi(\mu)\ \textit{outside of}\ K,\\&\phi(\mu)\geqq\phi(\mu_\Gamma^\circ)\ \textit{everywhere in}\ E.\end{aligned}\tag{3.5}$$

It remains us to prove the uniqueness of μ_Γ°; suppose now another μ_Γ' verifies the above conditions (3.4) and (3.5), and for an arbitrary $\phi(\nu)\in B^+(E)$ decompose it into $\phi(\nu)=\phi(\nu_K)+\phi(\nu_{E-K})$, then we have

$$\begin{aligned}\int\phi(\nu)d\mu_\Gamma^\circ&=\int\phi(\nu_K)d\mu_\Gamma^\circ+\int\phi(\mu_\Gamma^\circ)d\nu_{E-K}\\&=\int\phi((\nu_K)_\Gamma^\circ)d\mu+\int\phi(\mu)d\nu_{E-K}\\&=\int\phi(\nu_K)d\mu_\Gamma'+\int\phi(\mu_\Gamma')d\nu_{E-K}\\&=\int\phi(\nu)d\mu_\Gamma',\end{aligned}$$

from which it follows that $\int fd\mu_\Gamma^\circ=\int fd\mu_\Gamma'$ for every $f\in B(E)$. Owing to the property 5) of potential, we conclude $\mu_{\Gamma}^{\circ}=\mu'_{\Gamma}$, completing the proof.

We call such μ_{Γ}° the balayaged measure or, according to M. Brelot's terminology, the extremal measure of μ. Of course, by anȧlogue to μ_{Γ} we are enable to define the extremal measure of any μ of $\mathfrak{M}^{+}(K)$ and hence of $\mathfrak{M}(K)$.

A characterization of μ_{Γ}°, $\mu\in\mathfrak{M}^{+}(K)$, is contained in:

THEOREM 12. *μ_{Γ}° is the measure whose potential $\phi(\mu_{\Gamma}^{\circ})$ realizes the minimum among all such potentials $\phi(\nu)$, $\nu\in\mathfrak{M}^{+}(K)$, as satisfy the condition* (3. 5).

The proof is somewhat trivial, since μ_{Γ}° is considered as the balayaged measure of ν.

3. 2. *Stable boundary points.* A point $x\in\nabla_0$ (in other words, $\hat{\varepsilon_x}\in Ext.\ \overline{M}_0^{+}(K)$) is said to be *stable*; that is, a stable boundary point of K is nothing but a regular boundary point with respect to the balayage in the present sense.

THEOREM 13. *A point $x\in\partial K$ is stable, if and only if $(\varepsilon_x)_{\Gamma}^{\circ}=\varepsilon_x$. Thus, any stable point is characterized by another simple condition*;

$$\phi(\mu)(x)=\phi(\mu_{\Gamma}^{\circ})(x) \textit{ for every } \mu\in\mathfrak{M}^{+}(K). \tag{3.6}$$

We show before beginning the proof a useful Lemma:

LEMMA 6. *If $\mu^{\frown}=\nu^{\frown}$ in $M_0^{+}(K)$ and μ is a basic measure of $\mu^{\frown}$, then we have $\mu^{\wedge}=\nu^{\wedge}$.*

In fact, $\phi(\nu)(y)=\nu^{\wedge}(\phi(\varepsilon_y))=\mu^{\frown}(\phi(\varepsilon_y))=\mu^{\wedge}(\phi(\varepsilon_y))=\phi(\mu)(y)$, whatever y may be in $E-K$, so that for any $\phi(\tau)\in H(K)$ it holds

$$\int\phi(\tau)d\nu=\int\phi(\nu)d\tau=\int\phi(\mu)d\tau=\int\phi(\tau)d\mu,$$

which comletes the proof of Lemma 6.

Proof of Theorem 13. 1) Let first $x\in\nabla_0$, *i. e.* $\hat{\varepsilon_x}\in Ext.\ M_0^{+}(K)$, then since $(\varepsilon_x)_{\Gamma}^{\circ\wedge}=(\hat{\varepsilon_x})\in Ext.\ \overline{M}_0^{+}(K)$ the basic measure of $(\varepsilon_x)_{\Gamma}^{\circ\wedge}$ is uniquely determined, hence $=(\varepsilon_x)_{\Gamma}^{\circ}$ itself, and moreover it must be equal to a point measure $\varepsilon_y, y\in K$, which is a direct consequence of Theorem 2. By the property 4) of potential, we obtain $x=y$ and thus $(\varepsilon_x)_{\Gamma}^{\circ}=\varepsilon_x$.

2) Conversely, suppose that $(\varepsilon_x)_{\Gamma}^{\circ}=\varepsilon_x$ nevertheless $\hat{\varepsilon_x}$ were not an extreme point of $\overline{M}_0^{+}(K)$, and set now $\tilde{\varepsilon_x}=\frac{1}{2}(\tilde{\mu}+\tilde{\nu})$ for $\tilde{\mu}$, $\tilde{\nu}\in\overline{M}_0^{+}(K)$, $\tilde{\mu}\neq\tilde{\nu}$, with basic measure $\mu,\nu\in\mathfrak{M}_0^{+}(K)$ respectively. Owing to the above Lemma, we have $\hat{\varepsilon_x}=\frac{1}{2}(\mu^{\wedge}+\nu^{\wedge})$, and since the balayage is unique, it must be that $\varepsilon_x=(\varepsilon_x)_{\Gamma}^{\circ}=\frac{1}{2}(\mu_{\Gamma}^{\circ}+\nu_{\Gamma}^{\circ})$, so that $\mu_{\Gamma}^{\circ}=\nu_{\Gamma}^{\circ}=\varepsilon_x$ and consequently $\mu^{\wedge}=\nu^{\wedge}=\hat{\varepsilon_x}$. Since $\tilde{\mu}\neq\tilde{\nu}$ by hypothesis, there must exist such a $\phi(\mu)\in H(K)(\mu\in\mathfrak{M}^{+}(E-K))$ that

$$\tilde{\mu}(f)>\hat{\varepsilon_x}(f)>\tilde{\nu}(f),^{16)} \qquad f=\phi(\mu),$$

16) In fact, suppose that for every $\phi(\tau)\in B^{+}(E)\cap H(K)$ $\tilde{\mu}(\phi(\tau))=\tilde{\nu}(\varphi(\tau))$, then it follows $\tilde{\mu}(f)=\tilde{\nu}(f)$ for every $f\in H(K)$; thus, $\tilde{\mu}\neq\tilde{\nu}$ yields the assertion.

and hence $\varepsilon_x^{\frown}(f)>\nu^{\smile}(f)\geqq\nu^{\frown}(f)$,[17] which is a contradiction. Thus, $\varepsilon_x^{\frown}$ is extreme in $\overline{M}_0^+(K)$.

3) $(\varepsilon_x)_F^\circ=\varepsilon_x$ implies directly that $\phi(\mu)(x)=\int\phi(\mu)d(\varepsilon_x)_F^\circ=\int\phi(\mu_F^\circ)d\varepsilon_x=\phi(\mu_F^\circ)(x)$. Conversely, from (2.6) it follows that $\phi(\mu)(x)=\phi(\mu_F^\circ)(x)=\int\phi(\mu)d(\varepsilon_x)_F^\circ$ for all $\mu\in\mathfrak{M}^+(K)$; on the other hand, it is obvious that $\phi(\nu)(x)=\int\phi(\nu)d\varepsilon_x=\int\phi(\nu)d(\varepsilon_x)_F^\circ$ for every $\nu\in\mathfrak{M}^+(E-K)$. Combining these, we have

$$\int fd\varepsilon_x=\int fd(\varepsilon_x)_F^\circ \text{ for every } f\in B(E),$$

from which $(\varepsilon_x)_F^\circ=\varepsilon_x$. Thus, the proof of Theorem 13 is completed.

In the case of Newtonian potential in $R^n(n\geqq3)$, we assert:

THEOREM 14. *If K is a set of Lebesgue measure null, all points of K are stable.*

Proof. For each $y\in E$, consider the integral means of $\phi(\varepsilon_x)$ and of $\phi((\varepsilon_x)_F^\circ)$ on the sphere $\Sigma_{y,r}$ with center y and radius r, then since $\Sigma_{y,r}\cap K$ is a null set for n-dimensional Lebesgue measure and $\phi(\varepsilon_x)=\phi((\varepsilon_x)_F^\circ)$ in $E-K$, we have

$$\frac{1}{m_r}\int_{\Sigma_{y,r}}\phi(\varepsilon_x)dv=\frac{1}{m_r}\int_{\Sigma_{y,r}}\phi((\varepsilon_x)_F^\circ)dv,$$

where dv denotes the n-dimensional volume element and m_r the total volume of a sphere with radius r. Letting $r\to0$, we have $\phi(\varepsilon_x)(y)=\phi((\varepsilon_x)_F^\circ)(y)$ for every $y\in E$, so that it concludes $\varepsilon_x=(\varepsilon_x)_F^\circ$, as desired.

3.3. *The case of general closed sets.* For a non compact closed set F with the relatively compact complement $E-K$, we are able to define the balayage by analogy to **2.4** and **3.1**, in adding the unit to $H(F)$ and compactifying F. The most important difference from the compact case is that it may be $\int d\mu>\int d\mu_F^\circ$.

The case in which $E-F$ is not open or ∂F is not compact is of less interest, so that omitted here.

§4. Regular Points and Stable Points.

4.1. *Characterization of regular or stable points.* We are now in a position to investigate the regular or stable boundary points more critically: The next theorem is a summarization of the properties of them (however, the assertion about stable points is just only a version of Theorem 13). We treat here exclusively either relatively compact D or compact K.

THEOREM 15. *The following three conditions are mutually equivalent;*

17) If $\nu_\lambda^{\frown}\to\nu^{\smile}$ in $\overline{M}_0^+(K)$ and $\nu_\lambda\to\nu$ vaguely, we have $\nu^{\smile}(f)=\lim_\lambda\nu_\lambda^{\frown}(f)=\lim_\lambda\int fd\nu_\lambda\geqq\int fd\nu$.

i) $x \in \Gamma_0$ (*resp.* $x \in \nabla_0$) *or equivalently* $\varepsilon_x^\frown \in Ext.\overline{M}_0^+(D)$ (*resp.* $\varepsilon_x^\frown \in Ext.\overline{M}_0^+(K)$),

ii) $(\varepsilon_x)_{\overline{V}}^\circ=\varepsilon_x$ (*resp.* $(\varepsilon_x)_{\Gamma}^\circ=\varepsilon_x$)

iii) $f(x)=f_\Gamma^\circ(x)$ (*resp.* $f(x)=f_{\overline{V}}^\circ(x)$) *for every* $f \in B(E)$.*)

Before beginning the proof, we propose an important Lemma which will be available to prove the Theorem itself.

LEMMA 7. *If* $\mu_\lambda^\frown \to \mu^\frown$ *in* $\overline{M}_0^+(\overline{D})$, *then* $(\mu_\lambda)_\Gamma^\circ \to (\mu)_\Gamma^\circ$ *vaguely.*

In fact, for all continuous $f \in B(E)$ we have

$$\int f d\mu_\Gamma^\circ = \int f_\Gamma d\mu = \lim_\lambda \int f_\Gamma d\mu_\lambda = \lim_\lambda \int f d(\mu_\lambda)_\Gamma^\circ .$$

Refering to the property 5) of potential, we conclude the assertion.

We shall now prove Theorem 15 in such a direction that i)$\leftrightarrows$ii) and ii)$\leftrightarrows$iii).

i)$\leftrightarrows$ii): i)$\to$ii) is clear by Theorem 4, i). The proof of ii)$\to$i) is somewhat complicated. Suppose first that $(\varepsilon_x)_\Gamma^\circ=\varepsilon_x$ nevertheless $\varepsilon_x^\frown \bar{\in} Ext.\ \overline{M}_0^+(\overline{D})$; set then $\varepsilon_x^\frown=\frac{1}{2}(\mu^\frown+\nu^\frown)$ for $\mu^\frown,\nu^\frown \in \overline{M}_0^+(\overline{D})$, $\mu^\frown \neq \nu^\frown$, with basic measures μ, ν respectively. For the sequences $\{\mu_\lambda^\frown\}$ such that $\mu_\lambda^\frown \to \mu^\frown$ and $\{\nu_\lambda^\frown\}$ such that $\nu_\lambda^\frown \to \nu^\frown$ in $\overline{M}_0^+(\overline{D})$, we see by the above Lemma that $\frac{1}{2}(\mu_\lambda+\nu_\lambda)$ converges vaguely to $(\varepsilon_x)_\Gamma^\circ=\varepsilon_x$ since $\frac{1}{2}(\mu_\lambda^\frown+\nu_\lambda^\frown)$ converges to $\varepsilon_x^\frown$ in $\overline{M}_0^+(\overline{D})$. Therefore, one concludes that

$$\varepsilon_x=\frac{1}{2}(\mu+\nu)_\Gamma^\circ=\frac{1}{2}(\mu_\Gamma^\circ+\nu_\Gamma^\circ),$$

from which $\mu_\Gamma^\circ=\nu_\Gamma^\circ=\varepsilon_x$. By the same way as in the second part 2) of the proof for Theorem 13, we have finally $\mu^\frown=\nu^\frown=\varepsilon_x^\frown$, which is contradictory with the assumption.

ii)$\leftrightarrows$iii): ii)$\to$iii) is an immediate consquence of (3.4), while iii)$\to$ii) is also clear, because iii) implies $\int f d\varepsilon_x=\int f d(\varepsilon_x)_\Gamma^\circ$ for every $f \in B(E)$. Thus, Theorem 15 is completely proved.

THEOREM 16. *If* $x \in \partial\overline{D}$ *is stable with respect to* $\overline{D}$, *then it is regular with respect to D.*

Proof. It holds always

$$\phi(\mu) \geqq \phi(\mu_\Gamma^\circ) \geqq \phi(\mu_{\overline{V}}^\circ). \tag{4.1}$$

for all $\mu \in \mathfrak{M}^+(\overline{D})$, so that $\phi(\mu)(x)=\phi(\mu_{\overline{V}}^\circ)(x)$ implies $\phi(\mu)(x)=\phi(\mu_\Gamma^\circ)(x)$, which proves the Theorem.

4.2. *Simple sufficient condition for stable or regular point.* Hereafter on, we shall restrict ourselves within the case of Newtonian potentials in $R^n (n \geqq 3)$. From the property that x is stable if and only if $\varepsilon_x^\frown \in Ext.\ \overline{M}_0^+(K)$, we get shortly a simple criterion in order that x be a stable boundary point, as an application

*) $\phi(\nu)_{\overline{V}}^\circ=\phi((\nu_K)_{\overline{V}}^\circ)+\phi(\nu_{E-K})$

of Theorem 15; that is,

THEOREM 17. *If we can draw in the outside of K an osculating sphere Σ to ∂K at a point $x \in \partial K$, then x is a stable boundary point of K.*

In fact, let x_0 and r_0 be the center and radius of Σ respectively, and take now an inner point z on the segment combining x and x_0; suppose further $\hat{\varepsilon_x} = \frac{1}{2}(\mu^{\sim}+\nu^{\sim})$, $\mu^{\sim} \neq \nu^{\sim}$ for $\mu^{\sim}, \nu^{\sim} \in \overline{M}_0^+(K)$, having the basic measures μ, ν respectively, then owing to Lemma 6 we have

$$\hat{\varepsilon_x}(\phi(\varepsilon_z)) = \frac{1}{2}[\mu^{\wedge}(\phi(\varepsilon_z)) + \nu^{\wedge}(\phi(\varepsilon_z))]; \tag{4.2}$$

on the other hand, one sees $\phi(\varepsilon_z)(y) = N_n r^{2-n}(y, z) \leqq N_n r_0^{2-n} < N_n r^{2-n}(x, z) = \phi(\varepsilon_z)(x)$ whenever $y(\neq x) \in K$, so that (4.2) is held if and only if $\mu^{\wedge} = \nu^{\wedge} = \hat{\varepsilon_x}$. By the same fashion as in the second part 2) of the proof for Theorem 13, it must be $\mu^{\sim} = \nu^{\sim} = \hat{\varepsilon_x}$, from which follows a contradiction. Thus, Theorem 17 is proved.

Refering to Theorem 16, we assert also;

COROLLARY. *If we can draw an outer osculating sphere Σ to ∂D at any point $x \in \partial D$, then x is a regular boundary point of D.*

4.3. *Further characterization in $R^n (n \geqq 3)$.* From the above Theorem 17, we can deduce another characterization for stable or regular points, which will play an important rôle in the next paragraph concerning to Dirichlet's problem:

THEOREM 18. *A necessary and sufficient condition that $x \in \partial D (\partial K)$ be regular (stable) is that for any sequence of points $y, y \in \overline{D} (\in K)$, such that $y \to x$ we have $(\varepsilon_y)_{\Gamma}^{\circ} \to \varepsilon_x ((\varepsilon_y)_{\Gamma}^{\circ} \to \varepsilon_x)$ vaguely.*

Proof. We shall prove the assertion only in the case of compact K, however it is quite all the same to the case of open D.

1°). Assume x to be a stable point of K, then according to $y \to x$ one sees

$$\begin{aligned} f(x) = f_{\Gamma}^{\circ}(x) &\leqq \varliminf_{y \to x} f_{\Gamma}^{\circ}(y) = \varliminf_{y \to x} \int f \, d(\varepsilon_y)_{\Gamma}^{\circ} \\ &\leqq \lim_{y \to x} f(y) = f(x), \end{aligned}$$

for any continuous $f \in B^+(E)$, from which it comes that $(\varepsilon_y)_{\Gamma}^{\circ} \to \varepsilon_x$ vaguely owing to the property 5) of potential.

2°). Conversely, we shall next show by using a selfcontradiction that, if x is not stable, there exists such a sequence of points $\{y\}, y \to x$, in K that $(\varepsilon_y)_{\Gamma}^{\circ}$ does not converge vaguely to ε_x. Suppose now it were not so; since x is not stable by assumption, we can take a measure $\mu \in \mathfrak{M}_0^+(K)$ such that $\phi(\mu)(x) > \phi(\mu_{\Gamma}^{\circ})(x)$, and for such μ it would hold that

$$\varliminf_{y \to x} \phi(\mu_{\Gamma}^{\circ})(y) = \varliminf_{y \to x} \int \phi(\mu) d(\varepsilon_y)_{\Gamma}^{\circ} \geqq \phi(\mu)(x) > \phi(\mu_{\Gamma}^{\circ})(x),$$

whatever $\{y\}$ may be. For a suitable open sphere Σ_x^0 with center x, we would have $\phi(\mu_\Gamma^\circ)(z) > \phi(\mu_\Gamma^\circ)(x)$ for any $z \in \Sigma_x^0 \cap K$. On the other hand, $\phi(\mu)$ being lower semi-continuous, there exists a sphere Σ_x^1 such that $\phi(\mu_\Gamma^\circ)(z) = \phi(\mu)(z) > \phi(\mu_\Gamma^\circ)(x)$ for all $z \in \Sigma_x^1 \cap (E-K)$.[18] Then, for every sphere Σ_x with center x contained in $\Sigma_x^0 \cap \Sigma_x^1$, we obtain

$$\int_{\Sigma_x} \phi(\mu_\Gamma^\circ) dv \Big/ \int_{\Sigma_x} dv > \phi(\mu_\Gamma^\circ)(x), \tag{4.3}$$

where dv denotes the volume element in E^n, which however contradicts itself with the fact that $\phi(\mu_\Gamma^\circ)$ is superharmonic in E. Thus, Theorem 18 is completely proved.

Remark 1). The proof 1°) remains valid, as is easily seen, even if E is not thus restricted in Euclidean space R^n; that is, in general, we assert that *if x is stable to K, $(\varepsilon_y)_\Gamma^\circ$ converges vaguely to ε_x for any sequence of points $y, y \in K$, such that $y \to x$.*

Remark 2). The condition in Theorem 18 for regular points is well strengthened by taking exclusively a sequence $\{y\}$ consisting only of inner points, *i.e.* $y \in D$. Indeed, replacing $\Sigma_x^0 \cap K$ by $\Sigma_x^0 \cap D$, all the proof remains valid; (4.3) is also assured since $\partial D - \Gamma_0$ is a null set for n-dimensional Lebesgue measure, and $(\varepsilon_y)_\Gamma^\circ = \varepsilon_y$ for every $y \in \Gamma_0$.

§5. Representation Theorems and Applications to Dirichlet's Problem.

5.1. *Linear space $H(\Gamma_0)$, and the representation.* Though, considering $H(D)$ as an archimedian partially ordered vector space, we can represent $H(D)$ onto a linear subspace $H(\bar{\Lambda}_0)$ of $C(\bar{\Lambda}_0)$, $\Lambda_0 = Ext.\,\overline{M}_0^+(\overline{D})$, under a linear order isomorphism and isometry (see R. V. Kadison [6], Theorem 2.1), it is less fitted for our present position to apply it to the Dirichlet's problem, because $H(\bar{\Lambda}_0)$ is not necessarily dense in $C(\bar{\Lambda}_0)$.

Thus, we need to investigate the representation of some normed linear subspace $H(\Gamma_0)$ of $H(D)$, whose definition is given just below, and through this representation theory we approach to a new solution of Dirichlet's problem. For the actual purpose, assume always that D is a relatively compact set in $R^n (n \geqq 3)$ with Newtonian potential or in the unit circle $|z| < 1$ in Z^2 with logarithmic potential. In the case, H. Cartan has constructed such a continuous $\phi(\alpha) \in B^+(E)$ (such α shall be called *H. Cartan's measure*)[19] that $\int \phi(\alpha) d\mu = \int \phi(\alpha) d\nu$ implies

18) In fact, choose $\varepsilon > 0$ as smaller than $\phi(\mu)(x) - \phi(\mu_\Gamma^\circ)(x)$; then since $\phi(\mu)$ is continuous in $E-K$, for a suitable Σ_x' it holds $\phi(\mu)(z) > \phi(\mu)(x) - \varepsilon > \phi(\mu_\Gamma^\circ)(x)$ for all $z \in \Sigma_x' \cap (E-K)$.

19) H. Cartan [4], n° **21.**

$\mu=\nu$ for $\mu, \nu \in \mathfrak{M}^+(E)$. Using this measure α, we can state:

LEMMA 8. *Every regular (or stable) point in ∂D is characterized by a single condition;*

$$\phi(\alpha)(x)=\phi(\alpha_\Gamma^\circ)(x) \quad (resp.=\phi(\alpha_\nabla^\circ)(x)). \tag{5.1}$$

Moreover, any μ_Γ° (or μ_∇°), $\mu \in \mathfrak{M}^+(\overline{D})$, is distributed in Γ_0 (resp. ∇_0), and $\partial D - \Gamma_0$ is of capacity 0.

Indeed, $\phi(\alpha)(x)=\phi(\alpha_\Gamma^\circ)(x)=\int\phi(\alpha)d(\varepsilon_x)_\Gamma^\circ$ implies that $\varepsilon_x=(\varepsilon_x)_\Gamma^\circ$ and the converse is trivial; this is all the same for $(\varepsilon_x)_\nabla^\circ$. Thus, one sees that

$$\begin{cases} \phi(\alpha)>\phi(\alpha_\Gamma^\circ) & \text{in } \partial D-\Gamma_0, \\ \phi(\alpha)>\phi(\alpha_\nabla^\circ) & \text{in } \partial D-\nabla_0. \end{cases} \tag{5.2}$$

If μ_Γ° has a portion $(\mu_\Gamma^\circ)_*$ distributed in $\partial D-\Gamma_0$, then putting $(\mu_\Gamma^\circ)_0=\mu_\Gamma^\circ-(\mu_\Gamma^\circ)_*$ we have

$$\begin{aligned} \int\phi(\alpha_\Gamma^\circ)d\mu_\Gamma^\circ &= \int\phi(\alpha_\Gamma^\circ)d(\mu_\Gamma^\circ)_0+\int\phi(\alpha_\Gamma^\circ)d(\mu_\Gamma^\circ)_* \\ &< \int\phi(\alpha)d(\mu_\Gamma^\circ)_0+\int\phi(\alpha)d(\mu_\Gamma^\circ)_* \\ &= \int\phi(\alpha)d\mu_\Gamma^\circ, \end{aligned}$$

which is absurd since μ_Γ° is also the balayage of μ_Γ° itself. The same is true for μ_∇°. From the first inequality of (5.2) follows immediately the last assertion since $\phi(\alpha)=\phi(\alpha_\Gamma^\circ)$ in ∂D excepting a set of capacity 0. This completes the proof.

DEFINITION 3. *$H(D, \Gamma_0)$ (or $H(\overline{D}, \nabla_0)$) is a normed linear subspace of $H(D)$ (resp. $H(\overline{D})$) consisting of all such functions as are uniformly continuous in Γ_0 (resp. ∇_0), and $H(\Gamma_0)$ (or $H(\nabla_0)$) is a normed linear space of all the restrictions $\hat{f}$ of $f \in H(D, \Gamma_0)$ (resp. $\in H(\overline{D}, \nabla_0)$) on Γ_0 (resp. ∇_0) with respect to the norm*

$$\|\hat{f}\|_{\Gamma_0}=\sup_{x\in\Gamma_0}|\hat{f}(x)| \quad (\|\hat{f}\|_{\nabla_0}=\sup_{x\in\nabla_0}|\hat{f}(x)|). \tag{5.3}$$

LEMMA 9. *$H(D, \Gamma_0)$ is isometrically isomorphic to $H(\Gamma_0)$ in an order preserving fashion, and so is $H(\overline{D}, \nabla_0)$ to $H(\nabla_0)$.*

In fact, owing to Lemma 8, one has

$$|f(x)|=\left|\int\hat{f}d(\varepsilon_x)_\Gamma^\circ\right| \leqq \|\hat{f}\|_{\Gamma_0},$$

for every $x\in\overline{D}$, so that $\|f\|_D\leqq\|\hat{f}\|_{\Gamma_0}$. $\|\hat{f}\|_{\Gamma_0}\leqq\|f\|_D$ being clear, one concludes $\|f\|_D=\|\hat{f}\|_{\Gamma_0}$. It is all the same for $\|\hat{f}\|_{\nabla_0}$, and the rest is somewhat trivial.

THEOREM 19. *$H(\Gamma_0)$ is dense in $C_u(\Gamma_0)$.*

Proof. In fact, every $h \in C_u(\Gamma_0)$ has a uniquely determined continuous extension $\hat{h}$ up to $\overline{\Gamma}_0$ and a further arbitrary continuous extension $\tilde{h}$ with compact support in E. For such $\tilde{h}$, there exists a sequence of continuous $f_j \in B(E)$ which converges to $\tilde{h}$ uniformly in E; for each j $(f_j)^\circ_\Gamma$ is equal to f_j on Γ_0, hence uniformly continuous there since f_j is continuous in $\overline{\Gamma}_0$, that is, $(f_j)^\circ_\Gamma \in H(\Gamma_0)$. As is easily seen, such sequence $\{(f_j)^\circ_\Gamma\}$ converges to $h=\tilde{h}$ uniformly on Γ_0, in other words, with respect to the norm of $H(\Gamma_0)$. Thus, Theorem 19 is completely proved.

Now, $C_u(\Gamma_0)$ forms a commutative Banach algebra with respect to the pointwise product, and as is well known, $C_u(\Gamma_0)$ is isomorphically isometric to $C(\overline{\Lambda}_\Gamma)$, where $\overline{\Lambda}_\Gamma$ is compact and Γ_0 is homeomorphic to a dense part Λ_Γ of $\overline{\Lambda}_\Gamma$. Thus we can arrange these results into:

THEOREM 20 (Representation theorem). *A linear subspace $H(D, \Gamma_0)$ of $H(D)$ is isomorphic to a dense subspace of $C(\overline{\Lambda}_\Gamma)$ under a norm- and order-preserving fashion, where $\overline{\Lambda}_\Gamma$ is compact and Γ_0 is homeomorphic to a dense space Λ_Γ of it.*

Remark 1. Here, $\Lambda_\Gamma = Ext.\ \Xi^*_0$ for the unit sphere in the dual space $(C_u(\Gamma_0))^*$ to $C_u(\Gamma_0)$; in this case, since $C_u(\Gamma_0)$ is commutative, every functional in $Ext.\ \Xi^*_0$ is multiplicative and hence positive, so that with the same notation as in §**1** one sees

$$Ext.\ \Xi^*_0 = Ext.\ \overline{M}^+_0(\overline{\Gamma}_0).$$ [20]

Remark 2. Since every $\hat{f} \in H(\Gamma_0)$ is uniformly continuous in Γ_0, it is well prolonged up to the closure $\overline{\Gamma}_0$ of Γ_0 without raising the norm. Such prolonged ones $\bar{f}$ of $\hat{f} \in H(\Gamma_0)$ form a linear subspace $H(\overline{\Gamma}_0)$ of $C(\overline{\Gamma}_0)$. Obviously, we have

$$H(\Gamma_0) \cong H(\overline{\Gamma}_0) \quad \text{and} \quad C_u(\Gamma_0) \cong C(\overline{\Gamma}_0) \cong C(\overline{\Lambda}_\Gamma),$$ [21]

therefore, $H(\overline{\Gamma}_0)$ is dense in $C(\overline{\Gamma}_0)$. But, it should be noticed that, for a point $x \in \overline{\Gamma}_0 - \Gamma_0$, $\bar{f}(x)$ is not generally equal to $\hat{f}(x)$; in order to avoid any confusion of such a kind, we have adopted the representation of $H(D, \Gamma_0)$ into $C(\overline{\Lambda}_\Gamma)$ as in the above Theorem, although the latter (that is, representation into $C(\overline{\Gamma}_0)$) seems to be more simple.

5.2. *Application to Dirichlet's problem.* From Lemma 9 and Theorem 19, we obtain the solution of Dirichlet's problem as follows:

THEOREM 21. *For every $f \in C(\partial D)$, there exists the unique solution $\tilde{f}$ of Dirichlet's problem with respect to D, which satisfies;*

20) About these matter, see e.g. R. V. Kadison [6], S. Matsushita [9], etc.

21) The symbol $\cong$ indicates an isomorphism which preserves the order (and hence norm) structures.

a) $\tilde{f}$ *is harmonic in* D *and bounded in* $\bar{D}$,

b) $\lim_{x \to x_0} \tilde{f}(x) = f(x_0)$ *for* $x \in D$ *and* $x_0 \in \Gamma_0$, *that is, excepting a set of capacity* 0, *the boundary-values of* $\tilde{f}$ *coincide with* $f(x)$.

Proof. Existence of the solution: For each point $x \in D$ and any sphere Σ_x with center x, included in D, the integral operators $\varepsilon_x^{\sim}$ and $\lambda_x^{\sim}$, λ_x being the spherical measure on Σ_x,[22] such that

$$(5.4) \qquad \varepsilon_x^{\sim}(\hat{h}) = \int_{\Gamma_0} \hat{h} d(\varepsilon_x)_\Gamma^\circ, \qquad \lambda_x^{\sim}(\hat{h}) = \int_{\Sigma_x} \int_{\Gamma_0} \hat{h} d(\varepsilon_y)_\Gamma^\circ d\lambda_x(y),$$

are both bounded linear functionals (actually, of norm 1) on $H(\Gamma_0)$, which coincide with one another, since on $H(D, \Gamma_0)$ so are those that $\varepsilon_x^{\sim}(h) = h(x)$ $(=\varepsilon_x^{\sim}(h))$ and $\lambda_x^{\sim}(h) = \int_{\Sigma_x} h(y) d\lambda_x(y) (=\lambda_x^{\sim}(h))$. Since $H(\Gamma_0)$ is dense in $C_u(\Gamma_0)$, both $\varepsilon_x^{\sim}$ and $\lambda_x^{\sim}$ are uniquely prolonged to $C_u(\Gamma_0)$, on which $\varepsilon_x^{\sim} = \lambda_x^{\sim}$, too. Thus, for $f \in C(\partial D)$, the restriction f_{Γ_0} of f in Γ_0 is contained in $C_u(\Gamma_0)$ and

$$(5.5) \qquad \tilde{f}(x) = \int_{\Gamma_0} f_{\Gamma_0} d(\varepsilon_x)_\Gamma^\circ = \int_{\partial D} f\, d(\varepsilon_x)_\Gamma^\circ$$

is the desired one, because one has $\int_{\Sigma_x} \tilde{f} d\lambda_x = \lambda_x^{\sim}(f_{\Gamma_0}) = \varepsilon_x^{\sim}(f_{\Gamma_0}) = \tilde{f}(x)$. The condition $b)$ is an immediate consequence of Theorem 18, **4.3**.

Uniqueness of the solution follows from the general property of harmonic functions that if $\tilde{f}$ and $\tilde{g}$ are bounded harmonic in D and, excepting a set of capacity 0 in ∂D, their boundary-values are identical, then $\tilde{f} = \tilde{g}$ in D (a proof of this fact, however, shall be given in the later section independently from the present proof), which completes the proof of Theorem 21.

The case of non relatively compact domain D with compact ∂D is also analogically treated, but it is needful in this case to set up a further condition for $\tilde{f}$, in addition to $a)$, $b)$ in Theorem 21, such that

c) $\qquad f \in L_\infty(D), \qquad$ *i.e.* $\qquad \lim_{x \to \infty} f(x) = 0,$

since $(\varepsilon_x)_\Gamma^\circ$ converges vaguely to 0 as x runs to the infinite, as is easily seen.

An analogue of Theorem 21 for $C_u(\nabla_0)$ is easily obtained; employing the above obtained results, we can prove

THEOREM 22. *For every* $f \in C_u(\nabla_0)$ *(hence, for* $f \in C(\partial \bar{D})$*)*,

$$(5.6) \qquad f^*(x) = \int f d(\varepsilon_x)_\nabla^\circ$$

is a harmonic function in D, *which has the property*; $\lim_{x \to x_0} f^*(x) = f(x_0)$ *for* $x \in D$, $x_0 \in \nabla_0$.

22) About the definition of λ_x (spherical distribution on Σ_x), refer to **2.2.**

In fact, f has a continuous extension $\bar{f}\in C_u(\Gamma_0)$, then owing to Theorem 19, there exists a sequence of $\hat{g}\in H(\Gamma_0)$ which converges to $\bar{f}$ uniformly on Γ_0, and hence to f on ∇_0. The integral functionals (with the same notations as in (5.4)):

$$\varepsilon_x^*(f)=\int_{\Gamma_0} fd(\varepsilon_x)_\Gamma^\circ, \qquad \lambda_x^*(f)=\int_{\Sigma_x}\int_{\Gamma_0} fd(\varepsilon_y)_\Gamma^\circ d\lambda_x(y)$$

are both bounded linear on $C_u(\Gamma_0)$, and satisfy $\varepsilon_x^*(\hat{g})=\int_{\Gamma_0} gd(\varepsilon_x)_\Gamma^\circ=g_\Gamma^\circ(x)=\int_{\Sigma_x} g_\Gamma^\circ(y)d\lambda_x=\lambda_x^*(\hat{g})$ for every $\hat{g}\in H(\Gamma_0)$, since g_Γ° is harmonic in D. Thus, $\varepsilon_x^*=\lambda_x^*$ on $C_u(\Gamma_0)$, so that f^* is harmonic in D. The rest is clear by Theorem 18.

5.3. *Extension of boundary functions in Dirichlet's problem*: Here we consider only n-dimensional Euclidean space $R^n(n\geqq 3)$ with Newtonian potentials and assume always that D is relatively compact. Now, assume further ∂D to be a measure space with respect to a certain measure m such that 1°) $m(X)=0$ for any set $X\subset\partial D$ of capacity 0, 2°) every bounded potential is m-measurable on ∂D.

THEOREM 23. *For every essentially bounded m-measurable function f on ∂D, there corresponds a bounded harmonic function $\tilde{f}$ in D such that; if f is continuous in a neighborhood $U(x_0)$ of a regular boundary point $x_0(\in\Gamma_0)$, then $\tilde{f}(x)\to f(x_0)$ as $x\in D\to x_0$, and if f is continuous excepting a set of capacity 0, such $\tilde{f}$ is uniquely determined.*

Let us denote by $M(\partial D)$ the Banach space of all m-measurable essentially bounded functions with respect to the norm $\|f\|_M=\operatorname{ess.max}_{x\in\partial D}|f(x)|$ (*i.e.* essential maximum). All $f\in H(D)$ form a linear subspace $H(\partial D)$ of $M(\partial D)$. Then we see that $H(\partial D)$ is dense in $M(\partial D)$: in fact, suppose now this were not so, and take a non-trivial functional $\xi^\sim$ on $M(\partial D)$, which vanishes on $H(\partial D)$. From a general investigation about the conjugate space of $M(\partial D)$, we conclude that such $\xi^\sim$ defines a Radon measure ξ on ∂D, for which $\xi(X)=0$ on every set of X of capacity 0.

For H. cartan's measure α cited in **5.1**, we have by hypothesis $\int\phi(\alpha)d\xi=\int\phi(\alpha_\Gamma^\circ)d\xi=\xi^\sim(\phi(\alpha_\Gamma^\circ))=0$, since we have $\phi(\alpha)=\phi(\alpha_\Gamma^\circ)$ on ∂D excepting a set of capacity 0; so that, ξ itself must be null measure and $\xi^\sim=0$, contradicting with the hypothesis. Then, the bounded linear functionals $\lambda_x^\sim$ and $\varepsilon_x^\sim$, $x\in D$, defined similarly as in (5.4) on $M(\partial D)$ are coincident, since so on $H(\partial D)$, which implies that $f(x)=\int_{\partial D} fd(\varepsilon_x)_\Gamma^\circ$ is harmonic in D. Owing to the fact that if $x_0\in\Gamma_0$, $(\varepsilon_x)_\Gamma^\circ$ converges vaguely to ε_{x_0} as $x\in D\to x_0$ and hence so is the restriction of $(\varepsilon_x)_\Gamma^\circ$ in $U(x_0)$, and by the same reason appeared in the proof of uniqueness for Theorem 21, the rest is somewhat clear.

We remark that this Theorem involves Theorem 21 entirely and yet its proof is quite independent of the latter.

COROLLARY. *With the same f as above, $f^*(x)=\int fd(\varepsilon_x)^0_\Gamma$ is harmonic in D and if f is continuous in a neighborhood $U(x_0)$ of a stable point $x_0(\epsilon\nabla_0)$, $f^*(x)\to f(x_0)$ as $x\epsilon D\to x_0$.*

Appendix I. We consider exclusively $R^n(n\geqq 3)$ and Newtonian potentials. Suppose first that $\mu\epsilon\mathfrak{M}^+(E)$ has a compact support. Since $\phi(\mu)\epsilon L_\infty(E)$, for a suitable positive number k, $g=\phi(\mu)\frown k\epsilon B^+(E)$ and $f=\phi(\mu)-g$ vanishes outside of a relatively compact open set U. f being lower semi-continuous, there exists a sequence of continuous functions f_j such that $f_j\nearrow f$; then, putting $f^0_j=(f_j-\varepsilon/2^j)^+$,[23] we see easily $f^0_j\nearrow f$ again and f^0_j has a compact support wholly contained in U. Owing to the property 5) of potential, for each f^0_j there exists a continuous potential $\phi(\nu_j)\geqq 0$ such that

$$|\phi(\nu_j)-f^0_j|<\varepsilon/2^{j+2},$$

and $\phi(\nu_j)$ vanishes clearly outside of U. As is easily seen, $\phi(\nu_j)\leqq\phi(\nu_{j+1})$ for every j and $\phi(\nu_j)\nearrow f$, so that, putting $\phi_j=(g+\phi(\nu_j))$ which is continuous in E, $\phi_j\nearrow\phi(\mu)$ as desired.

If the support F of $\mu\epsilon\mathfrak{M}^+(E)$ is not compact, we can take a sequence of compact sets K_j such that $K_j\subset int.\ K_{j+1}$ and $\overline{\bigcup K_j}\supset F$; denote now the restriction of μ in K_j by μ_j and for each $\phi(\mu_j)$ consider ϕ_j in the above sense, then $\phi(\mu_j)\nearrow\phi(\mu)$ and hence $\phi_j\nearrow\phi(\mu)$ as desired.

Appendix II. We shall prove next that if two harmonic functions $\tilde{f}$ and $\tilde{g}$ have the same boundary-values excepting a set Θ of capacity 0 in ∂D, then $\tilde{f}=\tilde{g}$ in D; with the same notations as in **5. 2,** consider at fiirst the sequence of domains $D_j\subset D$ such that $D_j=D-\bigcup S_{\rho_j}(x)$ for spheres $S_{\rho_j}(x)$ with center x for all $x\epsilon\Theta_0=(\partial D-\Gamma_0)\cup\Theta$, which being the set of capacity 0, and radius $\rho_j=1/2^j$. Clearly, $D_j\subset D_{j+1}$, $\lim_j D_j=D$, and all the boundary points of each D_j are regular (refer to Theorem 17).[24] For an arbitrarily fixed $x\epsilon D$, there exists an index j_0 such that

23) $(\cdot)^+$ means the positive part of $(\cdot)$, *i.e.* $(\cdot)\cup 0$.

24) Let us prove each $x_0\epsilon\partial D_j-\partial(\bigcup S_{\rho j}(x))$ to be regular with respect to ∂D_j; take now a neighborhood $U(x_0)$ contained in $\partial D_j-\partial(\bigcup S_{\rho j}(x))$. For $f\epsilon C_0(U(x_0))$, we see that

$$\int_{\partial D_j}\tilde{f}d(\varepsilon_x)^0_j=\tilde{f}(x)=\int_{\partial D}fd(\varepsilon_x)^0_\Gamma\to f(x_0)\text{ as } x\epsilon D_j\to x_0,$$

since f is continuous in ∂D_j, so that $((\varepsilon_x)^0_j)_{U(x_0)}\to\varepsilon_{x_0}$ and hence $(\varepsilon_x)^0_j\to\varepsilon_{x_0}$ vaguely, from which the assertion.

$x \in D_j$ for every $j \geqq j_0$. Denote by $(\varepsilon_x)_j^0$ the balayage of ε_x, $x \in D_j$ in ∂D_j and by $(\varepsilon_x)_j^i$ $(i=1,2)$ the restriction of $(\varepsilon_x)_j^0$ in $\partial D_j \cap \partial D$ and that in $\partial D_j - \partial D$ respectively; a subsequence of $\{(\varepsilon_x)_j^2\}$ converges vaguely to a certain measure ν on ∂D and, as $\phi(\varepsilon_x) \geqq \varliminf_j \phi((\varepsilon_x)_j^2) \geqq \phi(\nu)$ on ∂D, $\phi(\nu)$ is bounded in E, while $\lim_j \int d(\varepsilon_x)_j^2 = \int d\nu < +\infty$. Therefore, ν must be distributed in $\partial D - \Theta_0$ and we can see that $|\tilde{f} - \tilde{g}|(\varepsilon_x)_j^2$ converges to $|\tilde{f} - \tilde{g}|\nu = 0$,[25] so that

$$|\tilde{f}(x) - \tilde{g}(x)| \leqq \int_{\partial D_j} |\tilde{f} - \tilde{g}| d(\varepsilon_x)_j^0$$
$$= \int_{\partial D_j - \partial D} |\tilde{f} - \tilde{g}| d(\varepsilon_x)_j^2 \to \int_{\partial D} |\tilde{f} - \tilde{g}| d\nu = 0,$$

and, $x \in D$ being arbitrary, $\tilde{f} = \tilde{g}$ in D.

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25) $\mu_j = |\tilde{f} - \tilde{g}|(\varepsilon_x)_j^0$ converges vaguely to a certain ν_0 on ∂D (in $\bar{D}$ and hence in $D - \Theta_0$), whose energy is evidently finite, so that ν_0 is placed on $\partial D - \Theta_0$, and hence $\mu_j \to \nu_0$ in $\bar{D} - \Theta_0$, which shows $\nu_0 = |\tilde{f} - \tilde{g}|\nu$.