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On the homology of classical Lie groups

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1. Introduction

We shall give, in this paper, cellular decompositions of the classical Lie groups $SO(n)$, $SU(n)$ and $Sp(n)$. The important role is to give the primitive cells by making use of cross-sections from cells such that spheres $S^{n-1}=SO(n)/SO(n-1)$, $S^{2n-1}=SU(n)/SU(n-1)$ and $S^{4n-1}=Sp(n)/Sp(n-1)$ minus one point, respectively, to $SO(n)$, $SU(n)$ and $Sp(n)$. The cells of $SO(n)$ are closely connected with the real projective space P [7], [10] and the cells of $SU(n)$ are closely connected with the suspended space $E(M)$ of the complex projective space M [11]. The cells of $Sp(n)$, however, have no connection with the quaternion projective space directly.

In the classical Lie groups, the cup products and the Pontrjagin products are calculated rather simply : the Pontrjagin products of cells, fortunately, are cellular in the almost cases. As for the Steenrod's reduced powers, since these operations are calculated in the projective spaces P and M (and hence $E(M)$), we can calculate some reduced powers in $SO(n)$ and $SU(n)$. In the case of $Sp(n)$, we shall obtain the aim by researching the connections between $SU(2n)$ and $Sp(n)$.

The cellular decompositions of the classical Lie groups follow cellular decompositions of the Stiefel manifolds $V_{n,m}=SO(n)/SO(n-m)$, $W_{n,m}=SU(n)/SU(n-m)$, $X_{n,m}=Sp(n)/Sp(n-m)$ and some homogeneous spaces $F_n=SO(2n)/SU(n)$, $X_n=SU(2n)/Sp(n)$. We shall compute their homological properties by making use of their cell structures.

2. Notations

Let X be a finite cell complex and Γ a coefficient commutative ring with a unit. We denote by $H(X; \Gamma)$ (resp. $H^*(X; \Gamma)$) the homology group (resp. cohomology algebra) of X with coefficient ring Γ. If $f: X\to Y$ is a continuous mapping, we denote by ${}_\Gamma f_{\#}$ (resp. ${}_\Gamma f^{\#}$) the chain (resp. cochain) homomorphism and by ${}_\Gamma f_* : H(X; \Gamma)\to H(Y; \Gamma)$ (resp. ${}_\Gamma f^* : H^*(Y; \Gamma)\to H^*(X; \Gamma)$) the homomorphism (resp. algebraic homomorphism) induced by f respectively. Throughout this paper, Γ will be Z or Z_p.[1] According as Γ is Z or Z_p, ${}_\Gamma f_{\#}$ (resp. ${}_\Gamma f^{\#}$) and

1) Z is a free cyclic group with one generator. Z_p is a cyclic group of order p, where p is a prime integer.

${}_rf_*$ (resp. ${}_rf^*$) will be denoted by f_{*} (resp. f^{*}) and f_* (resp. f^*) or ${}_pf_{*}$ (resp. ${}_pf^{*}$) and ${}_pf_*$ (resp. ${}_pf^*$). If e_X^k (where k denotes the dimension of e_X^k) is a cell of X, then (e_k^X) (resp. $({}_pe_k^X)$) denotes the integral chain (resp. integral chain reduced modulo p) which is represented by the cell e_X^k. If $({}_pe_k^X)$ (resp. $({}_pe_k^X)$) is a cycle, then $e_k^X \in H_k(X;\ Z)$ (resp. ${}_pe_k^X \in H^k(X;\ Z_p)$) denotes the homology class containing the chain (e_k^X) (resp. $({}_pe_k^X)$). Let $[e_X^k]$ (resp. $[{}_pe_X^k]$), analogously, be the integral cochain (resp. integral cochain reduced modulo p) which assigns 1 to only the cell e_X^k and 0 to the others. If $[e_X^k]$ (rese. $[{}_pe_X^k]$) is a cocycle, then $e_X^k \in H^k(X;\ Z)$ (resp. ${}_pe_X^k \in H^k(X;\ Z_p)$) denotes the homology class containing the cochain $[e_X^k]$ (resp. $[{}_pe_X^k]$). We shall omit, later on, the brackets ( ) and [ ] (occasionally, even the left sufix p) in the case there is no danger of confusion. $P_X(t)$ denotes the (usual) Poincaré polynomial of X and ${}_pP_X(t)$ denotes the Poincaré polynomial of X modulo p.

Let H be a (finite dimensional) free algebra (resp. algebra over Z_p), graded by the submodules H^i ($i \geqq 0$), anticommutative, with a unit which is a base of H^0. A set $(x_1, x_2, \cdots\cdot)$, where x_k is a positive dimensional homogeneous element of H, of H is a simple generator of H if the monomials $x_{k_1}x_{k_2}\cdots\cdot x_{k_j}$, where $k_1 > k_2 > \cdots\cdot > k_j$; $k=1,2,\cdots\cdot$, form with the unit an additive free base (resp. additive base over Z_p) of H, and denote by $H=\Delta(x_1,x_2,\cdots\cdot)$. An algebra H with a simple generator $(x_1, x_2,\cdots\cdot)$ is a free exterior algebra (resp. exterior algebra over Z_p) if $xx=0$, $x \in H$, and denote by $H=\Lambda(x_1, x_2,\cdots\cdot)$.

3. Classical Lie groups and Stiefel manifolds

We denote by F one of three fields of real numbers R, complex numbers C or quaternion numbers Q, and by $d = d(F)$ the dimension of F over R; $d(R)=1$, $d(C)=2$ and $d(Q)=4$. Let F^n be the right vector space of dimension n whose elements are ordered sets of n elements of F. Specifically $x=(x_1, x_2, \cdots\cdot, x_n)$ is in F^n if each $x_i \in F$, and, if a is in F, then $xa=(x_1a, x_2a,\cdots\cdot, x_na)$. Let e_i be the element of F^n whose i-th component is 1 and whose other components are 0. Define the inner product of $x = \sum_{i=1}^n e_ix_i$ and $y = \sum_{i=1}^n e_iy_i$ in F^n and the norm of x by

$$(x, y) = \sum_{i=1}^n \bar{x}_i y_i \quad (\bar{x}_i \text{ is the conjugate of } x_i),$$

and

$$x = \sqrt{(x, x)}$$

respectively. The elements $e_1, e_2, \cdots\cdot, e_n$ form an orthonormal base in F^n.

Let $G(n)$ be the group of linear transformations in F^n preserving the inner product. In matrix notation, $(n. n)$-matrix A with coefficient in F is in $G(n)$ if and only if

$$AA^* = A^*A = I_n.^{2)}$$

2) A^* is the transposed conjugate matrix of A. I_n is the unit (n, n)-matrix.

$G(n)$ is called the *orthogonal group* $O(n)$, *unitary group* $U(n)$ or *symplectic group* $Sp(n)$ according as the field F is real, complex or quaternionic.

Since the fields of real and complex numbers are commutative, the determinant of matrix can be considered. Let $SG(n)$ be the subgroup of $G(n)$ (except $G(n)=Sp(n)$) composed of all matrices in $G(n)$ whose determinants are 1. $SG(n)$ is called the *special orthogonal group* $SO(n)$ or *special unitary group* $SU(n)$ according as the scalars are real or complex.

Define a mapping $\zeta : G(n)\rightarrow SG(n)\times S^{d-1}$, where S^{d-1} is the 0-or 1-dimensional sphere of real or complex numbers respectively whose norms are 1, by

$$\zeta (A) = A \begin{pmatrix}\det A^{-1} & \\ & I_{n-1}\end{pmatrix} \times \det A,$$

then ζ is a homeomorphism. Consequently $O(n)$ is non-connected and $SO(n)$ is the connected component of $O(n)$.

F^{n-1} is embedded in F^n as a vector subspace whose last component is 0. Then $G(n-1)$ may be regarded as a subgroup of $G(n)$ by extending a matrix A of $G(n-1)$ to $G(n)$ by requirement that $Ae_n=e_n$. Thus we have sequences $(S_F^{d-1}=)$ $G(1)\subset G(2)\subset\cdots\subset G(n)$ and $I_n=SG(1)\subset SG(2)\subset\cdots\subset SG(n)$.

Let $S_F^{dn-1}=\{x\in F^n;|x|=1\}$ be the unit sphere in F^n. Then the embedding $F^{n-1}\subset F^n$ gives rise to an embedding $S_F^{d(n-1)-1}\subset S_F^{dn-1}$. For integers $n\geqq m\geqq 1$, let $S_{n,m}$ be the Stiefel manifold of ordered orthonormal m vecters $a=(a_1,a_2,\cdots, a_m)$ in F^n. $S_{n-1,m-1}$ is embedded in $S_{n,m}$ by regarding a point $a=(a_1,a_2,\cdots,a_{m-1})$ of $S_{n-1,m-1}$ as a point $a=(a_1,a_2,\cdots,a_{m-1},e_n)$ of $S_{n,m}$. Thus we have a sequence $S_F^{d(n-m+1)-1}=S_{n-m+1,1}\subset S_{n-m+2,2}\subset\cdots\subset S_{n,m}$. $S_{n,m}$ is called the *real Stiefel manifold* $V_{n,m}$, *complex Stiefel manifold* $W_{n,m}$ or *quaternion Stiefel manifold* $X_{n,m}$ according as the scalars are real, complex or quaternionic.

Define a projection $p_m=p_{m,G(n)} : G(n)\rightarrow S_{n,m}$ (resp. $p_m=p_{m,SG(n)} : SG(n)\rightarrow S_{n,m}$, for $n>m$) by

$$p_m(A) = (Ae_{n-m+1}, \cdots, Ae_{n-1}, Ae_m).$$

Then $G(n)$ (resp. $SG(n)$) operates transitively on $S_{n,m}$ and $G(n-m)$ (resp. $SG(n-m)$) is the subgroup of $G(n)$ (resp. $SG(n)$) leaving fix a point $(e_{n-m+1},\cdots,e_{n-1},e_n)$. Hence we have a fibre space $G(n)/G(n-m)=S_{n,m}$ (resp. $SG(n)/SG(n-m)=S_{n,m}$) with projection p_m. That is, we have $O(n)/O(n-m)=SO(n)/SO(n-m)=V_{n,m}$, $U(n)/U(n-m)=SU(n)/SU(n-m)=W_{n,m}$ and $Sp(n)/Sp(n-m)=X_{n,m}$. Especially $V_{n,n}=O(n)$, $V_{n,n-1}=SO(n)$, $W_{n,n}=U(n)$, $W_{n,n-1}=SU(n)$ and $X_{n,n}=Sp(n)$; $V_{n,1}=S_R^{n-1}$, $W_{n,1}=S_C^{2n-1}$ and $X_{n,1}=S_Q^{4n-1}$.

4. Primitive characteristic map $f_{G(n)}: E_F^{dn-1}\rightarrow G(n)$ (resp. $f_{SG(n)}: E_F^{dn-1}\rightarrow SG(n)$)

$E_F^{d(n-1)}$ be a closed cell in F^{n-1} consisting of all $x=(x_1,x_2,\cdots,x_{n-1})$ such that $|x|^2=|x_1|^2+\cdots+|x_{n-1}|^2\leqq 1$, and E_F^{d-1} a closed cell in F consisting of all pure imaginary numbers whose norms $\leqq 1$. Constract a closed cell E_F^{dn-1} with the dimension $dn-1$ over R by $E_F^{d-1}\times E_F^{dn-1}$. Define a mapping $f_{G(n)}: E_F^{dn-1}\to G(n)$ by setting

$$f_{G(n)}(q, x) = (\delta_{ij}+x_i p\bar{x}_j), \quad i, j = 1, 2, \cdots, n,$$

where $x = (x_1, \cdots, x_{n-1}) \in E_F^{d(n-1)}$, $x_n = \sqrt{1-|x|^2}$ and $q \in E_F^{d-1}$, $p=2\sqrt{1-|q|^2}(q-\sqrt{1-|q|^2})$. It is readily verified that $f_{G(n)}(q, x)$ is in $G(n)$. In fact, using that $\sum_{k=1}^n |x_k|^2 = 1$ and $p+\bar{p}+|p|^2=0$,

$$\sum_{k=1}^n \overline{(\delta_{ki}+x_k p\bar{x}_i)}(\delta_{kj}+x_k p\bar{x}_j)$$
$$= \sum_{k=1}^n (\delta_{ki}+x_i \bar{p}\bar{x}_k)(\delta_{kj}+x_k p\bar{x}_j)$$
$$= \delta_{ij}+x_i(p+\bar{p}+|p|^2)\bar{x}_j = \delta_{ij}.$$

When the scalars are commutative, it will be verified that the determinant of $f_{G(n)}(q, x)$ is $-(-q+\sqrt{1-|q|^2})^2$ for any x. So that if we define a mapping $f_{SG(n)}$: $E_F^{dn-1}\to SG(n)$ by

$$f_{SG(n)}(q, x) = f_{G(n)}(q, x)\begin{pmatrix} -(q+\sqrt{1-|q|^2})^2 & \\ & I_{n-1}\end{pmatrix},$$

then $f_{SG(n)}(q, x)$ is in $SG(n)$.

We shall call $f_{G(n)}$ (resp. $f_{SG(n)}$) the *primitive characteristic map* of E_F^{dn-1} into $G(n)$ (resp. E_F^{dn-1} into $SG(n)$).

REMARK 4. 1. Let $_+E_F^{d(n-1)}$ be a set of all $x=(x_1, \cdots, x_n)\in F^n$ such that $|x|=1$ and $x_n\in R$, $x_n\geqq 0$. This set is a subset of S_F^{dn-1}. Define a homeomorphism $g: E_F^{d(n-1)}\to {}_+E_F^{d(n-1)}$ by the formula

$$g(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{n-1}, x_n),$$

where $x_n = \sqrt{1-|x|^2}$. Now define a mapping $\tilde{f}_{G(n)}: E_F^{d-1}\times S_F^{dn-1}\to G(n)$ by setting

$$\tilde{f}_{G(n)}(q, x) = (\delta_{ij}+x_i p\bar{x}_j),$$

where $x = (x_1, \cdots, x_n) \in S_F^{dn-1}$ and $p = 2\sqrt{1-|q|^2}(q-\sqrt{1-|q|^2})$. Then the following diagram is commutative:

$$\begin{array}{ccc} E_F^{dn-1}=E_F^{d-1}\times E_F^{d(n-1)} & \xrightarrow{f_{G(n)}} & G(n) \\ \downarrow I\times g & & \uparrow \tilde{f}_{G(n)} \\ E_F^{d-1}\times {}_+E_F^{d(n-1)} & \xrightarrow{I\times i} & E_F^{d-1}\times S_F^{dn-1} \end{array}$$

i.e. $f_{G(n)}=\tilde{f}_{G(n)}\circ(I\times i)\circ(I\times g)$, where I is the identity map and i is the injection.

REMARK 4. 2. When the field is commutative, the primitive characteristic map $f_{G(n)}$ can be written by the form

$$f_{G(n)}(q, x) = I_n+p(x_i\bar{x}_j), \quad i, j = 1, 2, \cdots, n.$$

We shall remember that all of hermitian matrices X with properties $tr(X)=1$ and $X^2=X$ form the matrix form Ω_{n-1}^* of the $d(n-1)$-dimensional projective space Ω_{n-1} over F [11]. Hence a matrix $x = (x_i\bar{x}_j)$ in the last term of $f_{G(n)}(q, x)$ is a point of Ω_{n-1}^*. Therefore, we can exchange the anti-image E_F^{dn-1} of $f_{G(n)}$ for the familiar space.

4. 1) *Case* $O(n)$

Let P_{n-1} be the $(n-1)$-dimensional real projective space and P^*_{n-1} its matrix form. We identify a point $x=[x_1, \cdots, x_n]\in P_{n-1}$ such that $x_1^2+\cdots+x_n^2=1$ with a point $X=(x_ix_j)\in P^*_{n-1}$ and P_{n-1} with P^*_{n-1}. As is well known, P_{n-1} has a k-dimensional cell w^k for $n-1\geqq k\geqq 0$. A characteristic map $f_{P_{n-1}}: E^k_R\to w^k\subset P^*_{n-1}$ for the cell w^k is given by

$$f_{P_{n-1}}(x_1, \cdots, x_k)=\begin{pmatrix} x_ix_j & \vdots & \\ \cdots & \vdots & \\ & & I_{n-k-1}\end{pmatrix}, \quad i,j=1,2,\cdots k+1,$$

where $x_{k+1}=\sqrt{1-(x_1^2+\cdots+x_k^2)}$.

Now, the primitive characteristic map $f_{O(n)}$ is extendable to the mapping $f'_{O(n)}: P_{n-1}\to O(n)$ defined by

$$f'_{O(n)}(X)=I_n-2X,$$

where $X\in P^*_{n-1}$. That is, $f_{O(n)}=f'_{O(n-1)}\circ f_{P_{n-1}}$ on E^{n-1}_R.

4. 2) *Case* $SO(n)$.

The primitive characteristic map $f_{SO(n)}$ is extendable to the mapping $f'_{SO(n)}: P_{n-1}\to SO(n)$ defined by

$$f'_{O(n)}(X)=(I_n-2X)\begin{pmatrix}-1 & \\ & I_{n-1}\end{pmatrix}.$$

That is, $f_{SO(n)}=f'_{SO(n)}\circ f_{P_{n-1}}$ on E^{n-1}_R.

4. 3) *Case* $U(n)$

Let M_{n-1} be the $2(n-1)$-dimensional complex projective space and M^*_{n-1} its matrix form. We identify a point $x=[x_1,\cdots,x_n]\in M_{n-1}$ such that $|x_1|^2+\cdots+|x_n|^2=1$ with a point $X=(x_i\bar{x}_j)\in M^*_{n-1}$ and M_{n-1} with M^*_{n-1}. As is will known, M_{n-1} has a $2k$-dimensional cell u^{2k} for $n-1\geqq k\geqq 0$. A characteristic map $f_{M_{n-1}}: E^{2k}_C\to M_{n-1}$ for the cell u^{2k} is given by

$$f_{M_{n-1}}(x_1, \cdots, x_k)=\begin{pmatrix} x_i\bar{x}_j & \vdots & \\ \cdots & \vdots & \\ & & I_{n-k-1}\end{pmatrix}, \quad i,j=1,2,\cdots,k+1,$$

where $x_{k+1}=\sqrt{1-(|x_1|^2+\cdots+|x_k|^2)}$.

Let E^1_C be a closed cell of pure imaginary complex numbers whose norms are $\leqq 1$. The suspended space $E(M_{n-1})$ is the space formed from $E^1_C\times M_{n-1}$ by shrinking $-i\times M_{n-1}$ and $i\times M_{n-1}$[4] to two different points of $E(M_{n-1})$ respectively. In the detail: let $\mathfrak{M}(n,C)$ be the space consisting of all (n,n)-matrices with complex coefficients. $\mathfrak{M}(n,C)$ is a subspace of C^{n^2} by a correspondence $(x_{ij})\in\mathfrak{M}(n,C)\to(x_{11},x_{12},\cdots,x_{nn})\in C^{n^2}$. Define $E(M_{n-1})$ as the space $\{(q,\sqrt{1-|q|^2}X);\ q\in E^1_C$ and $X\in M^*_{n-1}\}\subset E^1_C\times\mathfrak{M}(n,C)$. $E(M_{n-1})$ has two 0-dimensional cells v^0_-, v^0_+ and a $(2k-1)$-dimensional v^{2k-1} for $n\geqq k\geqq 1$. A characteristic map $f_{E(M_{n-1})}: E^{2k-1}_C=E^1_C\times E^{2(k-1)}_C\to v^{2k-1}\subset E(M_{n-1})$ for v^{2k-1} is given by

$$f_{E(M_{n-1})}(q, (x_1, \cdots, x_{k-1})) = (q, \sqrt{1-|q|^2}\, f_{M_{n-1}}(x_1, \cdots, x_{k-1})).$$

Now, the primitive characteristic map $f_{U(n)}$ is extendable to the mapping $f'_{U(n)} : E(M_{n-1}) \to U(n)$ defined by

$$f'_{U(n)}(q, Y) = I_n + 2(q - \sqrt{1-|q|^2})\, Y,$$

where $(q, Y) \in E(M_{n-1})$, i.e. Y is experessed by the form $\sqrt{1-|q|^2}X$, $X \in M^*_{n-1}$. Then we have $f_{U(n)} = f'_{U(n)} \circ f_{E(M_{n-1})}$ on E^{2n-1}_C.

4, 4) *Case* $SU(n)$

The another suspended space $E(M_{n-1})$ is the space formed from $E^1_C \times M_{n-1}$ by shrinking $-\boldsymbol{i} \times M_{n-1}$, $\boldsymbol{i} \times M_{n-1}$ and $E^1_C \times [1, 0, \cdots, 0]$ to a single point of $E(M_{n-1})$. $E(M_{n-1})$ has a 0-dimensional cell v^0 and a $(2k-1)$-dimensional cell v^{2k-1} for $n \geqq k \geqq 2$.

The primitive characteritsic map $f_{SU(n)}$ is extendable to the mapping $f'_{SU(n)}$: $E(M_{n-1}) \to SU(n)$ defined by

$$f'_{SU(n)}(q, X) = (I_n + pX)\begin{pmatrix} -(q+\sqrt{1-|q|^2})^2 & \\ & I_{n-1} \end{pmatrix}.$$

where $X \in M^*_{n-1}$, $q \in E^1_C$ and $p = 2\sqrt{1-|q|^2}\,(q - \sqrt{1-|q|^2})$. If $q = \pm \boldsymbol{i}$ or $X = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$, then $f'_{SU(n)}(q, X) = I_n$. Therefore $f'_{SU(n)}$ is well defined as a mapping of $E(M_{n-1})$.

5. Cellular decompositions of $G(n)$, $SG(n)$ and $S_{n,m}$

LEMMA 5. 1. *Given $a \in F$ such that $Re(a) \neq 0$[3], from the equation*

$$px = a,$$

p and x are determined uniquely and continuously with respect to a under the conditions $p \in F$, $p + \bar{p} + |p|^2 = 0$ and x is a real number.

Proof. In fact, we have readily that

$$p = \frac{-2Re(a)a}{|a|^2} \qquad x = \frac{-|a|^2}{2Re(a)}. \qquad \text{q.e.d.}$$

Define a mapping $\xi_F = \xi^n_F : E^{dn-1}_F \to S^{dn-1}$ by

$$\xi_F = p_1 \circ f_{G(n)} \qquad (\text{or } \xi_F = p_1 \circ f_{SG(n)}).$$

LEMMA 5. 2. *ξ_F maps $(E^{dn-1}_F)^\bullet$ to a point e_n of S^{dn-1}_F* and *$E^{dn-1}_F - (E^{dn-1}_F)^\bullet$ homeomorphically onto $S^{dn-1}_F - e_n$.*

Proof. It is obvious that ξ_F maps $(E^{dn-1}_F)^\bullet$ to e_n. Given any point $a = (a_1, \cdots, a_n)$ of $S^{dn-1}_F - e_n$, it is sufficient to show the following equations can be solved continuously :

$$\begin{cases} x_1 p x_n = a_1, \\ \cdots\cdots\cdots\cdots \\ x_{n-1} p x_n = a_{n-1}, \\ 1 + x_n p x_n = a_n. \end{cases}$$

Using the preceding lemma and noting that $Re(a_n - 1) < 0$, $x_n \in R$, $x_n > 0$ and

3) $Re(a)$ is the real part of a.

$p \in F$ (also q) are determined from the last equation. From the other equations, $x_1, \cdots, x_{n-1}$ can be determined continuously. q.e.d.

REMARK 5. 1. If we define a mapping $\phi_{G(n)}: S_F^{dn-1}-e_n \to G(n)$ by $\phi_{G(n)}= f_{G(n)} \circ \xi_F^{-1}$, then we have

$$\phi_{G(n)}(a_1, \cdots, a_{n-1}, a_n)=\begin{pmatrix} \delta_{ij}-a_i(1-\bar{a}_n)^{-1}\bar{a}_j & \vdots & a_1 \\ \cdots\cdots\cdots\cdots\cdots & \vdots & \vdots \\ -(1-a_n)(1-\bar{a}_n)^{-1}\bar{a}_j & & a_n \end{pmatrix},$$

where $i, j=1, 2, \cdots, n-1$. This mapping gives a cross-section $\phi_{G(n)}: S_F^{dn-1}-e_n \to G(n)$ in a fibre space $G(n)/G(n-1)=S_F^{dn-1}$ with projection p_1 [9. pp. 119, 125. 130].

From lemma 5. 2, we see that $f_{G(k)}$ (resp. $f_{SG(k)}$) maps $\mathcal{E}_F^{dk-1}=E_F^{dk-1}-(E_F^{dk-1})'$ homeomorphically into $G(k)\subset G(n)$ for $n\geqq k\geqq 1$ (resp. $SG(k)\subset SG(n)$ for $n\geqq k \geqq 2$). This mapping $f_{G(k)}$ (resp. $f_{SG(k)}$) also will be written by the same letter $f_{G(n)}$ (resp. $f_{SG(n)}$), if there occurs no confusion. Now put

$$e_{G(n)}^{dk-1}=f_{G(n)}(E_F^{dk-1}) \qquad \text{for } n \geqq k \geqq 1$$

and

$$e_{SG(n)}^{dk-1}=f_{SG(n)}(E_F^{dk-1}) \qquad \text{for } n \geqq k \geqq 2.$$

We shall call $e_{G(n)}^{dk-1}$ (resp. $e_{SG(n)}^{dk-1}$) the *primitive cell* of $G(n)$ (resp. $SG(n)$).

For integers $n\geqq k_i \geqq 1$; $i=1, 2, \cdots, j$, define a mapping

$$\bar{f}_{G(n)}: E_F^{dk_1-1}\times\cdots\times E_F^{dk_j-1} \to G(n),$$

which is an extension of $f_{G(n)}$ by setting

$$\bar{f}_{G(n)}(y_1, \cdots, y_j)=f_{G(n)}(y_1)\cdots f_{G(n)}(y_j),$$

and for integers $n\geqq k_i\geqq 2$; $i=1, 2, \cdots, j$, define a mapping

$$\bar{f}_{SG(n)}: E_F^{dk_1-1}\times\cdots\times E_F^{dk_j-1} \to SG(n)$$

by setting

$$\bar{f}_{SG(n)}(y_1, \cdots, y_j)=f_{SG(n)}(y_1)\cdots f_{SG(n)}(y_j).$$

Put

$$e_{G(n)}^{dk_1-1, \cdots, dk_j-1}=\bar{f}_{G(n)}(E_F^{dk_1-1}\times\cdots\times E_F^{dk_j-1}),$$

$$\mathcal{E}_{G(n)}^{dk_1-1, \cdots, dk_j-1}=\bar{f}_{G(r)}(\mathcal{E}_F^{dk_1-1}\times\cdots\times \mathcal{E}_F^{dk_j-1})$$

and

$$e_{SG(n)}^{dk_1-1, \cdots, dk_j-1}=f_{SG(n)}(E_F^{dk_1-1}\times\cdots\times E_F^{dk-j1}),$$

$$\mathcal{E}_{SG(n)}^{dk_1-1, \cdots, dk_j-1}=f_{SG(n)}(\mathcal{E}_F^{dk_1-1}\times\cdots\times \mathcal{E}_F^{dk_j-1})$$

and

$$e^0=I_n.$$

REMARK 5. 2. $O(n)$ has two 0-dimensional cells $e_{O(n)}^0=I_n$ and $\tilde{e}_{O(n)}^0=\begin{pmatrix} -1 & \\ & I_{n-1} \end{pmatrix}$. In the above notation, hewever, we can not distinguish these cells since these are written by the same letter $e_{O(n)}^0$. Confusion, however, will not occur. Zero in the expression $e_{O(n)}^{k_1, \cdots, 0, \cdots, k_j}$ is zero of $\tilde{e}_{O(n)}^0$.

Now, we shall show that $G(n)$ is a cell complex composed of e^0 and $e_{G(n)}^{dk_1-1,\cdots,dk_j-1}$ with $n \geqq k_1 > \cdots > k_j \geqq 1$.

First of all, we shall show that $G(n)$ is the union of cells e^0 and $\mathcal{E}_{G(n)}^{dk_1-1,\cdots,dk_j-1}$ with $n \geqq k_1 > \cdots > k_j \geqq 1$. Since $G(1) = S_F^{d-1}$, we shall assume that above assertion is true for $G(m)$ where $m > n$. If $A \in G(n)$ but $A \notin G(n-1)$, namely $p_1(A) \neq e_n$, then we can choose a point $y \in \mathcal{E}_F^{dn-1}$ such that $\xi_F(y) = p_1(A)$ by lemma 5. 2. If we put $U = f_{G(n)}(y)$, then $U^*A \in G(n-1)$. Hence U^*A belongs to some cell $\mathcal{E}_{G(n)}^{dk_1-1,\cdots,dk_j-1}$ with $n-1 \geqq k_1 > \cdots > k_j \geqq 1$ of $G(n-1)$ by the assumption. Therefore A belongs to a cell $\mathcal{E}_{G(n)}^{dn-1,dk_1-1,\cdots,dk_j-1}$.

Next we shall show that $f_{G(n)}$ maps $\mathcal{E}_F^{dk_1-1} \times \cdots \mathcal{E}_F^{dk_j-1}$ homeomorphically onto $\mathcal{E}_{G(n)}^{dk_1-1,\cdots,dk_j-1}$ and these cells $\mathcal{E}_{G(n)}^{dk_1-1\cdots dk_j-1}$ are disjoint one another. In fact, if $U_1 U_2 \cdots U_s = V_1 V_2 \cdots V_t$, where $U_m \in \mathcal{E}_F^{dk_m-1}$ and if $m > m'$ then $k_m < k_{m'}$, and $V_l \in \mathcal{E}_F^{dk_j-1}$ is also similar one, then $p_1(U_1 U_2 \cdots U_s) = p_1(V_1 V_2 \cdots V_t)$. Since $p_1(U_1 U_2 \cdots U_s) = p_1(U_1)$ and $p_1(V_1 V_2 \cdots V_t) = p_1(V_1)$, we have $p_1(U_1) = p_1(V_1)$. Since ξ_F is homeomorphic, it follows $U_1 = V_1$. Hence $U_2 \cdots U_s = V_2 \cdots V_t$. Similarly $U_2 = V_2$ and so on. Consequently we have $s = t$. Therefore these cells are disjoint to each other. The above proof also gives that $f_{G(n)}$ is one-to-one. The fact that $f_{G(n)}$ is a homeomorphism is obvious from the continuity of the group multiplication and homeomorphism of ξ_F.

Finally, it will be easily verified that the boundary of $e_{G(n)}^{dk_1-1,\cdots,dk_j-1}$ belongs to the lower dimensional skelton than the dimension of $e_{G(n)}^{dk_1-1,\cdots,dk_j-1}$.

By the quite similar method, we see that $SG(n)$ is a cell complex composed of e^0 and $e_{SG(n)}^{dk_1-1,\cdots,dk_j-1}$ with $n \geqq k_1 > \cdots > k_j \geqq 2$.

The dimension of $e_{G(n)}^{dk_1-1,\cdots,dk_j-1}$ (resp. $e_{SG(n)}^{dk_1-1,\cdots,dk_j-1}$) is $(dk_1-1)+\cdots+(dk_j-1)$.

Thus we have the following results.

THEOREM 5. 1. *The orthogonal group $O(n)$ is a cell complex composed of 2^n cells* $e_{O(n)}^0$ and $e_{O(n)}^{k_1,\cdots,k_j}$ with $n > k_1 > \cdots > k_j \geqq 0$.

THEOREM 5. 2. *The special orthogonal group $SO(n)$ is a cell complex composed of 2^{n-1} cells* $e_{SO(n)}^0$ and $e_{SO(m)}^{k_1,\cdots,k_j}$ *with $n > k_1 > \cdots > k_j \geqq 1$. Especially, $e_{SO(n)}^{k-1}$ $(n \geqq k \geqq 2)$ is obtained as the image of the k-dimensional projective space P_{k-1} by the primitive characteristic map $f'_{SO(k)} : P_{k-1} \to SO(k) \subset SO(n)$.*

THEOREM 5. 3. *The unitary group $U(n)$ is a cell complex composed of 2^n cells $e_{U(n)}^0$ and $e_{U(n)}^{2k_1-1,\cdots,2k_j-1}$ with $n \geqq k_1 > \cdots > k_j \geqq 1$. Especially, $e_{U(n)}^{2k-1}$ $(n \geqq k \geqq 1)$ is obtained as the image of the suspended space $E(M_{k-1})$ of $2(k-1)$-dimensional complex projective space M_{k-1} by the primitive characteristic map $f'_{U(k)} : E(M_{k-1}) \to U(k) \subset U(n)$.*

THEOREM 5. 4. *The special unitary group $SU(n)$ is a cell complex composed of 2^{n-1} cells $e_{SU(n)}^0$* and $e_{SU(n)}^{2k_1-1,\cdots,2k_j-1}$ *with $n \geqq k_1 > \cdots > k_j \geqq 2$. Especially, $e_{SU(n)}^{k}$ $(n \geqq k \geqq 2)$ is obtained as the image of the another suspended space $E(M'_{k-1})$ of*

$2(k-1)$-dimensional complex projective space M_{k-1} by the primitive characteristic map $f'_{SU(k)}: E(M_{k-1})\to SU(k)\subset SU(n)$.

THEOREM 5. 5. *The symplectic group $Sp(n)$ is a cell complex composed of 2^n cells $e^0_{Sp(n)}$ and $e^{4k_1-1,\cdots,4k_j-1}_{Sp(n)}$ with $n\geqq k_1>\cdots>k_j\geqq 1$.*

To give a cell structure of the Stiefel manifold $S_{n,m}$ with $n>m$, we put

$$e^{dk_1-1,\cdots,dk_j-1}_{S_{n,m}}=p_m\,(e^{dk_1-1,\cdots,dk_j-1}_{G(n)})$$
$$=p_m\,(e^{dk_1-1,\cdots,dk_j-1}_{SG(n)})$$

and

$$e^0_{S_{n,m}}=(e_{n-m+1},\cdots,e_{n-1},\ e_n).$$

Then $S_{n,m}$ is a cell complex composed of the cells $e^0_{S_{n,m}}$ and $e^{dk_1-1,\cdots,dk_j-1}_{n,m}$ with $n\geqq k_1>\cdots>k_j\geqq n-m+1$. The proof will be analogously performed as in the proof of the case $G(n)$ [11].

THEOREM 5. 4. *The real Stiefel manifold $V_{n,m}=O(n)/O(n-m)=SO(n)/SO(n-m)$ is a cell complex composed of 2^m cells $e^0_{V_{n,m}}$ and $e^{k_1,\cdots,k_j}_{V_{n,m}}$ with $n>k_1>\cdots>k_j\geqq n-m$.*

THEOREM 5. 5. *The complex Stiefel manifold $W_{n,m}=U(n)/U(n-1)=SU(n)/SU(n-m)$ is a cell complex composed of 2^m cells $e^0_{W_{n,m}}$ and $e^{2k_1-1,\cdots,2k_j-1}_{W_{n,m}}$ with $n\geqq k_1>\cdots>k_j\geqq n-m+1$.*

THEOREM 5. 6. *The quaternion Stiefel manifold $X_{n,m}=SP(n)/Sp(n-m)$ is a cell complex composed of 2^m cells $e^0_{X_{n,m}}$ and $e^{4k_1-1,\cdots,4k_j-1}_{X_{n,m}}$ with $n\geqq k_1>\cdots>k_j\geqq n-m+1$.*

6. Cellular decompositions of $F_n=SO(2n)/U(n)$ and $X_n=SU(2n)/Sp(n)$

A complex number (resp. quaternion number) a may be represented in the form $a=a_1+\boldsymbol{i}a_2$[4], where a_1 and a_2 are real numbers (resp. $a=a_1+\boldsymbol{j}a_2$[4], where a_1 and a_2 are complex numbers). Define an isomorphic mapping $\varphi^n_{RC}: C^n\to R^{2n}$ (resp $\varphi^n_{CQ}: Q^n\to C^{2n}$) by the farmula

$$\varphi^n_{RC}\,(a_1,\cdots,a_n)=(x_1,x_2,\cdots,x_{2n-1},x_{2n}),$$

where $a_k=x_{2k-1}+\boldsymbol{i}\,x_{2k}$; x_{2k-1} and $x_{2k}\in R$,

$$\text{(resp. }\varphi^n_{CQ}\,(a_1,\cdots,a_n)=(x_1,x_2,\cdots,x_{2n-1},x_{2n}),$$

where $a_k=x_{2k-1}+\boldsymbol{j}\,x_{2k}$, x_{2k-1} and $x_{2k}\in C$) and also define $\varphi^n_{RQ}: Q^n\to R^{4n}$ by

$$\varphi^n_{RQ}\,(a_1,\cdots,a_n)=(x_1,x_2,x_3,x_4,\cdots,x_{4n-3},x_{4n-2},x_{4n-1},x_{4n}),$$

where $a_k=x_{4l-3}+\boldsymbol{i}\,x_{4k-2}+\boldsymbol{j}\,x_{4k-1}+\boldsymbol{k}\,x_{4k}$; $x_{4k-3},x_{4k-2},x_{4k-1}$ and $x_{4k}\in R$. Then these mappings induce homeomorphisms onto $\varphi^n_{RC}: C^n\to R^{2n}$, $\varphi^n_{CQ}: Q^n\to C^{2n}$, $\varphi^n_{RC}: Q^n\to R^{4n}$; $\varphi^n_{RC}: E^{2n}_C\to E^{2n}_R$, $\varphi^n_{CQ}: E^{4n}_Q\to E^{4n}_C$, $\varphi^n_{RQ}: E^{4n}_Q\to E^{4n}_R$ and $\varphi^n_{RC}: S^{2n-1}_C\to S^{2n-1}_R$, $\varphi^n_{CQ}: S^{4n-1}_Q\to S^{4n-1}_C$, $\varphi^n_{RQ}: S^{4n-1}_Q\to S^{4n-1}_R$.

By this isomorphism, a unitary (resp. symplectic) linear transformation of C^n (resp. Q^n) induces an orthogonal (resp. unitary) linear transformation of R^{2n}

4) $\{1,\boldsymbol{i}\}$ (resp. $\{1,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ is the usual base of C (resp. Q) over R

(resp. C^{2n}), that is, $U(n)$ (resp. $Sp(n)$) is a subgroup of $O(2n)$ (resp. $U(2n)$). In matrix notation, we assign $A=(a_{ij})\in U(n)$ (resp. $Sp(n)$)

$$A' = \left(\begin{pmatrix} x_{2i-1,\,2j-1} & -\bar{x}_{2i,\,2j} \\ x_{2i,\,2j} & \bar{x}_{2i-1,\,2j-1} \end{pmatrix}\right) \in O\,(2n)\ \big(\text{resp. } U(2n)\big)$$

where $a_{ij} = x_{2i-1,\,2j-1}+\boldsymbol{i}\,x_{2i,\,2j}$; $x_{2i-1,\,2j-1}$, $x_{2i,\,2j} \in R$ (resp. $a_{ij} = x_{2i-1,\,2j-1}+\boldsymbol{j}\,x_{2i,\,2j}$; $x_{2i-1,\,2j-1}$, $x_{2i,\,2j} \in C$). As is easily verified, A' satisfies the equality ${}^tA'JA'=J$ where $J=\begin{pmatrix} J' & & \\ & \ddots & \\ & & J' \end{pmatrix}$ with $J' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence we have readily $U(n)\subset SO(2n)$ (resp. $Sp(n)\subset SU(2n)$).

We define F_n to be $SO(2n)/U(n)$ and X_n to be $U(2n)/Sp(n)$ respectively. Denote by $p_{F_n}: SO(2n)\to F_n$ (resp. $p_{X_n}: SU(2n)\to X_n$) the projection.

Since $U(n-1)=SO(2n-1)\cap U(n)$ and $SO(2n)=SO(2n-1)U(n)$ (resp. $Sp(n-1)=SU(2n-1)\cap Sp(n)$ and $SU(2n) = SU(2n-1)Sp(n)$, the inclusion map $\psi: SO(2n-1)\to SO(2n)$ (resp. $\phi: SU(2n-1)\to SU(2n)$) induces a bundle isomorphism $\bar{\psi}$: $SO(2n-1)/U(n-1) \cong SO(2n)/U(n)=F_n$ (resp. $\bar{\phi}$: $SU(2n-1)/Sp(n-1) \cong SU(2n)/Sp(n)=X_n$). Hence we have a natural embedding $F_{n-1}\subset F_n$ (resp. $X_{n-1}\subset X_n$).

Now, to give a cell structure of F_n (resp. X_n), we shall define the primitive characteristic map $f_{F_n}: E_R^{2n-2}\to F_n$ (resp. $f_{X_n}: E_C^{4n-3}\to X_n$) by $f_{F_n}=p_{F_n}\circ\psi\circ f_{SO(2n-1)}$ (resp. $f_{X_n}=p_{X_n}\circ\phi\circ f_{SU(2n-1)}$).

Similarly, we define the characteristic map $\bar{f}_{F_n}: E_R^{2k_1}\times\cdots\times E_R^{2k_j}\to F_n$ (resp. $f_{X_n}: E_C^{4k_1-3}\times\cdots\times E_C^{4k_j-3}\to X_n$) by $f_{F_n} = p_{F_n}\circ\psi\circ f_{SO(2n-1)}$ (resp. $p_{X_n}\circ\phi\circ f_{SU(2n-1)}$). Put

$$e_{F_n}^{2k_1,\cdots,2k_j} = \bar{f}_{F_n}(E_R^{2k_1}\times\cdots\times E_R^{2k_j})$$

and

$$e_{F_n}^{0} = p_{F_n}\big(U(n)\big),$$

$$(\text{resp. } e_{X_n}^{4k_1-3,\cdots,4k_j-3} = \bar{f}_{X_n}(E_C^{4k_1-1}\times\cdots\times E_C^{4k_j-3})$$

and

$$e_{X_n}^{0} = p_{X_n}\big(Sp(n)\big).$$

Since $\bar{f}_{SO(2n-1)}$ (resp. $\bar{f}_{SU(2n-1)}$) is homeomorphic on $\mathcal{E}_R^{2k_1}\times\cdots\times\mathcal{E}_R^{2k_j}$, with $n>k_1>\cdots>k_j\geqq 1$ (resp. $\mathcal{E}_C^{4k_1-3}\times\cdots\times\mathcal{E}_C^{4k_j-3}$; $n\geqq k_1>\cdots>k_j\geqq 2$), we can see easily that $\bar{f}_{F_n} = \bar{\psi}\circ q_{F_n}\circ\bar{f}_{SO(2n-1)}$ (resp. $\bar{f}_{X_n} = \bar{\phi}\circ q_{X_n}\circ\bar{f}_{SU(2n-1)}$) is also homeomorphic on $\mathcal{E}_R^{2k_1}\times\cdots\times\mathcal{E}_R^{2k_j}$ (resp. $\mathcal{E}_C^{4k_1-3}\times\cdots\times\mathcal{E}_C^{4k_j-3}$), where $q_{F_n}: SO\,(2n-1)\to F_n$ (resp. $q_{X_n}: SU(2n-1)\to X_n$) the projection. Hence we have readily the following theorems, applying the same techniques that were used in the proof of the case $G(n)$.

THEOREM 7. 1. *$F_n = SO(2n)/U(n)$ is a cell complex composed of 2^{n-1} cells $e_{F_n}^0$ and $e_{F_n}^{2k_1,\cdots,2k_j}$ with $n>k_1>\cdots>k_j\geqq 1$.*

THEOREM 7. 2. *$X_n = SU(2n)/Sp(n)$ is a cell complex composed of 2^{n-1} cells $e_{X_n}^0$ and $e_{X_n}^{4k_1-3,\cdots,4k_j-3}$ with $n\geqq k_1>\cdots>k_j\geqq 2$.*

7. Homology and cohomology groups of $G(n)$, $SG(n)$, $S_{n,m}$ F_n and X_n

In order to determine the homology and cohomology groups of a cell complex, we have to select orientations of cells exist in it. For this purpose, we shall begin by selecting orientaion of $E^n=E^n_R$.

We recall that an orientation of E^n (resp. $S^n=S^n_R$) is simply a generator of the integral relative homology group $H_n(E^n, S^{n-1}; Z)=Z$ (resp. $H_n(S^n, e_{n+1}; Z)=Z$). We first orient E^1, that is, we select a generator E_1 of $H_1(E^1, S^0; Z)$ and fix this generator. We now suppose that orientations S_{n-1} of S^{n-1} and E_n of E^n have been selected and proceed to define inductively first S_n and then E_{n+1}. If a mapping $\eta^n : (E^n, S^{n-1})\to(S^n, e_{n+1})$ is defined by the formula

$$\eta^n(x) = (2x_1 x_{n+1}, \cdots, 2x_n x_{n+1}, 1-2x^2_{n+1}),$$

where $x_{n+1}=\sqrt{1-|x|^2}$, then η^n induces an isomorphism η^n_*; $H_n(E^n, S^{n-1}; Z)\to H_n(S^n; Z)$; we set $S_n=\eta^n_*(E_n)$. The boundary homomorphism $\partial=\partial_{n+1}$ maps $H_{n+1}(E^{n+1}, S^n; Z)$ isomorphically onto $H_n(S^n; Z)$, we set $E_{n+1}=\partial^{-1}(S_n)$. The choice of orientations is indicated by the diagram

$$H_1(E^1, S^0; Z)\to \cdots\cdots\to H_n(E^n, S^{n-1}; Z)\xrightarrow{\eta^n_*} H_n(S^n; Z)\xleftarrow{\partial}$$
$$H_{n+1}(E^{n+1}, S^n; Z)\to \cdots\cdots .$$

Let X be a cell complex, e^k_X a cell of X and $f_X : E^k\to e^k_X\subset X$ a characteristic map used to define the cell e^k_X. Then we orient e^k_X so that $f_*(E_k)=e^X_k$.

Define a homeomorphism

$$\tau_{n,m} : (E^n\times E^m, E^n\times S^{m-1}\cup S^{n-1}\times E^m) \to (E^{n+m}, S^{n+m-1})$$

by the formula

$$\tau_{n,m}(x, y) = (x\lambda, y\lambda),$$

where $\lambda=(max(|x|, |y|))/\sqrt{|x|^2+|y|^2}$. To orient $E^n\times E^m$, we shall use the mapping $\tau^{-1}_{n,m} : E^{n+m}\to E^n\times E^m$ as a characteristic map for $E^n\times E^m$.

E^{2n}_C, E^1_C and S^{2n-1}_C (resp. E^{4n}_Q, E^3_Q and S^{4n-1}_Q) are oriented by the mapping $\varphi^{n-1}_{RC} : E^{2n}_R\to E^{2n}_C$ etc. (resp. $\varphi^{n-1}_{RQ} : E^{4n}_R\to E^{4n}_Q$ etc.).

LEMMA 7. 1. *Let ξ^n_R, ξ^n_C and ξ^n_Q be the mappings defined in* § 5. *Then we have*

7. 1) $$\xi^n_{R*}(E^R_n) = (-1)^n S^R_n,$$

7. 2) $$\xi^n_{C*}(E^C_{2n-1}) = -S^C_{2n-1},$$

7. 3) $$\xi^n_{Q*}(E^Q_{4n-1}) = -S^Q_{4n-1}.$$

Proof. The first formula is trivial because

$$\xi^n_R(x) = \eta^n(x)\begin{pmatrix}-I_{n-1} & \\ & 1\end{pmatrix}.$$

To prove the second formula, we shall compute the mapping degree of the composition $\tau^{-1}_{1,2n-2}\circ\lambda^n_C$ of the mappings

$$E_R^{2n-1}\xrightarrow{\tau_{1,2n-2}^{-1}}E_R^1\times E_R^{2n-2}\xrightarrow{\varphi_{RC}^{1\,-1}\times\varphi_{RC}^{n\,-1}}E_C^1\times E_C^{2n-1}\xrightarrow{\xi_C^n}S_C^{2n-1}\xrightarrow{\varphi_{RC}}S_R^{2n-1}\xrightarrow{\eta^{-1}}E_R^{2n-1}.$$

we shall define an another mapping $\lambda: E_R^1\times E_R^{2n-2}\to E_R^{2n-1}$ by setting

$$\lambda(t,x)=(-x_1\sqrt{1-t^2}-x_1't,\ x_1t-x_1'\sqrt{1-t^2},\cdots$$
$$-x_{n-1}\sqrt{1-t^2}-x_{n-1}'t,\ x_{n-1}t-x_{n-1}'\sqrt{1-t^2},\ \sqrt{1-|x|^2t}),$$

where $(t,x)=(t,x_1,x_1',\cdots,x_{n-1},x_{n-1}')\in E_R^1\times E_R^{2n-2}$. If we compute the local degree at $(t,x)=0$, we have $+1$. However $\lambda_{C*}^n=-\lambda_*$ since last two terms of λ and λ_C^n are exchanged to each other. Hence we have the lemma easily.

The last formula is proved analogously as in the proof of the second formula.

The boundary homomorphism on the real projective space P_n is given as follows

$$\begin{cases}\partial u_{2k+1}=0,\\ \partial u_{2k}=2u_{2k-1}.\end{cases}$$

The boundary homomorphisms on spaces appeared in the preceeding sections are also easily calculated. As for $V_{n,m}$, we prefer the results of [7].

LEMMA 7. 2. *The boundary homomorphism ∂ on $V_{n,m}$ is given by*

$$\partial e_{k_1,\cdots,k_j}^{V_{n,m}}=\sum_{i=1}^{j}(-1)^{k_1+\cdots+k_{i-1}}((-1)^{k_i}+1)\,e_{k_1,\cdots,k_i-1,\cdots,k_j}^{V_{n,m}},$$

where $n>k_1>\cdots>k_j\geqq n-m$ and the symbol $e_{k_1,\cdots,k_i-1,\cdots,k_j}^{V_{n,m}}=0$ if $i>1$ and $k_i-1=k_{i+1}$, or if $i=1$ aud $j=n-m$.

The coboundary homomorphism δ is given by

$$\delta e_{V_{n,m}}^{k_1,\cdots,k_j}=\sum_{i=1}^{j}(-1)^{k_1+\cdots+k_{i-1}}((-1)^{k_i}+1)\,e_{V_{n,m}}^{k_1,\cdots,k_i+1,\cdots,k_j}$$

where $n>k_1>\cdots>k_j\geqq n-m$ and the symbol $e_{V_{n,m}}^{k_1,\cdots,k_i+1,\cdots k_j}=0$ if $i<j$ and $k_i+1=k_{i-1}$ or $i=j$ and $k_1=n-1$.

LEMMA 6. 3. *The boundary and coboundary homomorphisms are trivial in all dimensions for $U(n)$, $SU(n)$, $W_{n,m}$; $Sp(n)$, $X_{n,m}$; F_n and X_n.*

Therefore we have the following theorems. The details of theorem 7. 1 appear in [7].

THEOREM 7. 1. *$V_{n,m}$ has only torsion groups of order 2 and the Poincaré polynomial is*

$$P_{V_{n,m}}(t)=\begin{cases}(1+t^{2n-2m+1})(1+t^{2n-2m+5})\cdots(1+t^{2n-3})\\ \qquad \text{if } n \text{ is odd and } m \text{ is even},\\ (1+t^{2n-2m+1})(1+t^{2n-2m+5})\cdots(1+t^{2n-5})(1+t^{n-1})\\ \qquad \text{if } n \text{ is even and } m \text{ is odd},\\ (1+t^{n-m})(1+t^{2n-2m+3})(1+t^{2n-2m+7})\cdots(1+t^{2n-3})\\ \qquad \text{if } n \text{ and } m \text{ are odd},\\ (1+t^{n-m})(1+t^{2n-2m+3})(1+t^{2n-2m+7})\cdots(1+t^{2n-5})(1+t^{n-1})\\ \qquad \text{if } n \text{ and } m \text{ are even}.\end{cases}$$

Especially,

$$P_{SO(2n+1)}(t)=(1+t^3)(1+t^7)\cdots(1+t^{4n-1}).$$

and

$$P_{SO(2n)}(t) = (1+t^3)(1+t^7)\cdots(1+t^{4n-5})(1+t^{2n-1}).$$

THEOREM 7. 2. *$W_{n,m}$ has no torsion group and the Poincaré polynomial is*

$$P_{W_{n,m}}(t) = (1+t^{2n-2m+1})(1+t^{2n-2m+3})\cdots\cdots(1+t^{2n-1}).$$

Especially,

$$P_{U(n)}(t) = (1+t^1)(1+t^3)(1+t^5)\cdots\cdots(1+t^{2n-1})$$

and

$$P_{SU(n)}(t) = (1+t^3)(1+t^5)\cdots\cdots(1+t^{2n-1}).$$

THEOREM 7. 3. *$X_{n,m}$ has no torsion group and the Poincaré polynomoal is*

$$P_{X_{n,m}}(t) = (1+t^{4n-4m+3})(1+t^{4n-4m+7})\cdots\cdots(1+t^{4n-1}).$$

Especially,

$$P_{Sp(n)}(t) = (1+t^3)(1+t^7)\cdots\cdots(1+t^{4n-1}).$$

THEOREM 7. 4. *F_n and X_n have no torsion group and their Poincaré polynomials are*

$$P_{F_n}(t) = (1+t^2)(1+t^4)\cdots\cdots(1+t^{2n-2})$$

and

$$P_{X_n}(t) = (1+t^5)(1+t^9)\cdots\cdots(1+t^{4n-3})$$

respectively.

The following lemmas will be easily verified.

LEMMA 7. 4. *The projections $p_m: O(n)$ (resp. $SO(n)$)$\to V_{n,m}$, $p_m: U(n)$ (resp. $SU(n)$)$\to W_{n,m}$ and $p_m: Sp(n)\to X_{n,m}$ are cellular.*

7.4. 1) *If $n>k_1>\cdots\cdots>k_j\geqq 1$, then*

$$p_{m*}(e^{SO(n)}_{k_1,\cdots\cdots,k_j}) = \begin{cases} 0 & \text{for } k_j<n-m, \\ e^{V_{n,m}}_{k_1,\cdots\cdots,k_j} & \text{for } k_j\geqq n-m, \end{cases}$$

and if $n>k_1>\cdots\cdots>k_j\geqq n-m$, then

$$p^{*}_m(e^{k_1,\cdots\cdots,k_j}_{V_{n,m}}) = e^{k_1,\cdots\cdots,k_j}_{SO(n)}.$$

7.4. 2) *If $n\geqq k_1>\cdots\cdots>k_j\geqq 1$ (resp. $n\geqq k_1>\cdots\cdots>k_j\geqq 2$), then*

$$p_{m*}(e^{U(n)}_{2k_1-1,\cdots\cdots,2k_j-1}) = (p_{m*}(e^{SU(n)}_{2k_1-1,\cdots\cdots,2k_j-1})\ (n\neq m))$$

$$= \begin{cases} 0 & \text{for } k_j\leqq n-m, \\ e^{W_{n,m}}_{2k_1-1,\cdots\cdots,2k_j-1} & \text{for } k_j\geqq n-m+1, \end{cases}$$

and if $n\geqq k_1>\cdots\cdots>k\ \geqq n-m+1$, then

$$p^*_m(e^{2k_1-1,\cdots\cdots,2k_j-1}_{W_{n,m}}) = e^{2k_1-1,\cdots\cdots,2k_j-1}_{U(n)}$$

$$(= e^{2k_1-1,\cdots\cdots,2k_j-1}_{SU(n)}\ (n\neq m)).$$

Especially p_{m*} is onto and p^*_m is isomorphic into.

7.4. 3) *If $n\geqq k_1>\cdots\cdots>k_j\geqq 1$, then*

$$p_{m*}(e^{Sp(n)}_{4k_1-1,\cdots\cdots,4k_j-1}) = \begin{cases} 0 & \text{for } k_j\leqq n-m \\ e^{X_{n,m}}_{4k_1-1,\cdots\cdots,4k_j-1} & \text{for } k_j\geqq n-m+1, \end{cases}$$

and if $n\geqq k_1>\cdots\cdots>k_j\geqq n-m+1$, then

$$p^*_m(e^{4k_1-1,\cdots\cdots,4k_j-1}_{X_{n,m}}) = e^{4k_1-1,\cdots\cdots,4k_j-1}_{Sp(n)}$$

Especially, p_{m*} is ontc and p^*_m is isomorphic into.

LEMMA 7. 5. *The projections* $p_{F_n}: SO(2n)\rightarrow F_n$ *and* $p_{X_n}: SU(2n)\rightarrow X_n$ *are cellular.*

7.5. 1) *If* $n>k_1>\cdots>k_j\geqq 1$, *then*

$$p_{F_n*}(e^{SO(2n)}_{2k_1,\ldots,2_jk})=e^{F_n}_{2k_1,\ldots,2k_j},$$

and if $2n>k_1>\cdots>k_j\geqq 1$, *then*

$$p^*_{F_n}(e^{SO(n)}_{k_1,\ldots,k_j})=0 \quad \textit{for some } k_i \textit{ is odd,}$$

and if $n>k_1>\cdots>k_j\geqq 1$, *then*

$$p^*_{F_n}(e^{2k_1,\ldots,2k_j}_{F_n})=e^{2k_1,\ldots,2k_j}_{SO(2n)}.$$

7.5. 2) *If* $n\geqq k_1>\cdots>k_j\geqq 2$, *then*

$$p_{X_n*}(e^{SU(2n)}_{4k_1-3,\ldots,4k_j-3})=e^{X_n}_{4k_1-3,\ldots,4k_j-3},$$

and if $2n\geqq k_1>\cdots>k_j\geqq 2$, *then*

$$p_{X_n*}(e^{SU(2n)}_{k_1,\ldots,k_j})=0 \quad \textit{for some } k_i\not\equiv -3 \ (\textit{mod } 4)$$

and if $n\geqq k_1>\cdots>k_j\geqq 2$, *then*

$$p^*_{X_n}(e^{4k_1-3,\ldots,4k_j-3}_{X_n})=e^{4k_1-3,\ldots,4k_j-3}_{SU(2n)}.$$

Especially, p_{X_n*} *is onto and* $p^*_{X_n}$ *is isomorphic into.*

We shall use the following lemma [3].

LEMMA 7. 6. *Let* $q: E\rightarrow B$ *a be compact, connected fi re space with fibre* F. *Then the following two conditions are equivalent:*

7.6. 1) $i_*: H(F;\ Z_p)\rightarrow H(E;\ Z_p)$ *is isomorphic into, where* $i: F\rightarrow E$ *is an injection.*

7.6. 2) $${}_pP_E(t)={}_pP_B(t)\,{}_pP_F(t).$$

If E,B and F have no torsion group, then the lemma is also valid for the integral coefficient.

Using this lemma, we have

LEMMA 7. 7. $i_*: H(Sp(n);\ Z)\rightarrow H(SU(2n);\ Z)$ is isomorphic into.

Proof. $P_{SU(2n)}(t)=P_{X_n}(t)\,P_{Sp(n)}(t)=(1+t^3)(1+t^5)\cdots\cdots(1+t^{2n-3})(1+t^{2n-1})$ and $SU(2n)$, X_n and $Sp(n)$ have no torsion group.

REMARK 7. 1. Using that $p_{SU(n)}\circ i=p_{Sp(n)}$ in the diagram

$$\begin{array}{ccc} Sp(n) & \xrightarrow{\quad i \quad} & SU(2n) \\ & {\scriptstyle p_{Sp(n)}}\searrow \quad \swarrow{\scriptstyle p_{SU(2n)}} & \\ & S^{4n-1} & \end{array}$$

we can prove lemma 7.7 directly without lemma 7.6.

8. Pontrjagin product in $G(n)$ and $SG(n)$

For any topological group G and for any coefficient ring Γ, it is possible to define a multiplication in $H(G;\ \Gamma)$ in such a way that $H(G;\ \Gamma)$ becomes an associative algebra, called Pontrjagin algebra $H_*(G;\ \Gamma)$ of G with coefficient Γ. Pontrjagin product will be denote by the symbol.$*$ If G_1 and G_2 are two topological groups and $g: G_1\rightarrow G_2$ is a continuous (group)-homomprphism, then

$g_*: H_*(G_1; \Gamma) \to H_*(G_2; \Gamma)$ is also a (algebraic)-homomorphism (i. e. $g_*(a \ast b) = g_*(a) \ast g_*(b)$).

LEMMA 8. 1. *In $G(n)$ and $SG(n)$, we have*

$$e_{G(n)}^{dk_1-1, dk_2-1} = e_{G(n)}^{dk_2-1, dk_1-1} \quad \text{for } n \geqq k_1,\ k_2 \geqq 1,$$

and

$$e_{SG(n)}^{dk_1-1, dk_2-1} = e_{SG(n)}^{dk_2-1, dk_1-1} \quad \text{for } n \geqq k_1,\ k_2 \geqq 2.$$

Proof. Without loss of generality, we may assume that $k_1 = k_2 - 1 = n-1$. Give any point $AB \in e_{G(n)}^{d(n-1)-1, dn-1}$, where $A = \begin{pmatrix} a_{ij} & \vdots \\ \cdots & \vdots \\ & 1 \end{pmatrix} \in e_{G(n)}^{d(n-1)-1}$ and $B = (b_{ij}) \in e_{G(n)}^{dn-1}$ with $b_{ij} = \delta_{ij} + x_i p \bar{x}_j$, $\sum_{i=1}^{n} |x_i|^2 = 1$ and $x_i,\ p \in F$. If we choose $C = (c_{ij}) \in e_{G(n)}^{dn-1}$ with $c_{ij} = \delta_{ij} + z_i p \bar{z}_j$, where $z_i = \sum_{k=1}^{n-1} a_{ik} x_k$, $i = 1, 2, \cdots, n-1$ and $z_n = x_n$, then we have $AB = CA$. In fact,

$$AB = \begin{pmatrix} a_{ij} & \\ & 1 \end{pmatrix} \begin{pmatrix} & \\ & b_{ij} & \\ & \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n-1} a_{ij} b_{jk} \\ b_{n1}, \cdots\cdots, b_{nn} \end{pmatrix},$$

where $\sum_{j=1}^{n-1} a_{ij} b_{jk} = \sum_{j=1}^{n-1} a_{ij} (\delta_{jk} + x_j p \bar{x}_k) = a_{ik} + \sum_{j=1}^{n-1} a_{ij} x_j p \bar{x}_k$.
When $k = n$, we have furthermore

$$\sum_{j=1}^{n-1} a_{ij} b_{jn} = \delta_{in} + \sum_{j=1}^{n-1} a_{ij} x_j p \bar{x}_n = \delta_{in} + z_i p \bar{z}_n = c_{in}.$$

On the other hand,

$$CA = \begin{pmatrix} & \\ & c_{il} & \\ & \end{pmatrix} \begin{pmatrix} a_{lk} & \\ & 1 \end{pmatrix} = \begin{pmatrix} & c_{1n} \\ \sum_{l=1}^{n-1} c_{il} a_{lk} & \vdots \\ & c_{nn} \end{pmatrix},$$

where $\sum_{l=1}^{n-1} c_{il} a_{lk} = \sum_{l=1}^{n-1} (\delta_{il} + z_i p \bar{z}_l) a_{lk}$

$$= a_{ik} + \sum_{j,s,l=1}^{n-1} a_{ij} x_j p \bar{x}_s a_{ls} \bar{a}_{lk}$$

$$= a_{ik} + \sum_{j=1}^{n-1} a_{ij} x_j p \bar{x}_k \ \text{(because } \textstyle\sum_{l=1}^{n-1} \bar{a}_{ls} a_{lk} = \delta_{sk}\text{)}.$$

when $i = n$, we have furthermore

$$\sum_{l=1}^{n-1} c_{nl} c_{lk} = \delta_{nk} + \sum_{j=1}^{n-1} a_{nj} x_j p \bar{x}_k = \delta_{nk} + x_n p \bar{x}_k = b_{nk}.$$

Thus the first formula is proved.

It should be noted that this calculation is valid even if we take $A \in G(n-1)$ instead of $A \in e_{G(n)}^{d(n-1)-1}$.

The second formula is proved using the first formula, that is, given $A \in e_{SG(n)}^{d(n-1)-1}$ and $B \in e_{SG(n)}^{dn-1}$, then B is expressed by the form $B = B_1 T$, where $B_1 \in e_{G(n)}^{dn-1}$ and $T = \begin{pmatrix} -(q + \sqrt{1-|q|^2})^2 & \\ & I_{n-1} \end{pmatrix}$. For A and B_1, there exist $C_1 \in e_{G(n)}^{dn-1}$ such that $AB_1 = C_1 A$ by the note of the first formula. Hence

$$AB = AB_1 T = C_1 A T = (C_1 T)(T^{-1} \cdot AT),$$

where $C_1 T \in e_{SG(n)}^{dn-1}$ and $T^{-1} AT \in e_{SG(n)}^{d(n-1)-1}$. q.e.d.

LEMMA 8. 2. *If the field is commutative (i. e. except $G(n) = Sp(n)$), we have*

$$e_{G(n)}^{dk-1, dk-1} = e_{G(n)}^{dk-1, d(k-1)-1} \quad \text{for } n \geqq k \geqq 1,$$

and

$$e_{SG(n)}^{dk-1,\, dk-1} = e_{SG(n)}^{dk-1,\, d(k-1)-1} \qquad for\ n \geqq k \geqq 2.$$

Proof. Without loss of generality, we may assume $k=n$. Give any point $AB \in e_{G(n)}^{dn-1,\, dn-1}$, where $A=(a_{ij}) \in e_{G(n)}^{dn-1}$ with $a_{ij}=\delta_{ij}+p\, x_i \bar{x}_j$, $\sum_{i=1}^n |x_i|^2=1$, $p+\bar{p}+|p|^2=0$ and $B=(b_{ij}) \in e_{G(n)}^{dn-1}$ with $b_{ij}=\delta_{ij}+q\, y_i \bar{y}_j$, $\sum_{i=1}^n |y_i|^2=1$, $q+\bar{q}+|q|^2=0$. Choose $r \in F$, $r+\bar{r}+|r|^2=0$ and $z_n \in R$, $z_n>0$ which satisfy the equation (lemma 5. 1)

$$rz_n^2 = p|x_n|^2+q|y_n|^2+pq\, x_n \bar{y}_n \lambda, \text{ where } \lambda = \textstyle\sum_{i=1}^n \bar{x}_i y_i,$$

and determine z_l; $l=1, 2, \cdots, n-1$, from the equations

$$rz_l z_n = px_l \bar{x}_n+qy_l \bar{y}_n+pq\, x_l y_n, \quad l = 1, 2, \cdots, n-1.$$

Using lemma 5. 1 again, $s\,(s+\bar{s}+|s|^2=0)$, $t_1, t_2, \cdots, t_{n-2} \in F$ and $t_{n-1} \in R$, $t_{n-1}>0$ are determined from the equations

$$st_l\, t_{n-1} = \frac{pq\,(y_n x_l-x_n y_l)\, \overline{(y_n x_{n-1}-x_n y_{n-1})}}{rz_n^2}, \quad l = 1, 2, \cdots, n-1.$$

Then we have $st_i \bar{t}_k = \dfrac{st_i \bar{t}_{n-1} \bar{s}t_{n-1} \bar{t}_k}{\bar{s}t_{n-1}^2} = \dfrac{pq\,(y_n x_i-x_n y_i)\, \overline{(y_n x_{k-1}-x_n y_{k-1})}}{rz_n^2}$.

Now, if we take C and D as $C=(c_{ij}) \in e_{G(n)}^{dn-1}$, where $c_{ij}=\delta_{ij}+r\, z_i \bar{z}_j$, and $D=\begin{pmatrix} d_{ij} & \\ & 1 \end{pmatrix} \in e_{G(n)}^{d(n-1)-1}$, where $d_{ij}=\delta_{ij}+st_i \bar{t}_j$, then we have $AB=CD$ by the direct calculation.

The second formula is proved by the slight modification. Given a point $AB \in e_{SG(n)}^{dn-1,\, dn-1}$, then $A, B \in e_{SG(n)}^{dn-1}$ are expressed by the form $A=A_1 T$, $B=B_1 T$, where $A_1, B_1 \in e_{G(n)}^{dn-1}$. Since $TB_1T^{-1} \in e_{G(n)}^{dn-1}$, for A_1 and TB_1T^{-1} there exist $C_1 \in e_{G(n)}^{dn-1}$ and $D_1 \in e_{G(n)}^{d(n-1)-1}$ such that $A_1TB_1T^{-1}=C_1D_1$ by the first formula. Hence we have $AB=A_1TB_1T=A_1TB_1T^{-1}TT=C_1D_1TT=C_1T(T^{-1}D_1T)T$, where $C_1T \in e_{SG(n)}^{dn-1}$ and $(T^{-1}D_1T)T \in e_{SG(n)}^{d(n-1)-1}$. q.e.d.

LEMMA 8. 3. *Let* $\tilde{f}_{G(n)} : E_F^{d-1} \times S_F^{dn-1} \to G(n)$ *be the map defined in* §2. *Then we have*

$$A\, \tilde{f}_{G(n)}(q, x)\, A^{-1} = \tilde{f}_{G(n)}(q, Ax), \qquad for\ A \in G(n).$$

Proof. The (l, k)-element of $A\, \tilde{f}_{G(n)}(q, x)\, A^{-1} = \sum_{i,j=1}^n a_{li}(\delta_{ij}+x_i p \bar{x}_j)\, \bar{a}_{kj} = \delta_{ij}+(\sum_{i=1}^n a_{li} x_i)\, p\, \overline{(\sum_{j=1}^n a_{kj} x_j)}$ = the (l, k)-element of $\tilde{f}_{G(n)}(q, Ax)$.

LEMMA 8. 4. *Let* $*$ *denote the integral chain Pontrjagin product in* $G(n)$ *or* $SG(n)$. *Then we have*

8.4. 1) $\quad e_{dk_1-1}^{G(n)} * \cdots * e_{dk_j-1}^{G(n)} = e_{dk_1-1, \ldots, dk_j-1}^{G(n)}$

for $n \geqq k_1 > \cdots > k_j \geqq 1$,

8.4. 2) $\quad e_{dk_1-1}^{SG(n)} * \cdots * e_{dk_j-1}^{SG(n)} = e_{dk_1-1, \ldots, dk_j-1}^{SG(n)}$

for $n \geqq k_1 > \cdots > k_j \geqq 2$.

8.4. 3) $\quad e_0^{G(n)}$ (resp. $e_0^{SG(n)}$) *is a unit with respect to* $*$.

8.4. 4) $\quad e_{dk-1}^{G(n)} * e_{dk-1}^{G(n)} = 0$ (except $G(n)=Sp(n)$).

8.4. 5) $\quad e_{dk-1}^{SG(n)} * e_{dk-1}^{SG(n)} = 0$.

8.4. 6) $e^{G(n)}_{dk_1-1} * e^{G(n)}_{dk_2-1} = (-1)^{(dk_1-1)(dk_2-1)} e^{G(n)}_{sk_2-1} * e^{G(n)}_{dk_2-1}$.

8.4. 7) $e^{SO(u)}_{k_1} * e^{SO(n)}_{k_2} = (-1)^{k_1k_2+1} e^{SO(n)}_{k_2} * e^{SO(n)}_{k_1}$.

8.4. 8) $e^{SU(n)}_{2k_1-1} * e^{SU(n)}_{2k_2-1} =- e^{SU(n)}_{2k_2-1} * e^{SU(n)}_{2k_1-1}$.

Proof. The statements 9.4.1) –5) are trivial by the definitions of cells and lemma 8.2. By lemma 9.1, we see $e^{G(n)}_{dk_1-1} * e^{G(n)}_{dk_2-1} = \pm e^{G(n)}_{dk_2-1} * e^{G(n)}_{dk_1-1}$ for $n \geqq k_1 > k_2 \geqq 1$. In order to determine the sign, consider the diagram

$$\begin{array}{ccccc} E^{dk_1-1}_F \times E^{dk_2-1}_F & \xrightarrow{\rho} & E^{dk_2-1}_F \times E^{dk_1-1}_F & \xrightarrow{\theta} & E^{dk_2-1}_F \times E^{dk_1-1}_F \\ \downarrow f_{G(n)} \times f_{G(n)} & & & & \downarrow f_{G(n)} \times f_{G(n)} \\ G(n) \times G(n) & \xrightarrow{h} & G(n) & \xleftarrow{h} & G(n) \times G(n) \end{array}$$

where $\rho(z, x) = (x, z)$,

$$\theta(x; (q, y)) = (x, (q, (f_{G(n)}(x))^{-1}y)),$$

and h is the group multiplication in $G(n)$.

It is readily verified, using the rules of lemma 8. 3, that the diagram is commutative, that is, two mappings

$$\Phi_1 = h \circ (f_{G(n)} \times f_{G(n)}),$$
$$\Phi_2 = h \circ (f_{G(n)} \times f_{G(n)}) \circ \theta \circ \rho$$

agree : $\Phi_1 = \Phi_2$.

It is readily verified that each of the mappings in the diagram is cellular, at least in dimensions dk_1+dk_2-2 and dk_1+dk_2-3. In checking this for θ, one must remember that $k_1>k_2$. We hall show that θ is homotopic to the identity in such a way that during the homotopy it always remains celluler in dimensions dk_1+dk_2-2 and dk_1+dk_2-3. To see this, take a contraction $D_t(a)$ which contracts $E^{dk_2-1}_F$ into a point $a=i\times(0,\cdots,0)$; $D_0(x)=x$ and $d_1(x)=a$. Then define $\theta_t(x; (q, y)) = (x, (q, (f_{G(n)}\circ D_t(x))^{-1}y)))$. This gives the desired homotopy : indeed θ_1 is the identity, because $f_{G(n)}D_1(x))=f_{G(n)}(a)=I_n$.

Now, if we compute the chain mapping induced by our mappings; then, as is well known, we have

$$\rho_{\divideontimes}(E^F_{dk_1-1} \times E^F_{dk_2-1}) = (-1)^{(dk_1-1)(dk_2-1)} E^F_{dk_2-1} \times E^F_{dk_1-1},$$

and

$$\theta_{\divideontimes}(E^F_{dk_1-1} \times E^F_{dk_1-1}) = E^F_{dk_2-1} \times E^F_{dk_1-1}.$$

For the composition mappings, hence, we have

$$\Phi_{1\divideontimes}(E^F_{dk_1-1} \times E^F_{dk_2-1}) = e^{G(n)}_{dk_1-1, dk_2-1}$$
$$\Phi_{2\divideontimes}(E^F_{dk_1-1} \times E^F_{dk_2-1}) = (-1)^{(dk_1-1)(dk_2-1)} e^{G(n)}_{dk_2-1, dk_1-1}.$$

But $\Phi_1 = \Phi_2$. Hence we have formula 6).

In order to prove 7), 8), define a mapping $\tau_F : E^{dk_2-1}_I \to E^{dk_2-1}_F$ by

$$\tau_F(q, x_1, x_2, \cdots, x_{k_2-1})) = (q, -(q+\sqrt{1-|q|^2})^2 x_1, x_2, \cdots, x_{k_2-1}).$$

Then, as is readily verified, we have

$$\tau_{R\divideontimes}(E^R_{k_2-1}) = -E^R_{k_2-1},$$

$$\rho_{C*}(E^C_{2k_2-1}) = E^C_{2k_2-1}.$$

We can prove 7), 8), by the similar techniques as 6), if we replace Φ_1, Φ_2 in the proof of 6) by the following two mapping,

$$\Phi_1 = h\circ(f_{SG(n)}\times f_{SG(n)}),$$
$$\Phi_2 = h\circ(f_{SG(n)}\times f_{SG(n)})\circ\theta\circ\rho\circ(I\times\tau_F),$$

respectively, where $I:\ E_F^{dk_1-1}\to E_F^{dk_1-1}$ is the identity $(\Phi_1=\Phi_2)$.

THEOREM 8. 1. *The Pontrjagin algebras $H_*(G(n);\Gamma)$ and $H_*(SG(n);\Gamma)$ are given as follows. ($e_0^{G(n)}$ and $e_0^{SG(n)}$ are $*$- uuits).*

8.1. 1) $\quad H_*(O(n);\ Z_2) = \{\tilde{e}_0^{O(n)}\}_2\otimes\Lambda(e_1^{O(n)},\ e_2^{O(n)},\cdots,e_{n-1}^{O(n)}),$

where $\{\tilde{e}_0^{O(n)}\}_2$ is a group of order 2 which is composed of $e_0^{O(n)}$ and $\tilde{e}_0^{O(n)}$ and $e_0^{O(n)}$ is a unit, and

$$\begin{cases} e_{k_1}^{O(n)} * \cdots * e_{k_j}^{O(n)} = e_{k_1,\cdots,k_j}^{O(n)} & \text{for } n>k_1>\cdots>k_j\geqq 0.\\ \tilde{e}_0^{O(n)} * \tilde{e}_0^{O(n)} = e_0^{O(n)} & \\ e_{k_1,\cdots,k_j}^{O(n)} * \tilde{e}_0^{O(n)} = e_{k_1,\cdots k_j,0}^{O(n)} & \end{cases}$$

8.1. 2) $\quad H_*(SO(n);\ Z_2) = \Lambda(e_1^{SO(n)},\ e_2^{SO(n)},\cdots,e_{n-1}^{SO(n)}),$

and

$$e_{k_1}^{SO(n)} * \cdots * e_{k_j}^{SO(n)} = e_{k_1,\cdots,k_j}^{SO(n)} \qquad \text{for } n>k_1\cdots>k_j\geqq 1.$$

8.1. 3) $\quad H_*(O(2n+1)\ ;\ Z_p) = \{\tilde{e}_0^{O(2n+1)}\}_2\otimes\Lambda(e_{2,1}^{O(2n+1)},\ e_{4,3}^{O(2n+1)},\cdots,e_{2n,2n-1}^{O(2n+1)}),$

and

$$\begin{cases} e_{2k_1,2k_1-1}^{O(2n+1)} * \cdots * e_{2k_j,2k_j-1}^{O(2n+1)} = e_{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}^{G(2n+1)} \\ \qquad\qquad \text{for } n>k_1>\cdots>k_j\geqq 0,\\ \tilde{e}_0^{O(2n+1)} * \tilde{e}_0^{O(2n+1)} = e_0^{O(2n+1)} \\ e_{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}^{O(2n+1)} * e_0^{O(2n+1)} = e_{2k_1,2k_1-1,\cdots,2k_j,2k_j-1,0}^{O(2n+1)} \end{cases}$$

($e_{\cdots,0,-1}$ means $e_{\cdots,0}$). (also in 8.1. 5).

8.1. 4) $\quad H_*(SO(2n+1);Z_p) = \Lambda(e_{2,1}^{SO(2n+1)},\ e_{4,3}^{SO(2n+1)},\cdots,e_{2n,2n-1}^{SO(2n+1)}),$

and

$$e_{2k_1,2k_1-1}^{SO(2n+1)} * \cdots * e_{2k_j,2k_j-1}^{SO(2n+1)} = e_{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}^{SO(2n+1)} \qquad \text{for } n>k_1>\cdots>k_j\geqq 1.$$

8.1. 5) $\quad H_*(O(2n)\ ;\ Z_p) = \{\tilde{e}_0^{O(2n)}\}_2\otimes\Lambda(e_{2,1}^{O(2n)},\ e_{4,3}^{O(2n)},\cdots,e_{2n-2,2n-3}^{O(2n)},\ e_{2n-1}^{O(2n)})$

and

$$\begin{cases} e_{2k_1,2k_1-1}^{O(2n)} * \cdots * e_{2k_j,2k_j-1}^{O(2n)} = e_{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}^{O(2n)}, \\ \qquad\qquad \text{for } n>k_1>\cdots>k_j\geqq 0\\ e_{2n-1}^{O(2n)} * e_{2k_1,2k_1-1}^{O(2n)} * \cdots * e_{2k_j,2k_j-1}^{O(2n)} = e_{2n-1,2k_1,2k_1-1,\cdots,2k_j,2k_j-1}^{O(2n)}, \\ \qquad\qquad \text{for } n>k_1>\cdots>k_j\geqq 0\\ \tilde{e}_0^{O(2n)} * \tilde{e}_0^{O(2n)} = e_0^{O(2n)} \\ e_{k_1,\cdots,k_j}^{O(2n)} * \tilde{e}_0^{O(2n)} = {}_{k_1,\cdots,k_j,0}^{O(2n)} \end{cases}$$

8.1. 6) $\quad H_*(SO(2n)\ ;\ Z_p) = \Lambda(e_{2,1}^{SO(2n)},\ e_{4,3}^{SO(2n)},\cdots.\ e_{2n-2,2n-3}^{SO(2n)},\ e_{2n-1}^{SO(2n)}),$

and

$$\begin{cases} e^{SO(2n)}_{2k_1,2k_1-1} * \cdots * e^{SO(2n)}_{2k_j,2k_j-1} = e^{SO(2n)}_{2k_1,2k_1-1,\cdots,2k_j,2k_j-1} \\ \qquad\qquad \text{for } n>k_1>\cdots>k_j \geqq 1 \\ e^{SO(2n)}_{2n-1} * e^{SO(2n)}_{2k_1,2k_1-1} * \cdots * e^{SO(2n)}_{2k_j,2k_j-1} = e^{SO(2n)}_{2n-1,2k_1,2k_1-1,\cdots,2k_j,2k_j-1} \\ \qquad\qquad \text{for } n>k_1>\cdots>k_j \geqq 1 \end{cases}$$

8.1. 7) $H_*(U(n)\,;\,Z) = \Lambda\,(e_1^{U(n)},\ e_3^{U(n)},\ \cdots,\ e_{2n-1}^{U(n)})$,

and

$$e^{U(n)}_{2k_1-1} * \cdots * e^{U(n)}_{2k_j-1} = e^{U(n)}_{2k_1-1,\cdots,2k_j-1} \qquad \text{for } n \geqq k_1>\cdots>k_j \geqq 1.$$

8.1. 8) $H_*(SU(n)\,;\,Z) = \Lambda\,(e_3^{SU(n)},\ e_5^{SU(n)},\cdots,\ e_{2n-1}^{SU(n)})$,

and

$$e^{SU(n)}_{2k_1-1} * \cdots * e^{SU(n)}_{2k_j-1} = e^{SU(n)}_{2k_1-1,\cdots,2k_j-1} \qquad \text{for } n \geqq k_1>\cdots>k_j \geqq 2,$$

8.1. 9) $H_*(Sp(n)\,;\,Z) = \Lambda\,(e_3^{Sp(n)},\ e_7^{Sp(n)},\ \cdots,\ e_{4n-1}^{Sp(n)})$,

and

$$e^{Sp(n)}_{4k_1-1} * \cdots * e^{Sp(n)}_{4k_j-1} = e^{Sp(n)}_{4k_1-1,\cdots,4k_j-1} \qquad \text{for } n \geqq k_1>\cdots>k_j \geqq 1$$

Proof. The Statements 1) –6) are in [7]. The Statements 7), 8) are trivial by the lemma 8. 4. As for 9), the statement that $x*x=0$, where $x \in H_*(Sp(n); Z)$ is not yet proved. To see this, consider an isomorphism into (herein, one-to-one, $*$ homomorphism into) appeared in the lemma 7. 7,

$$i_*: H_*(Sp(n);\ Z) \to H_*(SU(2n);\ Z).$$

Since $H_*(SU(2n);\ Z)$ is an exterior algebra, $i_*(x*x) = i_*(x)*i_*(x) = 0$. Hence we have $x*x=0$. q.e.d.

9. Primitive element

Let X be a space and Γ a coefficient field. Denote by $D^*(X;\ \Gamma)$ the subgroup of the cohomology group $H^*(X;\ \Gamma)$ generated by the elements of the form $u \cup v$[5], where u and v are elements of dimension >0 in $H^*(X;\ \Gamma)$. Let a be a homogeneous element of the homological group $H(X;\ \Gamma)$ such that dim $a>0$. We shall a *homological primitive element* of $H(X;\ \Gamma)$ if a is orthogonal to $D^*(X;\ \Gamma)$.

LEMMA 9. 1. *If a is a homological primitive element of $H(X;\ \Gamma)$, then we have*

$$d_*\, a = a \otimes 1 + 1 \otimes a,$$

where $d_*: H(X;\Gamma) \to H(X;\Gamma) \otimes H(X;\Gamma)$ is the homomorphism induced by the diagonal mapping $d: X \to X \times X$ such that $d(x)=(x,\ x)$, and conversely.

LEMMA 9. 2. *Let $f: X \to Y$ be a mapping. Then for any homological primitive element a of $H(X;\ \Gamma)$, the image $f_*(a)$ is also a homological primitive element of $H(Y;\ \Gamma)$.*

LEMMA 9. 3. *Let f: $X \to Y$ be a mapping. If all cup products are trivial in $H^*(X;\ \Gamma)$, then the image $f_*(a)$, where a is any positive dimensional homogeneous*

5) $u \cup v$ is the cup product of u and v.

element of $H(X; \Gamma)$, *is a homological primitive element of* $H(Y; \Gamma)$.

Analogously, let G be a topological group and Γ a field. Denote by $D_*(G; \Gamma)$ the subgroup of $H(G; \Gamma)$ generated by the elements of the form $a * b$, where a and b are elements of dimension >0 in $H(G; \Gamma)$. If a homogeneous u of $H^*(G; \Gamma)$ such that dim $u>0$ is orthogonal to $D_*(G; \Gamma)$, then u is called a (*cohomological*) *primitive element* of $H^*(G; \Gamma)$.

LEMMA 9. 4. *If* u *is a primitive element of* $H^*(G; \Gamma)$, *then we have*

$$h^*(u) = u \otimes 1 + 1 \otimes u,$$

where $h^* : H^*(G; \Gamma) \to H^*(G; \Gamma) \otimes H^*(G; \Gamma)$ *is the homomorphism induced by the group multiplication* $h : G \times G \to G$, *and conversely.*

LEMMA 9. 5. *Let* G_1 *and* G_2 *be two topological groups and* $f : G_2 \to G_1$ *be a continuous homomorphism. Then for any primitive element* u *of* $H^*(G_1; \Gamma)$, $f^*(u)$ *is also a primitive element of* $H^*(G_2; \Gamma)$.

If X (resp. G) has no torsion, the above definition is also appicable to the case of the homological (resp. cohomological) primitive element of $H(X; Z)$ (resp. $H^*(G; Z)$) with integral coefficient.

THEOREM 9. 1. 9. 1. 1.) $e^{SO(2n+1)}_{2k,2k-1}$ *for* $n \geqq k \geqq 1$ *is a homological primitive element of* $H(SO(2n+1); Z_p)$, *where* $p \neq 2$.

9.1. 2) $e^{SO2(n)}_{2k,2k-1}$ *for* $n-1 \geqq k \geqq 1$ *and* $e^{SO(2n)}_{2n-1}$ *are homological primitive elements of* $H(SO(2n); Z_p)$, *where* $p \neq 2$.

In $O(n)$, *the results are similar as* $SO(n)$.

9.1. 3) $e^{U(n)}_{2k-1}$ *for* $n \geqq k \geqq 1$ (*resp.* $e^{SU(n)}_{2k-1}$ *for* $n \geqq k \geqq 2$) *is a homological primitive element of* $H(U(n); Z)$ (*resp.* $H(SU(n); Z)$).

9·1. 4) $e^{Sp(n)}_{4k-1}$ *for* $n \geqq k \geqq 1$ *is a homological primitive element of* $H(Sp(n), Z)$.

THEOREM 9. 2. 9. 2. 1) $e^k_{SO(n)}$ *for* $n-1 \geqq k \geqq 1$ *is a primitive element of* $H^*(SO(n); Z_2)$.

9.2. 2) $e^{2k,2k-1}_{SO(2n+1)}$ *for* $n \geqq k \geqq 1$ *is a primitive element of* $H^*(SO(2n+1); Z_p)$, *where* $p \neq 2$.

9.2. 3) $e^{2k,2k-1}_{SO(2n)}$ *for* $n-1 \geqq k \geqq 1$ *and* $e^{2n-1}_{SO(2n)}$ *are primitive elements of* $H^*(SO(2n); Z_p)$, *where* $p \neq 2$.

In $O(n)$, *the results are similar as* $SO(n)$.

9.2. 4) $e^{2k-1}_{U(n)}$ *for* $n \geqq k \geqq 1$ (*resp.* $e^{2k-1}_{SU(n)}$ *for* $n \geqq k \geqq 2$) *is a primitive element of* $H^*(U(n); Z)$ (*resp.* $H^*(SU(n); Z)$).

9.2. 5) $e^{4k-1}_{Sp(n)}$ *for* $n \geqq k \geqq 1$ *is a primitive element of* $H^*(Sp(n); Z)$.

Proof. These theorems are the direct consequences of the structures of the cup products (cf. Theorem 10. 1) and Pontrjagin algebra (cf. Theorem 8. 1.) of these groups.

10. Cup products in $G(n)$, $SG(n)$; $S_{n,m}$, F_n and X_n

Throughout sections 10 and 11, it is convenient to extend our notation for cells (cycles or cocycles) by requiring that, for example,

$$e_{U(n)}^{2k_1-1,\cdots,2k_j-1} = sign\ \omega\ e_{U(n)}^{2k_{\omega(1)}-1,\cdots,2k_{\omega(j)}-1}$$

$$_pe_{SO(n)}^{2k_1,2k_1-1,\cdots,2k_j,2k_j-1} = sign\ \omega\ _pe_{SO(n)}^{2k_{\omega(1)},2k_{\omega(1)}-1,\cdots,2k_{\omega(j)},2k_{\omega(j)}-1}$$

for all permutations ω of the indices $1,\cdots,j$ and that

$$e_{U(n)}^{2k_1-1,\cdots,2k_j-1} = 0\ ({}_pe_{SO(n)}^{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}=0)$$

if some $k_s = k_t$ for $s\neq t$, if some $k_s>n$, or if some $k_s<1$. We use similar notations for $e_{SO(n)}^{k_1,\cdots,k_j}$, $e_{SU(n)}^{2k_1-1,\cdots,2k_j-1}$, $e_{Sp(n)}^{4k_1-1,\cdots,4k_j-1}$, $e_{F_n}^{2k_1,\cdots,2k_j}$ and $e_{X_n}^{4k_1-3,\cdots,4k_j-3}$ etc.

THEOREM 10. 1. *The cohomology algebras $H^*(G(n);\Gamma)$ and $H^*(SG(n);\Gamma)$ are given as follows ($e_{G(n)}^0$ and $e_{SG(n)}^0$ are $\cup$-units).*

10.1. 1) $H^*(SO(n);Z_2)=\Delta(e_{SO(n)}^1, e_{SO(n)}^2,\cdots, e_{SO(n)}^{n-1})$,

and

$${}_2e_{SO(n)}^k \cup {}_2e_{SO(n)}^{k_1,\cdots,k_j} = {}_2e_{SO(n)}^{k,k_1,\cdots,k_j}+\textstyle\sum_{i=1}^j {}_2e_{SO(n)}^{k_1,\cdots,k_i+k,\cdots,k_j}$$

for $k, k_i\geqq 1$.

Especially we have

$${}_2e_{SO(n)}^k \cup {}_2e_{SO(n)}^k = \begin{cases} {}_2e_{SO(n)}^k & \text{if } 2k<n, \\ 0 & \text{if } 2k\geqq n. \end{cases}$$

10.1. 2) $H^*(SO(2n+1);Z_p) = \Lambda(e_{SO(2n+1)}^{2,1}, e_{SO(2n+1)}^{4,3},\cdots, e_{SO(2n+1)}^{2n,2n-1})$,

and

$$e_{SO(2n+1)}^{2k_1,2k_1-1}\cup\cdots\cup e_{SO(2n+1)}^{2k_j,2k_j-1} = e_{SO(2n+1)}^{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}.$$

10.1. 3) $H^*(SO(2n);Z_p) = \Lambda(e_{SO(2n)}^{2,1}, e_{SO(n)}^{4,3},\cdots, e_{SO(2n)}^{2n-2,2n-3}, e_{SO(2n)}^{2n-1})$,

and

$$e_{SO(2n)}^{2k_1,2k_1-1}\cup\cdots\cup e_{SO(2n)}^{2k_j,2k_j-1} = e_{SO(2n)}^{2k_1,2k_1-1,\cdots,2k_j,2k_j-1}$$

$$e_{SO(2n)}^{2n-1}\cup e_{SO(2n)}^{2k_1,2k_1-1}\cup\cdots\cup e_{SO(2n)}^{2k_j,2k_j-1} = e_{SO(2n)}^{2n-1,2k_1,2k_1-1,\cdots,2k_j,2k_j-1}$$

10.1. 4) $H^*(U(n);Z)=\Lambda(e_{U(n)}^1, e_{U(n)}^3,\cdots, e_{U(n)}^{2n-1})$,

and

$$e_{U(n)}^{2k_1-1}\cup\cdots\cup e_{U(n)}^{2k_j-1} = e_{U(n)}^{2k_1-1,\cdots,2k_j-1},$$

10.1. 5) $H^*(SU(n);Z) = \Lambda(e_{SU(n)}^3, e_{SU(n)}^5,\cdots, e_{SU(n)}^{2n-1})$

and

$$e_{SU(n)}^{2k_1-1}\cup\cdots\cup e_{SU(n)}^{2k_j-1} = e_{SU(n)}^{2k_1-1,\cdots,2k_j-1},$$

10.1. 6) $H^*(Sp(n);Z) = \Lambda(e_{Sp(n)}^3, e_{Sp(n)}^7,\cdots, e_{Sp(n)}^{4n-1})$

and

$$e_{Sp(n)}^{4k_1-1}\cup\cdots\cup e_{Sp(n)}^{4k_j-1} = e_{Sp(n)}^{4k_1-1,\cdots,4k_j-1}$$

Proof. These formulas essentially are due to that $e_{G(n)}^{dk-1}$ and $e_{SG(n)}^{dk-1}$ are primitive elements. We remember that dimensions of primitive elements are odd and $2x=0$ follows $x=0$ in $H^*(SO(n);Z_p)$, $H^*(U(n);Z)$, $H^*(SU(n);Z)$ and $H^*(Sp(n);Z)$. As the proof is performed anologously as in the case of the proposition 2.8, [2], we shall omit here.

THEOREM 10. 2. *The cohomology algebras* $H^*(F_n; Z_2)$ *and* $H^*(X_n; Z)$ *are given as follows*

10.2. 1) $H^*(F_n; Z_2) = \Delta(e^2_{F_n}, e^4_{F_n}, \cdots, e^{2n-2}_{F_n})$

and

$$e^{2k}_{F_n} \cup e^{2k_1,\cdots,2k_j}_{F_n} = e^{2k,2k_1,\cdots,2k_j}_{F_n} + \sum_{i=1}^{j} e^{2k_1,\cdots,2k_i+2k,\cdots,2k_j}_{F_n}.$$

10.2. 2) $H^*(X_n; Z) = \Lambda(e^5_{X_n}, e^9_{X_n}, \cdots, e^{4n-3}_{X_n})$

and

$$e^{4k_1-1}_{X_n} \cup e^{4k_2-1,\cdots,4k_j-1}_{X_n} = e^{4k_1-1,4k_2-1,\cdots,4k_j-1}_{X_n}.$$

Proof. We can see, by applying that $p^*_{F_n}$ and $p^*_{X_n}$ are isomorphic into, immedetaely.

11. Steenrod's reduced powers

Let p be a fixed prime number, K a finite complex and L a subcomplex of K. The Steenrod's reduced powers $\mathcal{P}^s_p$ are homomorphisms

$$\mathcal{P}^s_p : H^q(K, L; Z_p) \rightarrow H^{q+2s(p-1)}(K, L; Z_p)$$

defined for all two integers $s, t \geqq 0$ and all couples of K and L, where L is a subcomplex of K. On the other hand, if $p=2$, there exist, as is well knnwn, Steenrod's square homomorphisms Sq^s

$$Sq^s : H^q(K, L, Z_2) \rightarrow H^{q+s}(K, L; Z_2)$$

defined for all $s, t \geqq 0$ and all couples (K, L). These two operations $\mathcal{P}^s_p$ and Sq^s are combined by the relation $\mathcal{P}^s_2 = Sq^{2s}$.

We shall use only the following formulas.

11. 1) If $f: (K, L) \rightarrow (K', L')$ is a mapping, then $\mathcal{P}^s_p \circ f^* = f^* \circ \mathcal{P}^s_p$ (resp. $Sq^s \circ f^* = f^* \circ Sq^s$).

11. 2) $\mathcal{P}^0_p$ (resp. Sq^0) is the identity isomorphism.

11. 3) $\mathcal{P}^s_p$ is trivial for $q<2s$ (resp. Sq^s is trivial for $q<s$)

11. 4) $\mathcal{P}^s_p(x) = x^p$ [6] for $x \in H^{2s}(K, L; Z_p)$ (resp. $Sq^s(x) = x^2$ for $x \in H^s(K, L; Z_2)$).

11. 5) $\delta : H^q(L; Z_p) \rightarrow H^{q+1}(K, L; Z_p)$ be the coboundary homomorphism, then $\mathcal{P}^s_p \circ \delta = \delta \circ \mathcal{P}^s_p$ (resp. $Sq^s \circ \delta = \delta \circ Sq^s$).

11. 6) $\mathcal{P}^s_p(x \cup y) = \sum_{i+j=s} \mathcal{P}^i_p(x) \cup \mathcal{P}^j_p(y)$ (resp. $Sq^s(x \cup y) = \sum_{i+j=s} Sq^i(x) \cup Sq^j(y)$). (Cartan's formula)

Throughout this section, coefficients will continue to be taken exclusively in Z_p, Let $\binom{k}{j}$ be the binomial coefficient reduced modulo p. This symbol is to be zero when it makes no sense, that is, if either j or k is negative or if $k<j$.

The $(n-1)$-dimensional real projective space P_{n-1} has k-dimensional cell ω^k for $n-1 \geqq k \geqq 0$ and, as is well known, we have ${}_2\omega^k = ({}_2\omega^1)^k$ [7]. The $(n-1)$-dimensional complex projective space M_{n-1} has $2k$-dimensional cell u^{2k} for $n-1 \geqq k \geqq 0$ and, as is well known, we have $u^{2k} = (u^2)^k$ [7]

6) x^p denotes the p-fold cup product of x.

7) The expression in the right hand side is zero if it has no meansing.

LEMMA 11. 1. *In the real projective space P_{n-1}, we have*

$$Sq^s(\omega^k) = \binom{k}{s}\omega^{k+s}.$$

Proof. we proceed by an induction on s.

$$\begin{aligned} Sq^s(w^k) &= Sq^s((w^1)^k) = Sq^s(w^1 \cup w^{k-1}) = Sq^0 w^1 \cup Sq^s(w^{k-1}) + Sq^1 w^1 \cup Sq^{s-1}(w^{k-1}) \\ &= \omega^1 \cup \binom{k-1}{s}\omega^{k-1+s} + \omega^2 \cup \binom{k-1}{s-1}\omega^{k+s-2} \\ &= \binom{k}{s}\omega^{k+s}. \end{aligned}$$

LEMMA 11. 2. *In the complex projective spaee M_{n-1}, we have*

$$\mathcal{P}_p^s(u^{2k}) = \binom{k}{s} u^{2k+2s(p-1)},$$

and

$$\begin{cases} Sq^{2s}(u^{2k}) = \binom{k}{s} u^{2k+2s} \\ Sq^{2s+1}(u^{2k}) = 0. \end{cases}$$

Proof. The proof is similar as P_{n-1}. The formulas for Sq^s is a special case of $\mathcal{P}_p^s(u^{2k})$, since $Sq^{2s+1} = 0$.

In order to compute the reduced powers in $E(M_{n-1})$, put

$$\begin{aligned} E_+(M_{n-1}) &= \{(t\boldsymbol{i}, \sqrt{1-t^2}X);\ 0 \leqq t \leqq 1,\ X \in M_{n-1}^*\}, \\ M_{n-1} &= \{(0, X);\ X \in M_{n-1}^*\}, \\ E_-(M_{n-1}) &= \{(t\boldsymbol{i}, \sqrt{1-t^2}X);\ -1 \leqq t \leqq 0,\ X \in M_{n-1}^*\}. \end{aligned}$$

Define a mapping $g: E_+(M_{n-1}) \to E(M_{n-1})$ by $g(t\boldsymbol{i}, \sqrt{1-t^2}X) = ((2t-1)\boldsymbol{i}, 2\sqrt{t(1-t^2)}X)$. Using that $E_+(M_{n-1})$ is contractible and the excision of $(E_+(M_{n-1}), M_{n-1}) \subset (E(M_{n-1}), E_-(M_{n-1}))$,

$$H^q(M_{n-1}) \xrightarrow{\delta} H^{q+1}(E_+(M_{n-1}), M_{n-1}) \xleftarrow{g^*} H^{q+1}(E(M_{n-1})).$$

δ and g^* are isomorphisms and we have $v^{2k-1} = g^{*-1}\delta u^{2k-2}$.

LEMMA 11. 3. *In the suspended space* $E(M_{n-1})$ of M_{n-1} (*also* $E(M_{n-1})$) *we have*

$$\mathcal{P}_p^s(v^{2k-1}) = \binom{k-1}{s} v^{2k-1+2s(p-1)},$$

and

$$\begin{cases} Sq^{2s}(v^{2k-1}) = \binom{k-1}{s} v^{2k-1+2s}, \\ Sq^{2s+1}(v^{2k-1}) = 0. \end{cases}$$

Proof. $g^*\mathcal{P}_p^s(v^{2k-1}) = \mathcal{P}_p^s g^*(v^{2k-1}) = \mathcal{P}_p^s \delta(u^{2k-2}) = \delta\,\mathcal{P}_p^s(u^{2k-2}) = \delta\binom{k-1}{s}u^{2k-2+2s(p-1)}$ $= g^*\binom{k-1}{s}v^{2k-1+2s(p-1)}$. Since g^* is isomorphic, we have the first formula. The formulas for Sq^s are obtained as similar techniques.

Let $\bar{h}_{SO(n)}: P_{n-1} \times SO(n-1) \xrightarrow{f'_{SO(n)} \times i} SO(n) \times SO(n) \xrightarrow{h} SO(n)$
and

$$\bar{h}_{SU(n)} : E(M_{n-1})\times SU(n-1) \xrightarrow{f'_{SU(n)}\times i} SU(n)\times SU(n) \xrightarrow{h} SU(n)$$

be defined to be the compositions $h_{SO(n)} = h\circ(f'_{SO(n)}\times i)$ and $h_{SU(n)} = h\circ(f'_{SU(n)}\times i)$ respectively, where i is the inclusion map.

LEMMA 11. 4. 11. 4. 1) *$\bar{h}_{SO(n)}$ is cellular. ${}_2\bar{h}_{SO(n)*}$ is onto and*

$${}_2\bar{h}_{SO(n)*}({}_2\omega_{k_1}\times {}_2e^{SO(n-1)}_{k_2,\dots,k_j}) = {}_2e^{SO(n)}_{k_1,k_2,\dots,k_j} \quad \textit{for } n>k_1 \textit{ and } n-1>k_i$$

$${}_2\bar{h}_{SO(n)*}({}_2\omega_0\times {}_2e^{SO(n-1)}_{k_1,\dots,k_j}) = {}_2e^{SO(n)}_{k_1,\dots,k_j} \quad \textit{for } n-1>k_i.$$

11.4. 2) *$\bar{h}_{SU(n)}$ is cellular. $\bar{h}_{SU(n)*}$ is onto* and

$$\bar{h}_{SU(n)*}(v_{2k_1-1}\times e^{SU(n-1)}_{2k_2-1,\dots,2k_j-1}) = e^{SU(n)}_{2k_1-1,2k_2-1,\dots,2k_j-1}$$
$$\textit{for } n\geqq k_1 \textit{ and } n-1\geqq k_i\geqq 2.$$

$$\bar{h}_{SU(n)*}(v_0\times e^{SU(n-1)}_{2k_1-1,\dots,2k_j-1}) = e^{SU(n)}_{2k_1-1,\dots,2k_j-1}.$$

Proof. 11. 4. 2) : $\bar{h} = \bar{h}_{SU(n)}$ is certainly cellular since it is the composition of mappings we know to be cellular. Furthermore,

$$\bar{h}_*(v_{2k_1-1}\times e^{SU(n-1)}_{2k_2-1,\dots,2k_j-1}) = h_*(e^{SU(n)}_{2k_1-1}\times e^{SU(n)}_{2k_2-1,\dots,2k_j-1})$$
$$= e^{SU(n)}_{2k_1-1} * e^{SU(n)}_{2k_2-1,\dots,2k_j-1} = e^{SU(n)}_{2k_1-1,2k_2-1,\dots,2k_j-1},$$

and

$$\bar{h}_*(v_0\times e^{SU(n-1)}_{2k_1-1,\dots,2k_j-1}) = h_*(e^{SU(n)}_0\times e^{SU(n)}_{2k_1-1,\dots,2k_j-1})$$
$$= e^{SU(n)}_0 * e^{SU(n)}_{2k_1-1,\dots,2k_j-1} = e^{SU(n)}_{2k_1-1,\dots,2k_j-1}.$$

In any of the degenerate cases, these are valid.

11.4. 1) is similar as 11.4. 2).

LEMMA 11. 5. 11. 5. 1) *${}_2\bar{h}^*_{SO(n)}$ is isomorphic into. If $k_i\geqq 1$, then*

$${}_2\bar{h}^*_{SO(n)}({}_2e^{k_1,\dots,k_j}_{SO(n)}) = {}_2\omega^0\times {}_2e^{k_1,\dots,k_j}_{SO(n-1)} + \sum_{i=1}^{j} {}_2\omega^{k_i}\times {}_2e^{k_1,\dots,\widehat{k_i},\dots,\widehat{k_j}}_{SO(n-1)}$$

11. 5. 2) *$\bar{h}^*_{SU(n)}$ is isomorphic into. If $k_i\geqq 2$, then*

$$\bar{h}^*_{SU(n)}(e^{2k_1-1,\dots,2k_j-1}_{SU(n)}) = v^0\times e^{2k_1-1,\dots,2k_j-1}_{SU(n)}$$
$$+\sum_{i=1}^{j}(-1)^{i-1}\, v^{2k_i-1}\times e^{2k_1-1,\dots,\widehat{2k_i-1},\dots,2k_j-1}_{SU(n-1)}.$$

Proof. This is a corollary of lemma 11. 4, since $\bar{h}^*$ is the dual map to $\bar{h}_*$. These formulas are valid in the degenerate cases.

THEOREM 11. 1. *In the classical Lie groups, some reduced powers are given as follows.*

11. 1. 1) *In $H^*(SO(n); Z_2)$, we have*

$$Sq^s(e^k_{SO(n)}) = \binom{k}{s} e^{k+s}_{SO(n)}, \quad \textit{for } k\geqq 1.$$

$$Sq^s(e^{k_1,\dots,k_j}_{SO(n)}) = \sum_{i_1+\dots+i_j=s}\binom{k_j}{i_j}\cdots\binom{k_j}{i_j} e^{k_1,\dots,k_j}_{SO(n)} \quad \textit{for } k_i\geqq 1, i=1,\dots,j.$$

11. 1. 2) *In $SU(n)$, we have*

$$\mathcal{P}^s_p(e^{2k-1}_{SU(n)}) = \binom{k-1}{s} e^{2k-1+2s(p-1)}_{SU(n)},$$

$$\mathcal{P}_p^s\,(e_{SU(n)}^{2k_1-1,\cdots,2k_j-1})=\sum_{i_1+\cdots+i_j=s}\binom{k_1-1}{i_1}\cdots\binom{k_j-1}{i_1}e_{SU(n)}^{2k_1-1+2i_1(p-1),\cdots,2k_j+2i_j(p-1)}$$

and

$$Sq^{2s}\,(e_{SU(n)}^{2k-1})=\binom{k-1}{s}e_{SU(n)}^{2k-1+2s}$$

$$Sq^{s}\,(e_{SU(n)}^{2k_1-1,\cdots,2k_j-1})=\sum_{i_1+\cdots+i_j=s}\binom{k_1-1}{i_1}\cdots\binom{k_j-1}{i_j}e_{SU(n)}^{2k_1-1+2i_1,\cdots,2k_j-1+2i_j},$$

$$Sq^{s+1}=0$$

11. 1. 3) *In $Sp(n)$, we have*

$$\mathcal{P}_p^s\,(e_{Sp(n)}^{4k-1})=(-1)^{\frac{s(p-1)}{2}}\binom{2k-1}{s}e_{Sp(n)}^{4k-1+2s(p-1)},$$

$$\mathcal{P}_p^s\,(e_{Sp(n)}^{4k_1-1,\cdots,4k_j-1})=(-1)^{\frac{s(p-1)}{2}}\sum_{i_1+\cdots+i_j=s}\binom{2k_1-1}{i_1}\cdots\binom{2k_j-1}{i_j}e_{Sp(n)}^{4k_1-1+2i_1(p-1),\cdots,4k_j-1+2i_j(p-1)}$$

and

$$Sq^{4s}\,(e_{Sp(n)}^{4k-1})=\binom{2k-1}{2s}e^{4k-1+4s},$$

$$Sq^{4s}\,(e_{Sp(n)}^{4k_1-1,\cdots,4k_j-1})=\sum_{i_1+\cdots+i_j=s}\binom{2k_1-1}{2i_1}\cdots\binom{2k_j-1}{2i_j}e_{Sp(n)}^{4k_1-1+4i_1,\cdots,4k_j-1+4i_j},$$

$$Sq^s=0 \quad \text{for } s\not\equiv 0\ (mod\ 4)$$

Proof. 11. 1. 2) If $n=2$, the theorem is trivial. For $n>2$, we proceed inductively, supposing the theorem is valid for $SU(n-1)$. By making use of lemma 11. 5, we have

$$\begin{aligned}\bar{h}_{SU(n)}^*\,(\mathcal{P}_p^s\,(e_{SU(n)}^{2k-1}))&=\mathcal{P}_p^s\,(\bar{h}_{SU(n)}^*\,(e_{SU(n)}^{2k-1}))=\mathcal{P}_p^s\,(v^0\times e_{SU(n-1)}^{2k-1}+v^{2k-1}\times e_{SU(n-1)}^0)\\&=v_0\times\binom{k-1}{s}e_{SU(n-1)}^{2k-1+2s(p-1)}+\binom{k-1}{s}v^{2k-1+2s(p-1)}\times e_{SU(n-1)}^0\\&=\binom{k-1}{s}\bar{h}_{SU(n)}^*\,(e_{SU(n)}^{2k-1+2s(p-1)})\end{aligned}$$

Since $\bar{h}_{SU(n)}^*$ is isomorphic into, we have the first formula To see the second formula, we shall use the Cartan's formula by an induction on j.

$$\begin{aligned}&\mathcal{P}_p^s\,(e_{SU(n)}^{2k_1-1,2k_2-1,\cdots,2k_j-1})=\mathcal{P}_p^s\,(e_{SU(n)}^{2k_1-1}\cup e_{SU(n)}^{2k_2-1,\cdots,2k_j-1})\\&=\sum_{l+m=s}\mathcal{P}_p^l(e_{SU(n)}^{2k_1-1})\cup\mathcal{P}_p^m(e_{SU(n)}^{2k_2-1,\cdots,2k_j-1})\\&=\sum_{l+m=s}\binom{k_1-1}{l}e_{SU(n)}^{2k_1-1+2l(p-1)}\cup\sum_{i_2+\cdots+i_j=m}\binom{k_2-1}{i_2}\cdots\binom{k_j-1}{i_j}e_{SU(n)}^{2k_2-1+2i_2(p-1),\cdots,2k_j-1+2i_j(p-1)}\\&=\sum_{i_1+\cdots i_j=s}\binom{k_1-1}{i_1}\binom{k_2-1}{i_2}\cdots\binom{k_j-1}{i_j}e_{SU(n)}^{2k_1-1+2i_1(p-1),2k_2-1+2i_2(p-1),\cdots,2k_j-1+2i_j(p-1)}\end{aligned}$$

The other formulas are obtained quite similarly.

11.1. 1) is proved as similar as 11.1. 2).

To see 11.1. 3), we remember that the inclusion map $i: Sp(n)\to SU(2n)$ induces an isomorphism into: $i_*: H(Sp(n);\,\Gamma)\to H(SU(n);\,\Gamma)$, where Γ is Z or Z_p. By theoem 9.1. 3), $e_{4k-1}^{Sp(n)}$ $(n\geqq k\geqq 1)$ is a homological primitive element in $Sp(n)$ (for any coeffient Z or Z_p). $i_*(e_{4(k-1)}^{Sp(n)})$ is, hence, also a homological primitive element in $SU(2n)$ for Z and Z_p (lemma 9. 2). In $SU(n)$, however, $e_{4k-1}^{SU(n)}$ is the base of the $(4k-1)$-dimensional homological primitive element. So that we have $i_*(e^{Sp(n)}_{4k-1}) = \epsilon\, e^{SU(2n)}_{4k-1}$ where ϵ is 1 or -1.

To determine the sign ϵ, consider the diagram

$$\begin{array}{ccc} & E^3_R\times E^{4k-4}_R \to E^3_Q\times E^{4k-4}_Q = E^{4k-1}_Q \xrightarrow{f_{Sp(n)}} Sp(n) \xrightarrow{p_{Sp(n)}} S^{4k-1}_Q & \searrow \varphi_{RQ} \\ E^{4k-1}_R \nearrow\!\!\searrow & & S^{4k-1}_R. \\ & E^1_R\times E^{4k-2}_R \to E^1_C\times E^{4k-2}_C = E^{4k-1}_C \xrightarrow{f_{SO(n)}} SU(n) \xrightarrow{p_{SU(n)}} S^{4k-1}_C & \nearrow \varphi_{QR} \end{array}$$

We note that $\varphi_{CQ*}(S^Q_{4k-1}) = (-1)^k S^C_{4k-1}$, (bacause $\varphi_{RC}\circ\varphi_{CQ}(x) = (a, b, c, -d)$ for $Q \ni x = a+ib+jc+kd = (a+ib)+j(c-id)$). Now,

$p_{SU(n)}\circ i_*(e^{Sp(n)}_{4k-1}) = \varphi_{CQ*}\circ p_{Sp(n)*}(e^{Sp(n)}_{4k-1}) = \varphi_{CQ*}\circ \xi_{Q*}(E^Q_{4k-1}) = -\varphi_{CQ*}(S^Q_{4k-1})$ $= (-1)^{k+1}S^C_{4k-1}$. (lemma 7.1). On the other hand,

$$\epsilon\, p_{SU(n)*}(e^{SU(2n)}_{4k-1}) = \epsilon\, \xi_{C*}(E^C_{4k-1}) = -\epsilon\, S^C_{4k-1}. \quad \text{Hence } \epsilon = (-1)^k.$$

So that we have the following

LEMMA 11. 1. $i_*(e^{4k-1}_{Sp(n)}) = (-1)^k e^{4k-1}_{SU(n)}$.

We shall continue the proof of 11. 1. 3). Now

$$i^* : H^*(SU(2n);\ Z_p) \to H^*(Sp(n);\ Z_p)$$

is homomorphic onto and the kernel K is an ideal in $H^*(SU(2n); Z_p)$ generated by $e^{2k-1}_{SU(2n)}$ for $k\not\equiv 0$ (*mod* 2). Hence we have $H^*(SU(2n); Z_p)/K \cong H^*(Sp(n); Z_p)$, and

$$\begin{cases} i^*(e^{2k-1}_{SU(2n)}) = 0 & \text{for } k\not\equiv 0\ (mod\ 2) \\ i^*(e^{4k-1}_{SU(2n)}) = (-1)^k\, e^{4k-1}_{Sp(n)}. \end{cases}$$

It is readily verified that K is invariant by $\mathcal{P}^s_p$; $\mathcal{P}^s_p K\subset K$, using the formulas 11. 1. 2). Now we have

$$\mathcal{P}^s_p\, e^{4k-1}_{Sp(n)} = (-1)^k\, \mathcal{P}^s_p\, i^*(e^{4k-1}_{SU(2n)}) = (-1)^k i^*\mathcal{P}^s_p\, e^{4k-1}_{SU(2n)}$$

$$= (-1)^k i^*\binom{2k-1}{s} e^{4k-1+2s(p-1)}_{SU(2n)} = (-1)^k\ (-1)^{k+\frac{s(p-1)}{2}}\binom{2k-1}{s} e^{4k-1+2s(p-1)}_{Sp(n)}$$

$$= (-1)^{\frac{s(p-1)}{2}}\binom{k-1}{s} e^{4k-1+2s(p-1)}_{Sp(n)}.$$

By making use of Cartan's formula 11. 6 and the same technics as $\mathcal{P}^s_p$ in $SU(n)$, we have the other formulas. q.e.d.

REMARK 11.1. Using the isomorphisms into $p^*_m: H^*(S_{n,m}, Z_p)\to H^*(G(n); Z_p)$, we can easily compute the reduced powers in the Stiefel manifols $S_{n,m}$.

THEOREM 11. 2. *In the spaces F_n and X_n, some reduced powers are given as follows*

11. 2. 1) *In $H^*(F_n; Z_2)$, we have*

$$Sq^{2s}({}_2e^{2k}_{F_n}) = \binom{2k}{2s}\, {}_2e^{2k+2s}_{F_n},$$

$$Sq^{2s}({}_2e^{2k_1,\cdots,2k_j}_{F_n}) = \sum_{i_1+\cdots i_j=s}\binom{2k_1}{2i_1}\cdots\binom{2k_j}{2i_j}\, {}_2e^{2k_1+2i_1,\cdots,2k_j+2i_j}_{F_n}$$

$$Sq^{2s+1}=0.$$

11. 2. 2) *In $H^*(X_n;\ Z_p)$, we have*

$$\mathcal{P}_p^s\,(e_{X_n}^{4k-3})=\binom{2k-2}{\ }\,e_{X_n}^{4k-3+2s(p-1)}$$

$$\mathcal{P}_p^s\,(e_{X_n}^{4k_1-3,\cdots,4k_j-3})=\sum_{i_1+\cdots+i_j=s}\binom{2k_1-2}{i_1}\cdots\binom{2k_j-2}{i_j}\,e_{X_n}^{4k_1-3+2i_1(p-1),\cdots,4k_j-3+2i_j(p-1)}$$

and

$$Sq^{4s}\,(e_{X_n}^{4k-3})=\binom{2k-2}{2s}\,e_{X_n}^{4k-3+4s}$$

$$Sq^{4s}\,(e_{X_n}^{4k_1-3,\cdots,4k_j-3})=\sum_{i_1+\cdots+i_j=s}\binom{2k_1-2}{2i_1}\cdots\binom{2k_j-2}{2i_j}\,e_{X_n}^{4k_1-3+4i_1,\cdots,4k_j-3+2i_j(p-1)}$$

$$Sq^s=0,\qquad \textit{for } s\not\equiv 0 \pmod 4).$$

Proof. By applying that $p_{F_n}^*$ (resp. $p_{X_n}^*$) is isomorphic into, we obtain, in a trivial fashion, formulas from the formulas in $SO(2n)$ (resp. $(SU(2n))$.

12. Appendix

A cellular decomposition of a space determines how to attach a cell to the lower dimensional cells than it. In the lowest dimensions of the classical Lie groups, the attaching mappings are familiar ones.

THEOREM 12. 1 *In $SO(n)$, the 2-dimensional primitive cell $e_{SO(n)}^2$ is attached to the 1-dimensional primitive cell $e_{SO(n)}^1$ by the mapping $\mu: s^1\to s^1$ whose degree is 2.*

THEOREM 12. 2. *In $U(n)$ (resp. $SU(n)$), the 5-dimensional primitive cell $e_{U(n)}^5$ (resp. $e_{SU(n)}^5$) is attached to the 3-dimensional primitive cell $e_{U(n)}^3$ (resp. $e_{SU(n)}^3$) by the suspended Hopf map $E(\nu): S^4\to S^3$.*

THEOREM 12. 3. *In $Sp(n)$, the 7-dimensional primitive cell $e_{Sp(n)}^7$ is attached to the 3-dimensional primitive cell $e_{Sp(n)}^3$ by the Blaker-Massey's map $\rho: S^6\to S^3$.* This mapping ρ is a Hopf construction of the mapping $\rho': S^3\times S^2\to S^2$ such that $\rho'(x, y)=xy\bar{x}$, where S^3 is quaternion numbers whose norms are 1 and S^2 is pure imaginary quaternion numbers whose norms are 1.

proof. Theorem 12. 1 is obvious, because the real projective plane P_2 is attached to P_1 by the mapping whose degree is 2 and $f'_{SO(2)}$ is homeomorphic on P_2.

Theorem 12. 2. is also obvious, since the complex projective plane M_2 is attached to M_1 by the Hopf map ν (so that $\tilde{E}(M_2)$ is attached to $\tilde{E}(M_1)$ by $E(\nu)$) and $f'_{US(2)}$ is homeomorphic on $E(M_2)$.

In order to prove theorem 12. 3, we consider the formula

$$f_{Sp(2)}\,(q,\ x_1)=\begin{pmatrix}1+x_1p\bar{x}_1 & x_1px_2\\ x_2p\bar{x}_1 & 1+x_2px_2\end{pmatrix}\in Sp\,(2).$$

If $q\in S^2$, then $p=2\sqrt{1-|q|^2}(q-\sqrt{1-|q|^2})=0$. Hence $f_{Sp(2)}(q,\ x_1)=I_2$. Furthermore, if $x=x_1\in S^3$, then $x_2=\sqrt{1-|x|^2}=0$, Hence we have

$$f_{Sp(2)}\,(q,\ x)=\begin{pmatrix}1+xp\bar{x} & 0\\ 0 & 1\end{pmatrix},$$

where $1+xp\bar{x}=1+2x\sqrt{1-|q|^2}(q-\sqrt{1-|q|^2})\,\bar{x}$. If we put $q=y\sin\theta$, where $y\in S^2$ and $0\leqq\theta\leqq\pi/2$, then

$$1+xp\bar{x}=1+2x\cos\theta\,(y\sin\theta-\cos\theta)\,\bar{x}=-\cos 2\theta+\sin 2\theta\, xy\bar{x}.$$

This shows that a mapping $(q, x)\rightarrow 1+xp\bar{x}$ is nothing than a Hopf construction of a mapping $(x, y)\rightarrow xy\bar{x}$.

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