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A note on Hattori's theorems

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Recently Hattori [3] has given a characterization of Prüfer rings. In this short note we shall show that this characterization is valid for a commatative ring with a slightly weaker property than that of an integral domain.

Let Λ be a commutative ring and S be a set of non zero divisors which possesses the following properties:

1) $S \ni 1$; 2) S is closed under multiplication.

We shall denote Λ_S the ring of quotients of Λ with respect to S. We shall call an element a of a Λ-module A a *torsion element* if $sa=0$ for some $s \in S$. The torsion elements form a submodule tA of A. We can define a *torsion-free module, divisible element*, etc. similarly to the case of an integral domain.

We can easily obtain the following results:

1) *we have an exact sequence*

$$0 \to tA \to A \to A_S = A \otimes_{\Lambda} \Lambda_S$$

for any Λ-module A,

2) *Λ_S is Λ-flat,*

3) *$w.\dim_{\Lambda} A_S = w.\dim_{\Lambda_S} A_S$ for any Λ-module A,*

4) *we have*

$$\{Tor_n^{\Lambda}(A, C)\}_S \approx Tor_n^{\Lambda_S}(A_S, C_S)$$

for any Λ-modules A and C. (See Cartan and Eilenberg [1], VII, *Exer's* 9 *and* 10).

From now on we shall always assume that Λ_S is *regular*.

For instance, an integral domain has this property. Let Λ be a complete direct sum of integral domains, then Λ_S is regular, where S is the set of non-zero divisors.

PROPOSITION 1. *Let A be a torsion-free, divisible Λ-module. Then $w.\dim_{\Lambda} A = 0$.*

Proof. If A is torsion-free and divisible, we can regard A as a Λ_S-module by the following definition.

For $\frac{\mu}{\lambda} \in \Lambda_S (\lambda \in s)$, and $a \in A$, there exists a unique element b in A such that $\lambda b = a$. We define $\frac{\mu}{\lambda} \cdot a = \mu b$. From equalities: $\frac{\mu}{\lambda} \cdot a = \mu\left(\frac{1}{\lambda} a\right) = \frac{1}{\lambda}(\mu a)$ we can prove that A is a Λ_S-module. By an inclusion mapping: $\Lambda \to \Lambda_S$ and 2) we have

$$\text{w.}\dim_\Lambda A \leqq \text{w.}\dim_{\Lambda S} A=0$$

by the author [2], Th. 5.

PROPOSITION 2. *For any Λ-modules A and C $Tor_n^\Lambda(A,C)$ is a torsion module* ($n\geqq 1$). (Cf. [3], Th. 1).

Proof. We obtain from 4)

$$\{\mathrm{Tor}_n^\Lambda(A,C)\}_S \approx \mathrm{Tor}_n^{\Lambda_S}(A_S, C_S)=0 \quad n\geqq 1.$$

Hence we have $t\{\mathrm{Tor}_n^\Lambda(A,C)\}=\mathrm{Tor}_n^\Lambda(A,C)$ by 1).

THEOREM. *If Λ_S is regular, we have the following equivalent properties:*

a) *Λ is semi-hereditary,*

b) *$Tor_2^\Lambda(A,B)=0$ for any Λ-modules A and B,*

c) *$Tor_1^\Lambda(X,A)=0$ for any Λ-module X, if A is torsion free,*

d) *$A\underset{\Lambda}{\otimes} B$ is torsion-free, if both A and B are torsion-free.* (Cf. [3], Th. 2).

Proof. a)→b). See [1], VI, Prop. 2.9.

b)→c). If A is torsion-free, we have an exact sequence

$$0\to A\to A_S\to A_S/A\to 0$$

from 1). From this sequence we obtain an exact sequence

$$\mathrm{Tor}_2^\Lambda(X, A_S/A)\to\mathrm{Tor}_1^\Lambda(X,A)\to\mathrm{Tor}_1^\Lambda(X,A_S)$$

for any Λ-module X. By b) and a fact: $\text{w.}\dim_\Lambda A_S=\text{w.}\dim_{\Lambda_S} A_S=0$ we obtain $\mathrm{Tor}_1^\Lambda(X,A)=0$.

c)→d). See the proof of [1], VII, Prop. 4.5.

For the last two arguments we shall repeat the same ones of [3] in order to make the proof plainly.

d)→c). From an exact sequence: $0\to R\to P\to X\to 0$, with P projective, of a Λ-module X, we obtain an exact sequence:

$$0\to\mathrm{Tor}_1^\Lambda(X,A)\to R\otimes A\,.$$

From d) and Proposition 2, we have $\mathrm{Tor}_1^\Lambda(X,A)=0$.

c)→a). We have a homomorphism σ:

$$\mathrm{Hom}_\Lambda(B,C)\underset{\Lambda}{\otimes}A\to\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(A,B),C)$$

by setting $\sigma(f\otimes a)(g)=f(g(a))$, $f\in\mathrm{Hom}\ (B,C)$, $g\in\mathrm{Hom}(A,B)$ and $a\in A$, where A, B and C are Λ-modules.

Let F be a finitely generated Λ-projective module and C be a Λ-injective module. From an exact sequence: $0\to M\to F\to A\to 0$, we have a commutative diagram

$$(*)\quad \begin{array}{ccccccc} 0\to\mathrm{Hom}\ (Q,C)\otimes M & \to & \mathrm{Hom}\ (Q,C)\otimes F & \to & \mathrm{Hom}\ (Q,C)\otimes A & \to 0 \\ \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma & \\ 0\to\mathrm{Hom}\ (\mathrm{Hom}(M,Q),C) & \to & \mathrm{Hom}(\mathrm{Hom}(F,Q),C) & \to & \mathrm{Hom}\ (\mathrm{Hom}(A,Q),C) & \to 0 \end{array}$$

for any Λ-module Q. Since the middle σ is isomorphic, the right hand σ is epimorphic. If further Q is injective, Q is divisible by [1], VII, Prop. 1.2, and hence Hom (Q, C) is torsion-free. Therefore, from c), the two rows of (*) are exact, and so the left hand σ is monomorphic. Let Y be any Λ-module and I be a finitely generated ideal of Λ. From exact sequences: $0\to Y\to Q\to R\to 0$, with Q injective of Y and $0\to I\to \Lambda\to \Lambda/I\to 0$, we have a commutative diagram

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
\mathrm{Tor}_1(\mathrm{Hom}(Y, C), I) & \to & \mathrm{Hom}(R, C)\otimes I & \to & \mathrm{Hom}((Q, C)\otimes I \\
 & & \downarrow\sigma & & \downarrow\sigma \\
\mathrm{Hom}(\mathrm{Ext}^1(I, Y), C) & \to & \mathrm{Hom}(\mathrm{Hom}(I, R), C) & \to & \mathrm{Hom}(\mathrm{Hom}(I, Q), C) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

with exact rows and columns, by replacing A by I and then M by I in (*). Since I is torsion free, $\mathrm{Tor}_1(\mathrm{Hom}(Y, C), I)=0$. Hence $\mathrm{Hom}(\mathrm{Ext}^1(I, Y), C)=0$ for any injective module C. Since $\mathrm{Ext}^1(I, Y)=0$ for any Λ-module Y, I is projective.

Bibliography

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