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On the Singularity of the Peturbation-Term in the Field Quantum Mechanics

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Summary

Let H be a total Hamiltonian of a system consisting of two fields. When H is divided into two parts in two ways as $H=H_1^0+H_1'=H_2^0+H_2'$, where H_1^0 and H_2^0 are unperturbed terms and H_1' and H_2' are perturbation terms, then (i) two spaces $\mathfrak{H}(H_1^0)$ and $\mathfrak{H}(H_2^0)$ which are determined by the systems of eigenvectors of H_1^0 and H_2^0 respectively, are mutually orthogonal, and (ii) the zero point energy of H_1^0 differs from that of H_2^0 by infinity. The zero point energy of the total Hamiltonian of a system in which a fixed nucleon and a real scalar meson field are interacting, amounts to $-g^2c^2/4V\cdot\sum 1/\omega_k^2$ which diverges to minus infinity. The total Hamiltonian of a system electron plus photon field has the expectation value $(H\Psi_\beta, \Psi_\beta)=c_\beta+\sum(1/2-2\pi c^2e^2l_\lambda^2/hV\omega_\lambda^3)\,h\omega_\lambda$, where Ψ_β is a certain vector normalized to 4, c_β a finite constant depending on β, and l_λ the projection on the x-axis of the polarization vector e_λ of the λ-photon. The number of Ψ_β's is enumerably infinite and they are orthogonal with one another.

1. Introduction

In the previous paper[1] we have proved that the interaction term of a system which consists of a nucleon and a complex scalar meson field has no domain in a space, each of whose vectors is a superposition of states consisting of the nucleon and a finite number of mesons.

In a similar way, we can prove that the interaction term of a system electron field plus photon field has no domain in a space, each of whose vectors is a superposition of states which consits of a finite number of electrons and photons. The proof will be given elswhere.

Thus a vector representing a state in which an electron and a photon are in respective given state, does not belong to the domain of the total Hamiltonian. Here the zero point energy of the non-interacting term is not taken into account.

The total Hamiltonian operator is usually divided into two parts, the one is the principal part H^0 and the other is the perturbation term H'. H^0 can be transformed into a diagonal form by a suitable unitary transformation and the set of its eigenvectors determines an incomplete direct product space $\mathfrak{H}(H^0)$[1), 2)]. When the total Hamiltonian is divided into two parts in two different ways:

$$H = H_1^0 + H_1' = H^0 + H_2',$$

two spaces $\mathfrak{H}(H_1^0)$ and $\mathfrak{H}(H_2^0)$ are determined, and the energy of H_1' differs from that of H_2'. In the present paper, it will be proved that $\mathfrak{H}(H_1')$ and $\mathfrak{H}(H_2^0)$ are in general mutually orthogonal, and the energy difference of H_1' and H_2^1 is infinite. The orthogonality of the two spaces and the infinity of the energy difference seems to have some connections.

As mentioned above, the energy difference of H_1' and H_2' is infinity, so that even though we can cancel the singularity of H_1' by introducing a third field having negative probabilities, this newly introduced field will be unable to cancel the singularity of H_2'. Moreover, this mixed field theory has the following inadequateness. The field equation

$$ih\frac{d\psi}{dt} = H\psi, \quad (\psi)_{t=0} = \psi_0 \qquad (h \equiv \hbar)$$

has a unique solution when H is self-adjoint and ψ_0 belongs to the domain of H. When we take the negative probability into considerations, it is not clear whether the above existence theorem holds good or not. The details of this point will be discussed elswhere.

According to the analysis given in the present paper (§3), it will become clear that a nucleon not being accompanied by an infinite number of mesons does not exist.

It is difficult to obtain the exact eigenvalues of the Hamilton operator H of a total system electron plus photon field. However, it can be proved that there are infinitely many states Ψ_β's which are eigenstates of a part of the total Hamiltonian H and that the expectation value of H with respect to Ψ_β is equal to $c_\beta + \sum(1/2 - 2\pi c^2e^2l_\lambda^2/hV\omega_\lambda^3)\,h\omega_\lambda$, where c_β is a finite constant depending on β and l_λ the projection on the x-axis of the polarization vector e_λ of the λ-photon. The sum $\sum\frac{1}{2}h\omega_\lambda$ is the zero point energy of the free photon field and the series $\sum l_\lambda^2\omega_\lambda^{-3}$ diverges logarithmically (§4).

2. Orthogonality of Spaces $\mathfrak{H}(H_1^0)$ and $\mathfrak{H}(H_2^0)$

The Hamilton operator of the total system electron plus electromagnetic field, after elimination of the longitudinal parts of the electric field, is

$$H = c\left\{(\alpha, \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) + \beta mc\right\} + \frac{1}{8\pi}\int(\boldsymbol{E}^2 + \boldsymbol{H}^2)\,dV.$$

Expanding the vector potential $\boldsymbol{A}$ in Fourier series, we obtain[3)]

$$H = c\left\{(\alpha, \boldsymbol{p}-\sum_{s\lambda} \boldsymbol{a}_{s\lambda}[P_{s\lambda}\cos(\boldsymbol{k}_s, \boldsymbol{r})+Q_{s\lambda}\sin(\boldsymbol{k}_s, \boldsymbol{r})]+\beta mc\right\} + \frac{1}{2}\sum_{s\lambda}(P_{s\lambda}^2+Q_{s\lambda}^2)\,h\omega_s, \tag{1}$$

where

$$\boldsymbol{a}_{s\lambda} = 2e\left(\frac{\pi h}{V\omega_s}\right)^{\frac{1}{2}}\boldsymbol{e}_{s\lambda}, \tag{2}$$

and V is the volume within which the cyclical boundary conditions are applied, the summation index s characterizes the direction and circular frequency ω_s of the various waves with propagation vector $\boldsymbol{k}_s$, λ their state of polarization; and $\boldsymbol{e}_{s\lambda}$ is a unit vector in the direction of polarization. The dynamical variables $P_{s\lambda}$ and $Q_{s\lambda}$ obey the commutation laws

$$[P_{s\lambda}, Q_{s'\lambda'}] = -i\delta_{ss'}\delta_{\lambda\lambda'}, \quad [P_{s\lambda}, P_{s'\lambda'}] = [Q_{s\lambda}, Q_{s'\lambda'}] = 0.$$

In the usual perturbation method, the non-interacting part of H

$$H_1^0 = c\left\{(\alpha, \boldsymbol{p})+\beta mc\right\} + \frac{1}{2}\sum_{s\lambda}(P_{s\lambda}^2+Q_{s\lambda}^2)\,h\omega_s$$

is used as the umperterbed operator. The space $\mathfrak{H}(H_1^0)$ is then determined by the eigenvectors of H_1^0, i. e., the complete normalized orthogonal set $\{\varphi_\beta\}$, where

$$\varphi_\beta = \varphi(p)\otimes\prod_{s\lambda}\otimes_{\beta\in F}\varphi_{s\lambda,\ \beta(s\lambda)}(Q_{s\lambda}). \tag{3}$$

Here $\beta(s\lambda)$ is a function of s, λ and its range of values is $0, 1, 2, \ldots$. The notation $\beta\in F$ implies that $\beta(s\lambda)$ is zero for all but a finite number of s, λ. $\varphi(p)$ is an eigenvector of the operator

$$H(p) = c\left\{(\alpha, \boldsymbol{p})+\beta mc\right\},$$

and is written as $\varphi(p)=u(p)\exp(i\boldsymbol{p}\boldsymbol{r}/h)$ and $u(p)$ is a four-component vector.[1) $\varphi_{s\lambda,\ \beta(s\lambda)}(x)$ is the normalized solution of the oscillator equation

$$y''-x^2y+(2\beta(s\lambda)+1)\,y = 0.$$

Bloch and Nordsieck[3) has shown another powerful method in solving the eigenvalue problem

$$H\psi = E\psi.$$

They adopt the following H_2^0 as the principal term of H:

$$\left.\begin{aligned} H_2^0 &= (c\,\mu, \boldsymbol{p}-\sum_{s\lambda}\boldsymbol{a}_{s\lambda}[P_{s\lambda}\cos(\boldsymbol{k}_s, \boldsymbol{r})+Q_{s\lambda}\sin(\boldsymbol{k}_s, \boldsymbol{r})])+mc^2(1-\mu^2)^{\frac{1}{2}} \\ &+ \frac{1}{2}\sum_{s\lambda}(P_{s\lambda}^2+Q_{s\lambda}^2)\,h\omega_s, \end{aligned}\right\} \tag{4}$$

where $\mu=v/c$ and v stands for the constant velocity of the electron in its unperturbed motion. The space $\mathfrak{H}(H_2^0)$ is determined by the eigenvectors of H_2^0, i. e., the complete orthonormal set $\{\psi_\beta\}$, where

$$\left.\begin{aligned}\psi_\beta &= \gamma(\mu)\exp\left\{\frac{i}{h}\left(mc(1-\mu^2)^{-\frac{1}{2}}\mu, r\right)\right\}\otimes \prod_{s\lambda} \otimes \exp\left(i\sigma_{s\lambda}\cos(k_s, r)\right)\\ &\cdot\left[Q_{s\lambda}-\frac{1}{2}\sigma_{s\lambda}\sin(k_s, r)\right]\right)\varphi_{s\lambda},{}_{\beta(s\lambda)}(Q_{s\lambda}-\sigma_{s\lambda}\sin(k_s, r)),\\ &\sigma_{s\lambda} = (\mu, a_{s\lambda})/h(k_s-(\mu, k_s)),\end{aligned}\right\} \quad (5)$$

and $\gamma(\mu)$ is a normalized four-component amplitude.

Taking into account the condition $\beta, \beta' \in F$, we obtain

$$(\psi_\beta, \varphi_{\beta'}) = \text{const.}\exp\left(-\frac{1}{4}\sum\sigma_{s\lambda}^2\right),$$

and

$$\left.\begin{aligned}\frac{1}{4}\sum_{s\lambda}\sigma_{s\lambda}^2 &= \frac{e^2\pi c^2}{V}\sum_{s\lambda}\frac{(\mu, e_{s\lambda})^2}{(1-(\mu, e_s))^2}\frac{1}{h\omega_s^3},\\ e_s &= k_s/k_s.\end{aligned}\right\} \quad (6)$$

The last series diverges to plus infinity, so that ψ_β and $\varphi_{\beta'}$ are multually orthogonal. The two spaces $\mathfrak{H}(H_1^0)$ and $\mathfrak{H}(H_2^0)$ are thus mutally orthogonal.

The eigenvalue corresponding to ψ_β is

$$\left.\begin{aligned}E(\mu, \beta(s\lambda)) &= mc^2(1-\mu^2)^{\frac{1}{2}}+\sum_{s\lambda}\left(\beta(s\lambda)+\frac{1}{2}\right)h\omega_s\\ &-\frac{hc}{2}\sum_{s\lambda}\sigma_{s\lambda}^2(k_s-(\mu, k_s)).\end{aligned}\right\} \quad (7)$$

The last series on the right hand side diverges and it has a similar form to the series (6). This similarity will be made clearer in the next section.

3. Eigenvalues of the total Hamiltonian

It is not easy to obtain the eigenvalues of the total Hamiltonian exactly. But the following case in which a nucleon fixed in the space is interacting with a real scalar meson field.[4] Taking this extremely specialized case as an example, we shall examine the relation between $\mathfrak{H}(H_1^0)$ and $\mathfrak{H}(H)$, and the zero point energy of the total Hamiltonian.

The total Hamiltonian H is written as[4]

$$\left.\begin{aligned}H &= \sum H_k,\\ H_k &= \frac{1}{2}(P_k^2+\omega_k^2 Q_k^2)+\frac{gc}{\sqrt{2V}}\left(Q_k+\frac{i}{\omega_k}P_k\right)e^{ikX},\end{aligned}\right\} \quad (8)$$

where g is the coupling constant and X is the position of the nucleon. P_k, Q_k obey the commutation laws

$$[P_k, Q_{k'}] = -ih\delta_{kk'}, \quad [P_k, P_{k'}] = [Q_k, Q_{k'}] = 0,$$

In the representation in which Q_k is diagonal, P_k is written as

$$P_k = -ih\frac{d}{dQ_k}.$$

The eigenvalue problem

$$H_k u = E_k u$$

is therefore reduced to a differential equation and can easily be solved, and the n-th eigen-value and vector are

$$\left.\begin{aligned} E_{kn} &= \left(n+\frac{1}{2}-g^2c^2/4Vh\omega_k^3\right)h\omega_k \\ u_{kn} &= N_n \exp(-iF_kQ_k)\, H_n(\gamma_kQ_k+G_k)\exp\left\{-\frac{1}{2}(\gamma_kQ_k+G_k)^2\right\}, \end{aligned}\right\} \quad (9)$$

where

$$\gamma_k = \sqrt{\frac{\omega_k}{h}}, \quad N_n = \left(\frac{\gamma_k}{\Pi^{\frac{1}{2}}2^n n!}\right)^{\frac{1}{2}}, \quad F_k = \frac{gc}{\sqrt{2V}\,h\omega_k}\sin kX, \quad G_k = \frac{gc}{\sqrt{2V}\,h\omega_k}\cos kX.$$

u_{kn}'s satisfy the orthogonality relation:

$$\int_{-\infty}^{\infty}\tilde{u}_{km}u_{kn}dQ_k = \delta_{mn}.$$

When the coupling constant g is equal to zero, we obtain

$$\left.\begin{aligned} E_{kn}^0 &= \left(n+\frac{1}{2}\right)h\omega_k, \\ u_{kn}^0 &= N_nH_n(\gamma_kQ_k)\exp\left(-\frac{1}{2}\gamma_k^2Q_k^2\right). \end{aligned}\right\} \quad (10)$$

The spaces $\mathfrak{H}(H^0)$ and $\mathfrak{H}(H)$ corresponding to $H^0=\sum\frac{1}{2}(P_k^2+\omega_k^2Q_k^2)$ and H respectively, are determined by the complete orthonormal sets $\{\phi_\beta^0\}$ and $\{\phi_\beta\}$ respectively, where

$$\phi_\beta^0 = \prod_k \otimes_{\beta\in F} u_{k,\,\beta(k)}^0, \quad \phi_\beta = \prod_k \otimes_{\beta\in F} u_{k,\,\beta(k)}. \quad (11)$$

The inner product of ϕ_β^0 and $\phi_{\beta'}$ is

$$\left.\begin{aligned} (\phi_\beta^0 . \phi_{\beta'}) &= \prod_k (u_{k,\,\beta(k)}^0, u_{k,\,\beta'(k)}) \\ &= \text{const.} \prod_k (u_{k,\,0}^0, u_{k,\,0}) \\ &= \text{const.} \exp\left(-\frac{g^2c^2}{8Vh}\sum_k\frac{1}{w_k^3}\right) \end{aligned}\right\} \quad (12)$$

The series contained in (12) diverges logarithmically with ω_k. So that we obtain $(\phi_\beta^0, \phi_{\beta'})=0$ for arbitrary $\beta, \beta'\in F$. The orthogonality of spaces $\mathfrak{H}(H_0)$ and $\mathfrak{H}(H)$ has thus been proved. The zero point energy of H is less than that of H^0 by

$$\frac{g^2c^2}{4V}\sum\frac{1}{\omega_k^2} \quad (13)$$

Comparing (12) with (13), we can say that the divergence of the series contained in (12) securing the orthogonality of the two spaces is slower than that of the zero point energy difference (13).

Each factor $u_{k,\,\beta(k)}$ of ϕ_β is expanded in Fouries series of $u_{k,\,\beta'(k)}^0$'s, $\beta'(k)=0, 1, 2, \ldots$. In other words, each factor of ϕ_β is a superposition of states whose meson numbers are $0, 1, 2, \ldots$.

4. Zero point energy of the system electron plus photon field

It is difficult to transform the total Hamiltonian H into a diagonal form. However, we are able to find such states Ψ_β's that they are orthogonal to one another, their number is enumerably infinite and the expectation values $(H\Psi_\beta, \Psi_\beta)$'s are minus infinity, where the zero point energy $\sum \frac{1}{2} h\omega_\lambda$ of the free photon field is not taken into account.

The total Hamiltonian is written as

$$H = H(p)+H(r)+H(p, r), \tag{14}$$

where

$$H(p) = c(\alpha, p)+\beta mc^2, \tag{15}$$

$$H(r) = \sum_\lambda H_\lambda, \quad H_\lambda = \frac{1}{2}(P_\lambda^2+\omega_\lambda^2 Q_\lambda^2), \tag{16}$$

$$H(p, \lambda) = \sum_\lambda H(p, \lambda), \tag{17}$$

$$H(p, \lambda) = e\sqrt{\frac{h}{2\omega_\lambda}}\left\{a_\lambda(\alpha, A_\lambda)+a_\lambda^*(\alpha, \tilde{A}_\lambda)\right\}, \tag{18}$$

$$A_\lambda = \sqrt{\frac{4\pi c^2}{V}}\, e_\lambda \exp(ikr), \tag{19}$$

$$\left.\begin{aligned} a_\lambda &= \sqrt{\frac{\omega_\lambda}{2h}}\, Q_\lambda + \frac{i}{\sqrt{2k\omega_\lambda}} P_\lambda \\ a_\lambda^* &= \sqrt{\frac{\omega_\lambda}{2h}}\, Q_\lambda - \frac{i}{\sqrt{2h\omega_\lambda}} P_\lambda . \end{aligned}\right\} \tag{20}$$

The field variables P_λ, Q_λ obey the commutation laws

$$[P_\lambda, Q_\mu] = -ih\delta_{\lambda\mu}, \quad [P_\lambda, P_\mu] = [Q_\lambda, Q_\mu] = 0. \tag{21}$$

By using (19) and (20), $H(p, \lambda)$ is rewritten as

$$H(p, \lambda) = \frac{e}{2}\left\{g_\lambda Q_\lambda + \frac{i}{\omega_\lambda} f_\lambda P_\lambda\right\}(\alpha, e_\lambda) \tag{22}$$

where

$$g_\lambda = 2\sqrt{\frac{4\pi c^2}{V}}\cos(k_\lambda r), \quad f_\lambda = 2i\sqrt{\frac{4\pi c^2}{V}}\sin(k_\lambda r).$$

Let the polarization vector e_λ be decomposed into x-, y-, and z-components as

$$e_\lambda = (l_\lambda, m_\lambda, n_\lambda),$$

then

$$(\alpha_\lambda, e_\lambda) = l_\lambda \alpha_x + m_\lambda \alpha_y + n_\lambda \alpha_z,$$

and the operator $\bar{H}_\lambda \equiv H_\lambda + H(p, \lambda)$ has the following explicit form:

$$\bar{H}_\lambda = H_{\lambda x} + H_{\lambda yz},$$

where

$$H_{\lambda x} = \frac{1}{2}(P_\lambda^2+\omega_\lambda^2 Q_\lambda^2+2W_\lambda l_\lambda \alpha_x),$$

$$H_{\lambda yz} = W_\lambda(m_\lambda \alpha_y + n_\lambda \alpha_z),$$

$$W_\lambda = \frac{e}{2}\left(g_\lambda Q_\lambda + \frac{i}{\omega_\lambda} f_\lambda P_\lambda\right).$$

In the first place, by using a representation in which α_x is diagonal, we solve the eigenvalue problem

$$H_{\lambda x}\psi = E\psi . \tag{23}$$

The diagonal elements of α_x are $+1, +1, -1, -1$. Let the corresponding components of the eigenvector ψ be $\psi_1, \psi_2, \psi_3, \psi_4$, then $\psi_1=\psi_2$ and $\psi_3=\psi_4$, and ψ_1 and ψ_3 satisfy the equation

$$\frac{1}{2}\left\{P_\lambda^2+\omega_\lambda Q_\lambda^2+2\varepsilon l_\lambda W_\lambda\right\}\psi_\varepsilon = E\psi_\varepsilon , \tag{24}$$

where $\varepsilon=\pm1$, and $\varepsilon=+1$ and $\varepsilon=-1$ correspond to ψ_1 and ψ_3 respectively.

By using the relation $P_\lambda=-ih\,\partial/\partial Q_\lambda$, we can easily solve the equation (24). Let the $\beta(\lambda)$-th eigenvalue and function of Eq. (24) be $E_{\lambda,\,\beta(\lambda)}$ and $\psi_{\varepsilon,\,\beta(\lambda)}$ respectively, then we obtain

$$E_{\lambda,\,\beta(\lambda)} = \left(\beta(\lambda)+\frac{1}{2}-\frac{2\pi c^2}{V}\,\frac{e^2 l_\lambda}{h\omega_\lambda^3}\right)h\omega_\lambda ,$$

$$\psi_{\varepsilon,\,\beta(\lambda)} = N_\beta\sqrt{\gamma_\lambda}\exp\left(\frac{\varepsilon}{2}F_\lambda\gamma_\lambda Q_\lambda\right)H_{\beta(\lambda)}\left(\gamma_\lambda Q_\lambda+\frac{\varepsilon}{2}G_\lambda\right)\cdot\exp\left(-\frac{1}{2}\left(\gamma_\lambda Q_\lambda+\frac{\varepsilon}{2}G_\lambda\right)^2\right),$$

$$\beta(\lambda) = 0, 1, 2 \dots ,$$

where

$$\gamma_\lambda^2=\frac{\omega_\lambda}{h},\quad G_\lambda=\frac{el_\lambda g_\lambda}{\sqrt{h\omega_\lambda^3}},\quad F_\lambda=\frac{el_\lambda f_\lambda}{\sqrt{h\omega_\lambda^3}},\quad N_n=(\pi^{\frac{1}{2}}2^n n!)^{-\frac{1}{2}} .$$

$\psi_{\varepsilon,\,\beta(\lambda)}$'s satisfy the orthogonality relation

$$\int_{-\infty}^{\infty}\tilde{\psi}_{\varepsilon,\,\beta(\lambda)}\psi_{\varepsilon,\,\beta'(\lambda)}dQ_\lambda = \delta_{\beta\beta'} . \tag{25}$$

We represent the $\beta(\lambda)$-th eigenvector ψ_β of Eq. (24) as

$$\psi_\beta = \begin{pmatrix}\psi_{+,\,\beta(\lambda)}(Q_\lambda)\\ \psi_{+,\,\beta(\lambda)}(Q_\lambda)\\ \psi_{-,\,\alpha(\lambda)}(Q_\lambda)\\ \psi_{-,\,\beta(\lambda)}(Q_\lambda)\end{pmatrix} .$$

Then the eigenvector Ψ_β of the summed operator $\sum\limits_\lambda H_{\lambda x}$ is written as

$$\Psi_\beta = \begin{pmatrix}\Psi_{+,\,\beta}\\ \Psi_{+,\,\beta}\\ \Psi_{-,\,\beta}\\ \Psi_{-,\,\beta}\end{pmatrix} = \begin{pmatrix}\prod\limits_\lambda\otimes\psi_{+,\,\beta(\lambda)}(Q_\lambda)\\ \prod\limits_\lambda\otimes\psi_{+,\,\beta(\lambda)}(Q_\lambda)\\ \prod\limits_\lambda\otimes\psi_{-,\,\beta(\lambda)}(Q_\lambda)\\ \prod\limits_\lambda\otimes\psi_{-,\,\beta(\lambda)}(Q_\lambda)\end{pmatrix} \tag{27}$$

and the corresponding eigenvalue is

$$E_\beta = \sum E_{\lambda,\,\beta(\lambda)} .$$

According to (25) and (26), we obtain the orthogonality of Ψ_β's:

$$(\Psi_\beta, \Psi_{\beta'}) = 2(\Psi_{+,\,\beta}, \Psi_{+,\,\beta'})+2(\Psi_{-,\,\beta}, \Psi_{-,\,\beta'}) = 4\delta_{\beta\beta'} . \tag{28}$$

The above results are summalized as follows: *The operator $\sum_\lambda H_{\lambda x}$ has the eigenvalue E_β defined by* (27) *and the corresponding eigenvector Ψ_β defined by* (26), *the latter being normalized to 4.*

In the next place, we calculate the expectation values $(\sum_\lambda H_{\lambda yz}\Psi_\beta, \Psi_\beta)$ and $(H(p)\Psi_\beta, \Psi_\beta)$.

When α_x is diagonal, each of α_y, α_z and β has a matrix representation such as

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \tag{29}$$

where A is a 2-2 matrix. So that the operator $H_{\lambda yz}$ has a matrix form such as

$$H_{\lambda yz} = \begin{pmatrix} 0 & S_\lambda \\ S_\lambda{}^* & 0 \end{pmatrix} W_\lambda, \quad S_\lambda = \begin{pmatrix} s_{\lambda 1} & s_{\lambda 2} \\ s_{\lambda 3} & s_{\lambda 4} \end{pmatrix},$$

where the matrix S_λ depends on λ. Then we obtain

$$\left.\begin{aligned}(H_{\lambda yz}\Psi_\beta, \Psi_\beta) = a_\lambda \tilde{\Psi}_{+,\beta} W_\lambda \Psi_{-,\beta} \\ + a_\lambda \tilde{\Psi}_{-,\beta} W_\lambda \Psi_{+,\beta},\end{aligned}\right\} \tag{30}$$

where

$$a_\lambda = s_{\lambda 1} + s_{\lambda 2} + s_{\lambda 3} + s_{\lambda 4}$$

and

$$\tilde{\Psi}_{-,\beta} W_\lambda \Psi_{+,\beta} = (W_\lambda \psi_{+,\beta(\lambda)}, \psi_{-,\beta(\lambda)}) \prod_{\mu \neq \lambda} (\psi_{+,\beta(\mu)}, \psi_{-,\beta(\mu)}).$$

For a time, let us consider a special case in which $\beta(\mu)=0$ for all μ's. In this case, we obtain

$$(\psi_{+,0}, \psi_{-,0}) = \exp\left(-\frac{4\pi c^2}{V} \frac{e^2 l_\mu^2}{h\omega_\mu^3}\right), \tag{31}$$

so that

$$\left.\begin{aligned}\tilde{\Psi}_{-,0} W_\lambda \Psi_{+,0} = (W_\lambda \psi_{+,0}, \psi_{-,0}) \exp\left(\frac{4\pi c^2}{V} \frac{e^2 l_\lambda^2}{h\omega_\lambda^3}\right) \\ \cdot \exp\left(-\frac{4\pi c^2}{V} \frac{e^2}{h} \sum_\mu \frac{l_\mu^2}{\omega_\mu^3}\right).\end{aligned}\right\} \tag{32}$$

Corresponding to one wave vector $\boldsymbol{k}_\mu$, there are two polarization vectors $\boldsymbol{e}_{\mu,1}$ and $\boldsymbol{e}_{\mu,2}$, which we decompose into three components respectively as

$$\boldsymbol{e}_{\mu 1} = (l_{\mu 1}, m_{\mu 1}, n_{\mu 1}),$$
$$\boldsymbol{e}_{\mu 2} = (l_{\mu 2}, m_{\mu 2}, n_{\mu 2}).$$

The three vectors $\boldsymbol{e}_{\mu 1}$, $\boldsymbol{e}_{\mu 2}$, and $\boldsymbol{k}_\mu/k_\mu$ are all of unit length and orthogonal with one another, so that we have

$$(k_{\mu x}/k_\mu)^2 + l_{\mu 1}^2 + l_{\mu 2}^2 = 1,$$

where $k_{\mu x}$ is the x-component of $\boldsymbol{k}_\mu$. Consequently

$$\sum_\mu \frac{l_\mu^2}{\omega_\mu^3} = \frac{1}{c^3} \sum_\mu \frac{1}{k_\mu^3} \left\{1 - \left(\frac{k_{\mu x}}{k_\mu}\right)^2\right\}. \tag{33}$$

Let θ be a fixed positive angle smaller than $\pi/2$, then there is a fixed positive constant a^2 such that the inequality

$$1-\left(\frac{k_{\mu x}}{k_\mu}\right)^2 \geqq a^2 > 0$$

holds valid for an arbitrary vector $\boldsymbol{k}_\mu$ which satisfies the condition

$$\theta \leqq \text{angle between } \boldsymbol{k}_\mu \text{ and the } x\text{-axis} \leqq \pi-\theta\,. \tag{34}$$

Thus, from (33), we obtain

$$\sum_\mu \frac{l_\mu^2}{\omega_\mu^3} \geqq \frac{a^2}{c^3} \sum{}' \frac{1}{k_\mu^3} = +\infty\,, \tag{35}$$

where the prime on the right hand side implies a summation over all $\boldsymbol{k}_\mu$'s whose directions satisfy the condition (34). From (35), (32), and (30), we can conclude that

$$(\sum_\lambda H_{\lambda yz} \Psi_0\,,\ \Psi_0) = 0 \tag{36}$$

In the next place, we shall evaluate the expectation value $(H(p)\Psi_0\,,\ \Psi_0)$.

a) Evaluation of $(\alpha_x p_x \Psi_0\,,\ \Psi_0)$.

$$\begin{aligned}(\alpha_x p_x \Psi_0\,,\ \Psi_0) &= 2\tilde{\Psi}_{+\,,\,0}\, p_x \Psi_{+\,,\,0} - 2\tilde{\Psi}_{-\,,\,0}\, p_x \Psi_{-\,,\,0} \\ &= -\sum_\lambda (p_x F_\lambda) G_\lambda = -4\cdot\frac{4\pi c^2}{V} \sum_\lambda \frac{e^2 l_\lambda^2}{\omega_\lambda^3} k_{\lambda x} \cos^2(\boldsymbol{k}_\lambda \boldsymbol{r}) \\ &= -4\cdot\frac{4\pi c^2}{V}\frac{1}{c^3} \sum_\lambda \frac{e^2}{k_\lambda^3}\left(1-\left(\frac{k_{\lambda x}}{k_\lambda}\right)^2\right) k_{\lambda x} \cos^2(\boldsymbol{k}_\lambda \boldsymbol{r})\,.\end{aligned}$$

When $\boldsymbol{k}_{\lambda'} = -\boldsymbol{k}_\lambda$, we have $k_{\lambda' x} = -k_{\lambda x}$, so that the above sum is equal to zero.

b) Evaluation of $(\alpha_y p_y \Psi_0\,,\ \Psi_0)$, $(\alpha_z p_z \Psi_0\,,\ \Psi_0)$, and $(\beta\Psi_0\,,\ \Psi_0)$.

As the matrix representation of α_y has the form (29), we obtain

$$\left.\begin{aligned}(\alpha_y p_y \Psi_0\,,\ \Psi_0) &= b\tilde{\Psi}_{+\,,\,0}\, p_y \Psi_{-\,,\,0} \\ &\quad + \bar{b}\tilde{\Psi}_{-\,,\,0}\, p_y \Psi_{+\,,\,2}\,,\end{aligned}\right\} \tag{37}$$

where b is a constant depending on the matrix element of α_y. By using the same reasoning as that used in deducing the equation (36), it can be proved that the right hand side of (37) vanishes, i. e.,

$$(\alpha_y p_y \Psi_0\,,\ \Psi_0) = 0\,.$$

In the same way, we obtain

$$(\alpha_z p_z \Psi_0\,,\ \Psi_0) = (\beta\Psi_0\,,\ \Psi_0) = 0\,.$$

We can summalize the results obtained up to this place as follows: *In the state Ψ_0, the total Hamiltonian H has the expectation value*

$$(H\Psi_0\,,\ \Psi_0) = \sum_\lambda E_{\lambda\,,\,0} = \sum_\lambda \left(\frac{1}{2} - \frac{2\pi c^2}{V}\frac{e^2 l_\lambda^2}{h\omega_\lambda^3}\right) h\omega_\lambda\,. \tag{38}$$

When the zero point energy $\sum \frac{1}{2} h\omega_\lambda$ of the free photon field is not taken into account, the remainder of series (38) diverges logarithmically to minus infinity.

When $\beta(\lambda)$ is not identically zero, the equation (31) holds valid for all but a finite number of μ's. So that we obtain

$$(\sum_{\lambda} H_{\lambda yz}\Psi_\beta\,,\ \Psi_\beta) = 0\,.$$

The expectation values of $\alpha_x p_x$ is not necessarily zero, and it can be written as

$$(\alpha_x p_x \Psi_\beta\,,\ \Psi_\beta) = c_\beta\,,$$

where c_β is a finite constant depending on β.

On the other hand, the equations

$$(\alpha_y p_y \Psi_\beta\,,\ \Psi_\beta) = (\alpha_z p_z \Psi_\beta\,,\ \Psi_\beta) = (\beta\Psi_\beta\,,\ \Psi_\beta) = 0$$

hold valid as before.

In conclusion, we obtain

$$(H\Psi_\beta\,,\ \Psi_\beta) = c_\beta + \sum_{\lambda} E_{\lambda\,,\,0}$$

for all β's, where c_β is a certain constant depending on β, and $\sum_{\lambda} E_{\lambda\,,\,0}$ is given by (38).

Appendix

E. E. Salpeter[5] proved that the total Hamiltonian has an eigenvalue of minus infinity, by using a reasoning sketched below. His elegant method can be applied to any other field. However, his reasoning has a slight defect. We shall point out it.

In the first place, we shall sketch Salpeter's reasoning. H is the total Hamiltonian of a system electron plus electromagnetic field, K_0 a real parameter and $H(K_0)$ an operator depending on K_0 such that

$$\lim_{K_0 \to \infty} H(K_0) = H\,.$$

We consider an eigenvalue problem

$$H(K_0)\psi = E(K_0)\psi\,. \qquad (A,1)$$

The eigenvalue $E(K_0)$ and eigenfunction ψ depend on the parameter K_0. By using a method similar to the perturbation method, we obtain the first approximations of $E(K_0)$ and ψ. Let them be

$$E'(K_0)\,,\quad \Psi\,. \qquad (A,2)$$

The method to obtain them is not necessary for our purpose so that we shall omit it. $E'(K_0)$ satisfies the equation

$$\lim_{K_0 \to \infty} E'(K_0) = -\infty\,. \qquad (A,3)$$

Here, Salpeter uses the variation principle:

Variation principle. ψ is an arbitrary trial function. Then $\bar{E} = (H\psi, \psi)/(\psi, \psi)$ is always not smaller than the minimum eigenvalue E_0 of H, i. e.,

$$E_0 \leqq \bar{E}\,.$$

In our case, ψ_1 is used as a trial function. Then we can prove that $\bar{E} = E'(K_0)$. So that, from (A, 3) and (A, 4), we obtain $E_0 = -\infty$. Q. E. D.

The above reasoning, however, is not complete. For, in order to prove the variation principle (A, 4), it is necessary that[6] the operator H has so many eigenfunctions that an arbitrary trial function ψ_1 is able to be expanded in a series of these eigenfunctions. When it is not clear whether H has this property or not, we can not use this principle freely. The total Hamiltonian being considered here seems not be have this expansion property. According to the orthogonality of the two spaces $\mathfrak{H}(H^0)$ and $\mathfrak{H}(H)$ given in §3 of the present paper, it is not probable that the trial function ψ_1 can be expanded in Fourier series of eigenfunctions of H.

When H is Hermitian the variation principle does not hold valid in general. as shown below.

Let $P(\lambda)$ be a projection operator such that

$$P(\lambda)=\begin{cases}0 & \text{for } \lambda<\lambda_0, \\ P(\lambda_0) & \text{for } \lambda_0\leqq\lambda,\end{cases} \qquad 0<P(\lambda_0)<1.$$

Then the operator

$$H=\int_{-\infty}^{\infty}\lambda dP(\lambda)=\lambda_0 P(\lambda_0)$$

is Hermitian. Let $\mathfrak{H}$ be the whole space and be $P(\lambda_0)\mathfrak{H}=\mathfrak{M}_0$. Then, for an arbitrary vector $\varphi\in\mathfrak{M}_0$, we obtain

$$H\varphi=\lambda_0 P(\lambda_0)\varphi=\lambda_0\varphi,$$

and λ_0 is the minimum eigenvalve of H. On the other hand, for an arbitrary ψ such that $\psi\overline{\in}\mathfrak{M}_0$ and $\|\psi\|=1$, we obtain

$$(H\psi,\psi)=\lambda_0(P(\lambda_0)\psi,\psi)=\lambda_0\|P(\lambda_0)\psi\|^2<\lambda_0.$$

That is, the variation principle (A, 4) does not hold valid in this case.

When H is not necessarily symmetric, it is easy to give examples for which the variation principle can not be applied. For example, let

$$H=\frac{1}{2}\begin{pmatrix}3 & 2\\ 0 & 3\end{pmatrix}, \qquad \psi=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix},$$

then H has a double eigenvalue 1.5 and $(H\psi,\psi)/(\psi,\psi)=1$.

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