Daigle, D.
Osaka J. Math.
49 (2012), 53–80

TAME AND WILD DEGREE FUNCTIONS

DANIEL DAIGLE

(Received July 27, 2010)

Abstract

We give examples of degree functions deg: $R \to M \cup \{-\infty\}$, where R is $\mathbb{C}[X, Y]$ or $\mathbb{C}[X, Y, Z]$ and M is $\mathbb{Z}$ or $\mathbb{N}$, whose behaviour with respect to $\mathbb{C}$-derivations $D\colon R \to R$ is pathological in the sense that $\{\deg(Dx) - \deg(x) \mid x \in R \setminus \{0\}\}$ is not bounded above. We also give several general results stating that such pathologies do not occur when the degree functions satisfy certain hypotheses.

1. Introduction

Let B be a ring and $(G, +, \leq)$ a totally ordered abelian group. A map

$$\deg\colon B \to G \cup \{-\infty\}$$

is called a *degree function* if it satisfies, for all $x, y \in B$,
(1) $\deg(x) = -\infty$ iff $x = 0$;
(2) $\deg(xy) = \deg(x) + \deg(y)$;
(3) $\deg(x + y) \leq \max(\deg(x), \deg(y))$.

It is easy to see that if B admits a degree function then B is either the zero ring or an integral domain. Also, if deg: $B \to G \cup \{-\infty\}$ is a degree function and $x, y \in B$ are such that $\deg(x) \neq \deg(y)$, then $\deg(x + y) = \max(\deg(x), \deg(y))$.

Let B be an integral domain and $\deg\colon B \to G \cup \{-\infty\}$ a degree function, where G is a totally ordered abelian group. Given a derivation $D\colon B \to B$,

$$U = \{\deg(Dx) - \deg(x) \mid x \in B \setminus \{0\}\}$$

is a nonempty subset of the totally ordered set $G \cup \{-\infty\}$. If U has a greatest element, we define $\deg(D)$ to be that element; if U does not have a greatest element, we say that $\deg(D)$ is not defined. Note that if D is the zero derivation then $\deg(D)$ is defined and is equal to $-\infty$; in fact the condition $D = 0$ is equivalent to $\deg(D) = -\infty$. Also note that, in the special case $G = \mathbb{Z}$, $\deg(D)$ is defined if and only if the set U is bounded above.

Consider the associated graded ring $\mathrm{Gr}(B)$, which is a G-graded integral domain determined by the pair $(B, \deg)$ (see Paragraph 1.9 for details). It is well known that

2010 Mathematics Subject Classification. Primary 14R20; Secondary 13N15, 12J20, 13A18.
Research supported by grant RGPIN/104976-2005 from NSERC Canada.

each derivation $D\colon B\to B$ *such that* $\deg(D)$ *is defined* gives rise to a homogeneous derivation $\mathrm{gr}(D)\colon \mathrm{Gr}(B)\to\mathrm{Gr}(B)$. The technique of replacing D by $\mathrm{gr}(D)$, called "homogenization of derivations", is used quite systematically in the study of G_a-actions on affine algebraic varieties. We stress that homogenization requires prior verification that $\deg(D)$ is defined with respect to the given degree function. To clarify the discussion, we introduce the following notion:

DEFINITION 1.1. Let $A\subseteq B$ be integral domains of characteristic zero, and let G be a totally ordered abelian group. A degree function $\deg\colon B\to G\cup\{-\infty\}$ is said to be *tame over* A, or A*-tame*, if it satisfies:

$$\deg(D)\quad\text{is defined for all } A\text{-derivations}\quad D\colon B\to B.$$

If deg is not tame over A, we say that it is *wild over* A, or A*-wild*.

The present paper has two objectives:

I. To give examples of $\mathbf{k}$-wild degree functions $\deg\colon \mathbf{k}[X,Y]\to\mathbb{Z}\cup\{-\infty\}$ and $\deg\colon \mathbf{k}[X,Y,Z]\to\mathbb{Z}\cup\{-\infty\}$, where $\mathbf{k}$ is a field of characteristic zero;

II. to give results which state that degree functions satisfying certain hypotheses are tame.

There is a good measure of confusion in relation with degree functions. Consider the following statement:

($\times$) *If B is an integral domain and a finitely generated $\mathbb{C}$-algebra, then all degree functions on B are tame over $\mathbb{C}$.*

Assertion ($\times$) is *false*, as it is contradicted by either one of Propositions 1.2 and 1.3 (see below). However, ($\times$) has been used by several authors to justify the homogenization of derivations. Examples: [4, Proof of Lemma 1], [5, Proof of Lemma 5], [7, Proof of Theorem 3.1]; in [2], a variant[1] of ($\times$) is stated on p. 3 and implicitly used in the proof of Proposition 2; a (necessarily incorrect) proof of ($\times$) is given in [1, 6.2], and ($\times$) is then used to prove the following false statement [1, Corollary 6.3]: *for a $\mathbb{C}$-algebra B, if there exists a degree function* $\deg\colon B\to\mathbb{Z}\cup\{-\infty\}$ *such that* $\mathrm{Gr}(B)$ *is rigid, then B is rigid*[2] (Proposition 1.2 is a counterexample, as B is not rigid but $\mathrm{Gr}(B)=\mathbf{k}[t,t^{-1}]$ is rigid). We provide the correction: *if there exists a $\mathbb{C}$-tame degree function* $\deg\colon B\to G\cup\{-\infty\}$ *such that* $\mathrm{Gr}(B)$ *is rigid, then B is rigid.*

Also, one can find many examples in the literature where authors simply omit to raise the question whether $\deg(D)$ is defined, as if it were a priori clear that $\deg(D)$ is always defined. We hope that our examples will clear-up some of this confusion.

---

[1] Instead of assuming that B is finitely generated, the variant assumes that $\mathrm{Gr}(B)$ is finitely generated. This variant is false: in Proposition 1.2, both B and $\mathrm{Gr}(B)$ are finitely generated but deg is wild.

[2] One says that B is *rigid* if the only locally nilpotent derivation $D\colon B\to B$ is the zero derivation.

Sections 2 and 3 prove the following facts (the reader should compare these results to the statement of Theorem 1.7, below).

Proposition 1.2. *Let* $\mathbf{k}$ *be a field of characteristic zero and* $B = \mathbf{k}[X, Y] = \mathbf{k}^{[2]}$. *Then there exists a degree function* $\deg\colon B \to \mathbb{Z} \cup \{-\infty\}$ *satisfying*:
(a) $\deg(\lambda) = 0$ *for all* $\lambda \in \mathbf{k}^*$;
(b) $\mathrm{Gr}(B) \cong \mathbf{k}[t, t^{-1}]$;
(c) *the only* $\mathbf{k}$*-derivation* $D\colon B \to B$ *such that* $\deg(D)$ *is defined is the zero derivation.*

In the above statement and throughout this paper, we write $A = R^{[n]}$ to indicate that A is a polynomial ring in n variables over R. The proof of Proposition 1.2 is given in Section 2. The next fact is the special case "$A = \mathbf{k}^{[1]}$" of Corollary 3.8; it shows that wild degree functions with values in $\mathbb{N}$ do exist:

Proposition 1.3. *Let* $\mathbf{k}$ *be an uncountable field of characteristic zero and* $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$. *Then there exists a degree function* $\deg\colon B \to \mathbb{N} \cup \{-\infty\}$ *such that* $\deg(\lambda) = 0$ *for all* $\lambda \in \mathbf{k}^*$ *and with respect to which the degree of* $\partial/\partial X\colon B \to B$ *is not defined.*

We have a similar result for $B = \mathbf{k}^{[2]}$, but with more restrictions on $\mathbf{k}$:

Proposition 1.4. *Let* $\mathbf{k}$ *be a function field*[3] *over an uncountable field of characteristic zero, and let* $B = \mathbf{k}[X, Y] = \mathbf{k}^{[2]}$. *Then there exists a degree function* $\deg\colon B \to \mathbb{N} \cup \{-\infty\}$ *such that* $\deg(\lambda) = 0$ *for all* $\lambda \in \mathbf{k}^*$ *and with respect to which the degree of* $\partial/\partial X\colon B \to B$ *is not defined.*

Proposition 1.4 is an immediate consequence of part (e) of the next result, which exhibits some pathologies with respect to the process of extending degree functions:

Proposition 1.5. *Let* $\mathbf{k}_0$ *be an uncountable field of characteristic zero,* $\mathbf{k}_1$ *a function field over* $\mathbf{k}_0$ *and* $\mathbf{k}_2$ *the algebraic closure of* $\mathbf{k}_1$. *Consider the polynomial rings* $B_0 \subset B_1 \subset B_2$, *where* $B_i = \mathbf{k}_i[X, Y] = \mathbf{k}_i^{[2]}$. *Then there exist degree functions*

$$\deg_0\colon B_0 \to \mathbb{N} \cup \{\infty\}, \quad \deg_1\colon B_1 \to \mathbb{N} \cup \{\infty\} \quad \textit{and} \quad \deg_2\colon B_2 \to \mathbb{Z} \cup \{\infty\}$$

satisfying the following conditions:
(a) *if* $i \leq j$ *then* $\deg_i$ *is the restriction of* $\deg_j$;
(b) *for each* $i = 0, 1, 2$, $\deg_i(\lambda) = 0$ *for all* $\lambda \in \mathbf{k}_i^*$;
(c) $\deg_0$ *is determined by the grading* $B_0 = \bigoplus_{i \in \mathbb{N}} R_i$ *of* B_0 *defined by* $X \in R_2$ *and* $Y \in R_3$ *but, for each* $i = 1, 2$, $\deg_i$ *is not determined by a grading of* B_i;

[3]A *function field* is a finitely generated field extension of transcendence degree at least 1.

(d) $\mathrm{Gr}(B_i)$ *is affine over* $\mathbf{k}_i$ *if* $i \in \{0, 2\}$, *but not if* $i = 1$;
(e) $\deg_i(D_i)$ *is defined if* $i = 0$ *but not if* $i \in \{1, 2\}$, *where* $D_i = \partial/\partial Y : B_i \to B_i$.

See Paragraph 3.7 for the proof of Proposition 1.5. The notion of a degree function *determined by a grading* is defined in Paragraph 1.9. It may be worthwile to state the following consequence of Proposition 1.5:

Corollary 1.6. *Let* S *be the set of degree functions* $\deg : \mathbb{C}[X, Y] \to \mathbb{Z} \cup \{-\infty\}$ *satisfying* $\deg(\lambda) = 0$ *for all* $\lambda \in \mathbb{C}^*$, $\deg(X) = 2$ *and* $\deg(Y) = 3$. *Then there exist elements* d *and* d' *of* S *satisfying*:

$$d|_{\mathbb{Q}[X,Y]} = d'|_{\mathbb{Q}[X,Y]}, \quad d \textit{ is } \mathbb{C}\textit{-tame and } d' \textit{ is } \mathbb{C}\textit{-wild.}$$

See Paragraph 3.9 for the proof of Corollary 1.6.

The proof of Proposition 1.2 is quite simple, but those of Propositions 1.3–1.5 are more delicate because they involve constructing degree functions *with nonnegative values* and which are still wild. The crucial step is the proof, in Lemma 3.6.7, that $\mathrm{ord}_t(f) \leq 0$ for every nonzero element f of the subring $\mathbf{k}_1[x, y]$ of $\mathbf{k}_2((t))$. The idea that this inequality could be proved by using an expansion lemma such as Lemma 3.2 was inspired by past frequentations with expansion techniques à la Abhyankar-Sathaye.

Section 4 proves an array of results which assert that degree functions satisfying certain hypotheses are tame. Some of those facts are summarized in the following statement, but note that the results of Section 4 are stronger:

Theorem 1.7. *Suppose that* B *is an integral domain containing a field* $\mathbf{k}$ *of characteristic zero. Let* G *be a totally ordered abelian group and* $\deg : B \to G \cup \{-\infty\}$ *a degree function. Then, in each of the cases* (a)–(d) *below,* $\deg$ *is tame over* $\mathbf{k}$:
(a) B *is* $\mathbf{k}$*-affine and* $\deg$ *is determined by some* G*-grading of* B.
(b) $\mathrm{Gr}(B)$ *is* $\mathbf{k}$*-affine and* $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of* G.
(c) $\mathrm{trdeg}_{\mathbf{k}}(B) < \infty$, $\mathrm{Frac}(B)$ *is a one-dimensional function field over the field of fractions of the ring* $\{x \in B \mid \deg(x) \leq 0\}$, *and* $\deg$ *has values in* $\mathbb{N}$.
(d) $\mathrm{trdeg}_{\mathbf{k}}(B) < \infty$ *and* $\deg = \deg_\Delta$ *for some locally nilpotent derivation* $\Delta : B \to B$.

Here, $\mathrm{Frac}(B)$ denotes the field of fractions of B and “**k**-affine” means “finitely generated as a **k**-algebra”. Assertions (a), (b), (c) and (d) of Theorem 1.7 follow from Corollaries 4.8, 4.23, Proposition 4.24 and Corollary 4.12, respectively (also note that (d) is a special case of (c)).

Assertions (b) and (c) of Theorem 1.7 appear to be new. The case $G = \mathbb{Z}$ of Theorem 1.7 (a) is well known, and since the general case has the same proof we assume that it is also known. Assertion (d) of Theorem 1.7 appeared in [3, Theorem 2.11, p. 40], etc., with the mention that it was unpublished work of this author. The material in Definitions 4.15–Proposition 4.24 appears to be new. The results given in Setup 4.1–Definition 4.14 are generalizations and strengthenings of known results.

Let us also mention that most of the errors that we pointed out in the discussion between Definition 1.1 and Proposition 1.2 can be fixed by using the above Theorem 1.7 in conjunction with the following observation (Lemma 1.8 is an immediate consequence of Lemma 4.11, below):

Lemma 1.8. *Let B be an integral domain containing a field $\mathbf{k}$ of characteristic zero, $S \subset B$ a multiplicative set,* $\mathrm{DEG}\colon S^{-1}B \to G \cup \{-\infty\}$ *a degree function* (*where G is a totally ordered abelian group*) *and* $\deg\colon B \to G \cup \{-\infty\}$ *the restriction of* DEG. *If* DEG *is tame over* $\mathbf{k}$ *then so is* $\deg$.

1.9. Conventions, notations and terminologies. Given a totally ordered group G, it is understood that $G \cup \{-\infty\}$ is totally ordered and satisfies $-\infty < x$ for all $x \in G$. The same convention applies to $\mathbb{N} \cup \{-\infty\}$. In this article, $\mathbb{N}$ is the set of nonnegative integers, i.e., $0 \in \mathbb{N}$.

By a "domain", we mean an integral domain. If A is a domain then $\operatorname{Frac} A$ denotes its field of fractions. If $A \subseteq B$ are domains then $\operatorname{trdeg}_A(B)$ denotes the transcendence degree of $\operatorname{Frac}(B)$ over $\operatorname{Frac}(A)$. The symbol A^* denotes the set of units of a ring A. A polynomial ring in n variables over A is denoted $A^{[n]}$. A subring A of a domain B is said to be *factorially closed in B* if the conditions $x, y \in B$ and $xy \in A \setminus \{0\}$ imply that $x, y \in A$.

If $A \subseteq B$ are rings then $\operatorname{Der}(B)$ (resp. $\operatorname{Der}_A(B)$) is the set of derivations (resp. A-derivations) $D\colon B \to B$.

Let B be a domain and G a totally ordered abelian group. Then each G-grading $\mathfrak{g}$ of B determines a degree function $\deg_{\mathfrak{g}}\colon B \to G \cup \{-\infty\}$ as follows. Let $B = \bigoplus_{i \in G} B_i$ be the grading $\mathfrak{g}$. Given $x \in B$, write $x = \sum_{i \in G} x_i$ ($x_i \in B_i$) and consider the finite set $S_x = \{i \in G \mid x_i \neq 0\}$; then define $\deg_{\mathfrak{g}}(x)$ to be the greatest element of $S_x \cup \{-\infty\}$. This is what we mean by a degree function "determined by a grading".

Let B be a domain and $\deg\colon B \to G \cup \{-\infty\}$ a degree function, where G is a totally ordered abelian group. For each $i \in G$, let

$$B_i = \{x \in B \mid \deg(x) \leq i\}, \quad B_{i^-} = \{x \in B \mid \deg(x) < i\}, \quad B_{[i]} = B_i / B_{i^-}.$$

The direct sum $\operatorname{Gr}(B) = \bigoplus_{i \in G} B_{[i]}$ is a G-graded integral domain referred to as the *associated graded ring*; it is determined by $(B, \deg)$. Note that $\operatorname{Gr}(B)$ comes equipped with the degree function $\deg_{\mathfrak{g}}\colon \operatorname{Gr}(B) \to G \cup \{-\infty\}$ where $\mathfrak{g}$ denotes the grading $\operatorname{Gr}(B) = \bigoplus_{i \in G} B_{[i]}$. One also defines a set map $\operatorname{gr}\colon B \to \operatorname{Gr}(B)$ by $\operatorname{gr}(0) = 0$ and, for $x \in B \setminus \{0\}$, $\operatorname{gr}(x) = x + B_{i^-} \in B_{[i]} \setminus \{0\}$, where $i = \deg(x)$. The map gr preserves multiplication but, in general, not addition. For all $x \in B$, $\operatorname{gr}(x)$ is a homogeneous element of $\operatorname{Gr}(B)$ and $\deg(x) = \deg_{\mathfrak{g}}(\operatorname{gr}(x))$.

As mentioned in the introduction, each derivation $D\colon B \to B$ such that $\deg(D)$ is defined gives rise to a homogeneous derivation $\operatorname{gr}(D)\colon \operatorname{Gr}(B) \to \operatorname{Gr}(B)$; although this is not needed in the present paper, let us recall the definition. Let $d = \deg(D)$. If $d = -\infty$, set $\mathrm{gr}(D) = 0$. If $d \neq -\infty$ then $d \in G$ and (for each $i \in G$) D maps B_i into $B_{(i+d)}$ and B_{i^-} into $B_{(i+d)^-}$, and so a map $D_{[i]}\colon B_{[i]} \to B_{[i+d]}$ is defined by $x + B_{i^-} \mapsto D(x) + B_{(i+d)^-}$; then, given an element $y = \sum_{i \in G} y_i$ of $\mathrm{Gr}(B)$ (with $y_i \in B_{[i]}$), define $\mathrm{gr}(D)(y) = \sum_i D_{[i]}(y_i)$. If D is nonzero then so is $\mathrm{gr}(D)$, and if D is locally nilpotent then so is $\mathrm{gr}(D)$.

2. Proof of Proposition 1.2

Let $\mathbf{k}$ be a field of characteristic zero.

2.1. Consider the field $\mathbf{k}((t))$ of Laurent power series over $\mathbf{k}$ and the order valuation $\mathrm{ord}\colon \mathbf{k}((t)) \to \mathbb{Z} \cup \{+\infty\}$. Define

$$\deg\colon \mathbf{k}((t)) \to \mathbb{Z} \cup \{-\infty\}, \quad \deg(f) = -\,\mathrm{ord}(f) \quad \text{for all} \quad f \in \mathbf{k}((t)). \tag{1}$$

Then deg is a degree function on $\mathbf{k}((t))$ and it is easily verified that the associated graded ring $\mathrm{Gr}(\mathbf{k}((t)))$ is isomorphic to $\mathbf{k}[t, t^{-1}]$.

2.2. Note that if B is any ring such that $\mathbf{k} \subseteq B \subseteq \mathbf{k}((t))$ then the restriction

$$\deg\colon B \to \mathbb{Z} \cup \{-\infty\}$$

of the degree function (1) is a degree function on B satisfying $\deg(\lambda) = 0$ for all $\lambda \in \mathbf{k}^*$. Also, there is an injective $\mathbf{k}$-homomorphism $\mathrm{Gr}(B) \hookrightarrow \mathrm{Gr}(\mathbf{k}((t)))$. As any ring A satisfying $\mathbf{k} \subseteq A \subseteq \mathbf{k}[t, t^{-1}]$ is $\mathbf{k}$-affine, we see that $\mathrm{Gr}(B)$ is $\mathbf{k}$-affine.

Proof of Proposition 1.2. One can show that there exists $f(t) \in \mathbf{k}((t))$ such that $(t, f(t), f'(t))$ are algebraically independent over $\mathbf{k}$ and $\mathrm{ord}\, f(t) \geq 0$. Choose such an $f(t) = \sum_{j=0}^{\infty} a_j t^j$; let $x = t^{-1}$ and $y = f(t)$ and consider the subalgebra $B = \mathbf{k}[x, y]$ of $\mathbf{k}((t))$. Note that $B = \mathbf{k}^{[2]}$. Define $\deg\colon B \to \mathbb{Z} \cup \{-\infty\}$ as in Paragraph 2.2 and note (as in Paragraph 2.2) that $\mathrm{Gr}(B)$ is $\mathbf{k}$-affine and that $\deg(\lambda) = 0$ for all $\lambda \in \mathbf{k}^*$. Note that $\deg(y - a_0)$ is a negative integer; as $\deg(x) = 1$, it follows that $\{\deg(h) \mid h \in B \setminus \{0\}\} = \mathbb{Z}$. From this, it is easy to deduce that the natural embedding of $\mathrm{Gr}(B)$ into $\mathrm{Gr}(\mathbf{k}((t))) \cong \mathbf{k}[t, t^{-1}]$ is actually an isomorphism:

$$\mathrm{Gr}(B) \cong \mathbf{k}[t, t^{-1}].$$

For each $n \geq 1$,

$$x^n y = t^{-n} f(t) = \sum_{j=0}^{\infty} a_j t^{j-n} = \sum_{j=0}^{n-1} a_j t^{j-n} + \sum_{j=n}^{\infty} a_j t^{j-n} = \sum_{j=0}^{n-1} a_j x^{n-j} + \sum_{j=n}^{\infty} a_j t^{j-n}$$

so if we define $g_n \in B$ by $g_n = x^n y - \sum_{j=0}^{n-1} a_j x^{n-j}$, then $g_n = \sum_{j=n}^{\infty} a_j t^{j-n}$, so

$$\text{the set} \quad S = \{n \mid \deg(g_n) = 0\} \quad \text{is infinite.}$$

We have $\partial g_n / \partial y = x^n$ and

$$\begin{aligned}
\frac{\partial g_n}{\partial x} &= n x^{n-1} y - \sum_{j=0}^{n-1} (n-j) a_j x^{n-j-1} = n t^{1-n} \sum_{j=0}^{\infty} a_j t^j - \sum_{j=0}^{n-1} (n-j) a_j t^{j-n+1} \\
&= \sum_{j=0}^{n-1} j a_j t^{j-n+1} + \sum_{j=n}^{\infty} n a_j t^{j-n+1} = t^{2-n} \left[\sum_{j=0}^{n-1} j a_j t^{j-1} + \sum_{j=n}^{\infty} n a_j t^{j-1} \right] \\
&= t^{2-n} [\alpha_n + \varepsilon_n]
\end{aligned}$$

with $\alpha_n = \sum_{j=0}^{n-1} j a_j t^{j-1}$ and $\varepsilon_n = \sum_{j=n}^{\infty} n a_j t^{j-1}$. Notice that $\{\alpha_n + \varepsilon_n\}_{n=1}^{\infty}$ is a sequence in $\mathbf{k}((t))$ which converges to $f'(t)$ with respect to the (t)-adic topology.

Consider the $\mathbf{k}$-derivation $D = u\, \partial/\partial x - v\, \partial/\partial y \colon B \to B$, where $u, v \in B$, and assume that $\deg(D)$ is defined. Then there exists $d \in \mathbb{Z}$ satisfying $\deg(D(g_n)) - \deg(g_n) \le d$ for all $n \ge 1$, so in particular

$$\operatorname{ord}(t^n D(g_n)) \ge n - d \quad \text{for all} \quad n \in S. \tag{2}$$

On the other hand we have

$$t^n D(g_n) = t^n \left(u \frac{\partial g_n}{\partial x} - v \frac{\partial g_n}{\partial y} \right) = t^2 [\alpha_n + \varepsilon_n] u - v = [\alpha_n + \varepsilon_n] x^{-2} u - v. \tag{3}$$

The right hand side of (3) is a convergent sequence in $\mathbf{k}((t))$, with limit $f'(t) x^{-2} u - v$; so the sequence $\{t^n D(g_n)\}_{n=1}^{\infty}$ is convergent and, by (2), must converge to 0; so

$$f'(t) x^{-2} u - v = 0. \tag{4}$$

If $u \neq 0$ then (4) implies $f'(t) = x^2 v / u \in \mathbf{k}(x, y) = \mathbf{k}(t, f(t))$, which contradicts our choice of $f(t)$. So $u = 0$ and, by (4), $v = 0$. So $D = 0$. □

3. Wild degree functions with values in $\mathbb{N}$

NOTATIONS 3.1. For each finite subset $S = \{u_1, \dots, u_n\}$ of a ring A, define $\mu(S) = \prod_{i=1}^{n} u_i \in A$ (where $\mu(\emptyset) = 1$ by convention). If E is a set, $\mathcal{P}_{\text{fin}}(E)$ denotes the set of finite subsets of E and $\mathcal{P}^*_{\text{fin}}(E)$ is the set of nonempty finite subsets of E.

Lemma 3.2. *Let $(a_i)_{i \in \mathbb{N}}$ be a sequence of elements of a ring A. Define a sequence $(F_i)_{i \in \mathbb{N}}$ in $A[Y] = A^{[1]}$ by $F_0 = Y$ and, for each $i \in \mathbb{N}$, $F_{i+1} = F_i^2 - a_i$. Then* *each nonzero element of* $A[Y]$ *has a unique expression as a finite sum*

$$\alpha_1\mu(S_1) + \cdots + \alpha_N\mu(S_N),$$

where $N \geq 1$, $\alpha_i \in A \setminus \{0\}$ *and* $S_1, \dots, S_N$ *are distinct finite subsets of* $\{F_i \mid i \in \mathbb{N}\}$.

Proof. As F_i is monic of degree 2^i, we see that $\mu(S)$ is monic for each finite subset S of $\{F_i \mid i \in \mathbb{N}\}$, and $S \mapsto \deg(\mu(S))$ is a bijection from the set of finite subsets of $\{F_i \mid i \in \mathbb{N}\}$ to $\mathbb{N}$. The lemma follows from this. □

Lemma 3.3. *Let* L/K *be an extension of fields of characteristic* $\neq 2$ *and* $\mathcal{U}$ *a subset of* L *satisfying*:
(i) $u^2 \in K$ *for all* $u \in \mathcal{U}$;
(ii) $\mu(F) \notin K$ *for all* $F \in \mathcal{P}^*_{\text{fin}}\,\mathcal{U}$.
Then the family $(\mu(F))_{F \in \mathcal{P}_{\text{fin}}(\mathcal{U})}$ *of elements of* L *is linearly independent over* K.

Proof. This is certainly well known but, in lack of a suitable reference, we provide a proof. We imitate the proof that if $p_1, \dots, p_n$ are distinct prime numbers then $[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n}) : \mathbb{Q}] = 2^n$, see for instance [8]. The first step is to prove that the set

$$\Sigma = \{(F, G) \in \mathcal{P}_{\text{fin}}(\mathcal{U})^2 \mid F \neq \emptyset,\ F \cap G = \emptyset \text{ and } \mu(F) \in K[G]\}$$

is empty. Suppose the contrary, and choose $(F, G) \in \Sigma$ which minimizes $|G|$. Note that $G \neq \emptyset$ by (ii); pick $g \in G$ and let $G' = G \setminus \{g\}$. By minimality of $|G|$,

$$(F', G') \notin \Sigma \quad \text{for all} \quad F' \in \mathcal{P}_{\text{fin}}(\mathcal{U}). \tag{5}$$

Since $\mu(F) \in K[G] = K[G'][g]$ and $g^2 \in K$ by (i), we have $\mu(F) = a + bg$ for some $a, b \in K[G']$. Using (i) again gives $K \ni \mu(F)^2 = a^2 + 2abg + b^2g^2$; since $\operatorname{char} K \neq 2$, $abg \in K[G']$. If $ab \neq 0$ then $g \in K[G']$, so $(\{g\}, G') \in \Sigma$ contradicts (5). If $a = 0$ then $\mu(F \cup \{g\}) = \mu(F)g = bg^2 \in K[G']$, so $(F \cup \{g\}, G') \in \Sigma$ contradicts (5). If $b = 0$ then $\mu(F) = a \in K[G']$, so $(F, G') \in \Sigma$ contradicts (5). These contradictions show that $\Sigma = \emptyset$.

We now prove the assertion of the lemma, by contradiction. Suppose that $S_1, \dots, S_n$ are distinct elements of $\mathcal{P}_{\text{fin}}(\mathcal{U})$ such that $\mu(S_1), \dots, \mu(S_n)$ are linearly dependent over K, and suppose that n is the least natural number for which such sets exist. Observe that $n \geq 2$ and hence $\bigcup_{i=1}^n S_i \neq \bigcap_{i=1}^n S_i$. Pick $u \in \bigcup_{i=1}^n S_i \setminus \bigcap_{i=1}^n S_i$. Relabel the sets $S_1, \dots, S_n$ so as to have $u \in S_1 \cap \cdots \cap S_m$ and $u \notin S_{m+1} \cup \cdots \cup S_n$, and note that $1 \leq m \leq n-1$. Choose $a_1, \dots, a_n \in K$ not all zero such that $\sum_{i=1}^n a_i\mu(S_i) = 0$ and note that $a_1, \dots, a_n \in K^*$ by minimality of n. Let $S = \bigcup_{i=1}^n S_i$. We have

$$u \sum_{i=1}^m a_i\mu(S_i \setminus \{u\}) = -\sum_{i=m+1}^n a_i\mu(S_i),$$

where the two sums belong to $K[S\setminus\{u\}]$ and where $\sum_{i=1}^{m} a_i\mu(S_i\setminus\{u\})\neq 0$ by minimality of n. Thus $u\in K[S\setminus\{u\}]$ and consequently $(\{u\}, S\setminus\{u\})\in\Sigma$, a contradiction. $\square$

3.4. Consider the following conditions on a 4-tuple $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$:
(i) $\mathbf{k}_0\subset\mathbf{k}_1\subset\mathbf{k}_2$ are fields of characteristic zero and $\mathbf{k}_2$ is algebraic over $\mathbf{k}_1$;
(ii) $\mathcal{U}$ is an uncountable subset of $\mathbf{k}_2$;
(iii) $u^2\in\mathbf{k}_1$ for all $u\in\mathcal{U}$;
(iv) $\mu(S)\notin\mathbf{k}_1$ for each $S\in\mathcal{P}^*_{\text{fin}}(\mathcal{U})$;
(v) some element of $\mathcal{U}$ is transcendental over $\mathbf{k}_0$.
Note that, by Lemma 3.3, any 4-tuple $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$ satisfying (i)–(v) also satisfies:
(vi) the family $(\mu(F))_{F\in\mathcal{P}_{\text{fin}}(\mathcal{U})}$ of elements of $\mathbf{k}_2$ is linearly independent over $\mathbf{k}_1$.

Lemma 3.5. *Let $\mathbf{k}_0$ be an uncountable field of characteristic zero, $\mathbf{k}_1$ a function field over $\mathbf{k}_0$ and $\mathbf{k}_2$ the algebraic closure of $\mathbf{k}_1$. Then there exists a subset $\mathcal{U}$ of $\mathbf{k}_2$ such that $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$ satisfies the conditions of* Paragraph 3.4. *Moreover, if A is a ring such that $\mathbf{k}_0\subseteq A\subseteq\mathbf{k}_1$ and* $\operatorname{Frac}(A)=\mathbf{k}_1$, *then $\mathcal{U}$ can be chosen in such a way that $u^2\in A$ for all $u\in\mathcal{U}$.*

Proof. Choose a transcendence basis $\{t_1,\dots,t_n\}$ of $\mathbf{k}_1/\mathbf{k}_0$ such that $\{t_1,\dots,t_n\}\subset A$, let $R=\mathbf{k}_0[t_1,\dots,t_n]$ and $\mathbf{k}=\mathbf{k}_0(t_1,\dots,t_n)=\operatorname{Frac}R$. As $\mathbf{k}_1/\mathbf{k}_0$ is a function field, we have $n\geq 1$ and it makes sense to define $\mathcal{P}=\{t_1-\lambda \mid \lambda\in\mathbf{k}_0\}$, which is an uncountable set of prime elements of R satisfying:

$$\text{If}\quad p, q\quad\text{are distinct elements of}\quad \mathcal{P},\quad\text{then}\quad p\nmid q\quad\text{in}\quad R.$$

Choose a subset $\mathcal{U}_1$ of $\mathbf{k}_2$ such that $x\mapsto x^2$ is a bijection from $\mathcal{U}_1$ to $\mathcal{P}$. Then
- $u^2\in\mathbf{k}$ for all $u\in\mathcal{U}_1$;
- $\mu(S)\notin\mathbf{k}$ for each $S\in\mathcal{P}^*_{\text{fin}}(\mathcal{U}_1)$.

By Lemma 3.3, the family $(\mu(F))_{F\in\mathcal{P}_{\text{fin}}(\mathcal{U}_1)}$ of elements of $\mathbf{k}_2$ is linearly independent over $\mathbf{k}$; as $[\mathbf{k}_1:\mathbf{k}]<\infty$, it follows that $E=\{F\in\mathcal{P}_{\text{fin}}(\mathcal{U}_1)\mid \mu(F)\in\mathbf{k}_1\}$ is a finite set. Thus $C=\bigcup_{F\in E}F$ is a finite subset of $\mathcal{U}_1$ and $\mathcal{U}=\mathcal{U}_1\setminus C$ is uncountable. It is easily verified that $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$ satisfies the conditions of Paragraph 3.4. Moreover, $u^2\in A$ for all $u\in\mathcal{U}$. $\square$

3.6. We now fix $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$ satisfying the requirements of Paragraph 3.4. This is in effect throughout Paragraph 3.6.

3.6.1. Let $X_0, X_1, X_2, \dots$ be a countably infinite list of indeterminates over $\mathbf{k}_1$. For each $n\in\mathbb{N}$, let $E_n=\{f\in\mathbf{k}_1[X_0,\dots,X_n]\mid \deg_{X_n} f=1\}$. Note that the sets E_n are pairwise disjoint; when $f\in E_n$, we write $\operatorname{co}(f)\in\mathbf{k}_1[X_0,\dots,X_{n-1}]\setminus\{0\}$ for the coefficient of X_n in f.

For each $p \in \mathbb{N}$, let Σ_p be the set of series $\xi \in \mathbf{k}_1(X_0, X_1, \dots)((t))$ of the form

$$\xi = t^{-3}(f_p + f_{p+1}t^3 + f_{p+2}t^6 + \cdots) = t^{-3}\sum_{n=0}^{\infty} f_{p+n}t^{3n}, \tag{6}$$

such that $f_i \in E_i$ for all $i \geq p$. Given $\xi \in \Sigma_p$ with notation as in (6), define

$$V(\xi) = \Big\{(a_0, \dots, a_p) \in \mathbf{k}_2^{p+1} \ \Big|\ f_p(a_0, \dots, a_p) \neq 0 \\ \text{and } \forall_{i>p} \deg_{X_i}(f_i(a_0, \dots, a_p, X_{p+1}, \dots, X_i)) = 1\Big\}.$$

Lemma 3.6.2. *Let $p \in \mathbb{N}$ and $\xi = t^{-3}\sum_{n=0}^{\infty} f_{p+n}t^{3n} \in \Sigma_p$ (where $f_i \in E_i$ for all $i \geq p$). Let $\xi' = \xi^2 - f_p^2 t^{-6} \in \mathbf{k}_1(X_0, X_1, \dots)((t))$. Then*
(a) $\xi' \in \Sigma_{p+1}$.
(b) *If $(a_0, \dots, a_p) \in V(\xi)$ then there is a countable subset C of $\mathbf{k}_2$ such that, for all $a_{p+1} \in \mathbf{k}_2 \setminus C$, $(a_0, \dots, a_{p+1}) \in V(\xi')$.*

Proof. A straightforward calculation gives

$$\begin{aligned}\xi' = \xi^2 - f_p^2 t^{-6} &= t^{-3}(2 f_p f_{p+1} + (2 f_p f_{p+2} + f_{p+1}^2)t^3 + \cdots) \\ &= t^{-3}(g_{p+1} + g_{p+2}t^3 + g_{p+3}t^6 + \cdots) = t^{-3}\sum_{n=0}^{\infty} g_{p+1+n}t^{3n},\end{aligned}$$

where

$$g_{p+1+n} = \sum_{i=0}^{n+1} f_{p+i} f_{p+1+n-i} \quad \text{for all} \quad n \in \mathbb{N}. \tag{7}$$

Note that g_{p+1+n} is equal to $2 f_p f_{p+1+n}$ plus a sum of terms of the form $f_i f_j$ with $i, j < p+1+n$; this shows that

$$g_i \in E_i \quad \text{and} \quad \mathrm{co}(g_i) = 2 f_p \,\mathrm{co}(f_i) \quad \text{for all} \quad i \geq p+1. \tag{8}$$

In particular, $\xi' \in \Sigma_{p+1}$.

Suppose that $(a_0, \dots, a_p) \in V(\xi)$. Then $f_p(a_0, \dots, a_p) \neq 0$ and

$$\deg_{X_i} f_i(a_0, \dots, a_p, X_{p+1}, \dots, X_i) = 1 \quad \text{for all} \quad i \geq p+1. \tag{9}$$

Let $C = \{a_{p+1} \in \mathbf{k}_2 \mid (a_0, \dots, a_{p+1}) \notin V(\xi')\}$; we have to show that C is countable. Note that C is a countable union, $C = \bigcup_{i=p+1}^{\infty} C_i$, where

$$C_{p+1} = \{a_{p+1} \in \mathbf{k}_2 \mid g_{p+1}(a_0, \dots, a_{p+1}) = 0\}$$

and, for each $i \geq p + 2$,

$$C_i = \{a_{p+1} \in \mathbf{k}_2 \mid \deg_{X_i} g_i(a_0, \ldots, a_{p+1}, X_{p+2}, \ldots, X_i) < 1\}.$$

Since $g_{p+1}(a_0, \ldots, a_{p+1}) = 2 f_p(a_0, \ldots, a_p) f_{p+1}(a_0, \ldots, a_{p+1})$ where $f_p(a_0, \ldots, a_p) \neq 0$ and $f_{p+1}(a_0, \ldots, a_p, X_{p+1}) \in \mathbf{k}_2[X_{p+1}]$ has degree 1 by the case $i = p + 1$ of (9), we see that C_{p+1} is a finite set. Let $i \geq p + 2$. Then $\operatorname{co}(g_i) = 2 f_p \operatorname{co}(f_i)$ by (8), so

$$\begin{aligned}&\operatorname{co}(g_i)(a_0, \ldots, a_p, X_{p+1}, \ldots, X_{i-1})\\ &= 2 f_p(a_0, \ldots, a_p) \operatorname{co}(f_i)(a_0, \ldots, a_p, X_{p+1}, \ldots, X_{i-1}).\end{aligned}$$

Since $f_p(a_0, \ldots, a_p) \neq 0$ and, by (9), $\operatorname{co}(f_i)(a_0, \ldots, a_p, X_{p+1}, \ldots, X_{i-1}) \neq 0$, we have

$$\operatorname{co}(g_i)(a_0, \ldots, a_p, X_{p+1}, \ldots, X_{i-1}) \in \mathbf{k}_2[X_{p+1}, \ldots, X_{i-1}] \setminus \{0\}.$$

Consequently, there are only finitely many $a_{p+1} \in \mathbf{k}_2$ satisfying

$$\operatorname{co}(g_i)(a_0, \ldots, a_{p+1}, X_{p+2}, \ldots, X_{i-1}) = 0,$$

or equivalently

$$\deg_{X_i} g_i(a_0, \ldots, a_{p+1}, X_{p+2}, \ldots, X_i) < 1.$$

So C_i is a finite set (for each i) and it follows that C is countable. $\square$

3.6.3. For each $p \in \mathbb{N}$ we define a set map (well-defined by Lemma 3.6.2)

$$\Sigma_p \to \Sigma_{p+1}, \quad \xi \mapsto \xi'$$

by setting $\xi' = \xi^2 - f_p^2 t^{-6}$, where the notation for $\xi \in \Sigma_p$ is as in (6). Define a sequence $(\xi_p)_{p \in \mathbb{N}}$ by setting $\xi_0 = t^{-3} \sum_{n=0}^{\infty} X_n t^{3n} \in \Sigma_0$ and $\xi_{p+1} = \xi'_p$ for all $p \in \mathbb{N}$. Note that $\xi_p \in \Sigma_p$ for all $p \in \mathbb{N}$, and let the notation be as follows:

$$\xi_p = t^{-3} \sum_{n=0}^{\infty} f_{p,p+n} t^{3n} \quad (f_{p,p+n} \in E_{p+n}).$$

By (7) we have $f_{p+1,p+1+n} = \sum_{i=0}^{n+1} f_{p,p+i} f_{p,p+1+n-i}$ for all $p, n \in \mathbb{N}$, and in particular

$$f_{p+1,p+1} = 2 f_{p,p} f_{p,p+1} \quad \text{for all} \quad p \in \mathbb{N}. \tag{10}$$

Lemma 3.6.4. *For each $u_0 \in \mathcal{U}$, there exists a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of $\mathbf{k}_2$ satisfying the following conditions*:

(a) $a_0 = u_0$;

(b) $(a_0, \dots, a_p) \in V(\xi_p)$, *for each* $p \in \mathbb{N}$;
(c) $p \mapsto f_{p,p}(a_0, \dots, a_p)$ *is an injective map from* $\mathbb{N}$ *to* $\mathcal{U}$.

Proof. We define $(a_i)_{i\in\mathbb{N}}$ by induction. Define $a_0 = u_0$; note that $(a_0) \in V(\xi_0)$ and that $f_{0,0}(a_0) = a_0 = u_0 \in \mathcal{U}$.

Let $p \geq 0$ and assume that $(a_i)_{i=0}^p$ is such that $a_0 = u_0$, $(a_0, \dots, a_i) \in V(\xi_i)$ for all $i \in \{0, \dots, p\}$, and $i \mapsto f_{i,i}(a_0, \dots, a_i)$ is an injective map $\{0, \dots, p\} \to \mathcal{U}$.

Define $e_i = f_{i,i}(a_0,\dots,a_i) \in \mathcal{U}$, $0 \leq i \leq p$. By Lemma 3.6.2, there exists a countable set $C \subset \mathbf{k}_2$ such that, for each $a_{p+1} \in \mathbf{k}_2 \setminus C$, $(a_0, \dots, a_{p+1}) \in V(\xi_{p+1})$. By (10) we have $f_{p+1,p+1} = 2 f_{p,p} f_{p,p+1}$, so

$$\begin{aligned} f_{p+1,p+1}(a_0, \dots, a_p, X_{p+1}) &= 2 f_{p,p}(a_0, \dots, a_p) f_{p,p+1}(a_0, \dots, a_p, X_{p+1}) \\ &= 2e_p f_{p,p+1}(a_0, \dots, a_p, X_{p+1}) \\ &\in \mathbf{k}_2[X_{p+1}] \quad \text{is a polynomial of degree 1,} \end{aligned}$$

because $(a_0, \dots, a_p) \in V(\xi_p)$. Consequently, $x \mapsto f_{p+1,p+1}(a_0, \dots, a_p, x)$ is a bijective map $\mathbf{k}_2 \to \mathbf{k}_2$; as $\mathcal{U} \setminus \{e_0, \dots, e_p\}$ is uncountable, we may choose $a_{p+1} \in \mathbf{k}_2 \setminus C$ such that $f_{p+1,p+1}(a_0, \dots, a_{p+1}) \in \mathcal{U} \setminus \{e_0, \dots, e_p\}$. Then $(a_0, \dots, a_{p+1}) \in V(\xi_{p+1})$ and $i \mapsto f_{i,i}(a_0, \dots, a_i)$ is an injective map $\{0, \dots, p+1\} \to \mathcal{U}$. □

Corollary 3.6.5. *There exist sequences* $(a_i)_{i\in\mathbb{N}}$ *and* $(e_i)_{i\in\mathbb{N}}$ *of elements of* $\mathbf{k}_2$ *satisfying*:
(a) $f_{i,i}(a_0, \dots, a_i) = e_i$ *for each* $i \in \mathbb{N}$;
(b) $i \mapsto e_i$ *is an injective map from* $\mathbb{N}$ *to* $\mathcal{U}$;
(c) $a_0 = e_0$ *is transcendental over* $\mathbf{k}_0$.

Proof. By Paragraph 3.4 (v), we may pick $u_0 \in \mathcal{U}$ transcendental over $\mathbf{k}_0$; then choose $(a_i)_{i\in\mathbb{N}}$ satisfying conditions (a)–(c) of Lemma 3.6.4 and set $e_i = f_{i,i}(a_0,\dots,a_i)$ for each $i \in \mathbb{N}$. □

DEFINITION 3.6.6. Choose sequences $(a_i)_{i\in\mathbb{N}}$ and $(e_i)_{i\in\mathbb{N}}$ of elements of $\mathbf{k}_2$ satisfying the conditions of Corollary 3.6.5. Define $x = t^{-2}$ and $y = t^{-3}\sum_{n=0}^{\infty} a_n t^{3n} \in \mathbf{k}_2((t))$ and, for each $i \in \{0, 1, 2\}$, consider the subring $B_i = \mathbf{k}_i[x, y]$ of $\mathbf{k}_2((t))$ and the degree function $\deg_i \colon B_i \to \mathbb{Z} \cup \{-\infty\}$ defined by $\deg_i(f) = -\operatorname{ord}_t(f)$, for $f \in B_i$. Then $B_0 \subset B_1 \subset B_2$, $\deg_i$ is the restriction of $\deg_j$ when $i \leq j$, and (for each $i = 0,1,2$) $\deg_i(\lambda) = 0$ for all $\lambda \in \mathbf{k}_i^*$.

The notations of Definition 3.6.6 are fixed until the end of Paragraph 3.6. We will now show that x, y are algebraically independent over $\mathbf{k}_1$ and that $\deg_1$ has values in $\mathbb{N} \cup \{-\infty\}$. Let $\langle 2, 3\rangle$ denote the submonoid of $(\mathbb{Z}, +)$ generated by $\{2, 3\}$.

Lemma 3.6.7. $B_1 = \mathbf{k}_1^{[2]}$ *and* $\deg_1(f) \in \langle 2, 3\rangle$ *for all* $f \in B_1 \setminus \{0\}$.

Proof. Consider the subring R of $\mathbf{k}_1(X_0, X_1, \ldots)((t))$ whose elements are the series $\sum_{i\in\mathbb{Z}} f_i t^i$ satisfying $f_i \in \mathbf{k}_1[X_0, X_1, \ldots]$ for all $i \in \mathbb{Z}$ and $f_i = 0$ for $i \ll 0$, and the homomorphism of $\mathbf{k}_1$-algebras

$$\varphi\colon R \to \mathbf{k}_2((t)), \quad \sum_{i\in\mathbb{Z}} f_i(X_0, X_1, \ldots)t^i \mapsto \sum_{i\in\mathbb{Z}} f_i(a_0, a_1, \ldots)t^i.$$

As $\xi_p \in \Sigma_p \subset R$, we may define $y_p = \varphi(\xi_p) \in \mathbf{k}_2((t))$ for each $p \in \mathbb{N}$. Then $y_p = t^{-3}\sum_{n=0}^{\infty} f_{p,p+n}(a_0, \ldots, a_{p+n})t^{3n}$, so in particular

$$y_p = e_p t^{-3} + \text{higher powers of } t, \quad \text{for all} \quad p \in \mathbb{N}. \tag{11}$$

Note that $y_0 = t^{-3}\sum_{n=0}^{\infty} a_n t^{3n}$ and $y_{p+1} = \varphi(\xi_p^2 - f_{p,p}^2 t^{-6}) = y_p^2 - f_{p,p}(a_0,\ldots,a_p)^2 t^{-6}$, so

$$y_0 = y \quad \text{and} \quad y_{p+1} = y_p^2 - e_p^2 x^3 \quad \text{for all} \quad p \in \mathbb{N}. \tag{12}$$

As $e_p^2 \in \mathbf{k}_1$ for all p, this implies that $(y_p)_{p\in\mathbb{N}}$ is a sequence of elements of $B_1 = \mathbf{k}_1[x, y]$. Consider the polynomial ring $\mathbf{k}_1[X, Y] = \mathbf{k}_1^{[2]}$ and let $\pi : \mathbf{k}_1[X, Y] \to B_1$ be the $\mathbf{k}_1$-homomorphism sending X to x and Y to y. Also define the sequence $(F_p)_{p\in\mathbb{N}}$ of elements of $\mathbf{k}_1[X, Y]$ by $F_0 = Y$ and $F_{p+1} = F_p^2 - e_p^2 X^3$ ($p \in \mathbb{N}$). Then (12) implies that $\pi(F_p) = y_p$ for all $p \in \mathbb{N}$.

Given a finite subset $S = \{p_1, \ldots, p_r\}$ of $\mathbb{N}$ (with $p_1 < \cdots < p_s$), let $F_S = \prod_{i=1}^r F_{p_i} \in \mathbf{k}_1[X, Y]$, $y_S = \prod_{i=1}^r y_{p_i} \in \mathbf{k}_1[x, y]$, and $e_S = \prod_{i=1}^r e_{p_i} \in \mathbf{k}_2$ (in particular $F_\emptyset = 1$, $y_\emptyset = 1$ and $e_\emptyset = 1$). Then (11) implies that, given $\alpha(X) \in \mathbf{k}_1[X] \setminus \{0\}$,

$$\pi(\alpha(X)F_S) = \alpha(x)y_S = \lambda e_S t^m + \text{higher powers of } t, \tag{13}$$

for some $\lambda \in \mathbf{k}_1^*$ and $m \in \langle -2, -3\rangle$.

Let $G \in \mathbf{k}_1[X, Y] \setminus \{0\}$. By Lemma 3.2,

$$G = \alpha_1(X)F_{S_1} + \cdots + \alpha_N(X)F_{S_N},$$

where $N \geq 1$, $\alpha_i(X) \in \mathbf{k}_1[X] \setminus \{0\}$ for each i, and $S_1, \ldots, S_N$ are distinct finite subsets of $\mathbb{N}$. Then (13) gives

$$\pi(G) = \sum_{i=1}^{N} \alpha_i(x)y_{S_i} = \sum_{i=1}^{N} (\lambda_i e_{S_i} t^{m_i} + \text{higher powers of } t)$$

for some $\lambda_1, \ldots, \lambda_N \in \mathbf{k}_1^*$ and $m_1, \ldots, m_N \in \langle -2, -3\rangle$. By part (vi) of Paragraph 3.4 together with the fact that $p \mapsto e_p$ is injective, the elements $e_{S_1}, \ldots, e_{S_N}$ of $\mathbf{k}_2$ are linearly independent over $\mathbf{k}_1$; so $\pi(G) \neq 0$ and $\operatorname{ord}_t(\pi G) = \min\{m_1, \ldots, m_N\} \in \langle -2, -3\rangle$. It follows that $\pi : \mathbf{k}_1[X, Y] \to B_1$ is bijective, so $B_1 = \mathbf{k}_1^{[2]}$. We also obtain $\deg_1(f) = -\operatorname{ord}_t(f) \in \langle 2, 3\rangle$ for all $f \in B_1 \setminus \{0\}$, so the lemma is proved. $\square$

As $\mathbf{k}_2/\mathbf{k}_1$ is algebraic, Lemma 3.6.7 implies that x, y are algebraically independent over $\mathbf{k}_2$, so:

Corollary 3.6.8. $B_i = \mathbf{k}_i^{[2]}$ *for* $i = 0, 1, 2$.

Lemma 3.6.9. *Let A be a subring of $\mathbf{k}_1$ satisfying $u^2 \in A$ for all $u \in \mathcal{U}$. Consider the subring $A[x, y] = A^{[2]}$ of $B_1 = \mathbf{k}_1[x, y]$, the degree function* $\deg\colon A[x, y] \to \mathbb{N} \cup \{-\infty\}$ *defined by* $\deg(f) = -\operatorname{ord}_t(f)$, *and the A-derivation* $\partial/\partial y\colon A[x, y] \to A[x, y]$. *Then* $\deg(\partial/\partial y)$ *is not defined.*

Proof. Consider the sequence $(y_p)_{p\in\mathbb{N}}$ of elements of B_1 defined in the proof of Lemma 3.6.7. As $y_0 = y \in A[x, y]$ and $e_p^2 \in A$ for all $p \in \mathbb{N}$, (12) implies that $y_p \in A[x, y]$ for all $p \in \mathbb{N}$. Also, (11) shows that $\deg(y_p) = 3$ for all $p \in \mathbb{N}$. Write $D = \partial/\partial y$, then $D(y_{p+1}) = D(y_p^2 - e_p^2 x^3) = D(y_p^2) = 2y_p D(y_p)$, so $\deg(Dy_{p+1}) = 3 + \deg(Dy_p)$. Consequently, $\deg(Dy_p) = 3p$ and hence $\deg(Dy_p) - \deg(y_p) = 3p - 3$ for all $p \in \mathbb{N}$. So $\deg(D)$ is not defined. □

For each $i = 0,1,2$, define the $\mathbf{k}_i$-derivation $D_i = \partial/\partial y$: $B_i \to B_i$. By Lemma 3.6.9 we know that $\deg_1(D_1)$ is not defined, so in fact:

Corollary 3.6.10. $\deg_1(D_1)$ *and* $\deg_2(D_2)$ *are not defined.*

Lemma 3.6.11. $\{\deg_2(f) \mid f \in B_2 \setminus \{0\}\} = \mathbb{Z}$.

Proof. Consider the element $w = y^2 - a_0^2 x^3 - 2a_1 y - 2a_0 a_2 + a_1^2$ of $\mathbf{k}_2[x, y]$. Using $y = a_0 t^{-3} + a_1 + a_2 t^3 + \cdots$ and $x = t^{-2}$, we find $w = 2a_0 a_3 t^3 +$ higher powers of t, so $\operatorname{ord}_t(w) > 0$. Note that $w \neq 0$, since x, y are algebraically independent over $\mathbf{k}_2$. So $\deg(w)$ is a negative integer and consequently $\langle 2, 3, \deg(w)\rangle = \mathbb{Z}$, which proves the lemma. □

Lemma 3.6.12. $\operatorname{Gr}(B_1)$ *is not affine over* $\mathbf{k}_1$ *and* $\operatorname{Gr}(B_2)$ *is affine over* $\mathbf{k}_2$.

Proof. The fact that $\operatorname{Gr}(B_2)$ is affine over $\mathbf{k}_2$ follows from $\mathbf{k}_2 \subset B_2 \subset \mathbf{k}_2((t))$ and $\deg_2 = -\operatorname{ord}_t$, by Paragraph 2.2. Because $B_1 \not\subseteq \mathbf{k}_1((t))$, we cannot apply the same argument and show that $\operatorname{Gr}(B_1)$ is affine. In fact Theorem 1.7 (b) implies that $\operatorname{Gr}(B_1)$ is not affine over $\mathbf{k}_1$, because $\deg_1$ has values in $\mathbb{N}$ (Lemma 3.6.7) and $\deg_1(D_1)$ is not defined (Corollary 3.6.10). □

The fact that a_0 is transcendental over $\mathbf{k}_0$ (cf. Corollary 3.6.5 and Definition 3.6.6) played no role up to this point. It is needed for the following:

Lemma 3.6.13. *Let* $\mathfrak{g}$ *be the* $\mathbb{N}$*-grading* $B_0 = \mathbf{k}_0[x, y] = \bigoplus_{i\in\mathbb{N}} R_i$ *of* B_0 *defined by the conditions* $R_0 = \mathbf{k}_0$, $x \in R_2$ *and* $y \in R_3$. *Then* $\deg_0$ *is the degree function determined by* $\mathfrak{g}$. *Consequently,* $\deg_0(D_0)$ *is defined and* $\mathrm{Gr}(B_0)$ *is affine over* $\mathbf{k}_0$.

Proof. For each $i, j \in \mathbb{N}$,

$$x^i y^j = a_0^j t^{-2i-3j} + \text{higher powers of } t,$$

and a_0 is transcendental over $\mathbf{k}_0$. It easily follows that if S is a nonempty finite subset of $\mathbb{N}^2$ and $(\lambda_{ij})_{(i,j)\in S}$ is a family of elements of $\mathbf{k}_0 \setminus \{0\}$, then

$$\mathrm{ord}_t\Bigg(\sum_{(i,j)\in S} \lambda_{ij} x^i y^j\Bigg) = \min\{-2i - 3j \mid (i, j) \in S\},$$

or equivalently, $\deg_0\big(\sum_{(i,j)\in S} \lambda_{ij} x^i y^j\big) = \max\{2i+3j \mid (i, j) \in S\}$. So $\deg_0$ is the degree function determined by $\mathfrak{g}$. A straightforward calculation shows that $\deg_0(D_0)$ is defined and is equal to -3 (alternatively, $\deg_0(D_0)$ is defined by Theorem 1.7). Since $\deg_0$ is determined by a grading of B_0, we have $\mathrm{Gr}(B_0) \cong B_0$, so $\mathrm{Gr}(B_0)$ is affine. $\square$

3.7. Proof of Proposition 1.5. Let $\mathbf{k}_0$ be an uncountable field of characteristic zero, $\mathbf{k}_1$ a function field over $\mathbf{k}_0$ and $\mathbf{k}_2$ the algebraic closure of $\mathbf{k}_1$. By Lemma 3.5, there exists a set $\mathcal{U}$ such that $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$ satisfies the requirements of Paragraph 3.4; then all results of Paragraph 3.6 are valid when applied to $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$. Define the degree functions $\deg_i$ $(i = 0, 1, 2)$ as in Definition 3.6.6 and note that, by Lemma 3.6.7, $\deg_0$ and $\deg_1$ have values in $\mathbb{N} \cup \{-\infty\}$. Assertions (a) and (b) of Proposition 1.5 are clear, and (c), (d), (e) follow from Corollary 3.6.10, Lemmas 3.6.12 and 3.6.13 (note that, for each $i = 1, 2$, $\deg_i$ cannot be determined by a grading of B_i because that would imply that $\deg_i(D_i)$ is defined, by Theorem 1.7). $\square$

Corollary 3.8. *Let* A *be a domain which contains an uncountable field* $\mathbf{k}$ *of characteristic zero, and such that* $\mathrm{Frac}(A)$ *is a function field over* $\mathbf{k}$. *Consider* $A[X, Y] = A^{[2]}$ *and the* A*-derivation* $\partial/\partial Y : A[X, Y] \to A[X, Y]$. *Then there exists a degree function* $\deg : A[X, Y] \to \mathbb{N} \cup \{-\infty\}$ *such that* $\deg(a) = 0$ *for all* $a \in A \setminus \{0\}$, *and such that* $\deg(\partial/\partial Y)$ *is not defined.*

Proof. Let $\mathbf{k}_0 = \mathbf{k}$, $\mathbf{k}_1 = \mathrm{Frac}(A)$ and $\mathbf{k}_2$ the algebraic closure of $\mathbf{k}_1$. By Lemma 3.5, there exists a set $\mathcal{U}$ such that $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2, \mathcal{U})$ satisfies the requirements of Paragraph 3.4 and $u^2 \in A$ for all $u \in \mathcal{U}$. So we are done by Lemma 3.6.9. $\square$

3.9. Proof of Corollary 1.6. There exist an uncountable field $\mathbf{k}_0$ of characteristic zero and a function field $\mathbf{k}_1$ over $\mathbf{k}_0$ such that the algebraic closure of $\mathbf{k}_1$ is $\mathbb{C}$.

Then the triple $(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2 = \mathbb{C})$ satisfies the hypothesis of Proposition 1.5. Let us denote by d' the degree function $\deg_2\colon B_2 = \mathbb{C}[X, Y] \to \mathbb{Z} \cup \{-\infty\}$ given by Proposition 1.5. Also consider the grading of $\mathbb{C}[X, Y]$ defined by stipulating that X, Y are homogeneous of degrees 2 and 3 respectively, and let $d\colon \mathbb{C}[X, Y] \to \mathbb{Z} \cup \{-\infty\}$ be the degree function determined by this grading. Using Proposition 1.5, it is easily verified that $d|_{\mathbb{Q}[X,Y]} = d'|_{\mathbb{Q}[X,Y]}$ and that d' is wild over $\mathbb{C}$. By Theorem 1.7 (a), d is tame over $\mathbb{C}$. So d and d' satisfy the desired conditions. $\square$

4. Some positive results

We prove several results which assert that degree functions satisfying certain hypotheses are tame. The main results are Propositions 4.6, 4.21 and 4.24.

SETUP 4.1. Throughout Section 4 we consider a triple $(B, G, \deg)$ where B is a domain of characteristic zero, G be a totally ordered abelian group and $\deg\colon B \to G \cup \{-\infty\}$ a degree function.

4.2. Let $(B, G, \deg)$ be as in Setup 4.1. If $D\colon B \to B$ is a derivation, one defines an auxiliary map $\delta_D\colon B \to G \cup \{-\infty\}$ by $\delta_D(0) = -\infty$ and, given $x \in B \setminus \{0\}$, $\delta_D(x) = \deg(Dx) - \deg(x)$. Note that

$$\deg(Dx) = \delta_D(x) + \deg(x), \quad \text{for all} \quad x \in B.$$

We also define $\delta_D(S) \in G \cup \{-\infty\}$ for certain subsets S of B. If S is a nonempty subset of B such that the subset $U_S = \{\delta_D(x) \mid x \in S\}$ of $G \cup \{-\infty\}$ has a greatest element M, we define $\delta_D(S) = M$. If U_S does not have a greatest element, we leave $\delta_D(S)$ undefined. We also define $\delta_D(\emptyset) = -\infty$. Note in particular that $\delta_D(S)$ is defined for every finite subset S of B. If S_1, S_2 are subsets of B then the equality "$\delta_D(S_1) = \delta_D(S_2)$" is to be understood as meaning: either both $\delta_D(S_1)$ and $\delta_D(S_2)$ are undefined, or both are defined and are equal to the same element of $G \cup \{-\infty\}$. We also observe that the equality $\deg(D) = \delta_D(B)$ always holds (i.e., either both sides are undefined, or both sides are defined and are equal to the same element of $G \cup \{-\infty\}$).

Define the transitive relation $\preceq_D$ on the powerset $\mathcal{P}(B)$ of B by declaring that, for $S, S' \in \mathcal{P}(B)$,

$$S \preceq_D S' \iff \forall_{s \in S} \exists_{s' \in S'} \delta_D(s) \le \delta_D(s').$$

Then it is clear that

$$S \preceq_D S' \text{ and } S' \preceq_D S \Longrightarrow \delta_D(S) = \delta_D(S'). \tag{14}$$

Noting that $S \subseteq S'$ implies $S \preceq_D S'$, we obtain the following useful special case of (14):

$$S \subseteq S' \text{ and } S' \preceq_D S \Longrightarrow \delta_D(S) = \delta_D(S'). \tag{15}$$

DEFINITION 4.3. Let $(B, G, \deg)$ be as in Setup 4.1. By a 0-*subring* of B we mean a subring Z of B such that $\deg(x) = 0$ for all $x \in Z \setminus \{0\}$.

Lemma 4.4. *Let $(B, G, \deg)$ be as in* Setup 4.1. *Let $D\colon B \to B$ be a derivation and $x_1, \dots, x_n \in B$.*
(1) $\delta_D(x_1 x_2 \cdots x_n) \le \max_{1\le i\le n} \delta_D(x_i)$.
(2) *If* $\deg(x_1 + \cdots + x_n) = \max_{1\le i\le n} \deg(x_i)$, *then* $\delta_D(x_1 + \cdots + x_n) \le \max_{1\le i\le n} \delta_D(x_i)$.
(3) *If $x_1,\dots,x_n \in Z$ for some* 0*-subring Z of B, then* $\delta_D(x_1+\cdots+x_n) \le \max_{1\le i\le n}\delta_D(x_i)$.

Proof. We write $\delta = \delta_D$. Given $x, y \in B \setminus \{0\}$,

$$\begin{aligned}\delta(xy) &= \deg(D(xy)) - \deg(xy) = \deg(y\,Dx + x\,Dy) - \deg(xy)\\ &\le \max(\deg(y\,Dx), \deg(x\,Dy)) - \deg(xy)\\ &= \max(\deg(Dx) + \deg(y), \deg(Dy) + \deg(x)) - \deg(xy) = \max(\delta(x), \delta(y));\end{aligned}$$

assertion (1) follows by induction.

If $\deg(x_1 + \cdots + x_n) = \max_{1\le i\le n} \deg(x_i)$ then

$$\begin{aligned}\deg\left(D\left(\sum_i x_i\right)\right) &= \deg\left(\sum_i Dx_i\right) \le \max_i(\deg(Dx_i)) = \max_i(\delta(x_i) + \deg(x)_i)\\ &\le \max_i \delta(x_i) + \max_i \deg(x)_i = \max_i \delta(x_i) + \deg\left(\sum_i x_i\right),\end{aligned}$$

so assertion (2) holds. Assertion (3) immediately follows. □

Lemma 4.5. *Let $(B, G, \deg)$ be as in* Setup 4.1 *and let $A \subseteq Z$ be* 0*-subrings of B. Suppose that S is a subset of Z such that Z is algebraic over $A[S]$. Then, for all $D \in \mathrm{Der}_A(B)$, $\delta_D(Z) = \delta_D(S)$.*

Proof. Let $D \in \mathrm{Der}_A(B)$ and let $\delta = \delta_D$. Consider a product

$$\mu = ax_1 \cdots x_n \quad (\text{with } a \in A \text{ and } x_1, \dots, x_n \in S). \tag{16}$$

As $\delta(a) = -\infty$, we have $\delta(ax_1 \cdots x_n) \le \max(\delta(a), \delta(x_1), \dots, \delta(x_n)) = \max_i \delta(x_i)$ by Lemma 4.4, so $\delta(\mu) \le \delta(s)$ for some $s \in S$. Now consider an element $\xi \in A[S]$. Then ξ is a finite sum, $\xi = \mu_1 + \cdots + \mu_m$, where each μ_i is a product of the form (16); so, for each $i \in \{1, \dots, m\}$, there exists $s_i \in S$ such that $\delta(\mu_i) \le \delta(s_i)$; consequently we may choose $s \in S$ such that $\delta(\mu_i) \le \delta(s)$ holds for all $i \in \{1, \dots, m\}$. As $\mu_1, \dots, \mu_m \in Z$, part (3) of Lemma 4.4 gives $\delta(\xi) \le \max_i \delta(\mu_i)$, so $\delta(\xi) \le \delta(s)$. This shows that $A[S] \preceq_D S$.

Let $b \in Z \setminus \{0\}$. As b is algebraic over $R = A[S]$, we may choose a polynomial $\Phi(T) = \sum_i r_i T^i \in R[T] \setminus \{0\}$ (where T is an indeterminate and $r_i \in R$) of minimal degree such that $\Phi(b) = 0$. Then $0 = D\Phi(b) = \Phi^{(D)}(b) + \Phi'(b)Db$ and (using $\operatorname{char} B = 0$) $\Phi'(b) \in Z \setminus \{0\}$ imply $\deg(\Phi^{(D)}(b)) = \deg(Db) = \delta(b)$. Now $\Phi^{(D)}(b) = \sum_i D(r_i)b^i$, so

$$\delta(b) = \deg(\Phi^{(D)}(b)) = \deg\left(\sum_i D(r_i)b^i\right) \le \max_i \deg(D(r_i)b^i)$$

and $\deg(D(r_i)b^i) = \deg(Dr_i) = \delta(r_i)$ for each i, so $\delta(b) \le \max_i \delta(r_i)$. It follows that there exists $r \in A[S]$ such that $\delta(b) \le \delta(r)$, i.e., we have shown that $Z \preceq_D A[S]$. We get $Z \preceq_D S$ by transitivity and $\delta(Z) = \delta(S)$ by assertion (15) of Paragraph 4.2. $\square$

Proposition 4.6. *Let G be a totally ordered abelian group, $B = \bigoplus_{i \in G} B_i$ a G-graded integral domain of characteristic zero and* $\deg\colon B \to G \cup \{-\infty\}$ *the degree function determined by the grading. Assume that B is finitely generated as a B_0-algebra and let A be a subring of B_0 satisfying* $\operatorname{trdeg}_A(B_0) < \infty$. *Then* $\deg$ *is tame over A.*

More precisely, given any choice of $z_1, \dots, z_m \in B_0$ and homogeneous $x_1, \dots, x_n \in B$ such that B_0 is algebraic over $A[z_1, \dots, z_m]$ and $B = B_0[x_1, \dots, x_n]$,

$$\deg(D) = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\} \quad \text{for all} \quad D \in \operatorname{Der}_A(B).$$

Proof. Let $z_1, \dots, z_m \in B_0$ and $x_1, \dots, x_n \in B$ be as in the statement. Let $D \in \operatorname{Der}_A(B)$ and let $\delta = \delta_D$. Define $M = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\}$ (so $M \in G \cup \{-\infty\}$). It suffices to show that $\delta(x) \le M$ for all $x \in B \setminus \{0\}$. Indeed, if this is true then $\deg(D) = M$.

Lemma 4.5 (with $S = \{z_1, \dots, z_m\}$ and $Z = B_0$) implies that $\delta(B_0) = \max_{1 \le i \le m} \delta(z_i)$, so $\delta(b) \le M$ certainly holds for all $b \in B_0$.

Let $x \in B \setminus \{0\}$. Then x is a finite sum, $x = h_1 + \cdots + h_m$, where each h_i is homogeneous and $\deg(h_1) < \cdots < \deg(h_m)$. Then $\deg(h_1 + \cdots + h_m) = \max_i \deg(h_i)$, so part (2) of Lemma 4.4 implies that $\delta(h_1 + \cdots + h_m) \le \max_i \delta(h_i)$. So it's enough to show that $\delta(h_i) \le M$ for all i, i.e., we may assume that x is homogeneous.

Suppose that $x \in B_d \setminus \{0\}$, for some $d \in G$. Then x is a finite sum, $x = \mu_1 + \cdots + \mu_m$, where each $\mu_i \in B_d \setminus \{0\}$ is a monomial of the form $\mu_i = b_i x_1^{e_{i1}} \cdots x_n^{e_{in}}$ with $b_i \in B_0$ and $e_{ij} \in \mathbb{N}$. We have $\deg(\mu_1 + \cdots + \mu_m) = \max_i \deg(\mu_i)$, so part (2) of Lemma 4.4 implies that $\delta(x) = \delta(\mu_1 + \cdots + \mu_m) \le \max_i \delta(\mu_i)$. So it's enough to show that $\delta(\mu_i) \le M$ for all i. Part (1) of Lemma 4.4 gives $\delta(\mu_i) \le \max(\delta(b_i), \delta(x_1), \dots, \delta(x_n))$, so $\delta(\mu_i) \le M$ and we are done. $\square$

Corollary 4.7. *Let G be a totally ordered abelian group, $B = \bigoplus_{i \in G} B_i$ a G-graded integral domain of characteristic zero and* $\deg\colon B \to G \cup \{-\infty\}$ *the degree function determined by the grading. Assume*:

(1) *B has finite transcendence degree over a field* $\mathbf{k}$;
(2) *B is finitely generated as a B_0-algebra.*
Then deg *is tame over* $\mathbf{k}$.

More precisely, given any choice of $z_1,\dots,z_m \in B_0$ and homogeneous $x_1,\dots,x_n \in B$ such that B_0 is algebraic over $\mathbf{k}[z_1,\dots,z_m]$ and $B = B_0[x_1,\dots,x_n]$,

$$\deg(D) = \max\{\delta_D(z_1),\dots,\delta_D(z_m),\delta_D(x_1),\dots,\delta_D(x_n)\} \quad \textit{for all} \quad D \in \mathrm{Der}_{\mathbf{k}}(B).$$

Proof. As $\mathbf{k}$ is necessarily included in B_0, this is Proposition 4.6 with $A = \mathbf{k}$. □

The next two results are consequences of Corollary 4.7.

Corollary 4.8. *Let* $\mathbf{k}$ *be a field of characteristic zero, B a* $\mathbf{k}$*-affine integral domain and G a totally ordered abelian group. If* $\deg\colon B \to G \cup \{-\infty\}$ *is the degree function determined by some G-grading of B, then* deg *is tame over* $\mathbf{k}$.

More precisely, given any choice of homogeneous elements $x_1,\dots,x_n \in B$ satisfying $B = \mathbf{k}[x_1,\dots,x_n]$, we have $\deg(D) = \max_{1\le i\le n}\delta_D(x_i)$ *for all* $D \in \mathrm{Der}_{\mathbf{k}}(B)$.

Proof. Fix a grading $B = \bigoplus_{i\in G} B_i$ which determines deg and note that $\mathbf{k} \subseteq B_0$. Given homogeneous elements $x_1,\dots,x_n \in B$ satisfying $B = \mathbf{k}[x_1,\dots,x_n]$, it is certainly the case that $B = B_0[x_1,\dots,x_n]$. We may also choose $z_1,\dots,z_m \in B_0$ such that each z_i is a monomial of the form $x_1^{e_{i1}}\cdots x_n^{e_{in}}$ ($e_{ij} \in \mathbb{N}$) and B_0 is algebraic over $\mathbf{k}[z_1,\dots,z_m]$. Let $D \in \mathrm{Der}_{\mathbf{k}}(B)$, then $\deg(D) = \max\{\delta_D(z_1),\dots,\delta_D(z_m),\delta_D(x_1),\dots,\delta_D(x_n)\}$ by Corollary 4.7. Part (1) of Lemma 4.4 gives $\delta_D(z_i) \le \max_{1\le j\le n}\delta_D(x_j)$, so $\deg(D) = \max_{1\le i\le n}\delta_D(x_i)$. □

Corollary 4.9. *Let R be a domain of finite transcendence degree over a field* $\mathbf{k}$ *of characteristic zero and let $B = R[X_1,\dots,X_n] = R^{[n]}$. Let G be a totally ordered abelian group and define a G-grading on B by choosing $(d_1,\dots,d_n) \in G^n$ and declaring that the elements of $R \setminus \{0\}$ are homogeneous of degree* 0 *and that (for each i) X_i is homogeneous of degree d_i. Let* $\deg\colon B \to G \cup \{-\infty\}$ *be the degree function determined by this grading. Then* deg *is tame over* $\mathbf{k}$.

More precisely, if $z_1,\dots,z_m \in R$ are such that R is algebraic over $\mathbf{k}[z_1,\dots,z_m]$, then $\deg(D) = \max\{\delta_D(z_1),\dots,\delta_D(z_m),\delta_D(X_1),\dots,\delta_D(X_n)\}$ *for every* $D \in \mathrm{Der}_{\mathbf{k}}(B)$.

Proof. Let $B = \bigoplus_{i\in G} B_i$ be the grading and choose $\xi_1,\dots,\xi_N \in B_0$ such that each ξ_i is a monomial of the form $X_1^{e_{i1}}\cdots X_n^{e_{in}}$ ($e_{ij} \in \mathbb{N}$) and B_0 is algebraic over $R[\xi_1,\dots,\xi_N]$; then B_0 is algebraic over $\mathbf{k}[z_1,\dots,z_m,\xi_1,\dots,\xi_N]$ and $B = B_0[X_1,\dots,X_n]$. If $D \in \mathrm{Der}_{\mathbf{k}}(B)$ then, by Corollary 4.7,

$$\deg(D) = \max\{\delta_D(z_1),\dots,\delta_D(z_m),\delta_D(\xi_1),\dots,\delta_D(\xi_N),\delta_D(X_1),\dots,\delta_D(X_n)\}.$$

Part (1) of Lemma 4.4 gives $\delta_D(\xi_i) \le \max_{1\le j\le n}\delta_D(X_j)$, so we are done. □

Paragraph 4.10 and Lemma 4.11 are simple observations about localization of degree functions. These facts are used in the proofs of[4] Corollaries 4.12, 4.14 and 4.24. Note that Lemma 4.11 appeared in [6].

4.10. Let B be a domain, G a totally ordered abelian group and $\deg\colon B \to G \cup \{-\infty\}$ a degree function. If $S \subseteq B \setminus \{0\}$ is a multiplicative set, then deg has a unique extension to a degree function $\mathrm{DEG}\colon S^{-1}B \to G \cup \{-\infty\}$. Indeed, it is easily verified that the map DEG defined by $\mathrm{DEG}(0) = -\infty$ and $\mathrm{DEG}(x/s) = \deg(x) - \deg(s)$ (for $x \in B \setminus \{0\}$ and $s \in S$) is a well-defined degree function and is the unique extension of deg.

Lemma 4.11. *Let B be a domain of characteristic zero, $S \subseteq B \setminus \{0\}$ a multiplicative set, G a totally ordered abelian group, and $\deg\colon B \to G \cup \{-\infty\}$ and $\mathrm{DEG}\colon S^{-1}B \to G \cup \{-\infty\}$ degree functions such that* deg *is the restriction of* DEG. *Consider $D \in \mathrm{Der}(B)$ and its extension $S^{-1}D \in \mathrm{Der}(S^{-1}B)$. Then $\deg(D)$ is defined if and only if $\mathrm{DEG}(S^{-1}D)$ is defined, and if both degrees are defined then they are equal.*

Proof. As $\delta_D\colon B \to G \cup \{-\infty\}$ is the restriction of $\delta_{S^{-1}D}\colon S^{-1}B \to G \cup \{-\infty\}$, we have $U \subseteq U'$, where we define $U = \{\delta_D(x) \mid x \in B\}$ and $U' = \{\delta_{S^{-1}D}(x) \mid x \in S^{-1}B\}$. We first observe that if $s \in S$ then

(17)
$$\begin{aligned}\delta_{S^{-1}D}(1/s) &= \mathrm{DEG}((S^{-1}D)(1/s)) - \mathrm{DEG}(1/s)\\ &= \mathrm{DEG}(-D(s)/s^2) - \mathrm{DEG}(1/s) = \deg(Ds) - 2\deg(s) + \deg(s) = \delta_D(s).\end{aligned}$$

Applying part (1) of Lemma 4.4 to $\delta_{S^{-1}D}$ gives, for any $x \in B$ and $s \in S$,

$$\delta_{S^{-1}D}(x/s) = \delta_{S^{-1}D}(x(1/s)) \le \max(\delta_{S^{-1}D}(x), \delta_{S^{-1}D}(1/s)) = \max(\delta_D(x), \delta_D(s)) \in U.$$

This shows that $\forall_{u' \in U'} \exists_{u \in U} u' \le u$. This, together with $U \subseteq U'$, proves the lemma. $\square$

Recall that if B is a domain of characteristic zero then each locally nilpotent derivation $\Delta\colon B \to B$ determines a degree function $\deg_\Delta\colon B \to \mathbb{N} \cup \{-\infty\}$ (cf. for instance [3, 1.1.7]).

Corollary 4.12. *Let B be a domain of finite transcendence degree over a field* **k** *of characteristic zero. Let $\deg_\Delta\colon B \to \mathbb{N} \cup \{-\infty\}$ be the degree function determined by a locally nilpotent derivation $\Delta\colon B \to B$. Then $\deg_\Delta$ is tame over* **k**.

Moreover, if $t \in B$ is such that $\Delta(t) \neq 0$ and $\Delta^2(t) = 0$, and $z_1, \dots, z_m \in \ker\Delta$ are such that $\ker\Delta$ is algebraic over $\mathbf{k}[z_1, \dots, z_m]$, then for each $D \in \mathrm{Der}_{\mathbf{k}}(B)$

$$\deg_\Delta(D) = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(t)\}. \tag{18}$$

[4] Lemma 4.11 would also be used for proving Lemma 1.8, but this proof is omitted.

Proof. Let $A = \ker \Delta$, $\alpha = \Delta(t) \in A \setminus \{0\}$ and $S = \{\alpha^n \mid n \in \mathbb{N}\}$. Then $S^{-1}B = (S^{-1}A)[t] = (S^{-1}A)^{[1]}$ (cf. for instance [3, p. 27]). Moreover, if $\mathrm{DEG}\colon S^{-1}B \to \mathbb{N} \cup \{-\infty\}$ denotes the t-degree then $\deg_\Delta$ is the restriction of DEG, so the hypothesis of Lemma 4.11 is satisfied. Apply either Corollaries 4.7 or 4.9 to DEG and $S^{-1}D \in \mathrm{Der}_{\mathbf{k}}(S^{-1}B)$: as $S^{-1}A$ is algebraic over $\mathbf{k}[z_1, \dots, z_m]$, $\mathrm{DEG}(S^{-1}D) = \max\{\delta_{S^{-1}D}(z_1), \dots, \delta_{S^{-1}D}(z_m), \delta_{S^{-1}D}(t)\}$. We have $\deg_\Delta(D) = \mathrm{DEG}(S^{-1}D)$ by Lemma 4.11, so we are done. □

REMARK 4.13. Let the notations and assumptions be as in Corollary 4.12. Then (18) can be rewritten (thanks to Lemma 4.5) as

$$\deg_\Delta(D) = \max\{\delta_D(\ker \Delta), \delta_D(t)\}.$$

However, if we suppose that $\ker D \neq \ker \Delta$ then Corollary 2.16 on p. 42 of [3] asserts that $\deg_\Delta(D) = \delta_D(\ker \Delta)$; this last claim is not correct, as shown by the following example. Let $B = \mathbf{k}[z, t] = \mathbf{k}^{[2]}$, $\Delta = \partial/\partial t$ and $D = z\,\partial/\partial z + t^2\,\partial/\partial t$. As $\Delta(t) \neq 0$, $\Delta^2(t) = 0$ and $\ker \Delta = \mathbf{k}[z]$, (18) gives $\deg_\Delta(D) = \max\{\delta_D(z), \delta_D(t)\} = \max\{0, 1\} = 1$, while $\delta_D(\ker \Delta) = \delta_D(\mathbf{k}[z]) = 0$.

Here is another common situation where Lemma 4.11 is useful (compare with Lemma 1.8):

Corollary 4.14. *Let $L = \mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be the ring of Laurent polynomials in n variables over a field $\mathbf{k}$ of characteristic zero, let $\mathfrak{g}$ be a G-grading of L where G is some totally ordered abelian group, and let $\deg_{\mathfrak{g}}\colon L \to G \cup \{-\infty\}$ be the degree function determined by $\mathfrak{g}$. Let B be a ring such that $\mathbf{k}[X_1, \dots, X_n] \subseteq B \subseteq L$ and let $\deg\colon B \to G \cup \{-\infty\}$ be the restriction of $\deg_{\mathfrak{g}}$. Then deg is tame over $\mathbf{k}$. Moreover, if we also assume that each X_i is a $\mathfrak{g}$-homogeneous element of L then $\deg(D) = \max_{1 \le i \le n} \delta_D(X_i)$ for all $D \in \mathrm{Der}_{\mathbf{k}}(B)$.*

Proof. Let $S = \{X_1^{e_1} \cdots X_n^{e_n} \mid (e_1, \dots, e_n) \in \mathbb{N}^n\}$ and note that $S^{-1}B = L$. Let $D \in \mathrm{Der}_{\mathbf{k}}(B)$. By Corollary 4.8, $\deg_{\mathfrak{g}}(S^{-1}D)$ is defined; by Lemma 4.11, it follows that $\deg(D)$ is defined and $\deg(D) = \deg_{\mathfrak{g}}(S^{-1}D)$; in particular, deg is tame over $\mathbf{k}$. Under the additional assumption that each X_i is homogeneous, Corollary 4.8 gives

$$\begin{aligned} \deg_{\mathfrak{g}}(S^{-1}D) &= \max\{\delta_{S^{-1}D}(X_1), \delta_{S^{-1}D}(1/X_1), \dots, \delta_{S^{-1}D}(X_n), \delta_{S^{-1}D}(1/X_n)\} \\ &= \max\{\delta_D(X_1), \dots, \delta_D(X_n)\}, \end{aligned}$$

where for the last equality we used that $\delta_{S^{-1}D}(1/X_i) = \delta_D(X_i)$ for each i (see (17) in the proof of Lemma 4.11). We already noted that $\deg(D) = \deg_{\mathfrak{g}}(S^{-1}D)$, so we are done. □

Finite generation of the associated graded ring

We shall now study triples $(B, G, \deg)$ as in Setup 4.1 which satisfy the additional condition that $\mathrm{Gr}(B)$ is a finitely generated algebra over a zero-subring (as explained in Lemma 4.18, below). For this type of consideration, the following device is useful.

DEFINITION 4.15. Let $(B, G, \deg)$ be as in Setup 4.1.
(1) By a *subpair* of $(B, \mathrm{Gr}(B))$, we mean a pair $(A, \bar{A})$ where A is a subset of B, $1 \in A$, $\bar{A}$ is a homogeneous subring of $\mathrm{Gr}(B)$ and:

(†) Each homogeneous element of $\bar{A}$ is of the form $\mathrm{gr}(a)$ for some $a \in A$.

(2) Let $D \in \mathrm{Der}(B)$. By a *D-subpair* of $(B, \mathrm{Gr}(B))$, we mean a subpair $(A, \bar{A})$ of $(B, \mathrm{Gr}(B))$ such that $\delta_D(A)$ is defined.
(3) If $(A, \bar{A})$ is a subpair of $(B, \mathrm{Gr}(B))$ and $x \in B$, we define

$$(A, \bar{A})_x = (A_x, \bar{A}[\mathrm{gr}(x)]),$$

where A_x is the collection of all elements $b \in B$ which can be written in the form $b = \sum_{i=0}^m a_i x^i$ for some $m \in \mathbb{N}$ and $a_0, \dots, a_m \in A$ satisfying:

(‡) $$\deg(a_j x^j) = \deg(b) \quad \text{for all} \quad j \quad \text{such that} \quad a_j \neq 0.$$

REMARK 4.16. As in part (3) of Definition 4.15, consider a subpair $(A, \bar{A})$ of $(B, \mathrm{Gr}(B))$ and $x \in B$. Then $A \cup \{x\} \subseteq A_x \subseteq R[x]$, where R is the subring of B generated by A.

Lemma 4.17. *Let* $(B, G, \deg)$ *be as in* Setup 4.1.
(1) *If* $(A, \bar{A})$ *is a subpair of* $(B, \mathrm{Gr}(B))$, *then so is* $(A, \bar{A})_x$ *for each* $x \in B$.
(2) *Let* $D \in \mathrm{Der}(B)$. *If* $(A, \bar{A})$ *is a D-subpair of* $(B, \mathrm{Gr}(B))$, *then so is* $(A, \bar{A})_x$ *for each* $x \in B$. *Moreover,* $\delta_D(A_x) = \delta_D(A \cup \{x\})$.

Proof. Let $(A, \bar{A})$ be a subpair of $(B, \mathrm{Gr}(B))$, let $x \in B$, and consider $(A, \bar{A})_x = (A_x, \bar{A}[\mathrm{gr}(x)])$; we show that $(A, \bar{A})_x$ is a subpair of $(B, \mathrm{Gr}(B))$. We may assume that $x \neq 0$, because $(A, \bar{A})_0 = (A, \bar{A})$. As $1 \in A$ and $A \subseteq A_x$, we have $1 \in A_x$. We have to show that if $\bar{y}$ is a homogeneous element of $\bar{A}[\mathrm{gr}(x)]$ then $\bar{y} = \mathrm{gr}(y)$ for some $y \in A_x$. Note that this is clear if $\bar{y} = 0$ (because (†) implies $0 \in A$, hence $0 \in A_x$), so assume $\bar{y} \neq 0$. We have

$$\bar{y} = \sum_{i=0}^{m} \bar{a}_i \, \mathrm{gr}(x)^i$$

for some $m \in \mathbb{N}$ and some homogeneous elements $\bar{a}_0, \dots, \bar{a}_m \in \bar{A}$ satisfying

(19) $$\deg(\bar{a}_j \, \mathrm{gr}(x)^j) = \deg(\bar{y}) \quad \text{for all} \quad j \quad \text{such that} \quad \bar{a}_j \neq 0.$$

By (†), there exist $a_0, \dots, a_m \in A$ such that $\mathrm{gr}(a_i) = \bar{a}_i$ for all i. Since $\deg(a_j x^j) = \deg(\mathrm{gr}(a_j x^j)) = \deg(\bar{a}_j\, \mathrm{gr}(x)^j)$, (19) implies that $\deg(a_j x^j) = \deg(\bar{y})$ whenever $a_j \neq 0$. Consequently,

$$\bar{y} = \sum_{i=0}^{m} \mathrm{gr}(a_i x^i) = \begin{cases} \mathrm{gr}\left(\sum_{i=0}^{m} a_i x^i\right), & \text{if} \quad \deg\left(\sum_{i=0}^{m} a_i x^i\right) = \deg(\bar{y}), \\ 0, & \text{if} \quad \deg\left(\sum_{i=0}^{m} a_i x^i\right) < \deg(\bar{y}). \end{cases}$$

Since $\bar{y} \neq 0$, it follows that $\deg\left(\sum_{i=0}^{m} a_i x^i\right) = \deg(\bar{y})$ (so $\sum_{i=0}^{m} a_i x^i \in A_x$) and that $\bar{y} = \mathrm{gr}\left(\sum_{i=0}^{m} a_i x^i\right)$, so $\bar{y} = \mathrm{gr}(y)$ for some $y \in A_x$. So $(A, \bar{A})_x$ is indeed a subpair of $(B, \mathrm{Gr}(B))$, and assertion (1) is proved.

Let $D \in \mathrm{Der}(B)$, assume that $(A, \bar{A})$ is a D-subpair of $(B, \mathrm{Gr}(B))$ and let $x \in B$. To show that $(A, \bar{A})_x$ is a D-subpair of $(B, \mathrm{Gr}(B))$, we have to show that $\delta_D(A_x)$ is defined. We may assume that $x \neq 0$. Let $y \in A_x$; then we may write $y = \sum_{i=0}^{m} a_i x^i$ for some $m \in \mathbb{N}$ and $a_0, \dots, a_m \in A$ such that (‡) holds, i.e.,

$$\deg(a_j x^j) = \deg(y) \quad \text{whenever} \quad a_j \neq 0.$$

Write $f(X) = \sum_{i=0}^{m} a_i X^i$; then $y = f(x)$ and $Dy = f^{(D)}(x) + f'(x)\, Dx$.

If j is such that $a_j \neq 0$ then

$$\deg(D(a_j)x^j) = \delta_D(a_j) + \deg(a_j) + \deg(x^j) = \delta_D(a_j) + \deg(a_j x^j) = \delta_D(a_j) + \deg(y),$$

so $\deg(f^{(D)}(x)) \leq \delta_D(\alpha) + \deg(y)$ for some $\alpha \in A$. Also, if $j > 0$ is such that $a_j \neq 0$ then

$$\deg(j a_j x^{j-1} D(x)) = \deg(a_j x^j) - \deg(x) + \deg(D(x)) = \deg(y) + \delta_D(x),$$

so $\deg(f'(x)D(x)) \leq \deg(y) + \delta_D(x)$. Thus,

$$\begin{aligned} \delta_D(y) + \deg(y) = \deg(D(y)) &\leq \max(\deg(f^{(D)}(x)), \deg(f'(x)D(x))) \\ &\leq \max(\delta_D(\alpha) + \deg(y), \deg(y) + \delta_D(x)) \end{aligned}$$

and it follows that $\delta_D(y) \leq \max(\delta_D(\alpha), \delta_D(x))$. We have shown that $A_x \preceq_D A \cup \{x\}$. As $1 \in A$, we have $x \in A_x$ and hence $A \cup \{x\} \subseteq A_x$. Thus

$$\delta_D(A_x) = \delta_D(A \cup \{x\}), \tag{20}$$

by Paragraph 4.2. As $\delta_D(A)$ is defined, so is $\delta_D(A \cup \{x\})$; so, by (20), $\delta_D(A_x)$ is defined. This proves assertion (2). □

Lemma 4.18. *Given* $(B, G, \deg)$ *as in* Setup 4.1, *the following hold.*
(1) *For any* 0*-subring* Z *of* B (*cf.* Definition 4.3), $\mathrm{Gr}(B)$ *is a* Z*-algebra.*
(2) *If* $\deg(x) \geq 0$ *for all* $x \in B \setminus \{0\}$ *then the subring* $Z = \{x \in B \mid \deg(x) \leq 0\}$ *of* B *is in fact a* 0*-subring of* B, *and is factorially closed in* B. *By* (1), *it follows that* $\mathrm{Gr}(B)$ *is a* Z*-algebra.*
(3) *If* $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of* G *then* $\deg(x) \geq 0$ *for all* $x \in B \setminus \{0\}$, *i.e., the hypothesis of* (2) *is satisfied.*

Proof. Let Z be a 0-subring of B. If B_i, B_{i^-} and $B_{[i]}$ are defined as in Paragraph 1.9 then the composite $Z \hookrightarrow B_0 \to B_{[0]} \hookrightarrow \mathrm{Gr}(B)$ is an injective homomorphism of rings $Z \to \mathrm{Gr}(B)$, $z \mapsto \mathrm{gr}(z)$. This defines a structure of Z-algebra on $\mathrm{Gr}(B)$: if $z \in Z$ and $\xi \in \mathrm{Gr}(B)$, then $z\xi = \mathrm{gr}(z)\xi$. Assertions (2) and (3) are trivial. $\square$

Lemma 4.19. *Let* $(B, G, \deg)$ *be as in* Setup 4.1 *and let* Z *be a* 0*-subring of* B. *Assume that* $\mathrm{Gr}(B)$ *is finitely generated as a* Z*-algebra* (*cf.* Lemma 4.18) *and consider elements* $x_1, \dots, x_n \in B$ *satisfying* $\mathrm{Gr}(B) = Z[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$.
(1) *There exists a set* E *satisfying* $Z \cup \{x_1, \dots, x_n\} \subseteq E \subseteq Z[x_1, \dots, x_n]$ *and*:
(a) $\forall_{x \in B\setminus\{0\}} \exists_{e \in E} \deg(x - e) < \deg(x)$
(b) $\delta_D(E) = \delta_D(Z \cup \{x_1, \dots, x_n\})$ *for every* $D \in \mathrm{Der}(B)$ *such that* $\delta_D(Z)$ *is defined.*
(2) *If* $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of* G *then* $B = Z[x_1, \dots, x_n]$.

Proof. We have $\mathrm{Gr}(B) = Z[\bar{x}_1, \dots, \bar{x}_n]$, where we define $\bar{x}_i = \mathrm{gr}(x_i)$ for all i. Define $A_0 = Z \subseteq B$ and $\bar{A}_0 = \{\mathrm{gr}(z) \mid z \in Z\} \subseteq \mathrm{Gr}(B)$, and note that $(A_0, \bar{A}_0)$ is a subpair of $(B, \mathrm{Gr}(B))$. For $1 \leq i \leq n$, define $(A_i, \bar{A}_i) = (A_{i-1}, \bar{A}_{i-1})_{x_i}$; then set $E = A_n$ and note that $Z \cup \{x_1, \dots, x_n\} \subseteq E \subseteq Z[x_1, \dots, x_n]$, by Remark 4.16. Also, $(A_n, \bar{A}_n)$ is (by Lemma 4.17) a subpair of $(B, \mathrm{Gr}(B))$ and $\bar{A}_n = \bar{A}_0[\bar{x}_1, \dots, \bar{x}_n] = Z[\bar{x}_1, \dots, \bar{x}_n] = \mathrm{Gr}(B)$; it follows that each homogeneous element of $\mathrm{Gr}(B) = \bar{A}_n$ is of the form $\mathrm{gr}(e)$ for some $e \in E = A_n$; so E satisfies condition (a). Let $D \in \mathrm{Der}(B)$ be such that $\delta_D(Z)$ is defined; then $(A_0, \bar{A}_0)$ is a D-subpair of $(B, \mathrm{Gr}(B))$; so, by repeated application of Lemma 4.17, $(A_n, \bar{A}_n)$ is a D-subpair of $(B, \mathrm{Gr}(B))$ and $\delta_D(E) = \delta_D(A_n) = \delta_D(Z \cup \{x_1, \dots, x_n\})$. So E satisfies (b).

We prove (2) by contradiction: assume that $\{\deg(x) \mid x \in B \setminus \{0\}\}$ is well-ordered and $B \neq Z[x_1, \dots, x_n]$. Pick $b_0 \in B \setminus Z[x_1, \dots, x_n]$ such that $\deg(b_0)$ is the least element of $\{\deg(x) \mid x \in B \setminus Z[x_1, \dots, x_n]\}$. Then there exists $e \in E \subseteq Z[x_1, \dots, x_n]$ such that $\deg(b_0 - e) < \deg(b_0)$, and this leads to a contradiction. So $B = Z[x_1, \dots, x_n]$. $\square$

REMARK 4.20. The assumption that $\{\deg(x) \mid x \in B \setminus \{0\}\}$ is well-ordered, in Lemma 4.19 (2), is needed. Indeed, consider $B = \mathbf{k}[x, y]$ and $\deg \colon B \to \mathbb{Z} \cup \{-\infty\}$ as in the proof of Proposition 1.2. Then $\deg(x) = 1$ and $\deg(y - a_0) = -k$ where $k \geq 1$. Define $x_1 = x$ and $x_2 = x^{2k-1}(y - a_0)^2$, then $\deg(x_1) = 1$ and $\deg(x_2) = -1$, so $\mathrm{Gr}(B) = \mathbf{k}[\mathrm{gr}(x_1), \mathrm{gr}(x_2)]$ (because $\mathrm{Gr}(B) \cong \mathbf{k}[t, t^{-1}]$). However, $B \neq \mathbf{k}[x_1, x_2]$.

Proposition 4.21. *Let* $(B, G, \deg)$ *be as in* Setup 4.1 *and suppose that*
(1) $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of* G;
(2) $\mathrm{Gr}(B)$ *is finitely generated as a* Z*-algebra*,
where $Z = \{x \in B \mid \deg(x) \leq 0\}$ (*cf.* Lemma 4.18). *Then the following hold*:
(3) B *is finitely generated as a* Z*-algebra*;
(4) *if* A *is a subring of* Z *such that* $\mathrm{trdeg}_A(Z) < \infty$, *then* deg *is tame over* A.
More precisely, let A *be as in* (4) *and let* $z_1, \dots, z_m \in Z$ *and* $x_1, \dots, x_n \in B$ *be such that* Z *is algebraic over* $A[z_1, \dots, z_m]$ *and* $\mathrm{Gr}(B) = Z[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$; *then* $B = Z[x_1, \dots, x_n]$ *and*

$$\deg(D) = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\}, \quad \textit{for all} \quad D \in \mathrm{Der}_A(B).$$

Proof. In view of Lemma 4.18, assumption (1) implies that Z is a 0-subring of B and hence that $\mathrm{Gr}(B)$ is a Z-algebra, so assumption (2) makes sense. Let $z_1, \dots, z_m \in Z$ and $x_1, \dots, x_n \in B$ be such that Z is algebraic over $A[z_1, \dots, z_m]$ and $\mathrm{Gr}(B) = Z[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$; then $B = Z[x_1, \dots, x_n]$ by Lemma 4.19. Let $D \in \mathrm{Der}_A(B)$. To prove the proposition, we have to show that $\delta_D(x) \leq M$ for all $x \in B$, where we define

$$M = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\}.$$

Choose a subset $E \subseteq Z[x_1, \dots, x_n]$ satisfying the requirements of Lemma 4.19. By Lemma 4.5, $\delta_D(Z)$ is defined and is equal to $\max_{1 \leq i \leq m} \delta_D(z_i)$; so E satisfies:

$$\forall_{x \in B \setminus \{0\}} \exists_{e \in E} \deg(x - e) < \deg(x) \quad \text{and} \quad \delta_D(E) = \delta_D(Z \cup \{x_1, \dots, x_m\}) = M.$$

By contradiction, assume that some $x \in B$ satisfies $\delta_D(x) > M$; then the set $S_0 = \{i \in G \mid \exists_{x \in B}(\deg(x) = i \text{ and } \delta_D(x) > M)\}$ is not empty. By assumption (1), we may consider the least element i_0 of S_0. Now pick $x \in B$ such that $\deg(x) = i_0$ and $\delta_D(x) > M$; note in particular that $\delta_D(x) > M$ and $\delta_D(E) = M$ imply that $x \notin E$. Choose $e \in E$ such that $\deg(x - e) < \deg(x)$ and note that $x - e \neq 0$; so $\deg(x - e)$ is an element of G strictly less than i_0. By minimality of i_0, it follows that $\delta_D(x - e) \leq M$.

Note that $\deg(x) = \deg(e)$. If $\deg(Dx) = \deg(De)$, it follows immediately that $\delta_D(x) = \delta_D(e) \leq M$, a contradiction; so $\deg(Dx) \neq \deg(De)$ and consequently $\deg(Dx - De) = \max(\deg(Dx), \deg(De))$. Then

$$\begin{aligned}\delta_D(x) + \deg(x) = \deg(Dx) &\leq \max(\deg(Dx), \deg(De)) \\ &= \deg(D(x - e)) = \delta_D(x - e) + \deg(x - e) \leq M + \deg(x - e),\end{aligned}$$

so $\delta_D(x) \leq M + \deg(x - e) - \deg(x) < M$, a contradiction. □

Corollary 4.22. *Let* B *be an integral domain of finite transcendence degree over a field* $\mathbf{k}$ *of characteristic zero. Suppose that* $\deg\colon B \to G \cup \{-\infty\}$ (*where* G *is a totally ordered abelian group*) *is a degree function satisfying the conditions*

(1) $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of* G,
(2) $\mathrm{Gr}(B)$ *is a finitely generated algebra over the ring* $Z = \{x \in B \mid \deg(x) \leq 0\}$.
Then deg *is tame over* $\mathbf{k}$ *and* B *is finitely generated as a* Z*-algebra. More precisely, if* $z_1, \dots, z_m \in Z$ *and* $x_1, \dots, x_n \in B$ *are such that* Z *is algebraic over* $\mathbf{k}[z_1, \dots, z_m]$ *and* $\mathrm{Gr}(B) = Z[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$, *then* $B = Z[x_1, \dots, x_n]$ *and*

$$\deg(D) = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\}, \quad \textit{for all} \quad D \in \mathrm{Der}_{\mathbf{k}}(B).$$

Proof. In view of Lemma 4.18, assumption (1) implies that Z is a 0-subring of B and hence that $\mathrm{Gr}(B)$ is a Z-algebra, so assumption (2) makes sense. It is also noted in Lemma 4.18 that Z is factorially closed in B; this implies that $\mathbf{k} \subseteq Z$, so all hypotheses of Proposition 4.21 are satisfied with $A = \mathbf{k}$. The result follows from Proposition 4.21. □

Corollary 4.23. *Let* B *be an integral domain containing a field* $\mathbf{k}$ *of characteristic zero. Suppose that* $\deg\colon B \to G \cup \{-\infty\}$ (*where* G *is a totally ordered abelian group*) *is a degree function satisfying the conditions*
(1) $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of* G;
(2) $\mathrm{Gr}(B)$ *is a finitely generated* $\mathbf{k}$*-algebra.*
Then deg *is tame over* $\mathbf{k}$ *and* B *is a finitely generated* $\mathbf{k}$*-algebra. More precisely, if* $x_1, \dots, x_n \in B$ *are such that* $\mathrm{Gr}(B) = \mathbf{k}[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$, *then*:
(3) $B = \mathbf{k}[x_1, \dots, x_n]$;
(4) $\deg(D) = \max\{\delta_D(x_1), \dots, \delta_D(x_n)\}$ *for all* $D \in \mathrm{Der}_{\mathbf{k}}(B)$;
(5) $Z = \mathbf{k}[z_1, \dots, z_m]$, *where we define* $Z = \{x \in B \mid \deg(x) \leq 0\}$ *and where* $z_1, \dots, z_m$ *denote the elements of* $\{x_1, \dots, x_n\}$ *of degree* 0.

Proof. By Lemma 4.18, assumption (1) implies that $Z = \{x \in B \mid \deg(x) \leq 0\}$ is a 0-subring of B (so $\mathrm{Gr}(B)$ is a Z-algebra) and is factorially closed in B. The last condition implies that $\mathbf{k} \subseteq Z$, so $\mathrm{Gr}(B)$ is a $\mathbf{k}$-algebra and assumption (2) makes sense.

Let $x_1, \dots, x_n \in B \setminus \{0\}$ be such that $\mathrm{Gr}(B) = \mathbf{k}[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$. As $\mathbf{k}$ is a 0-subring of B (because $\mathbf{k} \subseteq Z$), Lemma 4.19 implies that $B = \mathbf{k}[x_1, \dots, x_n]$.

Write $\mathrm{Gr}(B) = \bigoplus_{i \in G} B_{[i]}$ with notation as in Paragraph 1.9. Let $\mu\colon Z \to B_{[0]}$ be the map $Z = B_0 \to B_0/B_{0^-} = B_{[0]}$, and note that μ is an isomorphism of $\mathbf{k}$-algebras and $\mu(z) = \mathrm{gr}(z)$ for all $z \in Z$. Let $z_1, \dots, z_m$ be the elements of $\{x_1, \dots, x_n\}$ of degree 0; as $\mathrm{Gr}(B) = \mathbf{k}[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$ where $\deg(\mathrm{gr}(x_i)) = \deg(x_i) \geq 0$ for each i, it follows that $B_{[0]} = \mathbf{k}[\mathrm{gr}(z_1), \dots, \mathrm{gr}(z_m)]$. So the composite $\mathbf{k}[z_1, \dots, z_m] \hookrightarrow Z \xrightarrow{\mu} B_{[0]}$ is surjective, and consequently $Z = \mathbf{k}[z_1, \dots, z_m]$. All hypotheses of Corollary 4.22 are satisfied, so

$$\deg(D) = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\} = \max\{\delta_D(x_1), \dots, \delta_D(x_n)\}$$

for every $D \in \mathrm{Der}_{\mathbf{k}}(B)$. □

Proposition 4.24. *Let B an integral domain of finite transcendence degree over a field $\mathbf{k}$ of characteristic zero and* $\deg\colon B \to G \cup \{-\infty\}$ *a degree function, where G is a totally ordered abelian group. Assume*:
(a) $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *is a well-ordered subset of G;*
(b) $\mathrm{Frac}(B)/\mathrm{Frac}(Z)$ *is a one-dimensional function field, where Z denotes the subring* $\{x \in B \mid \deg(x) \le 0\}$ *of B.*
Then $\deg$ *is tame over $\mathbf{k}$. Moreover, the ordered monoid* $\{\deg(x) \mid x \in B \setminus \{0\}\}$ *can be embedded in* $(\mathbb{N}, +, \le)$.

Proof. Let $S = Z \setminus \{0\}$, $B' = S^{-1}B$ and $Z' = S^{-1}Z = \mathrm{Frac}(Z)$. By Paragraph 4.10, deg extends to a degree function $\deg'\colon B' \to G \cup \{-\infty\}$. Note that

(21) $$\mathrm{trdeg}_{\mathbf{k}}(B') < \infty,$$

(22) $\{\deg'(x) \mid x \in B' \setminus \{0\}\}$ is equal to $\{\deg(x) \mid x \in B \setminus \{0\}\}$ and hence is a well-ordered subset of G,

(23) $$Z' = \{x \in B' \mid \deg'(x) \le 0\}.$$

Let $L = \mathrm{Frac}(B)$ and, using Paragraph 4.10 again, let $\mathrm{DEG}\colon L \to G \cup \{-\infty\}$ be the unique degree function which extends deg and deg′. Let $v\colon L \to G \cup \{\infty\}$ be the valuation of L defined by $v(x) = -\mathrm{DEG}(x)$. As $\deg(x) = 0$ for all $x \in Z \setminus \{0\}$, we note that v is a valuation over Z'; as L/Z' is a one-dimensional function field, it follows that v is a rank 1 discrete valuation; so the residue field κ of v is a finite extension of Z' and $\{v(x) \mid x \in L^*\} \cong \mathbb{Z}$. It follows that $\{\deg(x) \mid x \in B \setminus \{0\}\}$ can be embedded in $(\mathbb{N}, +, \le)$.

Consider the associated graded rings $\mathrm{Gr}(B)$, $\mathrm{Gr}(B') = \bigoplus_{i \in G} B'_{[i]}$ and $\mathrm{Gr}(L)$ determined by $(B, \deg)$, $(B', \deg')$ and (L, DEG) respectively, and note that $Z' = B'_{[0]} \subseteq \mathrm{Gr}(B')$. As DEG extends deg′ and deg′ extends deg, there are injective degree-preserving homomorphisms of graded rings, $\mathrm{Gr}(B) \hookrightarrow \mathrm{Gr}(B') \hookrightarrow \mathrm{Gr}(L)$. Using that v is a rank 1 discrete valuation, we get $\mathrm{Gr}(L) \cong \kappa[t, t^{-1}]$ where t is an indeterminate over κ. Thus

$$Z' \subseteq \mathrm{Gr}(B') \subseteq \kappa[t, t^{-1}].$$

Now $[\kappa\colon Z'] < \infty$, so $\kappa[t,t^{-1}]$ is a finitely generated Z'-algebra of transcendence degree 1 over Z'; it follows that

(24) $$\mathrm{Gr}(B') \text{ is finitely generated as a } Z'\text{-algebra.}$$

Let $D \in \mathrm{Der}_{\mathbf{k}}(B)$, and consider $S^{-1}D \in \mathrm{Der}_{\mathbf{k}}(B')$. By (21), (22), (23) and (24), $(B', G, \deg')$ and Z' satisfy the hypothesis of Corollary 4.22 and consequently $\deg'(S^{-1}D)$ is defined; by Lemma 4.11, $\deg(D)$ is defined. So deg is tame over $\mathbf{k}$. □

REMARK 4.25. Let us indicate how to compute the value of $\deg(D)$, in the above proof. We have $\mathbf{k} \subseteq Z$, because (Lemma 4.18) Z is factorially closed in B; so we may choose $z_1, \dots, z_m \in Z$ such that Z is algebraic over $\mathbf{k}[z_1, \dots, z_m]$. Also note that $\mathrm{Gr}(B) \hookrightarrow \mathrm{Gr}(B')$ is the localization: $\mathrm{Gr}(B') = S^{-1}\,\mathrm{Gr}(B)$; this and (24) imply that we can choose $x_1, \dots, x_n \in B$ satisfying $\mathrm{Gr}(B') = Z'[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$. As Z' is algebraic over $\mathbf{k}[z_1, \dots, z_m]$ and $\mathrm{Gr}(B') = Z'[\mathrm{gr}(x_1), \dots, \mathrm{gr}(x_n)]$, Corollary 4.22 gives

$$\begin{aligned}\deg'(S^{-1}D) &= \max\{\delta_{S^{-1}D}(z_1), \dots, \delta_{S^{-1}D}(z_m), \delta_{S^{-1}D}(x_1), \dots, \delta_{S^{-1}D}(x_n)\} \\ &= \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\}.\end{aligned}$$

Now Lemma 4.11 implies that $\deg(D) = \deg'(S^{-1}D)$, so we conclude that

$$\deg(D) = \max\{\delta_D(z_1), \dots, \delta_D(z_m), \delta_D(x_1), \dots, \delta_D(x_n)\}.$$

References

[1] A. Crachiola and S. Maubach: *Rigid rings and Makar-Limanov techniques*, arXiv: 1005.4949v1, 2010.

[2] H. Derksen: *More on the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$*, preprint, 1995.

[3] G. Freudenburg: Algebraic Theory of Locally Nilpotent Derivations, Invariant Theory and Algebraic Transformation Groups, VII, Springer, Berlin, 2006.

[4] L. Makar-Limanov: *Facts about cancellation*, preprint, 1997.

[5] L. Makar-Limanov: *Again $x + x^2y + z^2 + t^3 = 0$*; in Affine Algebraic Geometry, Contemp. Math. **369**, Amer. Math. Soc., Providence, RI., 2005, 177–182.

[6] A. Nur: *Locally nilpotent derivations and the cancellation problem*, Master's thesis, University of Ottawa, 2010.

[7] P.-M. Poloni: *Classification(s) of Danielewski hypersurfaces*, arXiv:0912.3241v1, 2009.

[8] R.L. Roth: *Classroom notes: on extensions of Q by square roots*, Amer. Math. Monthly **78** (1971), 392–393.

Department of Mathematics and Statistics
University of Ottawa
Ottawa, K1N 6N5
Canada
e-mail: ddaigle@uottawa.ca