On relations between lattices of finite uniform coverings of a metric space and the uniform topology of the space

By Jun-iti Nagata

(Received July 10, 1953)

We characterized a complete uniform space by the lattice of uniform coverings satisfying some two conditions in the previous paper. But for simplicity of the theory it is desirable to use a lattice consist of finite uniform coverings only. In the case of a totally bounded space the possibility of such a restriction is obvious.

In the case of a metric space the totality of finite uniform coverings are not uniform basis generally, but then we can use a lattice of finite uniform coverings for characterizing its uniform topology. In this paper we shall show that a lattice of finite uniform coverings of a complete metric space characterizes the uniform topology and that in the case of a general metric space the lattice characterizes the completion of the space.

We concern ourselves with a lattice $L(R)$ consist of open finite uniform coverings of a complete metric space $R$ satisfying the following conditions,

1) if $U, V \in L(R)$, then $U \vee V \in L(R),$ 
2) if $U, V$ are some open sets such that $U \cap V = \emptyset$, $V = \emptyset$, then there exists $M \in L(R)$ such that $U \subseteq M$, $V \subseteq M$, 
3) $L(R)$ is a basis of the totality of finite uniform coverings of $R$.

Remarks. The order $U < V$ between elements of $L(R)$ is the relation that $U$ is refiner than $V$. We denote by $U \vee V$ the uniform covering $\{W | W \in U \text{ or } W \in V\}$. In $L(R)$ we regard two equivalent coverings as the same element. Hence the notation $U \subseteq M$ means the fact that for some $U' \supseteq U$, $U' \subseteq M$ holds. In condition 2) we assume implicitly that $R$ has no isolated points.

Definition. We denote by $U < V$ the fact that $V \subseteq M \in L(R)$ implies $U \subseteq M$.

Definition. We mean by a max. family for $U \in L(R)$ a subset $\mu$ of $L(R)$ having the property that $\mathcal{P}_i \in \mu \ (i = 1 \ldots k)$ imply $U \subseteq \bigvee_{i=1}^k \mathcal{P}_i$ and for every $\mu' \supseteq \mu$ this condition does not hold.

---

2) If for every element $\mathcal{B}$ of a family $A$ of coverings of $R$ there exists $\mathcal{U} \in L(R)$ such that $\mathcal{U} < \mathcal{B}$, then we call $L(R)$ a basis of $A$.
3) If $\mathcal{U} < \mathcal{B}$, $\mathcal{B} < \mathcal{U}$ hold, then we say that $\mathcal{U}$ and $\mathcal{B}$ are equivalent.
Lemma 1. In order that a subset $\mu$ of $L(R)$ is a max. family for $\mathfrak{U}$ it is necessary and sufficient that $\mu = \{ \mathfrak{V} | U \in \mathfrak{V} \in L(R) \}$ for some $U \in \mathfrak{U}$ such that $V \in \mathfrak{U}, V > U$ imply $U > V$.

Proof. Let $\mu = \{ \mathfrak{V} | U \in \mathfrak{V} \}, U \in \mathfrak{U},$ and let $V \in \mathfrak{U}, V > U$ imply $U > V$. If $\mathfrak{V}_i \in \mu (i = 1, \ldots, k)$, then from $U \leq \bigvee_{i=1}^k \mathfrak{V}_i$ we get $\mathfrak{V} \leq \bigvee_{i=1}^k \mathfrak{V}_i$.

Next if $\mathfrak{V} \notin \mu$, then there exists $N \in \mathfrak{V}$ such that $N \supset U$. We denote by $V_i (i = 1, \ldots, l)$ all the elements of $\mathfrak{U}$. If $V_i \supset U (i = 1 \ldots l)$, then there exists $\mathfrak{V}_i \in L(R)$ such that $V_i \in \mathfrak{V}_i, U \notin \mathfrak{V}_i$; hence $\mathfrak{V}_i \in \mu (i = 1 \ldots l)$. If $V_i > U$, then from the property of $U, U > V_i$ holds. Since $U \in \mathfrak{V}$, we get $V_i \notin \mathfrak{V}$. Therefore we get $\mathfrak{U} \leq (\bigvee_{i=1}^l \mathfrak{V}_i) \mathfrak{V}, \mathfrak{V}_i \in \mu (i = 1 \ldots l)$, i.e. $\mu$ is a max. family.

In the contrary, let $\mu$ be a max. family for $\mathfrak{U}$, then there exists $U \in \mathfrak{U}$ such that $U \notin \mathfrak{V}_a$ for all $\mathfrak{V}_a \notin \mu$. Since $\mathfrak{U}$ is a finite covering, there exists some $V \in \mathfrak{U}$ such that $V > U; \mathfrak{V} \supset V' > V$ implies $V > V'$. Since $U \notin \mathfrak{V}_a$ for all $\mathfrak{V}_a$, too. Hence we get $\mu \subset \{ \mathfrak{V} | V \notin \mathfrak{V} \}$. Therefore from the maximum property of $\mu$ we get $\mu = \{ \mathfrak{V} | V \notin \mathfrak{V} \}$.

Definition. We mean by a chauchy sequence of $L(R)$ a sequence $\{ \mu_n | n = 1, 2, \ldots \}$ of max. families of $L(R)$ such that $\mu_n \supset \mu_{n+1}$, and for every $\mathfrak{U} \in L(R)$ and for some $\mu_n, \mathfrak{U} \notin \mu_n$ holds.

Remarks. By lemma 1 let us assume that $\mu_n = \{ \mathfrak{V} | U_n \notin \mathfrak{V} \} (n = 1, 2, \ldots)$. In order that $\mu_n \supset \mu_{n+1}$ it is necessary and sufficient that $U_n > U_{n+1}$. We note that the last formula implies $U_{n+1} \subset U_n$. For in the contrary case we get from the condition 2) of $L(R)$ an element $\mathfrak{U}$ of $L(R)$ such that $U_n \in \mathfrak{U}, U_{n+1} - U_n \notin \mathfrak{U}$, and accordingly $U_{n+1} \notin \mathfrak{U}$. This consequence contradicts the fact that $U_{n+1} < U_n$.

Lemma 2. If $\mu_n = \{ \mathfrak{V} | U_n \notin \mathfrak{V} \} (n = 1, 2, \ldots)$, then in order that $\{ \mu_n | n = 1, 2, \ldots \}$ is a chauchy sequence of $L(R)$ it is necessary and sufficient that $\{ U_n | n = 1, 2, \ldots \}$ is a chauchy sequence of $R$.

Proof. Since $U_n \in \mathfrak{U}$ implies $\mathfrak{U} \notin \mu_n$, the sufficiency of the condition is obvious.

Now assume that $\{ U_n | n = 1, 2, \ldots \}$ is no chauchy sequence of $R$, and assume that $U_n \subseteq S_m(x) \forall n$ and for all $x \in R$, where $S_m(x) = \{ y | \rho(x, y) < 1/2^m \}$; $\rho$ is the distance between $x$ and $y$. Then there exist $x_1, y_1 \in U_1 = U_{n_1}$ such that $y_1 \subseteq S_m(x_1)$. If $S_{m+1}(x_1) \cap U_n = \phi$ for all $n$, then for the uniform covering $\mathfrak{M} = \{ S_{m+1}(x_1) \} \cap U_n = \phi$ we can take a refinement $\mathfrak{U} \in L(R)$ of $\mathfrak{M}$ by condition 3) of $L(R)$. Since $U_n \notin \mathfrak{U}$ for all $n, U_n \notin \mathfrak{U}$ hold for all $n$; hence $\mathfrak{U} \notin \mu_n$, and

4) We mean by a chauchy sequence of $R$ a sequence $U_n (n = 1, 2, \ldots)$ of open sets of $R$ such that $U_n > U_{n+1}$, and the diameters of $U_n$ tend to zero.

5) We denote by $A^c$ the complement of $A$. Since $\{ S_{m+1}(x) | x \in R \} < \mathfrak{M}, \mathfrak{M}$ is a uniform covering of $R$. 

On relations between lattices of finite uniform coverings of a metric space and the uniform topology of the space

37

hence \( \{ \mu_n \} \) is no cauchy sequence of \( L(R) \). In the case that \( S_{n+1}(y_1) \cap U_n \neq \phi \) for all \( n \), we see analogously that \( \{ \mu_n | n = 1, 2, \ldots \} \) is no cauchy sequence of \( L(R) \).

If \( S_{n+1}(x_1) \cap U_n' = \phi \), \( S_{n+1}(y_1) \cap U_n'' = \phi \), then for \( n \geq \text{max} \left( n', n'' \right) = n_2 \) from \( U_n \subseteq U_n' \), \( U_n \subseteq U_n'' \) we get \( S_{n+1}(x_1) \cap U_n = \phi \) and \( S_{n+1}(y_1) \cap U_n = \phi \).

Then we can take \( x_2, y_2 \in U_n' \) such that \( x_2 \neq y_2 \). If \( S_{n+1}(x_2) \cap U_n \neq \phi \) for all \( n \) or \( S_{n+1}(y_2) \cap U_n \neq \phi \) for all \( n \), then we can conclude that \( \{ \mu_n | n = 1, 2, \ldots \} \) is no cauchy sequence of \( L(R) \) as in the previous manner. In the contrary case \( Sm+1(x_2) \cap U_n = \phi \), \( Sm+1(y_2) \cap U_n = \phi \) hold for some \( n_3 \) and for all \( n \geq n_3 \).

Then we take \( x_3, y_3 \in U_n' \) such that \( y_3 \notin S_m(x_3) \). By an inductive consideration we get the conclusion that \( \{ \mu_n | n = 1, 2, \ldots \} \) is no cauchy sequence of \( L(R) \) or the conclusion that there exists a sequence \( x_i, y_i (i = 1, 2, \ldots) \) of points of \( R \) such that \( x_i, y_i \in U_n' \); \( x_i \notin S_m(x_i), S_{m+1}(x_i) \cap U_n = \phi \), \( S_{m+1}(y_i) \cap U_n = \phi \) hold for \( j \geq i+1 \).

In the last case we get a finite uniform covering \( \mathcal{M} = \bigcup_{i=1}^{n_3} S_{m+1}(x_i), R - \bigcup_{i=1}^{n_3} x_i \), for which \( U_n \notin \mathcal{M} \) hold for all \( i \). For \( x_i \in U_n \) implies \( U_n \subseteq R - \bigcup_{i=1}^{n_3} x_i \) and \( y_i \notin S_m(x_i) \) combining with \( x_i \in U_n \) implies \( U_n \subseteq \bigcup_{i=1}^{n_3} S_{m+1}(x_i) \). By the condition 3) of \( \mathcal{L}(R) \), we take \( U \) such that \( \mathcal{M} > U \in \mathcal{L}(R) \). Then for an arbitrary \( U_n \), \( n \geq n \) implies \( U_n < U_n \); hence from \( U_n \notin \mathcal{M} \) we conclude that \( U_n \notin U \).

Hence \( \{ \mu_n | n = 1, 2, \ldots \} \) is no cauchy sequence of \( L(R) \) also in this case.

**Definition.** We denote by \( \{ \mu_n | n = 1, 2, \ldots \} \sim \{ \nu_n | n = 1, 2, \ldots \} \) the relation between two cauchy sequences of \( L(R) \) such that for every \( U \in \mathcal{L}(R) \) there exist two elements \( \mu_n, \nu_n \) of the sequence and some max. family \( \lambda \) such that \( \lambda \supset \mu \cup \nu \), \( U \notin \lambda \).

**Lemma 3.** In order that \( \{ \mu_n | q = 1, 2, \ldots \} \sim \{ \nu_n | n = 1, 2, \ldots \} \) it is necessary and sufficient that \( \{ U_n | n = 1, 2, \ldots \} \) and \( \{ V_n | n = 1, 2, \ldots \} \) are equivalent cauchy sequences of \( R \), where \( \mu_n = \{ U | U_n \notin \mathcal{M} \}, \nu_n = \{ V | V_n \notin \mathcal{M} \} \).

**Proof.** The sufficiency of the condition is obvious.

If \( \{ U_n \} \) and \( \{ V_n \} \) are not equivalent in \( R \), then for some \( m \) \( U_n \cup V_n \subset S_m(x) \) hold for all \( n \) and for all \( x \in R \). Hence in the same way as in the previous proof we get \( \mathbb{U} \in \mathcal{L}(R) \) such that \( U_n \cup V_n \notin \mathbb{U} \) for all \( n \). Take \( \mathbb{W} \in \mathcal{L}(R) \) such that \( \mathbb{W} \subset \mathbb{U} \). If \( \mathbb{W} \notin \mathbb{U} \) for some max. family \( \lambda = \{ W | W \notin \mathbb{W} \} \), and if \( \lambda \supset \mu_n \cup \nu_n \), then \( U_n < W, V_n < W \); hence from \( W \in \mathbb{W}, U_n \cup V_n \subset W \subset U \notin \mathbb{U} \), but this is impossible. Therefore the negation of \( \{ \mu_n | n = 1, 2, \ldots \} \sim \{ \nu_n | n = 1, 2, \ldots \} \) holds.

From lemma 3 we can classify all the cauchy sequences of \( L(R) \) by the relation \( \sim \). We denote by \( \mathbb{E}(R) \) the set of all such classes. From this lemma and the completeness of \( R \) we get a one-to-one correspondence between \( R \) and
Definition. We mean by a uniform covering of \( \mathcal{L}(R) \) a covering \( \{ \mathcal{L}(U_a) \mid a \in A \} \) of \( \mathcal{L}(R) \) such that there exists a definite covering \( \{ \mathcal{L}(U_a) \} : \{ \mathcal{L}(U_a) \} \triangleleft \{ \mathcal{L}(U'_a) \} \) and for an arbitrary binary covering \( \{ \mathcal{L}(U), \mathcal{L}(V) \} \rangle \{ \mathcal{L}(U_a) \}, \) there exists \( U \in L(R) \) such that \( \mu \in \{ \nu \mid \mu = 1, 2, \ldots \} \in \mathcal{L}(U), \nu \in \{ \nu \mid \nu = 1, 2, \ldots \} \in \mathcal{L}(V) \) imply \( U < U' \lor V' \) for some \( U' \in \mu \) and \( V' \in \nu \).

Lemma 4. In order that \( \{ \mathcal{L}(U_a') \} \) is a uniform covering of \( \mathcal{L}(R) \) it is necessary and sufficient that \( \{ \mathcal{L}(U_a') \} \) is a uniform covering of \( R. \)

Proof. Sufficiency. Let \( \{ U_a' \} \) be a uniform covering of \( R, \) then there exists a uniform covering \( \{ U_a \} \) of \( R \) such that \( \{ U_a \} \triangleleft \{ U_a' \}, \) i.e. \( \{ \mathcal{L}(U_a) \} \triangleleft \{ \mathcal{L}(U_a') \}. \) If \( \{ \mathcal{L}(U), \mathcal{L}(V) \} \rangle \{ \mathcal{L}(U_a) \}, \) then since \( \{ U_a \} \triangleleft \{ U, V \} \) in \( R, \) \( \{ U, V \} \) is a binary uniform covering of \( R. \) Hence from condition 3) of \( L(R) \) there exists \( U \in L(R) \) such that \( U < U, V. \) If \( \mu \in \{ \mu \} \notin \mathcal{L}(U), \) \( \nu \in \{ \nu \} \notin \mathcal{L}(V) \) and if \( \mu = \{ \nu \} U \notin \mathcal{L}(U), \) \( \nu = \{ \nu \} V \notin \mathcal{L}(V), \) then \( \{ U_a \} \) converges to \( a \notin U \) and \( \{ V_n \} \) converges to \( b \notin V. \) Let \( U' \in U, \) then from \( U < U, V, \) \( U' \subseteq U \) or \( U' \subseteq V \) holds. If \( U' \subseteq U, \) then from \( a \notin U \) and from \( a \in U_n \) we get \( U' \supseteq U_n. \) Hence from condition 2) of \( L(R) \) there exists \( U \subseteq U' \subseteq L(R) \) such that \( U' \subseteq U \subseteq U, U_n \notin \mathcal{L}(U'). \) If \( U' \subseteq V, \) then analogously there exists \( U \subseteq U' \subseteq L(R) \) such that \( U' \subseteq U \subseteq U \subseteq U, V_n \notin \mathcal{L}(U'). \) Hence \( \forall \{ \mathcal{L}(U') \} \) \( U' \subseteq U \subseteq \mathcal{L}(U'), \) \( V_n \notin \mathcal{L}(U') \). Therefore \( \{ \mathcal{L}(U_a') \} \) is a uniform covering of \( \mathcal{L}(R) \) by the above definition.

Necessity. Assume that \( \{ U_a' \} \) is no uniform covering of \( R \) and that \( \{ \mathcal{L}(U_a) \} \triangleleft \{ \mathcal{L}(U_a') \}, \) then \( \{ U_a \} \triangleleft \{ U_a' \}. \) We denote by \( \mathcal{C}_n \) the uniform covering \( \{ S_n(x) \mid x \in R \} \) of \( R. \) Putting \( \mathcal{A} = \{ U_a' \}, \) for every \( n \) we get \( S_n \in \mathcal{C}_n \) \( (n = 1, 2, \ldots) \) such that \( S_n \notin \mathcal{L}(U_a). \) For this \( S_1 \) we take \( x_1, y_1 \in S_1 \) such that \( y_1 \notin S^2(x_1, \mathcal{A}) \). If \( S(x_1, \mathcal{A}) \cap S_n = \emptyset \) hold for an infinite number of \( n \) \( (i = 1, 2, \ldots) \) then for \( x_n \in S(x_1, \mathcal{A}) \cap S_n \) \( (i = 1, 2, \ldots) \), \( \mathcal{A}' = \{ S^2(x_1, \mathcal{A}), R - \bigcup_{i=1}^{\infty} S_n \} \) is a binary covering of \( R \) such that \( \mathcal{A} \subset \mathcal{A}' \). Since \( \mathcal{C}_n \ni S_n \notin \mathcal{A}' \) \( (i = 1, 2, \ldots), \mathcal{A}' \) is no uniform covering of \( R \). If \( S(y_1, \mathcal{A}) \cap S_n = \emptyset \) hold for an infinite number of \( n \), then analogously there exists a binary non-uniform covering \( \mathcal{A}' \) of \( R \) such that \( \mathcal{A} \subset \mathcal{A}' \).

If \( n \geq n_2 \) implies \( S(x_1, \mathcal{A}) \cap S_n = \emptyset \) and \( S(y_1, \mathcal{A}) \cap S_n = \emptyset \) for some \( n_2, \) then

---

6) This notation is due to J. W. Tukey, Convergence and Uniformity in topology, 1940.
7) For \( S(x_1, \mathcal{A}) \cap S_n = \emptyset \) implies \( S_n \subseteq S^2(x_1, \mathcal{A}), \) and \( x_n \in S_n \) implies \( S_n \subseteq R - \bigcup_{i=1}^{\infty} x_n \).
On relations between lattices of finite uniform coverings of a metric space and the uniform topology of the space

we take \( x_2, y_2 \in S_{n_2} \) such that \( y_2 \notin S(x_2, \mathbb{W}) \). For these \( x_2, y_2; S_{n_2} \) in the same way as for \( x_1, y_1, S_{n_1} = S_1 \), we get a binary non-uniform covering \( \mathbb{W} \) of \( R \) such that \( \mathbb{W} \nless \mathbb{W}' \) or \( x_3, y_3; S_{n_3} (n_3 > n_2) \) such that \( x_3, y_3 \in S_{n_3}; S(x_2, \mathbb{W}) \cap S_n = \emptyset \), \( S(y_2, \mathbb{W}) \cap S_n = \emptyset (n > n_3) \), \( y_3 \notin S(x_3, \mathbb{W}) \). By such an argument we get a binary non-uniform covering \( \mathbb{W} \) of \( R \) such that \( \mathbb{W} \nless \mathbb{W}' \) or points \( x_i, y_i (i = 1, 2, \ldots) \) of \( R \) such that \( x_i, y_i \in S_{n_i}; S_i \subseteq S(x_i, \mathbb{W}) \). In the latter case, we get a binary non-uniform covering \( \mathbb{W}' = \bigcup_{i=1}^{\infty} S_i \), \( R - \bigcup_{i=1}^{\infty} x_i \). For this \( \mathbb{W} \nless \mathbb{W}' \) is obvious. Since \( x_i \in S_{n_i}, S_{n_i} \subseteq R - \bigcup_{i=1}^{\infty} x_i \). From \( y_i \in S_{n_i} \) and from \( y_i \notin S(x_j, \mathbb{W}) \) for all \( j \), \( S_{n_i} \subseteq \bigcup_{i=1}^{\infty} S(x_i, \mathbb{W}) \) holds. Hence \( \mathbb{W} \nless \mathbb{W}' \). Since this formula holds for every \( i, \mathbb{W}' \) is no uniform covering of \( R \). Therefore in every case we get a binary non-uniform covering \( \mathbb{W}' \) such that \( \mathbb{W}' \nless \mathbb{W} \).

Let \( \mathbb{W} \) be an arbitrary uniform covering in \( L(R) \), Then \( \mathbb{W} \nless \mathbb{W} \) holds for this \( \mathbb{W} \), i.e. there exists \( U \in \mathbb{W} \) such that \( U \nless A, B \) for both elements \( A, B \) of \( \mathbb{W} \). Take \( x, y \) so that \( x \in U \cap A, y \in U \cap B \), and let \( L(x) = \{v_n | n = 1, 2, \ldots\}, L(y) = \{v_m | m = 1, 2, \ldots\}; \mu_n = [\mathbb{W}|U_n \in \mathbb{W}]; \nu_m = [\mathbb{W}|V_m \in \mathbb{W}] \); then since \( U_n, V_n \) converge to \( x, y \) respectively in \( R \), there exist \( U_n, V_n \) such that \( U_n \subseteq U, V_n \subseteq U \). For every \( U' \in \mathbb{W} \), \( V' \in \mathbb{W} \) we get \( U_n \subseteq U', V_n \subseteq V' \). Combining these formulas with the above \( U_n \cup V_n \subseteq U, \) we get \( U \nless U', V \nless V' \). Hence \( \mathbb{W} \nless \mathbb{W}' \cap \mathbb{W} \) for such \( \mathbb{W}' \). Therefore \( \{[\mathbb{W}|U_n']\} \) is no uniform covering of \( L(R) \) by the above definition.

By this lemma \( R \) and \( L(R) \), the uniform space having the above defined uniform coverings are uniformly homeomorphic. Since points and uniform coverings of \( L(R) \) are defined by elements of \( L(R) \) and by relations \( \nless \) between elements of \( L(R) \), we get the following theorem.

Theorem 1. In order that two complete metric spaces \( R_1, R_2 \) are uniformly homeomorphic it is necessary and sufficient that \( L(R_1) \) and \( L(R_2) \) are lattice-isomorphic, where \( L(R_1), L(R_2) \) are lattices of finite uniform coverings of \( R_1, R_2 \) respectively and satisfy conditions 1), 2), 3).

Next we concern ourselves with a metric space having no completeness property. We denote by \( L(R) \) the lattice of all finite uniform coverings of \( R \). We define max. family of \( L(R) \) as in the above proof of Theorem 1, and we mean by chauchy sequence of \( L(R) \) a sequence of max. families of \( L(R) \), \( \{\mu_n | n = 1, 2, \ldots\} \) satisfying besides the above conditions the condition that there exists a max. family \( \mu \) such that \( \mu \supseteq \mu_n \) for all \( n \), and \( \nu \supseteq \mu \) is not valid but \( \nu = [\mathbb{W}|R \in \mathbb{W}] \). Thus we can characterize a converging chauchy sequence of \( R \) by such a chauchy sequence of \( L(R) \) and by an analogous argument to the case of complete metric space we get the following,

Corollary. In order that two metric spaces \( R_1, R_2 \) are uniformly homeomorphic it is necessary and sufficient that lattices \( L(R_1), L(R_2) \) of all finite
uniform coverings of $R_1$, $R_2$ respectively are lattice-isomorphic.

This corollary is obvious for totally bounded uniform spaces $R_1$, $R_2$, too.

Next let us consider relations between $L(R)$ and the completion $\widetilde{R}$ of $R$.

**Theorem 2.** If $R_1$, $R_2$ are metric spaces and if $\widetilde{R_1}$, $\widetilde{R_2}$ are the completions of $R_1$, $R_2$ respectively, then in order that $\widetilde{R_1}$ and $\widetilde{R_2}$ are uniformly homeomorphic it is necessary and sufficient that lattices of finite uniform coverings, $L(R_1)$, $L(R_2)$ satisfying conditions 1), 2), 3) are lattice-isomorphic.9)

**Proof.** For each $\mathbb{U} = \{U_a \in L(R_1)\}$ we denote by $\mathbb{U}$ the uniform covering \{$(U_a)^\wedge | U_a \in \mathbb{U}$\} of $\widetilde{R_1}$, where $U_a$, $U_b$, $\widetilde{U}$ mean complement in $R_1$, complement in $\widetilde{R_1}$, closure in $\widetilde{R_1}$ respectively. Putting $L(R_1) = \{\mathbb{U} | \mathbb{U} \in L(R_1)\}$, we see easily that $L(R_1)$ and $L(\widetilde{R_1})$ are isomorphic. For $\mathbb{U} \ni U \subset \mathcal{V} \ni \mathbb{W}$ implies $(\widetilde{U})^\wedge \subset (\mathcal{V})^\wedge$; hence $\mathbb{U} \lessdot \mathbb{W}$ implies $\widetilde{\mathbb{U}} \lessdot \widetilde{\mathbb{W}}$. If $\widetilde{\mathbb{U}} \lessdot \widetilde{\mathbb{W}}$, then for all $U \in \mathbb{U}$ there exists $V \in \mathcal{V}$ such that $(\widetilde{U})^\wedge \subset (\mathcal{V})^\wedge$; hence $(\widetilde{U})^\wedge \cap R_1 = U \subset \mathcal{V} = (\mathcal{V})^\wedge \cap R_1$. Therefore $\mathbb{U} \lessdot \mathcal{V}$ is obvious and since $L(R_1)$ satisfies condition 1), $L(\widetilde{R_1})$ satisfies condition 1). If $U'$, $V'$ are open sets in $\widetilde{R_1}$ such that $V' = \phi$, $U' \cap V' = \phi$, then denoting $U' \cap R_1 = U$, $V' \cap R_1 = V$, we get $U' \cup V' = \phi$, $V' = \phi$. Hence there exists $\mathbb{U} \in L(R_1)$, for which $U \subset U_0$ for some $U_0 \in \mathbb{U}$ and $V \subset U_a$ for every $U_a \in \mathbb{U}$. Then from $U_0 \subset U \subset U^\wedge$ we get $\widetilde{U_0} \supset U^\wedge$, and hence $\widetilde{U'} \subset (\widetilde{U_0})^\wedge \subset \widetilde{\mathbb{U}}$. Thus $L(\widetilde{R_1})$ satisfies condition 2) in $\widetilde{R_1}$.

Next we shall show that $L(\widetilde{R_1})$ satisfies condition 3) in $\widetilde{R_1}$. Let $\{U_i | i = 1, \ldots, k\}$ be an arbitrary finite uniform covering of $\widetilde{R_1}$, then taking a uniform covering $\mathbb{S}$ of $\widetilde{R_1}$ such that $\mathbb{S}^\wedge \subset \{U_i\}$, we get open sets $G_i = \cap \{S | S' \cap R_1 = S \supset F \in \mathbb{F} \text{ for some } S' \in \mathbb{S} \text{ and for some } \mathbb{F}, F \text{ such that } F \in \mathbb{F} \in U_i\}$ of $R_1$, where $\mathbb{F}$ is a maximum chauchy filter of closed sets of $R_1$, and $\mathbb{F}$ is also a point of $\widetilde{R_1}$. For example, $F \in \mathbb{F}$ means that the fiber $\mathbb{F}$ of $R_1$ contains the subset $F$ of $R_1$, and $\mathbb{F} \in U$ means that the point $\mathbb{F}$ of $\widetilde{R_1}$ is contained in the subset $U$ of $\widetilde{R_1}$. Now we show that $\mathbb{U} = \{G_i\} \supset \mathbb{S}$ in $R_1$, where $\mathbb{U}$ is not an open covering generally. Assume the contrary and assume that $S \in \mathbb{S}$, $S' = S \cap R_1$, $S' \cap G_i = \phi$ $(i = 1, \ldots, k)$, then there exist open sets $S_i$ of $\widetilde{R_1}$ and maximum chauchy filters $\mathbb{F}_i$ of $R_1$ such that $S \subset S_i \cap R_1 = S_i$, $S_i \supset F \in \mathbb{F}_i$, $\mathbb{F}_i \in U_i$. For these $\mathbb{F}_i$ taking $S_i' \subset \mathbb{S}$ such that $\mathbb{F}_i \in S_i'$, we see easily that $S_i \supset S_i' = \phi$. For $S_i \subset S_i^\wedge$ combining with $S_i \subset F \in \mathbb{F}_i$ implies $\mathbb{F}_i \subset S_i \subset S_i^\wedge$, which contradicts the fact that $\mathbb{F}_i \in S_i'$. Therefore $S_i' \subset S^\wedge (S, \mathbb{S})$ from

9) The completion $\widetilde{R}$ of $R$ consists of all the maximum chauchy filters of closed sets of $R$. The topology of $\widetilde{R}$ is defined by the closed basis $\{\mathbb{F} | \mathbb{F} = \{\mathbb{F} | \mathbb{F} \supset F\}, F \text{ is closed subset of } R\}$. The uniform topology of $\widetilde{R}$ is defined by the uniform coverings $\mathbb{U} = \{(\widetilde{U})^\wedge | U \in \mathbb{U}\}$ for uniform coverings $\mathbb{U}$ of $R$. 

---

40 Jun-iti NAGATA
On relations between lattices of finite uniform coverings of a metric space and the uniform topology of the space

$S \cap S_i \neq \emptyset$. Therefore $\mathfrak{F}_i \in U_i$, $\mathfrak{F}_i \in S_i' \subseteq S_i(\emptyset)\subseteq U_i$ ($i = 1, \ldots, k$), but this contradicts the fact that $\emptyset^{\text{fin}} \subseteq \{U_i\}$. This contradiction proofs the validity of $\emptyset \subseteq \mathfrak{F}$ in $R_1$. Hence $\mathfrak{F}$ is a finite uniform covering of $R_1$ and hence we can take $\mathfrak{F} \in L(R_1)$ such that $\mathfrak{F} \subseteq \mathfrak{F}$. Let $V \in \mathfrak{F}$ and let $V \subseteq G_1 \in \mathfrak{F}$. If $\mathfrak{F}$ is a maximum chauchy filter of $R_1$ or a point of $\mathfrak{F}$ such that $\mathfrak{F} \subseteq U_i$ in $R_1$, then taking $S \in \mathfrak{F}$ such that $S \supseteq F \in \mathfrak{F}$ for some $F$, from the definition of $G_i$ we get $R_1 \cap S \supseteq G_i \subseteq V^c$. Hence $\mathfrak{F} \subseteq V^c$, and hence $U_i \supseteq \mathfrak{F}$, i.e. $U_i \supseteq (\mathfrak{F})^c$. Since $V$ is an arbitrary element of $\mathfrak{F}$, $\mathfrak{F} \subseteq \{U_i\}$ for $\mathfrak{F} \in L(R_1)$. Thus we see that $L(R_1)$ is a basis of all the finite uniform coverings of $\mathfrak{F}$, i.e. $L(R_1)$ satisfies 3), too.

If $L(R_1)$ and $L(R_2)$ are isomorphic, then $L(R_1)$ and $L(R_2)$ are isomorphic; hence from the above conclusion and from Theorem 1 we get Theorem 2.

For completions of non-metric spaces we get the following propositions by the theorem of my previous paper\(^{10}\) and by analogous arguements.

**Corollary.** If we denote by $\mathfrak{R}_1$, $\mathfrak{R}_2$ the completions of totally bounded uniform spaces $R_1$, $R_2$ respectively, then in order that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are uniformly homeomorphic it is necessary and sufficient that lattices $L(R_1)$ and $L(R_2)$ of finite uniform coverings satisfying conditions 1), 2), 3) are lattice-isomorphic.

**Corollary.** If we denote by $\mathfrak{R}_1$, $\mathfrak{R}_2$ the completions of uniform spaces $R_1$, $R_2$ respectively, then in order that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are uniformly homeomorphic it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are lattice-isomorphic, where $L(R_1)$ and $L(R_2)$ are lattices of uniform coverings of $R_1$ and $R_2$ respectively and satisfy the following conditions.

1') $U \in L(R_1)$, $\mathfrak{F} \in L(R_1)$ imply $U \vee \mathfrak{F} \in L(R_1)$,

2') if $U \in L(R_1)$ and if $U = \phi$ is an open set of $R_1$, then there exists $\mathfrak{M} \in L(R_1)$ such that $U \subseteq \mathfrak{M}$; $U^c \supseteq U' \subseteq U$ implies $U' \subseteq \mathfrak{M}$,

3') $L(R_1)$ is a basis of the totality of uniform coverings of $R_1$.

\(^{10}\) Loc. cit.