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On Kronecker products of primitive algebras

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In this note we shall prove some supplementary results of Jacobson [5] concerning Kronecker products of primitive algebras and those of P.M.I. algebras (that is, algebras with faithful minimal one sided ideals) and study their applications.

Let A_i $(i=1, 2)$ be a primitive algebra over a field Φ and Δ_i be the division algebra of all A_i-endomorphisms of a faithful irreducible A_i-module (if A_i is a P.M.I. algebra, Δ_i is uniquely determined up to isomorphisms, and we shall call it the associated division algebra (denoted by *A.D.*) of A_i).

In section 1 we consider relations between semi-simplicity and primitivity of $A_1 \otimes A_2$ and those of $\Delta_1 \otimes \Delta_2$. In section 2, using results of section 1, for P.M.I. algebra A_i we study properties of Δ_i when $A_1 \otimes A_2$ is primitive or P.M.I., and give for a P.M.I. algebra A conditions under which $A \otimes A^*$ is primitive or P.M.I.. Further we prove that if B is central simple and $A \otimes B$ is P.M.I., then A is P.M.I. under special conditions. In section 3 we study the same problems as in section 2 in the case where primitivity is replaced by semi-simplicity. In section 4 we study Kronecker products of strongly dense algebras (see definition of section 4) and of closed irreducible algebras.

Throughout this note, we assume that algebras are all over a fixed ground field Φ, endomorphisms of right (left) A-module M act on the right (left) side of M, and that A^* means an anti-isomorphic algebra of an algebra A.

1. LEMMA 1. *Let A be a ring and e be an idempotent of A. If A is primitive (semi-simple in the sense of Jacobson [6]) then eAe is primitive (semi-simple). Further we assume that A is primitive, then A is a P.M.I. ring if and only if eAe is so.*

Proof. The first half is well known (cf. [5], Ch. 3, Pr. 7.1). Let A be a P.M.I. ring with the non zero socle $\mathfrak{S}$ and $\mathfrak{l}$ be a minimal left ideal of A such that $\mathfrak{l}e \neq 0$ for $\mathfrak{S}e \neq 0$, and $\mathfrak{l}e$ is a faithful minimal left ideal of A. For any non zero element exe of $e\mathfrak{l}e$, $eAe \cdot exe = e\mathfrak{l}e$ and $e\mathfrak{l}e$ is a faithful minimal left ideal of eAe, hence eAe is a P.M.I. ring. Conversely if A is primitive and eAe is a P.M.I. ring, then eAe has an idempotent e_0 such that e_0Ae_0 is a division ring, hence A is a P.M.I. ring.

PROPOSITION 1. *Let A be a right primitive algebra with a faithful irreducible module M and Δ be the associated division algebra of M, and let B be an algebra* *with unit element. If $\Delta\otimes B^*$ is left primitive then $A\otimes B$ is right primitive and the associated division algebra of any faithful irreducible $\Delta\otimes B^*$-module N is anti-isomorphic to the one of a faithful irreducible $A\otimes B$-module depending on N. Further we assume that A is a P.M.I. algebra, then $A\otimes B$ is a P.M.I. algebra if and only if $\Delta\otimes B^*$ is a P.M.I. algebra.*

Proof. Let $\{x_\tau\}_{\tau\in I}$ be a basis of M over Δ. Then $M\otimes B=\sum\oplus(x_\tau\otimes 1)(\Delta\otimes B^*)$ and we can easily see that $A\otimes B$ is a dense ring in the finite topology in the ring $\mathfrak{M}_I(\Delta^*\otimes B)$ of $\Delta\otimes B^*$-endomorphisms of $M\otimes B$ (cf. Azumaya and Nakayama [2], Th. 8). Since $M\otimes B$ is $\Delta\otimes B^*$-free, the lattice of left ideals of $\Delta\otimes B^*$ is isomorphic to the lattice of $\mathfrak{M}_I(\Delta^*\otimes B)$-submodules of $M\otimes B$, hence of $A\otimes B$-submodules of $M\otimes B$, ([2], Lemma 1). If $\Delta\otimes B^*$ is left primitive, there exists a modular maximal left ideal $\mathfrak{l}$ such that $(\mathfrak{l}:\Delta\otimes B^*)=0$. Hence $M\otimes B$ has a maximal $A\otimes B$-submodule $(M\otimes B)\mathfrak{l}$. Then $M\otimes B/(M\otimes B)\mathfrak{l}\cong\sum\oplus(x\otimes 1)(\Delta\otimes B^*/\mathfrak{l})$ is a faithful irreducible $\mathfrak{M}_I(\Delta^*\otimes B)$-module and $A\otimes B$ is a primitive algebra with a faithful irreducible module $M\otimes B/(M\otimes B)\mathfrak{l}$. Therefore the associated division algebra of $\Delta\otimes B^*/\mathfrak{l}$ is anti-isomorphic to the one of $M\otimes B/(M\otimes B)\mathfrak{l}$. The last statement is easily obtained from Lemma 1 and the first half statement.

We note that if A is a primitive algebra with a central associated division algebra Δ of a faithful irreducible module, $\sum$ is a subalgebra of Δ, and if Γ is the centralizer of $\sum$ in Δ, then observing that $\Delta\otimes\sum^*$ is a primitive algebra with the associated division algebra Γ^* of $\Delta\otimes\sum^*$-module Δ, $A\otimes\sum$ is primitive with an associated division algebra Γ. In particular if we replace $\sum$ by a maximal subfield of Δ, then $\sum$ is a splitting field for A, (cf. [5], Ch. 5, Th.'s 12.2 and 3).

THEOREM 1. *Let A_i $(i=1, 2)$ be a right primitive algebra and Δ_i be the associated division algebra of a faithful irreducible A_i-module M_i. Then we have*

1) If $\Delta_1\otimes\Delta_2$ is left primitive, then $A_1\otimes A_2$ is right primitive and for any left faithful irreducible $\Delta_1\otimes\Delta_2$-module M there exists a right faithful irreducible $A_1\otimes A_2$-module M' such that the associated division algebra of M' is anti-isomorphic to the one of M.

Moreover we assume that A_i is a P.M.I. algebra. Then we have

2) $A_1\otimes A_2$ is primitive if and only if $\Delta_1\otimes\Delta_2$ is primitive,

3) $A_1\otimes A_2$ is a P.M.I. algebra if and only if $\Delta_1\otimes\Delta_2$ is a P.M.I. algebra.

Proof. 1) Let A_2' be an algebra which is added the unit operator over M_2 to A_2, and if $A_1\otimes A_2'$ is primitive, $A_1\otimes A_2$ is so for $A_1\otimes A_2$ is an ideal of $A_1\otimes A_2'$, and the associated division algebra of a faithful irreducible $A_1\otimes A_2'$-module M coincides with the one of the faithful irreducible $A_1\otimes A_2$-module M (Azumaya and Nakayama [1], Lemma 26.5). If $\Delta_1\otimes\Delta_2$ is left primitive, $\Delta_1^*\otimes\Delta_2^*$ is right primitive. Hence $\Delta_1\otimes A_2'^*$ is left primitive and $A_1\otimes A_2'$ is right primitive by Proposition 1.

We have the same argument for the associated division algebras. 2) and 3) are clear by 1) and Lemma 1.

If we repeat the above argument to semi-simplicity, we have

LEMMA 2. *Let A be a primitive algebra and Δ be the associated division algebra of a faithful irreducible module M and let B be an algebra with unit element. If $\Delta\otimes B^*$ is semi-simple then $A\otimes B$ is so.*

Proof. We use the notations in the proof of Proposition 1. Since $\Delta\otimes B^*$ has unit element, there exists a maximal left ideal $\mathfrak{l}$ of $\Delta\otimes B^*$ and so a maximal right $A\otimes B$-module $\tilde{\mathfrak{l}}=\sum\oplus(x_\tau\otimes 1)\,\mathfrak{l}$ of $M\otimes B$ corresponds to $\mathfrak{l}$. $M\otimes B/\tilde{\mathfrak{l}}$ is an irreducible $A\otimes B$-module and if $\mathfrak{a}_{\mathfrak{l}}$ is the kernel of homomorphism of $A\otimes B$ to the ring of $\Delta\otimes B^*$-endomorphisms of $M\otimes B/\tilde{\mathfrak{l}}$, then $A\otimes B/\mathfrak{a}_{\mathfrak{l}}$ is primitive and $(M\otimes B)\cdot\mathfrak{a}_{\mathfrak{l}}\subseteqq\tilde{\mathfrak{l}}$. If $\Delta\otimes B^*$ is semi-simple, the intersection $\cap\,\mathfrak{l}$ of all maximal left ideals $\mathfrak{l}$ is zero, hence $0=\sum\oplus(x_\tau\otimes 1)\cdot(\cap\,\mathfrak{l})\supseteqq\bigcap_{\mathfrak{l}}(M\otimes B).\cdot\mathfrak{a}_{\mathfrak{l}}\supseteqq(M\otimes B)\cdot(\bigcap_{\mathfrak{l}}\mathfrak{a}_{\mathfrak{l}})$ and $\bigcap_{\mathfrak{l}}\mathfrak{a}_{\mathfrak{l}}=0$, that is, $A\otimes B$ is semi-simple.

THEOREM 2. *Let A_1, A_2 be primitive algebras and Δ_1, Δ_2 be as in Theorem 1. If $\Delta_1\otimes\Delta_2$ is semi-simple, then $A_1\otimes A_2$ is so, and further if A_1, A_2 are P.M.I. algebras, the converse holds.*

We can prove the theorem by Lemma 2 and the same way as in the proof of Theorem 1.

2. We shall apply results in section 1 to Kronecker products of P.M.I. algebras. First we have the following theorem whose first half is the converse of [5], Ch. 5, Th. 10.1.

THEOREM 3. *Let A_i $(i=1, 2)$ be a P.M.I. algebra and Δ_i be A.D. and let Σ_i be the center of Δ_i.*
If $A_1\otimes A_2$ is a P.M.I. algebra, then we have

1) *Σ_1 or Σ_2 is algebraic over Φ,*
2) *$\Delta_1\otimes\Delta_2$ satisfies the minimum condition,*
3) *there are isomorphisms φ_1, φ_2 such that $\Sigma_1^{\varphi_1}$, $\Sigma_2^{\varphi_2}$ are linearly disjoint over Φ.*

Concersely if 2) and 3) hold, then $A_1\otimes A_2$ is a P.M.I. algebra.

Proof. If $A_1\otimes A_2$ is a P.M.I. algebra, $\Delta_1\otimes\Delta_2$ is a P.M.I. algebra by Theorem 1. Since $\Sigma_1\otimes\Sigma_2$ is the center of $\Delta_1\otimes\Delta_2$ and $\Delta_1\otimes\Delta_2$ has the unique minimal ideal, $\Sigma_1\otimes\Sigma_2$ is integral and has a minimal ideal, hence $\Sigma_1\otimes\Sigma_2$ is a field. If Σ_1 and Σ_2 are not algebraic, they contain transcendental fields isomorphic to $\Phi(X)$, hence $\Sigma_1\otimes\Sigma_2$ is not a field by [1], Lemma 36.4. Since $\Sigma_1\otimes\Sigma_2$ is a field, $\Delta_1\otimes\Delta_2$ is a simple ring, hence $\Delta_1\otimes\Delta_2$ satisfies the minimum condition. From 1) we may assume

$\sum_1$ is algebraic and we can find an isomorphism φ of $\sum_1$ into an algebric closure of $\sum_2$ and $\sum_1^\varphi$, $\sum_2$ are linearly disjoint over Φ for $\sum_1 \otimes \sum_2 \simeq \sum_1^\varphi \otimes \sum_2$. Conversely if 3) holds, $\sum_1 \otimes \sum_2$ is integral, hence $\Delta_1 \otimes \Delta_2$ is simple by 2) and [5], Ch. 5, Th. 9.1 and $\Delta_1 \otimes \Delta_2$ is a P.M.I. algebra. From Theorem 1 $A_1 \otimes A_2$ is P.M.I..

By using the same argument as in Theorem 3 we obtain,

COROLLARY 1. *Let A_1, $\sum_1$ be as in Theorem 3 and further we assume $\sum_1$ is a algebraic over Φ, then $A_1 \otimes A_2$ is primitive if and only if 3) holds.*

COROLLARY 2. *Let A_i $(i=1, 2)$ be a P.M.I. algebra and $\mathfrak{S}_i$ be its socle. If $A_1 \otimes A_2$ is a P.M.I. algebra, then $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ is its socle. If $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ is a P.M.I. algebra then $A_1 \otimes A_2$ is a P.M.I. algebra with socle $\mathfrak{S}_1 \otimes \mathfrak{S}_2$.*

We can easily obtain Corollary 2 from 2) of Theorem 3, 3) of Theorem 2 and [5], Ch. 5, Th. 10.1.

We shall remark that conditions 2) and 3) of Theorem 3 are independent each other and they coincide with a condition that $\Delta_1 \otimes \Delta_2$ is a simple algebra with the minimum condition.

THEOREM 4. *Let A be a P.M.I. algebra and Δ be A.D. with center $\sum$. The following properties are equivalent.*

a) *Δ is a central division algebra with finite rank over Φ.*

b) *$A \otimes A^*$ is a P.M.I. algebra.*

c) *$A \otimes B$ is a P.M.I. algebra for any P.M.I. algebra B.*

In this case if C contains unit element and $A \otimes C$ is a P.M.I. algebra, then C is P.M.I.. Further we assume, $\sum$ is algebraic over Φ, then the following properties are equivalent.

a′) *Δ is central.*

b′) *$A \otimes A^*$ is primitive.*

c′) *$A \otimes B$ is primitive for any primitive algebra B.*

d′) *A^Σ is primitive.*

Proof. a)→b), c) are clear from Theorem 1 and c)→b) is obvious. If $A \otimes A^*$ is a P.M.I. algebra, $\Delta \otimes \Delta^*$ is so, and $\sum \otimes \sum$ is a field by the remark of Theorem 3, hence $\sum = \Phi$ by [1], Lemma 34.6. Further $\Delta \otimes \Delta^*$ satisfies the minimum condition, hence $[\Delta : \Phi] < \infty$ by [1], Th. 34.9. If $A \otimes C$ is a P.M.I. algebra, and C contains unit element, $\Delta \otimes C^*$ is so by Proposition 1. Since Δ is central and $[\Delta : \Phi] < \infty$, C is P.M.I. from Proposition 2 below. If A^Σ is primitive, then Δ^Σ is so by Theorem 1 and as above $\sum \otimes \sum$ is a field, therefore Δ is central. The remaining statements are clear by Theorem 1.

Next we shall study some properties of A_i when $A_1 \otimes A_2$ is P.M.I. and A_1 is central simple.

LEMMA 3. *Let A be an algebra without unit element and B be a central simple* *algebra with unit element, and let A' be an algebra which is added unit element by the natural way. If $A\otimes B$ is a P.M.I. algebra then $A'\otimes B$ is so.*

Proof. We can easily see that if A has no unit element, any non zero ideal $\mathfrak{a}$ of A' has non zero intersection with A[1]. If $A\otimes B$ is a P.M.I. algebra, $A\otimes B$ has a faithful irreducible right ideal $\mathfrak{r}$ and further $\mathfrak{r}$ is a faithful irreducible $A'\otimes B$-module, as $\mathfrak{r}(A'\otimes B)\subseteqq\mathfrak{r}$ and the annihilator ideal (of $A'\otimes B$ over $\mathfrak{r}$) $=\mathfrak{a}_0\otimes B$ where $\mathfrak{a}_0$ is the annihilator ideal of A' over $\mathfrak{r}$ and if $\mathfrak{a}_0\neq(0)$, $\mathfrak{a}_0\cap A\neq(0)$ by the above remark, and it is a contradiction. Hence $A'\otimes B$ is a P.M.I. algebra.

PROPOSITION 2. *Let B be a central simple algebra with $[B:\Phi]<\infty$. A is a P.M.I. algebra if and only if $A\otimes B$ is so.*

Proof. "Only if" part is clear by Theorem 1. By Lemma 3 and $[B:\Phi]<\infty$ we may assume that A has unit element and B is a central division algebra. If we regard $A\otimes B$ as a right $A\otimes B$- and left B-module, that is, a right $(A\otimes B)\otimes B^*$-module, $A\otimes B$ is a faithful $(A\otimes B)\otimes B^*$-module as in the proof of Lemma 3. By the assumption and Theorem 1 $(A\otimes B)\otimes B^*$ is a P.M.I. algebra and $A\otimes B$ has a faithful irreducible $(A\otimes B)\otimes B^*$-module $\mathfrak{r}$. $\mathfrak{r}$ is a right ideal of $A\otimes B$ and a left B-module. $A\otimes B$ is a completely reducible two sided B-module with B-basis $\{u_i\}$; $u_i\cdot b=b\cdot u_i$ for all $b\in B$, hence $\mathfrak{r}=\sum_i\oplus v_iB$, $v_i\in A$ and $\mathfrak{r}=\mathfrak{r}_0\otimes B$ where $\mathfrak{r}_0$ is the right ideal generated by $\{v_i\}$ of A. Therefore $\mathfrak{r}_0$ is a faithful irreducible right ideal of A and A is a P.M.I. algebra.

COROLLARY. *Let B be as in Proposition 2. If $A\otimes B$ is a semi-simple algebra all whose primitive images are P.M.I. algebras, then A is so. Conversely if B' is central simple with unit element and A is semi-simple, then $A\otimes B'$ is semi-simple.*

We note that we may assume A contains unit element by the remark in the proof of Lemma 3 and if B' is a central algebra with unit element, the radical of $A\otimes B'$ is contained in the Kronecker products of the radicals of A and B'.

PROPOSITION 3. *Let A be I_1-algebra (see Levitzki [7]) and B be a central simple algebra with unit element. If $A\otimes B$ is a P.M.I. algebra, then A and B are matrix algebras of finite degree over division algebras.*

Proof. First we assume A has unit element. If $A\otimes B$ is a P.M.I. algebra, its socle $\mathfrak{S}=\mathfrak{z}\otimes B$ where $\mathfrak{z}$ is the unique minimal ideal of A. By the assumption $\mathfrak{z}$ is I_1-algebra and has no non zero nilpotent ideal, hence $\mathfrak{z}$ is primitive. Further A is prime by the assumption, hence A is primitive by Goldie [3], Th. 1. If A has no

1) Let $\mathfrak{a}$ be an ideal of $A'=A+1\cdot K$ and $a'=a+1\cdot k$ be a non zero element of $\mathfrak{a}$. If $ab+bk=0$ for all $b\in A$, $-ka^{-1}$ is a left unit of A, similarly if $ba+bk=0$ for all $b\in A$, $-ka^{-1}$ is a right unit.

unit element, $A'\otimes B$ is P.M.I. by Lemma 3 and its socle is contained in $A\otimes B$, hence A is primitive in this case, too. Therefore A is a matrix algebra of finite degree over a divition algebra $\varDelta$ by [7], Coro. in p. 391 and $\varDelta\otimes B^*$ is a simple P.M.I. algebra, hence $\varDelta\otimes B^*$ satisfies the minimum condition, which proves the proposition.

3. We now consider a semi-simplicity of Kronecker products of P.M.I. algebras.

LEMMA 4. *Let A_i $(i=1, 2)$ be a simple algebra with unit element and $\sum_i$ be its center. If $\sum_1\otimes\sum_2$ is semi-simple then $A_1\otimes A_2$ is so.*

Proof. Let N be the radical of $A_1\otimes A_2$, then $N=(A_1\otimes A_2)\cdot\mathfrak{a}$, by [5], Ch. 5 Th. 9.1 where $\mathfrak{a}$ is a ideal of $\sum_1\otimes\sum_2$. $(\sum_1\otimes\sum_2)\cap N$ is a quasi-regular ideal of $\sum_1\otimes\sum_2$ since for any element x of $(\sum_1\otimes\sum_2)\cap N$ there exists an element y in N such that $(1-x)\cdot(1=y)=(1-y)\cdot(1-x)=1$ and as $(1-x)\in\sum_1\otimes\sum_2$, $(1-y)\in\sum_1\otimes\sum_2$, and $y\in(\sum_1\otimes\sum_2)\cap N$. Since $(0)\ =(\sum_1\otimes\sum_2)\cap N\supseteqq\mathfrak{a}$, $N=(0)$.

LEMMA 5. *Let A be primitive and $\varDelta$ be the associated division algebra of a faithful irreducible A-module. If the center $\sum$ of $\varDelta$ is algebraic separable over Φ, then $A\otimes B$ is semi-simple for any semi-simple algebra B.*

Proof. Let B be primitive and $\varDelta'$ be the associated division algebra of a faithful irreducible B-module and let $\sum'$ be the center of $\varDelta'$. Since $\sum$ is separable, $\sum\otimes\sum'$ is regular (in the sense of Neumann [8]) by [4], Pro. 3 and so semi-simple, hence $\varDelta\otimes\varDelta'$ is semi-simple and $A\otimes B$ is so by Lemma 4 and Theorem 2. If B is semi-simple, there exist primitive ideals $\mathfrak{l}_\alpha$ with $\cap\mathfrak{l}_\alpha=(0)$ and $(A\otimes B)/(A\otimes\mathfrak{l}_\alpha)\cong A\otimes B/\mathfrak{l}_\alpha$ are semi-simple, hence $A\otimes B$ is semi-simple.

THEOREM 5. *Let A be a P.M.I. algebra and $\varDelta$ be A.D., and let $\sum$ be the center of $\varDelta$. We assume $\sum$ is algebraic over Φ, then A^{Σ} is semi-simple if and only if $A\otimes A^*$ is semi-simple. In this case $A\otimes B$ is semi-simple for any semi-simple algebra B.*

Proof. If A^{Σ} is semi-simple, $\varDelta^{\Sigma}$ is so and since $\sum\otimes\sum$ is the center of $\varDelta^{\Sigma}$, it has no non zero nilpotent element, hence $\sum$ is separable and $A\otimes B$ is semi-simple for any semi-simple algebra B by Lemma 5. Conversely if $A\otimes A^*$ is semi-simple, $\varDelta\otimes\varDelta^*$ is so by Theorem 2 and $\sum\otimes\sum$ is the center of $\varDelta\otimes\varDelta^*$, hence $\sum$ is separable as above.

4. We shall prove some results of Krenecker products of endomorphism rings. Let A be an algebra and M be a faithful A-module. We shall call "A is strongly dense over M", if A is dense in the finite topology in the endomorphism ring $\bar{A}$ of A-endomorphism ring of M and any non zero ideal of $\bar{A}$ has the non zero intersection with A.

From the definition and the structure theorem of [5], Ch. 4 if A is primitive algebra with a faithful module M, A is strongly dense over M if and only If A is a P.M.I. ring.

PROPOSITION 4. *Let A_i $(i=1, 2)$ be primitive. A_i is strongly dense over a faithful irreducible A_i-module M_i if and only if $A_1 \otimes A_2$ is strongly dense over $M_1 \otimes M_2$.*

Proof. Let A_i be strongly dense over M_i and Δ_i be a centralizer of M_i and let $\{x_\tau\}_{\tau \in I}$ be Δ_1-basis of M_1 and $\{y_\mu\}_{\mu \in I'}$ be Δ_2-basis of M_2. Then $M_1 \otimes M_2 = \sum \oplus (x_\tau \otimes y_\mu)(\Delta_1 \otimes \Delta_2)$ and by [2], Th. 8 we know that the ring of $A_1 \otimes A_2$-endomorphisms of $M_1 \otimes M_2$ is equal to $\Delta_1 \otimes \Delta_2$ and $A_1 \otimes A_2$ is dense in the finite topology in the ring $\overline{A_1 \otimes A_2}$ of $\Delta_1 \otimes \Delta_2$-endomorphisms of $M_1 \otimes M_2$. Now let $\mathfrak{a}(\neq 0)$ be an ideal of $\overline{A_1 \otimes A_2}$ and $\sigma(\neq 0) \in \mathfrak{a}$, then there exists $x_\tau \otimes y_\mu$ such that $(x_\tau \otimes y_\mu)\sigma \neq 0$. Since A is a P.M.I. algebra, we have the following projections: $\varepsilon_\tau(\in A_1)$ and $\varepsilon_\mu'(\in A_2)$, $M_1 \cdot \varepsilon_\tau = x_\tau \Delta_1$, $x_\tau \varepsilon_\tau = x_\tau$, $M_2 \cdot \varepsilon_\mu' = y_\mu \Delta_2$, $y_\mu \varepsilon_\mu = y_\mu$. If we set $(x_\tau \otimes y_\tau)\sigma = \sum_{j=1}^{n} x_{\tau_j} \otimes z_{\tau_j}$, $z_{\tau_j}(\neq 0) \in M_2$, we can find $\rho_\tau \in A_1$ and $\rho_\mu' \in A_2$ such that $x_{\tau_1} \cdot \rho_\tau = x_{\tau_1}$, $x_{\tau_j} \cdot \rho_\mu = 0 (j \neq 1)$ and $z_{\tau_1} \rho_\mu' = y_\mu$. Then $(x_\tau \otimes y_\mu) \cdot (\varepsilon_\tau \otimes \varepsilon_\mu') \cdot \sigma \cdot (\rho_\tau \otimes \rho_\mu') = x_\tau \otimes y_\mu = (x_\tau \otimes y_\mu)(\varepsilon_\tau \otimes \varepsilon_\mu')$. From the properties of ε's and ρ's, $(\varepsilon_\tau \otimes \varepsilon_\mu') \cdot \sigma \cdot (\rho_\tau \otimes \rho_\mu') = \varepsilon_\tau \otimes \varepsilon_\mu' \in (A_1 \otimes A_2) \cap \mathfrak{a}$ and $A_1 \otimes A_2$ is strongly dense over $M_1 \otimes M_2$. Conversely let $A_1 \otimes A_2$ be strongly dense over $M_1 \otimes M_2$. The set $\mathfrak{F}(M_1 \otimes M_2)$ of elements σ of $\overline{A_1 \otimes A_2}$ such that $(M_1 \otimes M_2) \cdot \sigma$ is contained in a sum of submodules $(x_\tau \otimes y_\mu) \cdot (\Delta_1 \otimes \Delta_2)$, finite in number, is a two sided ideal of $\overline{A_1 \otimes A_2}$, hence $A_1 \otimes A_2 \cap \mathfrak{F}(M_1 \otimes M_2) \ni \sigma \neq 0$, $\sigma = \sum_i a_i^{(1)} \otimes a_i^{(2)}$, $a_i^{(j)} \in A_j$ and $(M_1 \otimes M_2) \cdot \sigma \subseteq \sum_{i=1}^{n} \otimes (x_i \otimes y_i)(\Delta_1 \otimes \Delta_2)$. There exists $\varepsilon^{(i)}$ in the set $\mathfrak{F}(M_i)$ of linear transformations of finite rank of M_i such that $x_j \varepsilon^{(1)} = x_j$ $y_j \varepsilon^{(2)} = y_j$ $j = 1, \cdots, n$. Then $\sigma = \sigma(\varepsilon^{(1)} \otimes \varepsilon^{(2)}) = \sum a_i^{(1)} \varepsilon^{(1)} \otimes a_i^{(1)} \varepsilon^{(2)} \in \mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2) \cap (A_1 \otimes A_2)$, hence $A_i \cap \mathfrak{F}(M_i) \neq 0$ and A_i is strongly dense over M_i.

COROLLARY. *Let $A_i (i=1, 2)$ be primitive and Δ_i be the associated division algebra of a faithful irreducible A_i-module M_i. If Δ_1 is central and $A_1 \otimes A_2$ is a P.M.I. algebra, then A_i is a P.M.I. algebra.*

Proof. By the assumptions $\Delta_1 \otimes \Delta_2$ is simple and $M_1 \otimes M_2$ has a faithful irreducible $A_1 \otimes A_2$ submodule, hence $\Delta_1 \otimes \Delta_2$ has a minimal left ideal and satisfies the minimum condition. By the same method as in the proof of [5] Ch. 5, Th. 10.1 $M_1 \otimes M_2 = \sum_{i=1}^{n} \oplus N_i$ where N_i is isomorphic to a faithful irreducible $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$-module N_1. Let Δ be the division ring of $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$-endomorphisms of N_1 then $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$ is the ring of Δ-endomorphisms of N_1, hence $A_1 \otimes A_2$ is strongly dense over $M_1 \otimes M_2$. Therefore we obtain Corollary by Proposition 4 and the remark above.

The following proposition is a generalization of [5] Ch. 5, Th. 3.1, (3) and the method of proof is quite analogous.

LEMMA 6. *Let Δ_i $(i=1, 2)$ be an algebra with unit element and M_i be a right*

Δ_i-module with Δ_i-basis. We may regard $M_1 \otimes M_2$ as right $\Delta_1 \otimes \Delta_2$-module. Then $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*) = \mathfrak{M}(\Delta_1^*) \otimes \mathfrak{M}(\Delta_2^*)$ if and only if there exists i such that $[M_i : \Phi] < \infty$ or $[M_j : \Delta_j] < \infty$ $(j=1, 2)$.

Proof. $M_1 \underset{\Phi}{\otimes} M_2 \approx M_1 \underset{\Delta_1}{\otimes} (\Delta_1 \underset{\Phi}{\otimes} M_2) = M_1 \underset{\Delta_1}{\otimes} \overline{M}_2$, where $\overline{M}_2 = \Delta_1 \otimes M_2$. We may regard $\overline{M}_2$ as a left Δ_1, right $\Delta_1 \otimes \Delta_2$-module by the natural way. Clearly we have a $\Delta_1 \otimes \Delta_2$-isomorphism φ of $M_1 \otimes M_2$ to $M_1 \underset{\Delta_1}{\otimes} \overline{M}_2$. Let $\{x_\alpha\}_{\alpha \in I}$ be a Δ_1-basis of M_1. We may identify $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$ with the ring of $\Delta_1 \otimes \Delta_2$-endomorphisms of $M_1 \otimes \overline{M}_2$. For any element σ of $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$ $(x_i \otimes \overline{m}_2) \cdot \sigma = \sum x_j \otimes f_{j,i}(\overline{m}_2)$ where $\overline{m}_2, f_{j,i}(\overline{m}_2) \in \overline{M}_2$, we can easily examine that mapping $f_{j,i} : \overline{m}_2 \to f_{j,i}(\overline{m}_2)$ are $\Delta_1 \otimes \Delta_2$-endomorphisms of $\overline{M}_2$ and $\sum_i f_{i,j}$ is summable and that conversely if $\sum_i f_{i,j}$ is summable for each j, then $(x_i \otimes \overline{m}_2) \cdot \sigma^* = \sum x_j f_{j,i}(\overline{m}_2)$ is a $\Delta_1 \otimes \Delta_2$-endomorphism of $M_1 \otimes \overline{M}_2$. Hence we can represent any element of $\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$ by a matrix $(f_{i,j})$, where $f_{i,j}$'s are elements of the ring $\overline{\mathfrak{M}(\Delta_1 \otimes \Delta_2)}$ of $\Delta_1 \otimes \Delta_2$-endomorphisms of $\overline{M}_2$ and $\sum_i f_{i,j}$ is summable for each j. If $a = (\delta_{k,l}) \in \mathfrak{M}(\Delta_1^*)$, $b \in \mathfrak{M}(\Delta_2^*)$, $(x_i \otimes \overline{m}_2)(a \otimes b) = \sum_j x_j \otimes (\overline{m}_2(\delta_{i,j})_l \otimes b)$ hence $a \otimes b = ((\delta_{i,j})_l \otimes b)$ and the matrix of $\sum_i a_i \otimes b_i$ has a property that the dimensionality of the space spanned by the linear transformations $f_{i,j}$ is finite over Δ_1. If $\mathfrak{M}(\Delta_1^*) \otimes \mathfrak{M}(\Delta_2^*) = \mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$, there exist the following two cases,

1) $[M_1 : \Delta_1] < \infty$ then $\Delta_{1l} \otimes \mathfrak{M}(\Delta_2^*) = \overline{\mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)}$, or
2) $[M_1 : \Delta_1] = \infty$ then $[\mathfrak{M}(\Delta_2^*) : \Phi] < \infty$, hence $[M_2 : \Phi] < \infty$, $[\Delta_2 : \Phi] < \infty$.

In case 1 if we replace $M_1 \otimes M_2$ by $\Delta_1 \otimes M_2$, we obtain as above either $[M_2 : \Delta_2] < \infty$, or $[\Delta_1 : \Phi] < \infty$. The converse is clear.

COROLLARY. *Let $A_i (i=1, 2)$ be a closed irreducible algebra. If $A_1 \otimes A_2$ is closed irreducible, then each of them satisfies the minimum condition or one of them is of finite rank over Φ.*

Proof. Let M_i be an irreducible A_i-module and Δ_i be A.D.. If $A_1 \otimes A_2$ is closed irreducible, $\Delta_1 \otimes \Delta_2$ is a simple ring with minimum condition by Theorem 3 and the remark following it. Using the same notations as in the proof of Corollary of Propositlon 4, we can obtain $A_1 \otimes A_2 \subseteqq \mathfrak{M}(\Delta_1^* \otimes \Delta_2^*) \subseteqq \mathfrak{M}(\Delta^*)$ and by the assumption $A_1 \otimes A_2 = \mathfrak{M}(\Delta^*)$, hence $A_1 \otimes A_2 = \mathfrak{M}(\Delta_1^* \otimes \Delta_2^*)$, which proves the corollary.

THEOREM 6. *Let M, N be vector spaces over a field Φ and $\mathfrak{a}$, $\mathfrak{b}$ be distinguished homogeneous algebras of linear transformations in M and N, respectively. Then $\mathfrak{a} \otimes \mathfrak{b}$ is a distinguished homogeneous of linear transformations in $M \otimes N$ if and only if the Kronecker product of the centers of $\mathfrak{a}$ and $\mathfrak{b}$ is a field, and either of the following conditions holds.*

1) $[\mathfrak{a} : \Phi] < \infty$ *or* $[\mathfrak{b} : \Phi] < \infty$,
2) $\mathfrak{a} \otimes \mathfrak{b}$ *has a minimum condition, (cf.* [5], *Ch.* 6 *Th.* 5.1).

Proof. Let $\mathfrak{a}\otimes\mathfrak{b}$ be a distinguished homogeneous ring in $M\otimes N$, then from [5], Ch. 5, Th. 6.2 $\mathfrak{a}\otimes\mathfrak{b}$ is a closed irreducible ring of a module. If the condition 1) of the theorem holds, we have immediately the first part. If the condition 1) does not hold, we obtain that $\mathfrak{a}$ and $\mathfrak{b}$ satisfy minimum conditions from Corollary of Lemma 6. In this case we have $\mathfrak{a}=\Delta_n$, $\mathfrak{b}=\Gamma_m$ where Δ, Γ are division algebras over Φ. Since $\Delta\otimes\Gamma$ has the minimum condition by Theorem 3, $\mathfrak{a}\otimes\mathfrak{b}$ has it and its center is a field. Conversely we may assume that M (resp. N) is irreducible $\mathfrak{a}$- (resp. $\mathfrak{b}$-) module. Let Γ (resp. Δ) be the centralizer of $\mathfrak{a}$ (resp. $\mathfrak{b}$) in M (resp. N). Then by the assumptions the centralizer of Γ (resp. Δ) in M (resp. N) is $\mathfrak{a}$ (resp. $\mathfrak{b}$) and $\Gamma\otimes\Delta$ is simple. If either of conditions 1) and 2) holds, $\Gamma\otimes\Delta$ has the minimum condition, hence $\Gamma\otimes\Delta$ is distinguished homogeneous in $M\otimes N$. On the other hand the centralizer of $\Gamma\otimes\Delta$ in $M\otimes N$ is $\mathfrak{a}\otimes\mathfrak{b}$ by Lemma 6. Therefore $\mathfrak{a}\otimes\mathfrak{b}$ is distinguished homogeneous from [5], Ch. 6, Th. 2.2.

Proposition 5. *Let A be a ring. Then A is distinguished homogeneous if and only if so is A_n where A_n is the total matrix ring over A.*

("Only if" part is readily obtained from Th. 6, but we shall give a direct proof.)

Proof. Let A be a distinguished homogeneous ring of a commutative group M, then A has unit element. We can easily show that $M^n=M\underset{A}{\otimes}A'$ is a faithful competely reducible A_n-module (cf. [1], Th. 47.1) where $A'=Ae_{11}+Ae_{12}+\cdots+Ae_{1n}$ and e_{ij} are matrix units. We can represent the endomorphism ring of M^n by the total matrix ring over the endomorphism ring of M. Let $\varphi=(\varphi_i)\in cl\,A_n$, and x_i $(i=1,\cdots,m)$ be arbitrary elements of M and $\tilde{x}_i=x_i\otimes e_{1k}$. Then there exists an element $a=(a_{ij})$ of A_n such that $\tilde{x}_i\varphi=\tilde{x}_i a$, i.e. $x_i\varphi_{kj}=x_i a_k$. Since $\varphi_{kj}\in cl\,A=A$, A_n is closed. Conversely if A_n is distinguished homogeneous in M, then A_n has unit element. Hence A has also unit element. M has the following decomposition:

$$(*)\qquad M=Me_{11}\oplus\cdots\oplus Me_{1n}\approx Me_{11}\otimes A'.$$

Me_{11} is obviously a faithful completely reducible A-module. We define an endomorphism $\bar{\varphi}$ of M from an endomorphism φ of Me_{11} by setting $(\sum m_i e_{1i})\,\bar{\varphi}=\sum (m_i e_{11}\varphi)\,e_{1j}$. Let $\varphi\in cl\,A$ and let x_i $(i=1,\cdots,m)$ be arbitrary elements of M. In virtue of the decomposition $(*)$ $x_i=\sum x_j e_{1j}$. From the assumption there exists $a\in A$ such that $x_{ij}e_{11}\varphi=x_{ij}e_{11}a$. Then $x_i\bar{\varphi}=\sum_j (x_{ij}e_{11})\,ae_{1j}=\sum_j (x_{ij}e_{11}a)\,e_{1j}=x_i\breve{a}$, $(\breve{a}=ae_{11}+\cdots+ae_{nn})$, $\bar{\varphi}\in A_n$ and so $\varphi\in A$, which proves the proposition.

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