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The weak dimension of algebras and its applications

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In this note we shall define the weak dimension of algebras Λ, analogous to the dimension of algebras in Cartan and Eilenberg [6], Ch. IX. In section 1 we shall characterize the algebras with the weak dimension zero, and study some properties of the weak dimension of the tensor product of two algebras, and we shall completely determine the weak dimension of fields. If an algebra Λ has a finite degree over a field K, it is well known that Λ is separable if and only if $\Lambda\otimes\Lambda^*$ $(=\Lambda^e)$ is semi-simple, where Λ^* is anti-isomorphic to Λ. Rosenberg and Zelinsky [15] proved that if Λ^e is a semi-simple algebra with minimum conditions, then $[\Lambda:K]<\infty$. Therefore if we want to define some generalized separability of algebras with infinite degree over K, then we may restrict ourselves to the case where Λ^e is semi-simple in the sense of Jacobson. In section 2 we shall call Λ R-separable if Λ^e is regular, and Λ has the property E, if $\Lambda\otimes L$ is regular for any field $L\geqq K$. We shall consider these algebras and relations between these two algebras. In section 3 we shall study some properties of tensor products of separable fields and algebras. In this note we always assume an algebra Λ has a unit element and that Λ-modules are unitary. We use [6] as a reference source for homological algebras.

1. The weak dimension of algebras

Let Λ be an algebra over a commutative ring K. We shall define the weak dimension of Λ (notation w. dim Λ), analogous to Cartan and Eilenberg [6], Ch. IX. 7.

DEFINITION 1. *w. dim Λ=the minimal integer n such that*

$$H_{n+1}(\Lambda,\ A)=Tor_{n+1}^{\Lambda^e}(A,\Lambda)=0$$

for any two sided Λ-module A.

First we state some remarks about the definition. Let Λ be an algebra over a field K. If Λ^e is Noetherian or if Λ is semi-primary with radical N such that $[\Lambda/N:K]<\infty$, then we have

$$\text{w. dim } \Lambda=\text{w. dim}_{\Lambda^e}\Lambda=\dim_{\Lambda^e}\Lambda=\dim\Lambda$$

from [6], Ch. VI, Exer. 3, and Auslander [2], Coro. 8 and [3], Th. 5.

In general we have clearly by the definition

$$\dim \Lambda \geqq \mathrm{w.}\dim \Lambda,$$

and there exists an algebra Λ in which the above equality is not satisfied.

Let K be a commutative ring, and Λ, Γ and Σ be K-algebras. We consider the functor

$$T(A, C)=A \underset{\Lambda\otimes\Gamma}{\otimes} (B \underset{\Sigma}{\otimes} C)=(A \underset{\Lambda}{\otimes} B) \underset{\Gamma\otimes\Sigma}{\otimes} C^{1)}$$

for the symbol $(A_{\Lambda,\Gamma},\ {}_{\Lambda}B_{\Sigma},\ {}_{\Gamma,\Sigma}C)$. According to Eilenberg, Rosenberg and Zelinsky [9], we have the spectral sequence when Γ is K-flat:

1) $\operatorname{Tor}_{p}^{\Lambda\otimes\Gamma}(A, \operatorname{Tor}_{q}^{\Sigma}(B, C)) \underset{p}{\Rightarrow} L_n T(A, C)$,

2) $\operatorname{Tor}_{q}^{\Gamma\otimes\Sigma}(\operatorname{Tor}_{q}^{\Lambda}(A, B), C) \underset{q}{\Rightarrow} L_n T(A, C)$.

If $\operatorname{Tor}_{q}^{\Sigma}(B, C)=0=\operatorname{Tor}_{p}^{\Lambda}(A, B)$ for $p, q>0$, 1) and 2) collapse and we have

(*) $\operatorname{Tor}_{q}^{\Lambda\otimes\Gamma}(A, B \underset{\Sigma}{\otimes} C) \approx \operatorname{Tor}_{p}^{\Gamma\otimes\Sigma}(A \underset{\Lambda}{\otimes} B, C)$, (cf. [6], Ch. IX, Th. 2.8).

If we replace Σ by Γ^{*} and C by Γ in (*), we have

LEMMA 1. *If Λ is a regular K-algebra and Γ is a K-flat K-algebra, we obtain*

$$H_p(\Gamma, A \underset{\Lambda}{\otimes} B) \approx \operatorname{Tor}_{p}^{\Lambda\otimes\Gamma}(A, B) \qquad \text{for } (A_{\Lambda\Gamma},\ {}_{\Lambda\Gamma}B).$$

If we replace Λ by K in the lemma 1, we obtain

LEMMA 2. *If K is commutative regular and Γ is a K-algebra, we have isomorphisms*

$$H_p(\Gamma, A \otimes B) \approx \operatorname{Tor}_{p}^{\Gamma}(A, B)$$

for $(A_{\Gamma},\ {}_{\Gamma}B)$.

If we replace Λ by Γ^{*} in the lemma 1, we obtain

LEMMA 3. *If Γ is a K-flat regular algebra, we have isomorphisms*

$$H_p(\Gamma, A \underset{\Gamma}{\otimes} B) \approx \operatorname{Tor}_p{}^{\Gamma^e}(A, B)$$

for $({}_{\Gamma}A_{\Gamma},\ {}_{\Gamma}B_{\Gamma})$.

We can obtain the analogous theorem to [6], Ch. IX, Prop. 7.10.

THEOREM 1. *Let K be a commutative regular ring and Λ be a K-algebra, then the following conditions are equivalent:*

a) *$w.dim\ \Lambda=0$,*

b) *$\Lambda\otimes\Lambda^{*}$ is regular.*

Proof. If Λ^e is regular we have immediately $\mathrm{w.}\dim\Lambda=0$ by the definition and the author [10], Th. 5. Conversely if $\mathrm{w.}\dim\Lambda=0$, we obtain by the lemma 2

1) Unadorned $\otimes$ is always taken over K.

$$0=\mathrm{w.\,dim}\Lambda\geqq\mathrm{w.\,gl.\,dim}\Lambda\,,$$

hence Λ is regular. We have, therefore, by the lemma 3

$$0=\mathrm{w.\,dim}\Lambda\geqq\mathrm{w.\,gl.\,dim}\Lambda^e.$$

Hence Λ^e is regular.

COROLLARY. *Let L be a commutative regular extension ring of K. If w. dim $\Lambda=0$ then $\Lambda\otimes L$ is regular.*

Proof. It is clear that $(\Lambda\underset{K}{\otimes}L)\underset{L}{\otimes}(\Lambda\underset{K}{\otimes}L)^*\approx(\Lambda\underset{K}{\otimes}\Lambda^e)\underset{K}{\otimes}L$. On the other hand, if we replace (Λ,Γ,Σ) by (L,Λ^e,K) and (B,C) by (L,Λ) in (*) we have

$$\mathrm{Tor}_p^{(L\otimes\Lambda)^e}(A,L\otimes\Lambda)\approx\mathrm{Tor}_p^{\Lambda^e}(A,\Lambda)\qquad({}_{L\cdot\Lambda}A_{L,\Lambda}).$$

Therefore it follows from the lemma 2 that

$$\mathrm{Tor}_p^{\Lambda\otimes L}(C,D)\approx\mathrm{Tor}_p^{(L\otimes\Lambda)^e}(C\underset{L}{\otimes}D,L\underset{L}{\otimes}\Lambda)\approx\mathrm{Tor}_p^{\Lambda^e}(C\underset{L}{\otimes}D,\Lambda)=0$$

for $p>0$ and $(C_{L,\Gamma},\ {}_{L,\Gamma}D)$. Hence $L\otimes\Lambda$ is regular.

We can obtain the following lemma from the spectral sequences 1) and 2) analogously to [9], Prop. 3.

LEMMA 4. *Let Λ be a K-flat K-algebra and let K be an L-algebra. Then we have*

$$L\text{-}w.\ dim\ \Lambda\leqq L\text{-}w.\ dim\ K+K\text{-}w.\ dim\ \Lambda.$$[2]

If further Λ is K-projective and contains a K-direct summand K' isomorphic with K, then

$$L\text{-}w.\ dim\ K\leqq L\text{-}w.\ dim\ \Lambda.$$

REMARK. Let K be a field. We assume $\Lambda\otimes L$ is regular for any commutative regular ring L containing K. If we replace L by the center Z of Λ, since $Z\otimes Z$ is the center of $\Lambda\otimes Z$, $Z\otimes Z$ is regular. Hence K-w. dim $Z=0$. Further if L' is any commutative regular ring containing Z, $\Lambda\underset{Z}{\otimes}L'$ is regular since $\Lambda\underset{Z}{\otimes}L'$ is a homomorphic image of $\Lambda\otimes L'$. Therefore in the consideration of the converse of the corollary, we may restrict ourselves to the case of a central algebra by the lemma 4, (cf. Prop. 3 below).

The following theorems have been proved independently by Eilenberg, Rosenberg and Zelinsky [9], using the above spectral sequences.

THEOREM 2. *Let K be a field and Λ and Γ be K-algebras. Then we have*

$$w.\,l.\,dim_{\Lambda\otimes\Gamma}A\otimes B=w.\,l.\,dim_{\Lambda}A+w.\,l.\,dim_{\Gamma}B,$$

for $({}_{\Lambda}A,{}_{\Gamma}B)$.

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2) L-w. dim means w. dim Λ where Λ is considered as an L-algebra.

Especially $w.dim\,\Lambda\otimes\Gamma = w.dim\,\Lambda + w.dim\,\Gamma$, (*cf.* [9], Prop. 10).

We use the following two lemmas to prove the theorem.

LEMMA 5. *Let K be a field. Then we have*

$$w.\,l.\,dim_{\Lambda\otimes\Gamma}\,A\otimes B \geqq w.\,l.\,dim_{\Lambda}A + w.\,l.\,dim_{\Gamma}B,$$

for $({}_{\Lambda}A, {}_{\Gamma}B)$

We can easily prove this lemma by using T-product of [6], Ch. XI.

LEMMA 6. *For an exact sequence:* $0\rightarrow A''\rightarrow A\rightarrow A'\rightarrow 0$ *of* Λ*-modules, we have*

$$w.l.dim_{\Lambda}A' \leqq max(w.l.dim_{\Lambda}A,\ \ w.l.dim_{\Lambda}A'')+1.$$

This is clear by the exactness of Tor.

Proof of the theorem. By the lemma 5 we may assume $\mathrm{w.\,l.\,dim}_{\Lambda}A + \mathrm{w.\,l.\,dim}_{\Gamma}B < \infty$. Hence we can prove the theorem by the induction with respect of $\mathrm{w.\,l.\,dim}_{\Lambda}A + \mathrm{w.\,l.\,dim}_{\Gamma}B$. If $\mathrm{w.\,l.\,dim}_{\Lambda}A = \mathrm{w.\,l.\,dim}_{\Gamma}B = 0$, replacing Σ by K in (*) we obtain

$$\mathrm{Tor}^{\Lambda\otimes\Gamma}(C, A\otimes B) \approx \mathrm{Tor}^{\Gamma}(C\otimes A, B),\ \ (C_{\Lambda,\Gamma}),$$

hence $\mathrm{w.\,l.\,dim}_{\Lambda\otimes\Gamma}A\otimes B = 0$. Assume now that the theorem is true for any left Λ-module A' and left Γ-module B' with $\mathrm{w.\,l.\,dim}_{\Lambda}A' + \mathrm{w.\,l.\,dim}_{\Gamma}B' \leqq m$, $(0 \leqq m < \infty)$, and that $\mathrm{w.\,l.\,dim}_{\Lambda}A + \mathrm{w.\,l.\,dim}_{\Gamma}B = m+1$. We may assume $\mathrm{w.\,l.\,dim}_{\Lambda}A = n \geqslant 1$. From a Λ-exact sequence: $0\rightarrow R\rightarrow P\rightarrow A\rightarrow 0$, of A with P projective, we obtain the exact sequence:

$$0\rightarrow R\otimes B\rightarrow P\otimes B\rightarrow A\otimes B\rightarrow 0.$$

By the induction hypothesis and the lemmas 5 and 6 we obtain $\mathrm{w.\,l.\,dim}_{\Lambda\otimes\Gamma}A\otimes B = \mathrm{w.\,l.\,dim}_{\Lambda}A + \mathrm{w.\,l.\,dim}_{\Gamma}B$.

We can prove similarly the following theorem.

THEOREM 3. *Let K be a commutative ring, and* Λ *and* Γ *be K-algebra. If* Γ *is K-flat, then*

$$w.\,gl.\,dim\Lambda\otimes\Gamma \leqq w.\,dim\Lambda + w.\,gl.\,dim\Gamma.$$

REMARK. Let K be a field. If $\mathrm{w.\,dim}\,\Lambda = 0$ we have

$$\mathrm{w.\,gl.\,dim}\Lambda\otimes\Gamma = \mathrm{w.\,gl.\,dim}\Gamma,$$

from the theorem 3 and lemma 5. If Λ is a semi-primary K-algebra with radical N such that dim Λ and $[\Lambda/N:K]$ are finite, then

$$\mathrm{w.\,gl.\,dim}\Lambda\otimes\Gamma = \mathrm{w.\,gl.\,dim}\Lambda + \mathrm{w.\,gl\,dim}\Gamma,$$

for any K-algebra Γ. Because, by the assumption and Auslander [3], we obtain $\mathrm{w.\,dim}\Lambda = \mathrm{dim}\Lambda = \mathrm{gl.\,dim}\Lambda = \mathrm{w.\,gl.\,dim}\Lambda$.

Next we shall consider the weak dimension of algebras which are represented as the direct limit of sub-algebras.

PROPOSITION 1. *Let K be a commutative ring and Λ be a K-algebra. Assume that Λ is a union of a family $\{\Lambda_\alpha\}(\alpha\in I)$ of subalgebras Λ_α such that if $\alpha<\beta$, $(\alpha,\beta\in I)$, $\Lambda_\alpha\leqq\Lambda_\beta$, where I is a direct set. Then we have*

$$w.\,gl.\,dim\ \Lambda\leqq sup\ w.gl.dim\ \Lambda_\alpha,$$

$$and\ w.\,dim\ \Lambda\leqq sup\ w.dim\ \Lambda_\alpha, \quad \text{(cf. [15], Prop. 3).}$$

Proof. Λ is the direct limit of system $\{\Lambda_\alpha, \pi_\alpha^\beta\}$ (π_α^β are inclusions). Since $\Lambda=U\Lambda_\alpha$, unit element of Λ is contained in all Λ_α for $\alpha\geqq$ sufficently large α_0. For any Λ-module A we have a Λ_α-module $A\cdot\Lambda_\alpha=A_\alpha$. It is clear that if $\alpha<\beta$, $A_\alpha\leqq A_\beta$, hence $A=\varinjlim A_\alpha$ by the above remark. Form [6], Ch. VI, Exer. 17, we have

$$\mathrm{Tor}_n^\Lambda(A,C)=\varinjlim \mathrm{Tor}_n^{\Lambda_\alpha}(A_\alpha,C_\alpha), \qquad (A_\Lambda, {}_\Lambda C).$$

This proves the first part of the proposition. We can prove similarly the second part.

If Λ is a commutative algebra over a field K with minimum condition, and Λ is not semi-simple, then w. dim.$\Lambda=\infty$ by [2], Prop. 15 and the lemma 2. Hence we can restrict ourselves to the semi-simple case, and further we may restrict ourselves to the case where Λ is itself a field.

The following arguments are slight modifications of [15], 5.

PROPOSITION 2. *Let K be a field and Λ the field $K(t_1\cdots t_n)$ of rational functions in n indeterminates over K. Then K-w. dm$\Lambda=n$.*

Since $\Lambda\otimes\Lambda$ is Noetherian, we have the proposition 2 from the remark of the definition 1 and [15], Th. 7.

LEMMA 7. *Let Λ be a locally separable algebra,*[3] *then Λ^e is regular.*

Proof. Let Λ be locally separable and $a=\sum b_i\otimes c_i^*$ be an element of Λ^e $(b_i, c_i\in\Lambda)$, then there exists a separable subalgebra Λ' of finite order over K, containing all b_i, c_i. Therefore a is regular in Λ'^e and hence in Λ^e.

PROPOSITION 3. *Let Λ be a field of transcendental degree $n\leqq\infty$ over K with separable basis. Then K-w.dim $\Lambda=n$.*

Proof. Let B be a separable basis with n elements, then Λ is algebraic separable over $K(B)$. By the lemmas 4 and 7, and the proposition 2, we have

$$n=K\text{-w.dim}\,K(B)\leqq K\text{-w. dim}\Lambda$$
$$\leqq K\text{-w. dim}K(B)+K(B)\text{-w. dim}\Lambda=n.$$

PROPOSITION 4. *If Λ is a finitely generated extension field of K with no separable basis over K, then K-w. dim $\Lambda=\infty$.*

Proof. Let $\Lambda=K(x_1,\cdots,x_r)$ and let s be the largest integer such that $S=K(x_1,\cdots,x_s)$ can be separably generated over K. Let $t_1,\cdots,t_n$ be a separable basis of

3) Every finite subset can be embedded in a separable subalgebra of finite order.

S and $L=K(t_1,\cdots,t_n)$. Then there exists a finite extension field G of K such that $L(x_{s+1})\otimes G$ is not semi-simple, (see the proof of the theorem 9 in [15]). Hence K-w. dim $\Lambda\geqq K$-w. dim$L(x_{s+1})=G$-w. dim$(L(x_{s+1})\otimes G)=\infty$, ([6], Ch. IX, Coro. 7.2).

PROPOSITION 5. *Let Λ be a field over L of transcendental degree $n<\infty$. Then w. dim $\Lambda=n$ if and only if Λ is locally separably generated.*[4]

Proof. If Λ is locally separably generated, there exists, for any elements $\lambda_1,\cdots,\lambda_n$, a separably generated extension $F_{\{\lambda_1,\dots\lambda_n\}}$ containing λ_i. The proposition 2 implies w. dim $F_{\{\lambda_1\dots\lambda_n\}}=$the transcendental degree of $F_{\{\lambda_1\dots\lambda_n\}}$, hence w. dim$\Lambda\leqq n$ by the proposition 1. On the other hand, w. dim $\Lambda\geqq n$ is an immediate consequence of the proposition 2 and the lemma 4.

Conversely if $\{\lambda_1,\cdots,\lambda_m\}$ is any sub-set of Λ, then $K(\lambda_1,\cdots,\lambda_n)$ has a separable basis by the proposition 4 and the lemma 4. Hence Λ is locally separably generated.

COROLLARY *Let Λ be a field over K of transcendental degree $n<\infty$. Then if w. dim $\Lambda>n$, w. dim $\Lambda=\infty$.*

This is clear from the lemma 4 and the propositions 4 and 5. From above propositions we obtain

THEOREM 4. *Let Λ be a field over K.*

If w.dim $\Lambda=n<\infty$, Λ is a locally separably gederated field of transcendental degree n.

If w.dim $\Lambda=\infty$, we have either case a) or b):

a) *Λ is of finite transcendental degree over K and is not locally separably generated,*

b) *Λ is of infinite transcendental degree over K. Further the converse holds.*

2. R-separable algebras.

We shall always consider algebras over a fixed field K.

DEFINITION 2. *Let Λ be an algebra over K. Λ is called R-separable if $\Lambda^e=\Lambda\otimes\Lambda^*$ is regular, i.e. w. dim$\Lambda=0$.*

We obtain immediatly the following theorem from the theorems 1 and 2, and the remark of the theorem 3 and [10], Th. 5.

THEOREM 5. Let *Λ and Γ be algebras over K. Then $\Lambda\otimes\Gamma$ is R-separable if and only if Λ and Γ are R-separable. If Λ is R-separable, then $\Lambda\otimes\Gamma$ is regular if and only if so is Γ.*

PROPOSITION 6. *Let e be an idempotent of Λ. If Λ is R-separable then $e\Lambda e$ is R-separable and any homomorphic image of Λ is so.*

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4) A field is locally separably generated if every finite subset can be embedded in a finitely separably generated extension of K.

This is clear by definitions.

We shall make conveniently the following definition.

DEFINITION 3. *We call "Λ has the property E" if $\Lambda\otimes L$ is regular for any extension field L of K.*

PROPOSITION 7. *If Λ is an R-separable algebra, then it has the property E.*

It is an immediate consequence of the corollary of the theorem 1.

From the remark of the lemma 4 we obtain the converse of the proposition 7 under special assumptions.

PROPOSITION 8. *Let Λ be a directly indecomposable algebra over K. If Λ has the property E and Λ is of finite degree over its center, then Λ is R-separable.*

LEMMA 8. *Let Λ be a semi-simple algebra over K (in the sense of Jacobson [12]). We assume that Λ is a sub-direct sum of R-separable algebras. Then $\Lambda\otimes\Gamma$ is semi-simple for any regular algebra Γ. Next if Σ is a semi-simple algebra which is a sub-direct sum of primitive algebra with one sided minimal ideals, and further we assume Σ has the property E, then $\Sigma\otimes\Delta$ is semi-simple for any semi-simple algebra Δ.*

Proof. By the assumption there exist two sided ideals $\mathfrak{a}_\alpha$ such that $\Lambda/\mathfrak{a}_\alpha$ are R-separable and that $\bigcap_\alpha \mathfrak{a}_\alpha=(0)$.

$\mathfrak{a}_\alpha\otimes\Gamma$ are two sided ideals of $\Lambda\otimes\Gamma$, and since $\Lambda/\mathfrak{a}_\alpha\otimes\Gamma$ are regular by the theorem 5, it is semi-simple. On the other hand $\bigcap_\alpha(\mathfrak{a}_\alpha\otimes\Gamma)=(0)$. Therefore $\Lambda\otimes\Gamma$ is semi-simple. Next let $\bar{\Sigma}$ be any primitive image with one sided minimal ideals of Σ and A be its associated division algebra (see [13], Ch. IV) with center Z. By the assumption $\bar{\Sigma}\otimes Z$ is regular, and $(\bar{e}\otimes 1)(\bar{\Sigma}\otimes Z)(\bar{e}\otimes 1)\approx A^*\otimes Z$ is regular, where $\bar{e}$ is an idempotent of $\bar{\Sigma}$ such that $e\bar{\Sigma}e\approx A^*$. Since $Z\otimes Z$ is the center of $A^*\otimes Z$, it is regular, hence Z is algebraic separable by the theorem 4 and $\bar{\Sigma}\otimes\Delta$ is semi-simple by [11], Lemma 5. Therefore it follows by the similar reason above mentioned that $\Sigma\otimes\Delta$ is semi-simple.

PROPOSITION 9. *Let Λ be a commutative algebraic algebra over K. If Λ has the property E, then Λ is locally separable.*

Proof. Since all primitive images are fields by the assumption, Λ^e is semi-simple by the lemma 8. Moreover since Λ^e is commutative algebraic, for any finite elements $x_i (i=1,\cdots,m)$ of Λ, $[K[x_i]:K]<\infty$ and $K[x_i]\otimes K[x_i]$ is a semi-simple algebra with minimum conditions, hence $K[x_i]$ is separable.

PROPOSITION 10. *If Λ is an integral K-algebra and has the property E, then Λ is locally separable.*

Proof. Let Λ' be the field of quotients of Λ, and L any extension field of K. Since $\sum\frac{\mu_i}{\lambda_i}\otimes l_i=\sum\frac{\mu_i'}{\lambda}\otimes l_i$ in $\Lambda'\otimes L$, there exists an element $\sum\nu_j\otimes l_j'$ of $\Lambda\otimes L$ such that $(\Sigma\mu'_i\otimes l_i)(\Sigma\nu_j\otimes l'_j)(\Sigma\mu'_i\otimes l_i)=(\Sigma\mu'_i\otimes l_i)$. Hence $(\Sigma\frac{\mu'_i}{\lambda}\otimes l_i)(\Sigma\lambda\nu_j\otimes l'_j)(\Sigma\frac{\mu'_i}{\lambda}\otimes l_i)=\Sigma\frac{\mu'_i}{\lambda}\otimes l_i$. Therefore since Λ' has the property E, Λ' is R-separable. By the theorem 4 and the proposition 9 Λ is locally separable.

COROLLARY. *An integral R-separable algebra is locally separable.*

PROPOSITION 11. *If Λ has the property E, then the tensor product of its center Z and itself is semi-simple.*

Proof. By the assumption Λ is regular, and so Z is regular, too. Therefore Z is a subdirect sum of fields L_α. Since $Z\otimes L_\alpha$ is the center of $\Lambda\otimes L_\alpha$, $Z\otimes L_\alpha$ is regular. Hence $L_\alpha\otimes L_\alpha$ is regular, which proves the proposition by the lemma 8.

PROPOSITION 12. *Let Λ be an algebra with minimum or maximum conditions. If Λ has the property E, then its center is a direct sum of algebraic separable fields, and Λ^e is semi-simple.*

Proof. By the assumptions Λ has minimum conditions, hence $\Lambda=(D_1)_{n_1}\oplus\cdots\oplus(D_m)_{n_m}$, where D_i are division rings. Since D_i have the property E, their center Z_i are all algebraic separable by the theorem 4. Therefore, since $Z_i\otimes Z_j$ are semi-simple, we have the proposition by [11], Lemma 4.

EXAMPLES: 1. Let $\tilde{\Lambda}$ be the algebra of all column-finite matrices over an R-separable algebra Λ_0 of degree M, and let $\bar{\Lambda}$ be the algebra of all finite matrices. Then the algebra Λ generated by $\bar{\Lambda}$ and $\Lambda_0\cdot 1$ in $\tilde{\Lambda}$ is R-separable. Because, any finite sub-set of Λ is contained in a sub-algebra $\Lambda'=\Lambda_1+\Lambda_0\cdot 1$ where Λ_1 is the sub-algebra of all matrices whose all but fixed finite components are zero.

Since $\Lambda'\approx(\Lambda_0)_n\oplus\Lambda_0$, Λ' is R-separable by the theorems 1 and 2, hence Λ is R-separable.

2. Let π be a locally finite group. The group algebra $\Lambda=K(\pi)$ is a supplemented algebra with the augmentation map $\varepsilon:\Lambda\to K$ given by $\epsilon x=1$ for all $x\in\pi$. We assume that $r\cdot K=K$ for order r of any element of π. Then we can easily see that Λ is locally separable, hence R-separable. If we assume that $rK\neq K$ for order r of some central element, then Λ is not R-separable from [8], Th. 12 and the lemma 4. On the other hand if π is a free group, it follows from [6], Ch. X, 5, and the analogous theorem to [6], Ch. X, Th. 6. 1 that $\text{w.dim}\Lambda=\text{w.l.dim}_\Lambda K=1$.

REMARK. If Λ is a K-algebra with finite degree over K, and $\Lambda\otimes L$ is semi-simple (regular) for any algebraic extension field L of K, then $\Lambda\otimes\Lambda^*$ is semi-simple (regular). But if $[\Lambda:K]=\infty$, this is not true. For instance, a purely transcendental field $K(x)$ preserves regularity for algebraic extension fields of the coefficient field K, but $K(x)$ is not R-separable.

3. S-separable algebras.

We shall now define an algebra which has a weaker property than R-separable.

DEFINITION 4. *Let Λ be an algebra over K. Λ is called S-separable if and only if $\Lambda^e=\Lambda\otimes\Lambda^*$ is semi-simple (in the sense of Jacobson* [12]).

It is clear that R-separable algebras are all S-separable, and the following theorem shows that the converse is not true in general.

An argument of the proof of this theorem essentially owes to that of Amitsur [1], Lemma 1 J.

THEOREM 6. *Let $K(x_\alpha)$ be a purely transcendental field over K with finite or infinite indeterminantes x_α. If R is an algebra over K which has no nil ideals $\neq 0$, then $K(x_\alpha)\otimes R$ is semi-simple.*

Proof. If R has not unit element, then the algebra R' adjoined freely unit element to R has no nil ideals $\neq 0$ and R is an ideal of R'. Hence we may assume R has unit element. All elements $\neq 0$ of $K[x_\alpha]$ are not zero divisors in $K(x_\alpha)\otimes R$. Hence we have an isomorphism of $K(x_\alpha)\otimes R$ to the ring of quotients of $R[x_\alpha]$ with respect to $K[x_\alpha]$ (cf. [7], p. 80 Lemma 4). We shall denote this homomorphic image by $R^*[x_\alpha]$ and the Jacobson radical of a ring T by $J(T)$. We shall first show $J(R^*[x_\alpha])\cap R\leqq J(R)$. Let $r\in J(R^*[x_\alpha])\cap R$, then there exists a quasi-inverse element $f(x_\alpha)/k(x_\alpha)$ of r, where $f(x_\alpha)\in R[x_\alpha]$, $k(x_\alpha)\in K[x_\alpha]$, and

$$rk(x_\alpha)+f(x_\alpha)-rf(x_\alpha)=0.$$

From this equality we have d=total degree of $k(x_\alpha)\leqq$total degree of $f(x_\alpha)$. Comparing coefficients of a monomial of degree d of this equality we have

$$r+s-rs=0, \quad s\in R.$$

Since $J(R^*[x_\alpha])\cap R$ is an ideal of R, $J(R^*[x_\alpha])\cap R\leqq J(R)$.

Next we shall show $J(R^*[x_\alpha])\cap R$ is a nil ideal. Let $r\in J(R^*[x_\alpha])\cap R$, then $rx\in J(R^*[x_\alpha])$ where $x=x_1$. Hence there exists an element $f(x_\alpha)/t(x_\alpha)\in R^*[x_\alpha]$ such that $rx+f(x_\alpha)/t(x_\alpha)-rx\cdot f(x_\alpha)/t(x_\alpha)=0$, where $f(x_\alpha)\in R[x_\alpha]$, $t(x_\alpha)\in K[x_\alpha]$. As above we have m'=degree of $t(x_\alpha)$ on $x\leqq$degree of $f(x_\alpha)$ on $x=m$. Let

$$t(x_\alpha)=g_0(y)x^{m'-1}+\cdots+g_{m'}(y), \qquad g_i(y)\in K[x_2, x_3,\cdots].$$

From the above equality we obtain

$$f(x_\alpha)=rx\cdot f(x_\alpha)-rx\cdot t(x_\alpha).$$

Substitute $f(x_\alpha)$ on the right by the whole expression of the right-hand side of this equality. Repeating this process yields

$$f(x_\alpha)=(rx)^n\cdot f(x_\alpha)-(rx)^n t(x_\alpha)-(rx)^{n-1}t(x_\alpha)\cdots-rx\cdot t(x_\alpha).$$

If we replace n by $m+2$ and we compare the coefficients of degree $m+1$ on x

in the equality, then we obtain

$$r^{m'+1}\beta_0+\cdots+r^{m-m'+1}\beta_{m'}=0,$$

where β_j are coefficients of a fixed monomial in $g_{m'}(y)$ and β_j are not all zero. Hence r is algebraic over K and r is nilpotent by [13], p. 19, Th. 1. Therefore by the assumption, $J(R^*[x_\alpha])\cap R=(0)$. Now let $g(x_\alpha)\in J(R^*[x_\alpha])$ and $g(x_\alpha)=f(x_\alpha)/k(x_\alpha)$, $k(x_\alpha)\in K[x_\alpha]$, $f(x_\alpha)\in R[x_\alpha]$, then $g(x_\alpha)\cdot k(x_\alpha)\in J(R^*[x_\alpha])\cap R[X_\alpha]$, hence

$$J(R^*[x_\alpha])=(J(R^*[x_\alpha])\cap R[x_\alpha])\cdot R^*[x_\alpha].$$

In virtue of this equalty, it is sufficient for the proof of the theorem to prove $J(R^*[x_\alpha])\cap R[x_\alpha]=(0)$. First we assume the number of indeterminates is one. If $J(R^*[x])\cap R[x]\neq(0)$ there is a non zero polynomial $f(x)$ of minimal degree in it. Then $f(x)$ is not constant by the above. We have an automorphism of $R[x]$ sending $g(x)$ to $g(x+k)$, where $g(x)\in R[x]$, $k\in K$. Hence we obtain an automorphism of $R^*[x]$ by which $J(R^*[x])\cap R[x]$ is sent onto itself. Therefore $f(x)-f(x+k)\in J(R^*[x])\cap R[x]$. Since its degree is less than $f(x)$, we obtain $f(x)=f(x+k)$. If we represent $f(x)$ by using a basis u_i of R over K:

$$f(x)=\sum u_i g_i(x),\ g_i(x)\in K[x],$$

we have

$$g_i(x)=g_i(x+k).$$

If K is an infinite field we have immediately $g_i(x)\equiv$constant from this equality. Hence $f(x)$ is a constant, which is a contradiction. If K is a finite field of characteristic $p\neq 0$ we can easily prove by the induction on the degree of $f(x)$ that $f(x)\in R[x^p-x]$ (see [1], p. 356). Hence we may write $f(x)=h(x^p-x)$, $h(x)\in R[x]$. We shall now show that $f(x)\in J(R^*[x^p-x])$. Let $k(x)$ be any element of $f(x)\cdot R^*[x^p-x](\subset f(x)R^*[x]\subset J(R^*[x]))$, then $k(x)$ has a unique quasi-inverse $k'(x)$ in $R^*[x]$,

$$k(x)+k'(x)-k(x)k'(x)=0.$$

By using a mapping: $x\to x+1$ we obtain an automorphism of $R^*[x]$ and

$$k(x+1)+k'(x+1)-k(x+1)k'(x+1)=0.$$

Since $k(x)=k(x+1)$ has the unique quasi-inverse, we obtain

$$k'(x)=k'(x+1).$$

If we represent $k'(x)$ in terms of u_i:

$$k'(x)=\sum u_i\frac{f_i(x)}{t_i(x)},\qquad (f_i(x),\ t_i(x))=1,\qquad f_i(x),\ t_i(x)\in K[x],$$

then we obtain

$$\frac{f_i(x)}{t_i(x)}=\frac{f_i(x+1)}{t_i(x+1)} \quad \text{and } (f_i(x+1),\ t_i(x+1))=1.$$

From this equality we can easily see that

$$f_i(x)=(x+1),\ t_i(x)=t_i(x+1).$$

Hence as above $t_i(x)$, $f_i(x)\in K[x^p-x]$. Hence $k'(x)\in R^*[x^p-x]$, which proves $f(x)\in J(R^*[x^p-x])$. Finally by using a mapping: $x\to x^p-x$ we obtain an isomorphism of $R^*[x]$ to $R^*[x^p-x]$, and a inverse image of $f(x)$ is $h(x)$, and since $f(x)\in J(R^*[x^p-x])$ we have $h(x)\in J(R^*[x])$. But the degree of $h(x)$ is lower than $f(x)$, which is a contradiction. Now we shall prove the theorem in a general case. If $J(R^*[x_\alpha])\frown R[x_\alpha]\neq 0$ there exists a non zero polynomial $f(x_{\alpha'}, x_\beta)$ of minimum degree with respect to an indeterminate x_β. By using a mapping: $x_\beta\to x_\beta+g(x_{\alpha'})$ we obtain an automorphism of $R^*[x_\alpha]$, where $g(x_{\alpha'})\in K[x_{\alpha'}]$, and we have

$$f(x_{\alpha'},\ x_\beta)\equiv f(x_{\alpha'},\ x_\beta+g(x_{\alpha'})).$$

Hence since $K(x_{\alpha'})$ is an infinite field, we have a contradiction as above. This proves the theorem.

Corollary 1. *A purely transcendental field $K(x_\alpha)$ over K is S-separable, but not R-separable.*

This is an immediate consequence of Theorems 4 and 6.

Corollary 2. *Let Λ be an algebraic separable extension over a subfield Λ_0 and Λ_0 be purely transcendental over K. Then for any algebra R which has no nil ideal$\neq(0)$, $\Lambda\otimes R$ is semi-simple, and hence Λ is S-separable.*

Proof. It is clear that $\Lambda\otimes R=(\Lambda\underset{\Lambda_0}{\otimes}\Lambda_0)\otimes R=\Lambda\underset{\Lambda_0}{\otimes}(\Lambda_0\otimes R)$. From the theorem 6 $\Lambda_0\otimes R$ is semi-simple, and hence $\Lambda\underset{\Lambda_0}{\otimes}(\Lambda_0\otimes R)$ is semi-simple by the assumption and the lemma 8.

Proposition 13. *If Λ is finitely separably generated, $\Lambda\otimes R$ is semi-simple for any algebra which has no nil ideal$\neq(0)$. Conversely if $\Lambda'\otimes K^{p^{-1}}$ has no nilpotent elements, then Λ' is separable (not necessarily finitely generated) in the sence of Bourbaki* [5], *where p is the characteristic of K.*

Proof. The first part is clear from [5], p. 141 Th. 2 and the corollary 2. If $\Lambda\otimes K^{p^{-1}}$ has no nilpotent elements $\neq 0$ for any basis $\{b_\lambda\}$ of Λ', $\{b_\lambda^p\}$ is linearly independent over K. Otherwise we have

$$\sum b_i^p\alpha_i=0,\quad \alpha_1\neq 0,\quad \alpha_i\in K,$$

and hence $0\neq\sum b_i\otimes\alpha_i^{p^{-1}}\in(\Lambda'\otimes K^{p^{-1}})$ is nilpotent, which is a contradiction. Hence we obtain the proposition from [5], p. 129, Coro.

Remark. If K is a field of transcendental degree 1 over a perfect subfield P, and Λ is a S-separable extension field of K of finite transcendental degree, then Λ has a separating transcendental basis over K (see [14], p. 384, Coro.).

The corollary 1 and the following example show that S-seprable algebras are not necessarily algebraic.

Let Λ be a complete dircet sum of an infinite number of infinite fields K. Then we can easily show that Λ is not algebraic and is S-separable by the lemma 8.

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