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Some remarks on E-sequences in Noetherian rings

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In this short note we shall always assume that a ring A is a commutative Noetherian ring with unit element and A-modules are unitary. We use [2] as a reference source for homological algebra. Let $\mathfrak{a}$ be an ideal of A and E be a finitely generated A-module with $\mathfrak{a}E \neq E$.

A sequence $\{a_1, \cdots, a_k\}$ of elements in $\mathfrak{a}$ is called an $\mathfrak{a}$-E sequence if

$$\mathfrak{a}_i E : (a_{i+1}) = \mathfrak{a}_i E, \quad \text{for} \quad i=0, \cdots, k-1,$$

where $\mathfrak{a}_i$ is the ideal generated by $a_1, \cdots, a_i$, and $\mathfrak{a}_0$ means the zero ideal of A. An $\mathfrak{a}$-E sequence $\{a_1, \cdots, a_k\}$ is called a maximal $\mathfrak{a}$-E sequence if any sequence $\{a_1, \cdots, a_k, a_{k+1}\}$ containing $\{a_1, \cdots, a_k\}$ can not be an $\mathfrak{a}$-E sequence.

We shall prove in this note the following theorem:

Let A be a commutative Noetherian ring with an ideal $\mathfrak{a}$, and E be a finitely generated A-module. Then every $\mathfrak{a}$-E sequence can be extended to a maximal $\mathfrak{a}$-E sequence and all maximal $\mathfrak{a}$-E sequences have the same length.

Auslander and Buchsbaum [1], and Serre [6] have proved this theorem in the case of local ring with maximal ideal $\mathfrak{m}$ by using the structure theorem of Cohen.

If a sequence $\{a_1, \cdots, a_k\}$ is an $\mathfrak{a}$-E sequence, from the definition and $\mathfrak{a}E \neq E$ $\mathfrak{a}_i \subsetneqq \mathfrak{a}_{i+1}, (i=0, \cdots, k-1)$. Hence the length of every $\mathfrak{a}$-E sequence is finite. The following arguments are modifications of the section 1 of [5].

LEMMA 1. *Let $\{a_1, \cdots, a_i, a_{i+1}, a_{i+2}\}$ be an $\mathfrak{a}$-E sequence. Then $\{a_1, \cdots, a_i, a_{i+2}, a_{i+1}\}$ is an $\mathfrak{a}$-E seqnence if and only if*

$$(a_1, \cdots, a_i)E : (a_{i+2}) = (a_1, \cdots, a_i)E.$$

See the lemma 1.1 of [5].

LEMMA 2. *Let $\mathfrak{a}$ and $(\mathfrak{a} \supseteq)\mathfrak{b}$ be two ideals of A and let a, a' be two elements of $\mathfrak{a}$ which satisfy the conditions:*

$$\mathfrak{b} \cdot E : (a) = \mathfrak{b} \cdot E : (a') = \mathfrak{b} \cdot E.$$

Then the $A/\mathfrak{a}$-modules $\{(\mathfrak{b}, a)E : \mathfrak{a}\}/(\mathfrak{b}, a)E$ and $\{(\mathfrak{b}, a')E : \mathfrak{a}\}/(\mathfrak{b}, a')E$ are isomorphic.

Proof. By replacing E by $E/\mathfrak{b} \cdot E$, we can reduce the proof to the case $\mathfrak{b} \cdot E = (0)$, in which case a, a' are not zero-divisors in E. Let G be the set of elements in A which are not zero-divisors in E. We denote by E^* the module of quotients of E

with respect to G.

If we denote by $(E:\mathfrak{a})_{E^*}$ the subset of elements x of E^* which satisfy a condition $\mathfrak{a}x \subseteqq E$, $(E:\mathfrak{a})_{E^*}$ is an A-module. For any element x of $(E:\mathfrak{a})_{E^*}$, $ax=z \in E$ and $\mathfrak{a}z=\mathfrak{a}\cdot ax \subseteqq aE$, hence $z \in (aE:\mathfrak{a})$. Therefore the mapping: $x \to z$ induces an isomorphism of $(E:\mathfrak{a})_{E^*}$ onto $(aE:\mathfrak{a})$. From this isomorphism we obtain $(E:\mathfrak{a})_{E^*}/E \approx (aE:\mathfrak{a})/aE$. Similarly we can obtain an isomorphism $(E:\mathfrak{a})_{E^*}/E \approx (a'E):\mathfrak{a}/a'E$, which proves the lemma.

The following property (a) is well known.

Let E be a finitely generated A-module. And $\mathfrak{p}_i$ be the ideals belonging to a A-submodule F of E. For any ideal $\mathfrak{a}$ of A, $(F:\mathfrak{a})=F$ if and only if $\mathfrak{a}$ is not contained in any $\mathfrak{p}_i$.

LEMMA 3. *Let $\{a_1, a_2, \cdots, a_k\}$, $\{b_1, b_2, \cdots, b_k\}$ be two $\mathfrak{a}$-E sequences of the same length. Then the $A/\mathfrak{a}$-modules $((a_1, \cdots, a_k)E:\mathfrak{a})/(a_1, \cdots, a_k)E$ and $((b_1, \cdots, b_k)E:\mathfrak{a})/(b_1, \cdots, b_k)E$ are isomorphic.*

We can prove the lemma similarly to [5], Th. 1.3 by Lemmas 1 and 2, and the above property (a).

Proof of Theorem. Let $\{a_1, \cdots, a_k\}$ and $\{b_1, \cdots, b_t\}$ be two $\mathfrak{a}$-E sequences. If $k<t$ we have an isomorphism

$$(*) \qquad \{(a_1, \cdots, a_k)E:\mathfrak{a}\}/(a_1, \cdots, a_k)E \approx \{(b_1, \cdots, b_k)E:\mathfrak{a}\}/(b_1, \cdots, b_k)E$$

by Lemma 3. Since $\{b_1, \cdots, b_k, b_{k+1}\}$ is an $\mathfrak{a}$-E sequence, the right side of (*) is equal to (0). Hence $(a_1, \cdots, a_k)E:\mathfrak{a}=(a_1, \cdots, a_k)E$. From the property (a), therefore, we can find an element a_{k+1} in $\mathfrak{a}$ such that $\{a_1, \cdots, a_k, a_{k+1}\}$ is an $\mathfrak{a}$-E sequence. Thus we can obtain the theorem.

PROPOSITION 1. *Let A be a MC-ring*[1] *and $\mathfrak{a}$ be an ideal of A. Then the length of the maximal $\mathfrak{a}$-A sequences is equal to the rank of $\mathfrak{a}$ (in the sense of Krull).*

Proof. If the rank of $\mathfrak{a}$ is equal to r, there exist elements $x_i (1 \leqq i \leqq r)$ in $\mathfrak{a}$ such that rank $(x_1, \cdots, x_i)=i$ $(1 \leqq i \leqq r)$. Since A is a MC-ring, and rank $\mathfrak{a}$=rank $(x_1, \cdots, x_r)$, $\{x_1, \cdots, x_r\}$ is a maximal $\mathfrak{a}$-A sequence.

REMARK. We can obtain [6], Th. 2, [1], Prop. 3.4, and results of section 1 of [5], if we replace $\mathfrak{a}$ by the maximal ideal $\mathfrak{m}$ of a local ring, and E by A, respectively. From Proposition 1 we obtain [1], Prop. 2.7 and a fact that a MC-ring is a U-ring (see [4], p. 36).

The following lemma is stated in [3].

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1) A Noetherian ring A is called a MC-ring, (see [5]), if A satisfies a condition: if $(a_1, \cdots, a_r)$ is an ideal of rank r with a basis of r elements, then all prime ideals of $(a_1, \cdots a_r)$ have rank r.

It is clear that a regular local ring is a MC-ring.

LEMMA 4. *Let E be an A-module and $\{a_1, \cdots, a_k\}$ be an $\mathfrak{a}$-E sequence. For any integer $n \geqq 0$, we have*

$$Tor_m^A(A/\mathfrak{a}, E)=0 \quad \text{for all} \quad m>n$$

if and only if

$$Tor_{m'}^A(A/\mathfrak{a}, E/\mathfrak{a}_k E)=0 \quad \text{for all} \quad m'>n+k.$$

In this case we have

$$Tor_{n+k}^A(A/\mathfrak{a}, E/\mathfrak{a}_k E) \approx Tor_n^A(A/\mathfrak{a}, E).$$

Hence we have $dim_A\ A/\mathfrak{a} \geqq k$, where $\mathfrak{a}_k$ is an ideal genreated by $a_1, \cdots, a_k$.

We can easily obtain the lemma by the induction on k and the same method as that of the proof of [6], Prop. 3.

If $\{x_1, \cdots, x_n\}$ is a maximal $\mathfrak{a}$-E sequence, then there exists some prime dieal $\mathfrak{p}$ belonging to $(x_1, \cdots, x_n)E$ such that $\mathfrak{p} \supseteqq \mathfrak{a}$. Hence $\mathfrak{p}$ contains some prime ideal $\mathfrak{p}'$ belonging to (0) in E. We denote by $R_\mathfrak{p}$ the ring of quotients of R with respect to $R-\mathfrak{p}$. Then the functor $\otimes R_\mathfrak{p}$ is exact and $E \otimes R_\mathfrak{p} = E_\mathfrak{p}$ does not vanish from [2], VII, Exer. 10 and [1], Lemma 1.1. Since $\mathfrak{p}_\mathfrak{p} \supseteq \mathfrak{a}_\mathfrak{p}$, $\mathfrak{a}_\mathfrak{p} E_\mathfrak{p} \neq E_\mathfrak{p}$. By the same method as that of [1], Prop. 3.3 we can prove that $\{x_1, \cdots, x_n\}$ is also a maximal $\mathfrak{a}_\mathfrak{p}$-$E_\mathfrak{p}$ sequence. Hence we have

PROPOSITION 2. *$codim_\mathfrak{a} E = codim_{\mathfrak{a}_\mathfrak{p}} E_\mathfrak{p} \leqq dim_{R_\mathfrak{p}}\ R_\mathfrak{p}/\mathfrak{a}_\mathfrak{p} \leqq dim_R R/\mathfrak{a}$ and $codim_\mathfrak{a} E = codim_{\mathfrak{a}_\mathfrak{p}} E_\mathfrak{p} \leqq codim_{\mathfrak{p}_\mathfrak{p}} E_\mathfrak{p} \leqq dim\ R_\mathfrak{p} \leqq dim\ R$, where $\mathfrak{p}$ is some prime ideal such that $E_\mathfrak{p} \neq 0$, $\mathfrak{a}_\mathfrak{p} E_\mathfrak{p} \neq E_\mathfrak{p}$ and $codim_\mathfrak{a} E$ is the length of maxinal $\mathfrak{a}$-E sequences.*

If A is a regular local ring and E is a finitely generated A-projective module, and if we replace $\mathfrak{a}$ by the maximal ideal $\mathfrak{m}$ in Proposition 2, then the equalities in Proposition 2 hold by [6]. Coro. 1 of Th.1.

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