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Some Relations in Homotopy Groups of Spheres

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Introduction

It is well known that the suspension (*Einhängung*) homomorphism $E : \pi_n(S^r) \to \pi_{n+1}(S^{r+1})$ is isomorshism if $n<2r-1$ [3] [1].* In recent years, G. W. Whitehead has shown that the kernel of the suspension homomorphism E is the subgroup generated by whitehead product, if $n=2r-1$ [9, §7].

In this paper we shall calculate some special whitehead products, and indicate some non-trivial suspension homomorphisms. For example, in cases where $n=r+4\,(r=2, 4, 5)$ and $n=r+5\,(r=2, 4, 5, 6)$ E is not isomorphic, and also we obtain non-zero elements of $\pi_{4n+10}(S^{2n+4})$ and $\pi_{4n+22}(S^{2n+8})$ $(n=0, 1, \dots .)$, whose suspension vanish.

1. Notations

We shall use the notations analogous to those of G. W. Withehead [9, §1]. Define

$$S^n=\{(x_1, \dots, x_{n+1}) \mid \sum x_i^2=1\},$$
$$E_+^n=\{x \in S^n \mid x_{n+1} \geqq 0\}, \qquad E_-^n=\{x \in S^n \mid x_{n+1} \leqq 0\},$$
$$I^n=\{(x_1, \dots, x_n) \mid -1 \leqq x_i \leqq 1\},$$
$$\dot{I}^n=\{(x_1, \dots, x_n) \mid \Pi\,(1-x_i^2)=1\},$$
$$J_+^n=\{x \in \dot{I}^{n+1} \mid x_{n+1} \geqq 0\}, \qquad J_-^n=\{x \in \dot{I}^{n+1} \mid x_{n+1} \leqq 0\},$$
$$y_*=(1, 0, \dots, 0), \qquad 0=(0, \dots, 0),$$
$$S^n \vee S^n=S^n \times y_* \cup y_* \times S^n \subset S^n \times S^n,$$

as sub-spaces in the euclidean spaces of suitable dimensions.

Define the mapping $d_n : S^n \times I^1 \to S^{n+1}$ as in [9, §1], which is characterized by the following properties:

$$d_n \text{ maps } (S^n-y_*)\times[0, 1) \text{ topologically on } E_+^n-y_*,$$
$$d_n \text{ maps } (S^n-y_*)\times(-1, 0] \text{ topologically on } E_-^n-y_*,$$
$$d_n(S^n \times \dot{I}^1 \cup y_* \times I^1)=y_*, \quad \text{and} \quad d_n(x, 0)=(x, 0).$$

We also define the mapping $\varphi_n : S^n \to S^n \vee S^n$ as in [9, §1], which maps subspaces $S_0^{n-1}=\{x \in S^n \mid x_2=0\}$ to the point $y_* \times y_*$ and elsewhere topologically preserving orientation.

We denote the point $(tx_1, \dots, tx_n)$ by tx, where $x=(x_1, \dots, x_n)$ and t is a real number.

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* Numbers in blackets refer to the references cited at the end of the paper.

Let $\rho_n : S^{n-1}\rightarrow \dot{I}^n$ be the *central projection* such that $\rho_n(x)=x/r$, where $r=\mathrm{Max}(x_1, \dots, x_n)$. Clearly we have $\rho_n(S^{n-2})=\dot{I}^{n-1}$, $\rho_n(E_+^{n-1})=J_+^{n-1}$, $\rho_n(E_-^{n-1})=J_-^{n-1}$ and $\rho_n(y_*)=y_*$.

Define $\Phi_{p,q}: \dot{I}^p\times\dot{I}^q\times I^1\rightarrow\dot{I}^{p+q}=(I^p\times I^q)^{\cdot}$ by

$$\Phi_{p,q}(x, y, t,)=\begin{cases} ((1-t)x, y) & 0\leqq t\leqq 1, \\ (x, (1+t)y) & -1\leqq t\leqq 0, \end{cases}$$

Then $\Phi_{p,q}$ is continuous and topological for $t\in\mathrm{Int}. I^1$, and satisfies the conditions $\Phi_{p,q}(\dot{I}^p\times\dot{I}^q\times[0,1])\subset I^p\times\dot{I}^q$, $\Phi_{p,q}(\dot{I}^p\times\dot{I}^q\times[-1,0])\subset\dot{I}^p\times I^q$, $\Phi_{p,q}(x, y, -1)=(x, y)$,

(1.1) $$\Phi_{p,q}(x, y, 1)=(0, y), \qquad \Phi_{p,q}(x, y, -1)=(x, 0).$$

With our notations we can construct some mappings:

i) *Suspension of* $f : S^n\rightarrow S^r$ is given by $Ef(d_n(x, t))=d_r(f(x), t)$, $x\in S^n$.

ii) *Join* of maps $f: I^p\rightarrow I^r$ and $g: I^q\rightarrow I^s$ is given by

$$(f*g)(\Phi_{p,q}(x, y, t))=\Phi_{r,s}(f(x), g(y), t), \qquad x\in I^p,\ y\in I^q.$$

iii) *Hopf construction of* $f:\dot{I}^p\times\dot{I}^q\rightarrow S^r$ is given by

(1.2) $$Gf(\Phi_{p,q}(x, y, t))=d_r(f(x, y), t), \qquad x\in\dot{I}^p,\ y\in\dot{I}^q.$$

iv) *Whitehead product* of $f:(I^p, \dot{I}^p)\rightarrow(X, x_*)$ and $g:(I^q, \dot{I}^q)\rightarrow(X, x_*)$ is given by

$$[f, g]\ (\Phi_{p,q}(x, y, t))=\begin{cases} f((1-t)x) & 0\leqq t\leqq 1, \quad x\in\dot{I}^p, \\ g((1+t)y) & -1\leqq t\leqq 0, \quad y\in\dot{I}^q, \end{cases}$$

It is easily verified that the above constructions are single valued and hence continuous, and that they coincide with those of [9, §3].

It was shown in [9, §3] that

(1.3) $$(f*g)\circ(f'*g')=(f\circ f')*(g\circ g'),$$

(1.4) $$[f, g]\circ(f'*g')=[f\circ E(f'), g\circ E(g')].$$

We shall use the following theorems due to G. W. Whitehead [8].

(1.5) $$E[\alpha, \beta]=0.$$

(1.6) If $f:\dot{I}^{p+q}\rightarrow X$ satisfy the condition $f(\dot{I}^p\times\dot{I}^q)=x_*$, then f is homotopic to the map $f_1+f_2+[g_1, g_2]$, where

$$f_1(\Phi_{p,q}(x, y, t))=f(\Phi_{p,q}(x, y, (t+1)/2))$$
$$f_2(\Phi_{p,q}(x, y, t))=f(\Phi_{p,q}(x, y, (t-1)/2))$$

and $g_1:(I^p, \dot{I}^p)\rightarrow(X, x_*)$, $g_2:(I^q, \dot{I}^q)\rightarrow(X, x_*)$ are given by

$$g_1((1-t)x)=f(\Phi_{p,q}(x, y_0, t)) \qquad 0\leqq t\leqq 1,\ x\in\dot{I}^p,$$
$$g_2((1+t)y)=f(\Phi_{p,q}(x_0, y, t)) \qquad -1\leqq t\leqq 0,\ y\in\dot{I}^q,$$

for fixed points $y_0\in\dot{I}^q$ and $x_0\in\dot{I}^p$.

2. Hopf and Frendenthal invariants

For all values of $n, r>1$, we can construct a Hopf homomorphism H_1: $\pi_n(S^r)\rightarrow\pi_{n+1}(S^{2r})$. According to [9, §4] we have direct sum decomposition $\pi_n(S^r\vee S^r)\approx\pi_n(S^r)\oplus\pi_n(S^r)\oplus\pi_{n+1}(S^r\times S^r, S^r\vee S^r)$. Let

$$(2.1)\qquad Q:\pi_n(S^r\vee S^r)\underset{\partial}{\overset{Q}{\rightleftarrows}}\pi_{n+1}(S^r\times S^r, S^r\vee S^r)$$

be the projection, then its right inverse is boundary operator ∂ in the sense that $Q\circ\partial=$*identity.*

Let $\Psi_r:(E^{2r}, \dot{E}^{2r})\rightarrow(S^r\times S^r, S^r\vee S^r)$ be the map given in [9, §1], such that Ψ_r maps Int. E^{2r} topologically onto $S^r\times S^r-S^r\vee S^r$. Since $S^r\times S^r-S^r\vee S^r$ is an open cell, we can construct a map $\theta_r:(S^r\times S^r, S^r\vee S^r)\rightarrow(S^{2r}, y_*)$ such that θ_r maps $S^r\times S^r-S^r\vee S^r$ topologically onto $S^{2r}-y_*$, and the composite mapping $\theta_r\circ\Psi_r:(E^{2r}, \dot{E}^{2r})\rightarrow(S^{2r}, y_*)$ presserves orientation.

Then the composite homomorphism

$$(2.2)\qquad \theta_r\circ\Psi_r\circ\partial^{-1}:\pi_n(\dot{E}^{2r})\rightarrow\pi_{n+1}(E^{2r}, \dot{E}^{2r})\rightarrow\pi_{n+1}(S^r\times S^r, S^r\vee S^r)\rightarrow\pi_{n+1}(S^{2r}, y_*).$$

represents the suspension hommorphism.

Now we define the *Hopf homomorphism* $H_1:\pi_n(S^r)\rightarrow\pi_{n+1}(S^{2r})$ by the composite homomorphism

$$(2.3)\qquad H_1=\theta_r\circ Q\circ\varphi_r:\pi_n(S^r)\rightarrow\pi_n(S^r\vee S^r)\rightarrow\pi_{n+1}(S^r\times S^r, S^r\vee S^r)\rightarrow\pi_{n+1}(S^{2r}).$$

Let $H=\Psi_r^{-1}\circ Q\circ\varphi_r$ be the Hopf homomorphism in the sense of [9], [10], then we have $H_1=E\circ H$ by (2.2). Since E is isomorphic for $n\leqq 4r-4$, *H_1 is equivalent to H.*

Also we can define *Freudenthal invariants* for all values of $n, r>1$. Consider the element ξ of triad homotopy group $\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r-1})$ as a null-homotopy of suspension $\Delta(\xi)=\partial\beta_+(\xi)\in\pi_n(S^r)$, where β_+ and ∂ are boundary operators of triad and relative homotopy groups [1]. According to §6 of [9] we define two homomorphisms

$$\Lambda_0', \Lambda_0'':\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r+1})\rightarrow\pi_{n+3}(S^{r+1}\times S^{r+1}, S^{r+1}\vee S^{r+1}),$$

and define Freudenthal invariants of ξ by

$$(2.4)\qquad \Lambda_1'(\xi)=\theta_{r+1}\circ\Lambda_0'(\xi)\quad\text{and}\quad\Lambda_1''(\xi)=\theta_{r+1}\circ\Lambda_0''(\xi)$$

Then $\Lambda_1', \Lambda_1'':\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r+1})\rightarrow\pi_{n+3}(S^{2r+2})$ are Freudenthal homomorphisms of our sense.

We shall use the following theorems similar to those of §§5, 7 of [9] without restriction of dimension.

(2.5) *Let $f:\dot{I}^p\times\dot{I}^q\rightarrow S^{n-1}$ be a map of type (α, β), then the Hopf invariant of $G(f)$ is given by*

$$H_1(\{G(f)\})=(-1)^r E(\alpha*\beta),$$

(2.6) $$H_1(E(\alpha))=0.$$

(2.7) *Let* $\alpha\in\pi_n(S^r)$, $\beta\in\pi_r(S^s)$, *and let* $\alpha=E(\alpha')$ *for some* $\alpha'\in\pi_{n-1}(S^{r-1})$ (*more generally if* $Q(\alpha)=0$), *then we have*

$$H_1(\beta\circ\alpha)=H_1(\beta)\circ E(\alpha).$$

(2.8) *If* $\alpha\in\pi_n(S^r)$, $\beta\in\pi_m(S^r)$, *then*

$$H_1[E(\alpha), E(\beta)]=\begin{cases}0, & \textit{if } r \textit{ is even},\\ 2E(\alpha*\beta), & \textit{if } r \textit{ is odd}.\end{cases}$$

(2.9) *If* $\alpha\in\pi_n(S^r)$, *and* $i_{2r}\in\pi_{2r}(S^{2r})$ *represents the identity map, then*

$$H_1(\alpha)=(-1)^r i_{2r}\circ H_1(\alpha).$$

(2.10) *If* $\xi\in\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r+1})$, *then*

$$\Lambda_1'(\xi)-\Lambda_1''(\xi)=(-1)^r EEH_1(\Delta(\xi)).$$

(2.11) *If* $\xi\in\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r+1})$, *then*

$$\Lambda_1'(\xi)=(-1)^{r+1}i_{2r+2}\circ\Lambda_1''(\xi).$$

(2.5) follows from the similar argument as the proof of Theorem 5.1 of [9] and (2.2). (2.6) is a direct consequence of the proof of Theorem 5.11 of [9].

To prove (2.7), we calculate $\varphi_s(\beta\circ\alpha)$ according to the proof of Theorem 5.19 of [9], and get the following equation

$$(\partial' Q'\varphi_s(\beta))\circ\alpha=\partial Q\varphi_s(\beta\circ\alpha),$$

where ∂', Q', ∂, Q are the corresponding operations in (2.1). Let $\partial_0:\pi_{n+1}(E^{r+1}, \dot{E}^{r+1})\to\pi_n(S^r)$ be the boundary homomorphism, then

$$(\partial' Q'\varphi_s(\beta))\circ\alpha=\partial(Q'\varphi_s(\beta)\circ\partial_0^{-1}(\alpha)).$$

Since ∂ is an isomorphism, we have

$$Q\varphi_s(\beta\circ\alpha)=Q'\varphi_s(\beta)\circ\partial_0^{-1}(\alpha),$$

so that by (2.3) we have

$$H_1(\beta\circ\alpha)=\theta_s Q\varphi_s(\beta\circ\alpha)=\theta_s(Q'\varphi_s(\beta)\circ\partial_0^{-1}(\alpha)).$$

Another direct calculation shows

$$\theta_s(Q'\varphi_s(\beta)\circ\partial_0^{-1}(\alpha))=\theta_s Q'\varphi_s(\beta)\circ E(\alpha)=H_1(\beta)\circ E(\alpha).$$

To prove (2.8), we use the fact $\alpha*\beta=E(\gamma)$ for some γ, and $H_1[i_r, i_r]=2i_{2r}$. Then (2.8) follows from (1.4) and (2.7).

Let $\sigma_r:(S^r\times S^r, S^r\vee S^r)\to(S^r\times S^r, S^r\vee S^r)$ be the map given by $\sigma_r(x,y)=(y,x)$, then by (4.22) of [9] we have

$$Q(\sigma_r(\alpha))=\sigma_r Q(\alpha),$$

where $\alpha\in\pi_n(S^r\vee S^r)$. And further calculations show

$$(-1)^r i_{2r}\circ\theta_r Q(\alpha)=\theta_r\sigma_r Q(\alpha)=\theta_r Q(\sigma_r(\alpha)).$$

Then (2.9) and (2.11) are verified by the similar arguments of Theorem 5.49 and 7.28 of [9] respectively.

To prove the formula of (2.10) we consider the relation between the opera tion A in [9, §7] and θ. We can show the equation

$$\theta_{r+1}\circ A=(-1)^r EE\circ\theta_r,$$

so that (2.10) is a direct consequence of Theorem 7.8 of [9].

3 Lemmas

If $f:\dot{I}^p\times\dot{I}^q\to S^{r-1}$ is given, we construct a *right suspension* $E'f:\dot{I}^p\times\dot{I}^{q+1}\to S^r$ by the rule

$$E'f(x, y_1, \dots\dots, y_{q+1}))=\begin{cases}(d_{r-1}(f(x,(y_1,\dots\dots,y_q)), y_{q+1}) & if\ y\in\dot{I}^q\times I^1,\\ y_* & if\ y\in I^q\times\dot{I}^1.\end{cases}$$

If f is homotopic to g, then $E'f$ is homotopic to $E'g$. Also we have

(3.1) $\qquad E'f(\dot{I}^p\times J^q_+)\subset E^r_+,\ E'f(\dot{I}^p\times J^q_-)\supset E^r_-,\ E'f\,|\,\dot{I}^p\times\dot{I}^q=f,$

and any map satisfying the condition (3.1) is homotopic to $E'f$.

LEMMA (3.2) $\qquad -G(E'f)\simeq E(G(f))$.

LEMMA (3.3) *If* $\qquad f(x,y)\equiv F(x)$, *then* $G(f)\simeq 0$.

Proof of (3.2). Let

$$K^{p+q-1}=I^p\times\dot{I}^q\times(1)\cup\dot{I}^p\times\dot{I}^q\times I^1\cup\dot{I}^p\times I^q\times(-1),$$
$$H^{p+q}_+=I^p\times I^q\times(1)\cup\dot{I}^p\times I^q\times I^1,$$
$$H^{p+q}_-=I^p\times\dot{I}^q\times I^1\cup I^p\times I^q\times(-1),$$

be the subspaces of $\dot{I}^{p+q+1}=(I^p\times I^q\times I^1)^{\cdot}$, then H^{p+q}_+, H^{p+q}_- are closed $(p+q)$-cells and we have $H^{p+q}_+\cup H^{p+q}_-=\dot{I}^{p+q+1}$, $H^{p+q}_+\cap H^{p+q}_-=K^{p+q-1}$.

Let us give the homeomorphism $\eta:\dot{I}^{p+q}\to K^{p+q-1}$ by

(3.4) $$\eta(\Phi_{p,q}(x,y,t))=\begin{cases}((-2t+2)x, y, 1) & 1/2\leq t\leq 1,\\ (x, y, 2t) & -1/2\leq t\leq 1/2,\\ (x,(2t+2)y,-1) & -1\leq t\leq -1/2,\end{cases}$$

Then we can extend η throughout $\dot{I}^{p+q+1}$ homeomorphically such that

(3.5) $\qquad \eta(J^{p+q}_+)\subset H^{p+q}_+,\quad \eta(J^{p+q}_-)\subset H^{p+q}_-,$

because K^{p+q-1} bounds the cells H^{p+q}_+ and H^{p+q}_-. As is easily seen, the map $\eta:\dot{I}^{p+q+1}\to\dot{I}^{p+q+1}$ preserves the orientation.

Let $G(E'f): \dot{I}^{p+q+1} \to S^{r+1}$ be given. Define the map $g: \dot{I}^{p+q+1} \to S^{r+1}$ as follows,

$$g | \dot{I}^p \times I^{q+1} = G(E'f) | \dot{I}^p \times I^{q+1},$$
$$g(I^p \times I^q \times (1)) = y^*.$$

Then g is defined on H_+^{p+q} such that $g(H_+^{p+q}) \subset E_-^{r+1}$ and $g(K^{p+q-1}) \subset S^r$. Since H_-^{p+q} is a cell bound by the sphere K^{p+q-1}, we can extend g over H_-^{p+q} such that $g(H_-^{p+q}) \subset E_+^{r+1}$.

Then g is homotopic to $G(E'f)$, because g and $G(E'f)$ coincide on $\dot{I}^p \times \dot{I}^{q+1}$, and map $I^p \times \dot{I}^{q+1}$ and $\dot{I}^p \times I^{q+1}$ to E_+^{r+1} and E_-^{r+1} respectively.

In another point of view, concider the map $g \circ \eta: \dot{I}^{p+q+1} \to S^{r+1}$, then we have by (3.5) $g \circ \eta(J_+^{p+q}) \subset E_-^{r+1}$ and $g \circ \eta(J_+^{p+q}) \subset E_+^{r+1}$.
Therefore $-(g \circ \eta)$ is homotopic to the suspension of $h = g \circ \eta | \dot{I}^{p+q}$. Since η is homotopic to the identity map, g is homotopic to $-E(h)$. h is also given by

$$h(\Phi_{p,q}(x, y, t)) = \begin{cases} y_* & 1/2 \leqq t \leqq 1, \\ d_{r-1}(f(x, y), 2t) & -1/2 \leqq t \leqq 1/2, \\ y_* & -1 \leqq t \leqq -1/2, \end{cases}$$

Then h is homotopic to the Hopf construction of f, and

$$E(G(f)) \simeq E(h) \simeq -g \simeq -G(E'f) \qquad q.e.d.$$

Proof of (3.3). Give the homotopy $f_\tau: \dot{I}^{p+q} \to S^r$ by

$$f_\tau(\Phi_{p,q}(x, y, t)) = d_{r-1}(F(x), t + \tau - t\tau), \quad 0 \leqq \tau \leqq 1.$$

Since $\Phi_{p,q}(x, y, -1) = (x, 0)$, f_τ is single valued, continuous and gives the nullhomotopy of $f_0 = G(f)$. *q.e.d.*

4 Theorem

In this paragraph, we assume that $n=4$ or 8, and regard the points of S^{n-1} as *quarternions* ($n=4$) or *Cayley numbers* ($n=8$). Also we may regard the points of $\dot{I}^n$ as that of S^{n-1}, relating by the central projection $\rho_n: S^{n-1} \to \dot{I}^n$.

Then the multiplication $\dot{I}^n \times \dot{I}^n \to \dot{I}^n$ (or S^{n-1}) can be defined, and denoted by $x \cdot y$. Let $h_n = G(f)$ be the Hopf construction of $f(x, y) = x \cdot y$, then h_n is so-called Hopf fibra map, and in our cases we have the direct sum decomposition [2] [6],

$$\pi_{2n-1}(S^n) \approx \pi_{2n-1}(S^{2n-1}) \oplus \pi_{2n-2}(S^{n-1}).$$

Let $i_n \in \pi_n(S^n)$ be the element represented by the idetity map, then whitehead product $[i_n, i_n]$ belongs to $\pi_{2n-1}(S^n)$ and has the direct sum decomposition as above. The following theorem is the main result of this paper.

THEOREM (4.1) $\quad [i_n, i_n] = 2\{h_n\} - E(\alpha_{n-1}),$

where $\alpha_{n-1}\in\pi_{2n-2}(S^{n-1})$ *has nonzero Hopf invariant.*

More precisely $\pm\alpha_{n-1}$ are the elements given in [1, §5] ($n=4$) and in [9, §8] ($n=8$).

It was proved in [8] that $[i_n, i_n]$ generated the kernel of suspension, E: $\pi_{2n-1}(S^n)\rightarrow\pi_{2n}(S^{n+1})$ (see also (3.49) of [9]). Hence we have

COROLLARY (4 2) $\quad 2E\{h_n\}=EE(\alpha_{n-1})\neq 0$.

COROLLARY (4.3) *For some* $k\geqq 2$, *k-fold suspinsion* $E^k:\pi_6(S^3)\rightarrow\pi_{6+k}(S^{s+k})$ *is an isomorphism into, but the image of* E^k *is not a direct summand.*

COROLLARY (4.4) $\quad (ki_n)\circ\{h_n\}=k^2\{h_n\}-k(k-1)/2\cdot E(\alpha_{n-1})$

$$(k=0,\ \pm 1,\ \pm 2,\ \dots\dots)$$

Proof of Theorem. Consider the map $\chi:\dot{I}^{2n}\rightarrow\dot{I}^{2n}$ by the rule

$$\chi(\Phi_{n,n}(x, y, t))=\begin{cases}\Phi_{n,n}(x\cdot y,\ y^{-1},\ 2t-1) & 0\leqq t\leqq 1,\\ \Phi_{n,n}(x\cdot y,\ x^{-1},\ -2t-1) & -1\leqq t\leqq 0.\end{cases}$$

It is seen from (1.1) that we have $\Phi_{n,n}(x\cdot y,\ y^{-1},\ -1)=\Phi_{n,n}(x\cdot y,\ x^{-1},\ -1)$ and that $\Phi_{n,n}(x, y, 1)$ and $\Phi_{n,n}(x\cdot y, y^{-1}, 1)$ depend only y, and $\Phi_{n,n}(x, y, -1)$ and $\Phi_{n,n}(x\cdot y, x^{-1}, 1)$ depend only x. Therefore χ is single valued, hence continuous.

The composite map $h_n\circ\chi:\dot{I}^{2n}\rightarrow S^n$ is given by

$$h_n\circ\chi(\Phi_{n,n}(x, y, t))=\begin{cases}d_{n-1}((x\cdot y)\cdot y^{-1},\ 2t-1) & 0\leqq t\leqq 1,\\ d_{n-1}((x\cdot y))x^{-1},\ -2t-1) & -1\leqq t\leqq 0,\end{cases}$$

Then $h_n\circ\chi$ satisfies the condition of (1.6), and therefore it is homotopic to the sum $F_1+F_2+[g_1, g_2]$, were

$$F_1(\Phi_{n,n}(x, y, t))=d_{n-1}((x\cdot y)\cdot y^{-1},\ t),$$
$$F_2(\Phi_{n,n}(x, y, t))=d_{n-1}((x\cdot y)\cdot x^{-1},\ -t).$$

To determine g_1, g_2, we choose $x_0=y_0=y_*$ in (1.6), then g_1, g_2 represent the elements i_n, $-i_n$ respectively. Therefore we have

$$h_n\circ\chi\simeq G(f_1)+(-i_n)\circ G(f_2)-[i_n, i_n].$$

where $f_1(x, y)=(x\cdot y)\cdot y^{-1}$, $f_2(x, y)=(x\cdot y)\cdot x^{-1}$.

The following properties of quarternion and Cayley number were established,

(4.5) $\qquad (x\cdot y)\cdot y^{-1}=x$,

(4.6) $\qquad$ If $y=(y_1,\ \dots\dots,\ y_n)$ and $(x\cdot y)\cdot x^{-1}=(y_1',\ \dots\dots,\ y_n')$, then $y_1=y_1'$.

According to Lemma (3.3) and (4.5), we have $F_1=G(f_1)\simeq 0$.

To apply the Lemma (3.2) to $G(f_2)$, we must take some permutations of the coordinates of I^{2n}, but such permutations only chang the sign of $G(f_2)$.

Therefore (4.6), (3.1) and Lemma (3.3) show $G(f_2)\simeq EG(f_0)$, where f_0 is given by $f_0(x, y)=(x\cdot y)\cdot x^{-1}$ for $y\in\dot{I}^{n-1}$ and $x\in\dot{I}^n$ (in the multiplication we regard $y=(y_1,\ \dots,\ y_{n-1})$ as $(0,\ y_1,\ \dots\dots,\ y_{n-1})$ in $\dot{I}^n$).

Now $G(f_0)$ was given in [1, §5] ($n=4$), and [9, §8] ($n=8$), and it is shown that the Hopf invariants of $G(f_0)$ are essential elements of $\pi_{2n-1}(S^{2n-2})$.

Consequently we get the following equation for $\alpha_{n-1}=\{G(f_0)\}$

$$\{h_n\}\cdot\chi=0+(-i_n)\circ E(\alpha_{n-1})-[i_n, i_n]=-E(\alpha_{n-1})-[i_n, i_n],$$
$$H_1(\alpha_{n-1})\neq 0.$$

If the degree of χ is d, then $H_1[i_n, i_n]=2i_{2n}$, $H_1(E(\alpha_{n-1}))=0$ and $H_1\{h_n\circ\chi\}=di_{2n}$. Therefore $d=-2$, and hence $\{h_n\circ\chi\}=-2\{h_n\}$. *q.e.d.*

5 Non-isomorphic suspensions

It is already known that the suspension homomorphisms $E:\pi_{2r-1}(S^r)\to\pi_{2r}(S^{r+1})$ are not isomorphic in the cases $r\equiv 0$ (mod. 2) and $r\equiv 1$ (mod. 4) ($r>1$), because the whitehead products $[i_n, i_n]$ of the identity mps $i_n: S^n\to S^n$ are essential [9, §9], and $E[i_n, i_n]=0$ by (1.5).

We shall show that *the suspension homomorphisms*

$$E:\pi_n(S^r)\to\pi_{n+1}(S^{r+1})$$

are not isomorphic for the following values of n and r (hence $\pi_n(S^r)\neq 0$).

n	6	7	8	8	9	10	16	17	22	$4k+10$	$4k+22$	$8k+2$	$8k+3$
r	2	2	2	4	4	4	8	8	8	$2k+4$	$2k+8$	$4k+1$	$4k+1$

($k=1, 2, \ldots$).

In other words the boundary homomorphisms of triads

$$\beta_+:\pi_{n+2}(S^{r+1}; E_+^{r+1}, E_-^{r+1})\to\pi_{n+1}(E_+^{r+1}, S^r)$$

are non-trivial. (*Cf.* Theorem II of [1, §4]).

Let $\nu_2=\{h_2\}$ be the generator of $\pi_3(S^2)$, then $\nu_n=E^{n-2}(\nu_2)$ is the generator of $\pi_{n+1}(S^n)\approx I_2$.

Let $\nu_4'=\{h_4\}$ be the element given by Hopf map $\{h_4\}$, and let $\nu_n=E^{n-4}(\nu_4')$ be the (n-4)-fold suspension of ν_4'.

$\nu_8''=\{h_8\}$ and $\nu_n''=E^{n-8}(\nu_8'')$ can be also defined.

It is verified in [10] using the Theorems (2.10), (2.11), that the suspension $E:\pi_4(S^2)\to\pi_5(S^3)$ is isomorphism onto, and $\pi_{n+2}(S^n)\approx I_2(n\geqq 2)$. We denote the generator $\nu_n\circ\nu_{n+1}$ of $\pi_{n+2}(S^n)$ by η_n, then we have $\eta_n=E^{n-2}(\eta_2)$.

Now consider the suspension $E:\pi_5(S^2)\to\pi_6(S^3)$. We have $\pi_5(S^2)\approx I_2$ and its generator is given by $\nu_2\circ\eta_3$, and $H_1(\nu_2\circ\eta_3)=E(\eta_3)=\eta_4$ by (2.7).

If $E\nu_2\circ\eta_3=0$, there corresponds Freudenthal invariants $\Lambda_1', \Lambda_1''\in\pi_6(S^6)$, and by (2.10), (2.11) we have

$$\Lambda_1'-\Lambda_1''=(-1)^2E^2H_1(\nu_2\circ\eta_3)=\eta_6,$$
$$\Lambda_1'=(-1)i_6\circ\Lambda_1''.$$

Since $\Lambda_1''=E(\gamma)$ for some $\gamma\in\pi_7(S^5)$, we have $(-1)i_6\circ\Lambda_1''=(-i_6)\circ E(\gamma)=-E(\gamma)$ $=-\Lambda_1'$ and $2\Lambda_1'=\eta_6$. This contradicts the fact that η_6 generates $\pi_8(S^6)$, and therefore $E(\nu_2\circ\eta_3)=\nu_3\circ\eta_4\neq 0$. Denote $\nu_n\circ\eta_{n+1}=\eta_n'(n\geqq 2)$, then η_n' is a non-zero element of $\pi_{n+3}(S^n)$ by (4.3).

Let α_3, α_7 be the elements of $\pi_6(S^3)$ and $\pi_{14}(S^7)$ given in Theorem (4.1), then we have $H_1(\alpha_3)=\nu_6$, $H_1(\alpha_7)=\nu_{14}$, $E^2(\alpha_3)=2\nu_5'$ and $E^2(\alpha_7)=2\nu_9''$.

i) For case $r=2$.

Consider the elements $\nu_2\circ\alpha_3\in\pi_6(S^2)$, $\nu_2\circ\alpha_3\circ\nu_6\in\pi_7(S^2)$ and $\nu_2\circ\alpha_3\circ\eta_6\in\pi_8(S^2)$. By (2.7), we have $H_1(\alpha_3\circ\nu_6)=\nu_6\circ\nu_7=\eta_6\neq 0$, $H_1(\alpha_3\circ\eta_6)=\nu_6\circ\eta_7=\eta_7'\neq 0$. Since ν_2 induces isomorphism onto, we have $\nu_2\circ\alpha_3\neq 0$, $\nu_2\circ\alpha_3\circ\nu_6\neq 0$, $\nu_2\circ\alpha_3\circ\eta_6\neq 0$.

We have $E^2(\nu_2\circ\alpha_3)=\nu_4\circ E^2(\alpha_3)=\nu^4\circ(2\nu_5')=2\nu_4\circ\nu_5'=0$.

Since $E:\pi_n(S^3)\to\pi_{n+1}(S^4)$ is an isomorphism, we have

$$E(\nu_2\circ\alpha_3)=0,$$

and also $$E(\nu_2\circ\alpha_3\circ\gamma)=0 \qquad \text{for any } \gamma\in\pi_n(S^6).$$

Remark. P. Serre announced in [7] that $\pi_{2p+k-3}(S^k)$, for odd $k\geqq 3$, and for prime p, has the element whose order is p. It follows directly that the suspension $E:\pi_{2p}(S^2)\to\pi_{2p+1}(S^3)$ is not isomorphic.

In the following cases it is sufficient to show the existence of non-zero whitehead roducts, because $E[\alpha,\beta]=0$.

ii) The cases $r=4, 8$.

Consider the whitehead product $[\nu_4, i_4]\in\pi_8(S^4)$. By (1.3), (4.1),

$$[\nu_4, i_4]=[i_4, i_4]\circ(\nu_3 * i_3)=(2\nu_4'-E(\alpha_3))\circ E^4\nu_3=2\nu_4'\circ E^4\nu_3-E(\alpha_3)\circ E^4\nu_3$$
$$=\nu_4'\circ 2E^4\nu_3-E(\alpha_3\circ E^3\nu_3)=E(\alpha_3\circ\nu_6).$$

Since $H_1(\alpha_3\circ\nu_6)=\eta_6\neq 0$ and $E:\pi_n(S^3)\to\pi_{n+1}(S^4)$ is an isomorphism into, we have $[\nu_4, i_4]\neq 0$. Similarly we have $[\eta_4, i_4]=E(\alpha_3\circ\eta_6)\neq 0$, $[\nu_8, i_8]=E(\alpha_7\circ\nu_{10})\neq 0$ and $[\eta_8, i_8]=E(\alpha_7\circ\eta_{10})\neq 0$.

Consider the whitehead product $[\nu_4', i_4]\in\pi_{10}(S^4)$. If $[\nu_4', i_4]=0$, by (3.72) of [9, §3] there exists a map $f: S^7\times S^4\to S^4$ of *type* (ν_4', i_4). Therefore by (2.5) $H_1(\{G(f)\})=\nu_{10}'$, but by (2.9) $2\nu_{10}'=0$. This contradicts to (4.2). Hence $[\nu_4', i_4]\neq 0$. Similarly $[\nu_8''. i_8]\neq 0$.

iii) The other cases.

By (2.8), (4.2), $$H_1[\nu_{2k+4}', i_{2k+4}]=2\nu_{4k+8}'\neq 0,$$
$$H_1[\nu_{2k+8}'', i_{2k+8}]=2\nu_{4k+16}''\neq 0, \quad (k=1, 2, \dots).$$

This shows that the suspension E referred to above is not isomorphic in the cases $r=2k+4$ and $r=2k+8$.

It is shown in §9 of [9] that there exists an element γ of $\pi_{8k}(S^{4k})$ such that $H_1(\gamma)=\gamma_{4k}$, $E(\gamma)=[i_{4k+1}, i_{4k+1}]$ for $k\geqq 1$. By (1.2), $[\nu_{4k+1}, i_{4k+1}]=E(\gamma\circ\nu_{8k})$. From (2.10), (2.11) and $H_1(\gamma\circ\nu_{8k})=\eta_{8k}\neq 0$, we have $E(\gamma\circ\nu_{8k})\neq 0$. Similarly $[\eta_{4k+1}, i_{4k+1}]=E(\gamma\circ\eta_{8k})\neq 0$.

REFERENCES

1) A. L. Bleakers and W. S. Massey. Ann. of Math. 53 (1951), pp. 161-205.
2) B. Eckmann. Comment. Math. Helv. 14 (1941), pp. 141-192.
3) H. Freudenthal. Comp. Math. 5 (1937), pp. 299-314.
4) H. Hopf. Math. Ann. 104 (1931), pp. 637-665.
5) H. Hopf. Fund. Math. 25 (1935), pp. 427-440.
6) W. Hurewicz and N. E. Steenrod. Proc. Nat. Akad. Sci. U. S. A. 27 (1941). pp. 60-64.
7) P. Serre. C. R. Paris 232 (1951), pp. 142-144.
8) G. W. Whitehead. Ann. of Math. 47 (1946), pp 460-475.
9) G. W. Whitehead. Ann. of Math. 51 (1950), pp. 192-237.
10) G. W. Whitehead. Ann. of Math. 52 (1950), pp. 245-247.
11) J. H. C. Whitehead. Ann. of Math. 42 (1941), pp. 409-428.