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CONVERGENCE OF RECURRENCE OF BLOCKS FOR MIXING PROCESSES

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Abstract

Let $R_n(x)$ be the first return time of the initial sequence $x_1 \cdots x_n$ of $x = x_1 x_2 \cdots$. For mixing processes, sharp bounds for the convergence of $R_n(x)P_n(x)$ to exponential distribution are presented, where $P_n(x)$ is the probability of $x_1 \cdots x_n$. As a corollary, the limit of the mean of $\log(R_n(x)P_n(x))$ is obtained. For exponentially ϕ-mixing processes, $-E[\log(R_n P_n)]$ converges exponentially to the Euler's constant. A similar result is observed for the hitting time.

1. Introduction

Convergence of the logarithm of the first return time (recurrence time) of the initial block normalized by the block length has been investigated in relation to estimation of entropy or data compression methods such as the Ziv–Lempel algorithm [21]. Let $\{X_n : n \in \mathbb{N}\}$ be a stationary ergodic process on the space of infinite sequences $(\mathcal{A}^{\mathbb{N}}, \Sigma, \mathbb{P})$, where $\mathcal{A}$ is a finite set, Σ is the σ-field generated by finite dimensional cylinders, and $\mathbb{P}$ is a shift invariant ergodic probability measure.

Define R_n to be the first return time of the initial n-block $x_1^n = x_1 \cdots x_n$, i.e.,

$$R_n(x) := \min\bigl\{j \geq 1 \colon x_1^n = x_{j+1}^{j+n}\bigr\}.$$

Ornstein and Weiss [15] showed that for an ergodic process with entropy h

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(x) = h$$

almost surely. This convergence was first considered by Wyner and Ziv [18] as convergence in probability related do data compression algorithms. For a comprehensive introduction to the relationship among the first return time, entropy, and data compression algorithm, refer to [17] and [19].

The waiting time (hitting time) is defined by $W_n(x, y) := \min\bigl\{j \geq 1 \colon x_1^n = y_j^{j+n-1}\bigr\}$. A.D. Wyner and Ziv [18] proved that for Markov chains $(\log W_n)/n$ converges to entropy in probability with respect to the product probability measure of x and y. Shields

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[16] showed the almost sure convergence for Markov chains with respect to the product measure. He also showed that for a general ergodic case, $(\log W_n)/n$ may not converge to entropy. Also refer to [13] and [11] for related results.

Let $P_n(x)$ be the probability of the initial sequence $x_1^n := x_1 x_2 \cdots x_n$, i.e., $P_n(x) = \mathbb{P}(\{y\colon y_1^n = x_1^n\}) = \mathbb{P}(x_1^n)$. Then, the Shannon–Breiman–McMillan theorem [17] states that for ergodic processes, $-(\log P_n(x))/n$ converges to entropy h in L^1 and almost surely. This suggests that $\log R_n$ and $-\log P_n$ are closely related.

A process is called ψ-mixing if

$$\sup_{A\in\Sigma_0^n, B\in\Sigma_{n+l}^\infty} \frac{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|}{\mathbb{P}(A)\mathbb{P}(B)} \leq \psi(l)$$

for a decreasing sequence $\psi(l)$ converging to 0, and it is called ϕ-mixing if

$$\sup_{A\in\Sigma_0^n, B\in\Sigma_{n+l}^\infty} \frac{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|}{\mathbb{P}(A)} \leq \phi(l)$$

for a decreasing sequence $\phi(l)$ converging to 0, where Σ_i^j denotes the σ-algebra generated by $X_i^j := X_i X_{i+1} \cdots X_j$.

For any $\beta > 0$, Kontoyiannis [10] showed that for Markov chains, $\log(R_n(x)P_n(x)) = o(n^\beta)$ almost surely, and for ψ-mixing processes, $\log(W_n(x, y)P_n(x)) = o(n^\beta)$ almost surely with respect to the product measure. In fact, $R_n P_n$ and $W_n P_n$ converge to the exponential distribution with mean 1 for Markov chains and ψ-mixing processes [20]. We refer to [1], [2], [5], [6], [7], and [8] for more information on the convergence to exponential distribution.

For each block $B \in \mathcal{A}^n$, let $[B] = \{x\colon x_1^n = B\}$ denote the cylinder set defined by B. Define the waiting time (hitting time) to the cylinder set $[B]$ by

$$\tau_B(x) = \inf\{i \geq 1\colon T^i(x) \in [B]\},$$

where T is the left shift map defined by $(Tx)_k = x_{k+1}$ on $\mathcal{A}^{\mathbb{N}}$. Note that $R_n(x) = \tau_{x_1^n}(x)$ and $W_n(x, y) = \tau_{x_1^n}(y)$. For each block $B \in \mathcal{A}^n$, we denote $\mathbb{P}(\{x\colon \tau_B(x) = k\})$ and $\mathbb{P}([B])$ by $\mathbb{P}(\tau_B = k)$ and $\mathbb{P}(B)$, respectively. Let $\mathbb{P}_B(\tau_B = k)$ be the conditional probability of $\mathbb{P}(\tau_B(x) = k, x_1^n = B)/\mathbb{P}(B)$. Kac [9] showed that $E_B[\tau_B] = 1/\mathbb{P}(B)$, where E_B is the conditional expectation on the cylinder set $[B]$. Abadi [2] gave an exponential bound of $\mathbb{P}(\tau_B \mathbb{P}(B) < t)$ for ψ-mixing and ϕ-mixing processes with summable ϕ.

In this article, for each block $B \in \mathcal{A}^n$, we have an exponential bound of the conditional probability distribution $\mathbb{P}_B(\tau_B \mathbb{P}(B) < t)$ in the case of ψ-mixing and ϕ-mixing processes with summable ϕ; this bound enables us to obtain the limit of the mean of $\log(R_n P_n)$. In Section 2, we present a lemma for demonstrating the relationship between $\mathbb{P}_B(\tau_B \mathbb{P}(B) < t)$ and $\mathbb{P}(\tau_B \mathbb{P}(B) < t)$ and a theorem for determining the bound of $\mathbb{P}_B(\tau_B\mathbb{P}(B) < t)$ for ψ-mixing and ϕ-mixing processes with summable ϕ. In Section 3, the bounds of the expectation value $E[\log \tau_B]$ and $E_B[\log \tau_B]$ for each block B are obtained for ψ-mixing and ϕ-mixing processes with summable ϕ. Finally, in Section 4, we show that for exponentially ϕ-mixing processes

$$\lim_{n\to\infty} E[\log(W_n(x, y)P_n(x))] = -\gamma$$

and

$$\lim_{n\to\infty} E[\log(R_n(x)P_n(x))] = -\gamma.$$

For an earlier work for Bernoulli processes, see [8].

Maurer [12] studied the nonoverlapping first return time for i.i.d. processes in order to test pseudorandom number generators. His testing algorithm employed the nonoverlapping first return time $R_{(n)}(x) := \min\{j \geq 1 \colon x_1^n = x_{jn+1}^{jn+n}\}$. He showed that the convergence speed of $\log R_{(n)}/n$ to its entropy is asymptotically proportional to $1/n$ on average, and he conjectured that a similar result would hold for Markov chains; however, a correction term is necessary ([3], [4]). In [3], Abadi and Galves showed the exponential bound of the nonoverlapping return time and hitting for ψ-mixing processes and discussed the difference between the nonoverlapping return time and the overlapping one. see also [14] for the distributional convergence to the normal distribution.

2. Estimation of the distribution of the recurrence time

The relationship between the distribution of the first return time and the waiting time is expressed as follows (e.g. [7]):

Lemma 1. *In the case of stationary processes, we have*

$$\mathbb{P}(\tau_B = i + 1) = \mathbb{P}(\tau_B = i) - \mathbb{P}(B)\mathbb{P}_B(\tau_B = i)$$

for any integer $i \geq 1$, therefore, we have

$$\mathbb{P}(B)\mathbb{P}_B(\tau_B \geq i) = \mathbb{P}(\tau_B = i) = \mathbb{P}(\tau_B \geq i) - \mathbb{P}(\tau_B \geq i + 1)$$

for $i \geq 1$.

From the following lemma, we have determined the bound of $\mathbb{P}_B(\tau_B > t)$ using the bound of $\mathbb{P}(\tau_B > t)$.

Lemma 2. *For each integer $k \geq 0$ and real number $d_1 > 0$, we have*

$$\mathbb{P}_B(\tau_B > k) \geq \frac{\mathbb{P}(\tau_B > k) - \mathbb{P}(\tau_B > k + d_1)}{d_1\mathbb{P}(B)}.$$

For any integer k and real number d_2, where $0 < d_2 \le k$, we have

$$\mathbb{P}_B(\tau_B > k-1) \le \frac{\mathbb{P}(\tau_B > k-d_2) - \mathbb{P}(\tau_B > k)}{d_2 \mathbb{P}(B)}.$$

Proof. Let i, j be integers, where $1 \le i < j$. Since

$$\mathbb{P}_B(\tau_B \ge j-1) \le \mathbb{P}_B(\tau_B \ge j-2) \le \cdots \le \mathbb{P}_B(\tau_B \ge i+1) \le \mathbb{P}_B(\tau_B \ge i),$$

from Lemma 1, we have

$$\begin{aligned}\mathbb{P}(\tau_B \ge j-1) - \mathbb{P}(\tau_B \ge j) &\le \mathbb{P}(\tau_B \ge j-2) - \mathbb{P}(\tau_B \ge j-1) \\ &\le \cdots \le \mathbb{P}(\tau_B \ge i+1) - \mathbb{P}(\tau_B \ge i+2) \\ &\le \mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge j) &= \mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1) + \mathbb{P}(\tau_B \ge i+1) - \mathbb{P}(\tau_B \ge i+2) \\ &\quad + \cdots + \mathbb{P}(\tau_B \ge j-1) - \mathbb{P}(\tau_B \ge j) \\ &\le \mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1) + \mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1) \\ &\quad + \cdots + \mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1) \\ &= (j-i)(\mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1))\end{aligned}$$

and similarly,

$$\mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge j) \ge (j-i)(\mathbb{P}(\tau_B \ge j-1) - \mathbb{P}(\tau_B \ge j)).$$

Therefore, from Lemma 1

$$\mathbb{P}_B(\tau_B \ge i) = \frac{\mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge i+1)}{\mathbb{P}(B)} \ge \frac{\mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge j)}{(j-i)\mathbb{P}(B)} \tag{1}$$

and

$$\mathbb{P}_B(\tau_B \ge j-1) = \frac{\mathbb{P}(\tau_B \ge j-1) - \mathbb{P}(\tau_B \ge j)}{\mathbb{P}(B)} \le \frac{\mathbb{P}(\tau_B \ge i) - \mathbb{P}(\tau_B \ge j)}{(j-i)\mathbb{P}(B)} \tag{2}$$

for $1 \le i < j$.

If $0 < d_1 < 1$, then

$$\mathbb{P}_B(\tau_B > k) \ge 0 = \frac{\mathbb{P}(\tau_B > k) - \mathbb{P}(\tau_B > k + d_1)}{d_1 \mathbb{P}(B)}$$

for any $k \geq 0$. When $d_1 \geq 1$, let $d_1 = m_1 + \alpha$ where $m_1 \in \mathbb{N}$, $m_1 \geq 1$ and $0 \leq \alpha < 1$. Substituting i and j with $k+1$ and $k+m_1+1$, respectively, in (1), for each integer $k \geq 0$, we have

$$\begin{aligned}\mathbb{P}_B(\tau_B > k) = \mathbb{P}_B(\tau_B \geq k+1) &\geq \frac{\mathbb{P}(\tau_B \geq k+1) - \mathbb{P}(\tau_B \geq k+m_1+1)}{m_1\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(\tau_B > k) - \mathbb{P}(\tau_B > k+m_1+\alpha)}{m_1\mathbb{P}(B)} \geq \frac{\mathbb{P}(\tau_B > k) - \mathbb{P}(\tau_B > k+d_1)}{d_1\mathbb{P}(B)}.\end{aligned}$$

For the upper bound, let $d_2 = m_2 - \alpha$, where $m_2 \in \mathbb{N}$, $m_2 \geq 1$, and $0 \leq \alpha < 1$. Substituting i and j with $k - m_2 + 1$ and $k+1$, respectively, in (2), for any integer k, where $k \geq m_2 \geq d_2 > 0$, we have

$$\begin{aligned}\mathbb{P}_B(\tau_B > k-1) = \mathbb{P}_B(\tau_B \geq k) &\leq \frac{\mathbb{P}(\tau_B \geq k-m_2+1) - \mathbb{P}(\tau_B \geq k+1)}{m_2\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(\tau_B > k-m_2+\alpha) - \mathbb{P}(\tau_B > k)}{m_2\mathbb{P}(B)} \leq \frac{\mathbb{P}(\tau_B > k-d_2) - \mathbb{P}(\tau_B > k)}{d_2\mathbb{P}(B)}. \quad \square\end{aligned}$$

In [2], Abadi showed th following bound of the waiting time:

Fact 3 ([2], Theorem 1). *For ψ-mixing or ϕ-mixing with ϕ summable processes, there exist constants $C > 0$, Ξ_1, Ξ_2, and n_0 where $0 < \Xi_1 < 1 < \Xi_2 < \infty$ such that for all $B \in \mathcal{A}^n$, $n \geq n_0$, and $t > 0$ there exists $\xi_B \in [\Xi_1, \Xi_2]$ for which we have*

$$\left|\mathbb{P}\left(\tau_B > \frac{t}{\xi_B\mathbb{P}(B)}\right) - e^{-t}\right| \leq C\varepsilon(B)e^{-t}(t \vee 1), \tag{3}$$

where $\varepsilon(B) = \inf_{n \leq \Delta \leq 1/\mathbb{P}(B)}[\Delta\mathbb{P}(B) + *(\Delta)]$ and $*$ represents ψ or ϕ.

For any ψ-mixing or ϕ-mixing processes, it is known that the maximum probability of n-blocks decreases exponentially as n increases to infinity ([1], [6]). Therefore, for large n, $\varepsilon(B) = \inf_{n \leq \Delta \leq 1/\mathbb{P}(B)}[\Delta\mathbb{P}(B) + *(\Delta)] \leq n\mathbb{P}(B) + *(n)$ is defined and bounded by a decreasing function of n converging to 0. Moreover, for exponentially ϕ-mixing processes, constants C_0 and $\Gamma > 0$ exist such that for all $B \in \mathcal{A}^n$, $n \geq n_0$

$$\varepsilon(B) \leq n\mathbb{P}(B) + \phi(n) \leq C_0 e^{-\Gamma n}. \tag{4}$$

Let

$$\rho(B) = \frac{2\sqrt{C\varepsilon(B)}}{\sqrt{1 + C\varepsilon(B)} + \sqrt{C\varepsilon(B)}}.$$

Note $0 < \rho(B) < 1$. We have the following theorem on the distribution of the first return time τ_B. We assume that $B \in \mathcal{A}^n$, $n \geq n_0$.

Theorem 4. *For ψ-mixing or ϕ-mixing with summable ϕ processes, we have*

$$\mathbb{P}_B\left(\tau_B > \frac{t}{\xi_B \mathbb{P}(B)}\right) > \xi_B e^{-t}\left(1 - 2\sqrt{C\varepsilon(B)(t \vee 1)}\right) \quad \text{for} \quad t > 0$$

and

$$\mathbb{P}_B\left(\tau_B > \frac{t}{\xi_B \mathbb{P}(B)}\right) < \xi_B e^{-t}(1 + 2\sqrt{C\varepsilon(B)(t \vee 1)(1 + C\varepsilon(B)(t \vee 1))} + 2C\varepsilon(B)(t \vee 1))$$

for $t \geq \rho(B)$, where ξ_B and C are the same constants as those used in Fact 3.

Proof. Let $c_B = C\varepsilon(B)$ and $p_B = \mathbb{P}(B)$ for notational simplicity.
First, we shall prove the lower bound. For all $t > 0$, let

$$\frac{t}{\xi_B p_B} = \frac{s}{\xi_B p_B} + \alpha, \quad \text{where} \quad \frac{s}{\xi_B p_B} \in \mathbb{N} \cup \{0\} \quad \text{and} \quad 0 \leq \alpha < 1.$$

From Lemma 2 and Fact 3, for any $d_1 = \delta_1/(\xi_B p_B) > 0$, we have

$$\begin{aligned} \mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B}\right) &\geq \frac{\mathbb{P}(\tau_B > s/(\xi_B p_B)) - \mathbb{P}(\tau_B > s/(\xi_B p_B) + d_1)}{d_1 p_B} \\ &\geq \frac{\xi_B e^{-s}}{\delta_1}(1 - c_B(s \vee 1) - e^{-\delta_1}(1 + c_B((s + \delta_1) \vee 1))) \\ &= \frac{\xi_B e^{-s}}{\delta_1}(1 - e^{-\delta_1} - c_B((s \vee 1) + e^{-\delta_1}((s + \delta_1) \vee 1))) \\ &> \xi_B e^{-s}\left(\frac{1 - e^{-\delta_1}}{\delta_1} - c_B(s \vee 1)\frac{1 + e^{-\delta_1}}{\delta_1} - c_B\right). \end{aligned} \tag{5}$$

Let

$$\delta_1 = 2\sqrt{c_B(s \vee 1)} + \frac{4}{3}c_B(s \vee 1).$$

Then, we have

$$\sqrt{c_B(s \vee 1)} = \frac{3}{4}\left(\sqrt{1 + \frac{4}{3}\delta_1} - 1\right) = \frac{\delta_1}{\sqrt{1 + (4/3)\delta_1} + 1}.$$

Since

$$\begin{aligned} e^{\delta_1} &> 1 + \delta_1 + \frac{\delta_1^2}{2} + \frac{\delta_1^3}{6} + \frac{\delta_1^4}{24} > 1 + \delta_1 + \frac{\delta_1^2}{2} + \frac{\delta_1^3}{6} + \frac{\delta_1^4(4 - 4\delta_1)}{24(6 - 4\delta_1 + \delta_1^2)} \\ &= \frac{6 + 2\delta_1}{6 - 4\delta_1 + \delta_1^2}, \end{aligned}$$

we have

$$1-e^{-\delta_1} > 1-\frac{6-4\delta_1+\delta_1^2}{6+2\delta_1} = \frac{6\delta_1-\delta_1^2}{6+2\delta_1} = \delta_1\left(1-\frac{3}{4}\frac{2\delta_1}{3+\delta_1}\right).$$

Also, we have

$$\begin{aligned}\frac{2\delta_1}{3+\delta_1} &= \frac{3(1+\delta_1)}{3+\delta_1}-1 = \frac{\sqrt{9+18\delta_1+9\delta_1^2}}{3+\delta_1}-1 < \frac{\sqrt{9+18\delta_1+9\delta_1^2+(4/3)\delta_1^3}}{3+\delta_1}-1\\ &= \frac{\sqrt{(1+(4/3)\delta_1)(9+6\delta_1+\delta_1^2)}}{3+\delta_1}-1 = \sqrt{1+\frac{4}{3}\delta_1}-1 = \frac{4}{3}\sqrt{c_B(s\vee 1)}.\end{aligned}$$

Therefore, we have

$$\frac{1-e^{-\delta_1}}{\delta_1} > 1-\sqrt{c_B(s\vee 1)}. \tag{6}$$

If $0 < c_B(s\vee 1) \le 1/4$, then $0<\delta_1\le 4/3$,

$$\begin{aligned}e^{-\delta_1} &< 1-\delta_1+\frac{\delta_1^2}{2}-\frac{\delta_1^3}{6}+\frac{\delta_1^4}{24} = 1-\frac{5}{8}\delta_1-\frac{\delta_1}{6}\left(\delta_1-\frac{3}{2}\right)^2+\frac{\delta_1^4}{24}\\ &\le 1-\frac{5}{8}\delta_1+\frac{\delta_1^4}{24} \le 1-\frac{5}{8}\delta_1+\frac{\delta_1}{24}\left(\frac{4}{3}\right)^3 < 1-\frac{1}{2}\delta_1 = \sqrt{\left(1+\frac{\delta_1}{2}\right)^2}-\delta_1\\ &= \sqrt{1+\delta_1+\frac{\delta_1^2}{4}}-\delta_1 \le \sqrt{1+\delta_1+\frac{\delta_1}{4}\left(\frac{4}{3}\right)}-\delta_1 = \sqrt{1+\frac{4}{3}\delta_1}-\delta_1,\end{aligned}$$

and

$$\begin{aligned}c_B(s\vee 1)\frac{1+e^{-\delta_1}}{\delta_1} &< c_B(s\vee 1)\left(\frac{1+\sqrt{1+(4/3)\delta_1}}{\delta_1}-1\right)\\ &= \frac{c_B(s\vee 1)}{\sqrt{c_B(s\vee 1)}}-c_B(s\vee 1).\end{aligned} \tag{7}$$

Therefore, by substituting (6) and (7) in (5), we get

$$\mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B}\right) > \xi_B e^{-s}(1-2\sqrt{c_B(s\vee 1)}).$$

Note that if $c_B(s\vee 1) > 1/4$, then the right-hand side of this inequality is negative and the inequality still holds. Since $t\ge s$, we have

$$\mathbb{P}_B\left(\tau_B > \frac{t}{\xi_B p_B}\right) = \mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B}\right) > \xi_B e^{-s}(1-2\sqrt{c_B(s\vee 1)}).$$

Since $e^{-s}(1-2\sqrt{c_B(s\vee 1)})$ is a decreasing function of s, when it is positive, we have

$$\mathbb{P}_B\left(\tau_B > \frac{t}{\xi_B p_B}\right) > \xi_B e^{-t}(1-2\sqrt{c_B(t\vee 1)}).$$

For the upper bound, let

$$\frac{t}{\xi_B p_B} = \frac{s}{\xi_B p_B} - \alpha, \quad \text{where} \quad \frac{s}{\xi_B p_B} \in \mathbb{N} \quad \text{and} \quad 0 < \alpha \le 1.$$

Then, from Lemma 2, it can be noted that for any $d_2 = \delta_2/(\xi_B p_B)$, where $0 < \delta_2 \le s$, we have

$$\begin{aligned}
\mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B} - 1\right) &\le \frac{\mathbb{P}(\tau_B > s/(\xi_B p_B) - d_2) - \mathbb{P}(\tau_B > s/(\xi_B p_B))}{d_2 p_B} \\
&\le \frac{\xi_B e^{-s}}{\delta_2}(e^{\delta_2}(1 + c_B((s-\delta_2)\vee 1)) - 1 + c_B(s\vee 1)) \\
&\le \frac{\xi_B e^{-s}}{\delta_2}(e^{\delta_2} - 1 + c_B(s\vee 1)(e^{\delta_2}+1)).
\end{aligned}$$

Let

$$\delta_2 = \frac{2\sqrt{c_B(s\vee 1)}}{\sqrt{1+c_B(s\vee 1)} + \sqrt{c_B(s\vee 1)}} = 2\sqrt{c_B(s\vee 1)(1+c_B(s\vee 1))} - 2c_B(s\vee 1).$$

Then, $0 < \delta_2 < 1$. Since

$$e^{\delta_2} < 1 + \delta_2 + \frac{{\delta_2}^2}{2} + \frac{{\delta_2}^3}{4} < 1 + \delta_2 + \frac{3}{4}{\delta_2}^2 \quad \text{for} \quad 0 < \delta_2 < 1, \tag{8}$$

we have

$$\begin{aligned}
\mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B} - 1\right) &\le \frac{\xi_B e^{-s}}{\delta_2}(e^{\delta_2} - 1 + c_B(s\vee 1)(e^{\delta_2}+1)) \\
&< \xi_B e^{-s}\left(1 + \frac{\delta_2}{2} + \frac{{\delta_2}^2}{4} + c_B(s\vee 1)\left(\frac{2}{\delta_2} + 1 + \frac{3\delta_2}{4}\right)\right)
\end{aligned}$$

for $s \ge \delta_2$. Since $\delta_2^2 = 4c_B(s\vee 1)(1-\delta_2)$, we have

$$\begin{aligned}
\mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B} - 1\right) &< \xi_B e^{-s}\left(1 + \frac{\delta_2}{2} + \frac{2c_B(s\vee 1)}{\delta_2} + 2c_B(s\vee 1) - c_B(s\vee 1)\frac{\delta_2}{4}\right) \\
&< \xi_B e^{-s}\left(1 + \frac{\delta_2}{2} + \frac{2c_B(s\vee 1)}{\delta_2} + 2c_B(s\vee 1)\right) \\
&= \xi_B e^{-s}(1 + 2\sqrt{c_B(s\vee 1)(1+c_B(s\vee 1))} + 2c_B(s\vee 1))
\end{aligned}$$

for $s \ge \delta_2$.

If $s \geq \rho(B)$, then either $s \geq 1 > \delta_2$ or

$$1 > s \geq \rho(B) = \frac{2\sqrt{c_B}}{\sqrt{1+c_B}+\sqrt{c_B}} = \frac{2\sqrt{c_B(s \vee 1)}}{\sqrt{1+c_B(s\vee 1)}+\sqrt{c_B(s \vee 1)}} = \delta_2;$$

therefore, the condition $s \geq \delta_2$ is satisfied when $s \geq \rho(B)$.

Since $s/(\xi_B p_B) - 1 \leq t/(\xi_B p_B) < s/(\xi_B p_B)$, for $t \geq \rho(B)$, we have

$$\begin{aligned}\mathbb{P}_B\left(\tau_B > \frac{t}{\xi_B p_B}\right) &= \mathbb{P}_B\left(\tau_B > \frac{s}{\xi_B p_B} - 1\right) \\ &< \xi_B e^{-s}(1 + 2\sqrt{c_B(s \vee 1)(1 + c_B(s \vee 1))} + 2c_B(s \vee 1)) \\ &< \xi_B e^{-t}(1 + 2\sqrt{c_B(t \vee 1)(1 + c_B(t \vee 1))} + 2c_B(t \vee 1)).\end{aligned}$$

The last inequality results from the fact that $e^{-t}\sqrt{(t \vee 1)(1 + c(t \vee 1))}$ and $e^{-t}(t \vee 1)$ are decreasing functions for any $c > 0$. □

Using the lower bound of $\mathbb{P}_B(\tau_B > t/(\xi_B \mathbb{P}(B)))$, we have the following corollary:

Corollary 5. *For ψ-mixing or ϕ-mixing with summable ϕ processes, we have*

$$\xi_B \leq \frac{1}{1 - 2\sqrt{C\varepsilon(B)}}$$

for $0 < 2\sqrt{C\varepsilon(B)} < 1$.

Proof. Letting $t \to 0$ in the lower bound of Theorem 4, we have

$$1 \geq \lim_{t \to 0}[\xi_B e^{-t}(1 - 2\sqrt{C\varepsilon(B)(t \vee 1)})] = \xi_B(1 - 2\sqrt{C\varepsilon(B)}).$$ □

Note that for an exponentially ϕ-mixing system, it is shown [2] that there are some constants such as C and c such that $\xi_B \leq 1 + Ce^{-cn}$ for all $B \in \mathcal{A}^n$, which can also be derived from Corollary 5 and (4).

3. Bounds for the expectation of the logarithm of return time

For $r \geq 0$, define

$$h(r) := -\int_0^r \log \xi e^{-\xi}\, d\xi = \int_r^\infty \log \xi e^{-\xi}\, d\xi + \gamma,$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n 1/k - \log n\right) = 0.5771\cdots$ is Euler's constant.

For $0 < x < 1$, we have $-x \log x \le e^{-1}$; therefore, for $0 < x < 1$, we have

$$1 + (e \log x)^{-1} \le 1 - x < e^{-x} < 1$$

and

$$-\log x - e^{-1} < -e^{-x} \log x < -\log x.$$

By taking integral from 0 to r we have

$$-r \log r < -r \log r + (1 - e^{-1})r < h(r) < -r \log r + r \quad \text{for} \quad 0 < r < 1. \tag{9}$$

Lemma 6. *Let X be a positive random variable. Suppose*

$$F_1(t) \le \mathbb{P}(X > t) \le F_2(t), \quad t \ge 0,$$

for absolutely continuous functions F_i with $\lim_{t\to 0^+} F_i(t) = 1$ and $\lim_{t\to\infty} F_i(t) = 0$, $i = 1, 2$. If the derivative $f_i = F_i'$ satisfies

$$\lim_{t\to 0^+} t(\log t)^{2+\varepsilon} f_i(t) = \lim_{t\to\infty} t(\log t)^{2+\varepsilon} f_i(t) = 0$$

for some $\varepsilon > 0$, then, we have

$$-\int_0^\infty f_1(t) \log t \, dt \le E[\log X] \le -\int_0^\infty f_2(t) \log t \, dt.$$

Proof. Since $F_1(e^t) \le \mathbb{P}(\log X > t) \le F_2(e^t)$, we have

$$\begin{aligned} \int_1^\infty F_1(s)\frac{ds}{s} = \int_0^\infty F_1(e^t)\, dt &\le \int_0^\infty \mathbb{P}(\log X > t)\, dt \\ &\le \int_0^\infty F_2(e^t)\, dt = \int_1^\infty F_2(s)\frac{ds}{s}. \end{aligned}$$

By l'Hospital's theorem, $\lim_{t\to\infty} t(\log t)^2 f_i(t) = 0$ implies $\lim_{t\to\infty} F_i(t)\log t = 0$. Using integration by parts

$$\int_1^\infty F_i(s)\frac{ds}{s} = -\int_1^\infty f_i(s) \log s \, ds$$

and

$$-\int_1^\infty f_1(s) \log s \, ds \le \int_0^\infty \mathbb{P}(\log X > t)\, dt \le -\int_1^\infty f_2(s) \log s \, ds.$$

Similarly, by $\lim_{t\to 0^+} t(\log t)^2 f_i(t) = 0$ we have $\lim_{t\to 0^+}(1 - F_2(t)) \log t = 0$ and

$$\int_0^1 f_2(s) \log s \, ds = \int_0^1 (1 - F_2(s))\frac{ds}{s} = \int_{-\infty}^0 (1 - F_2(e^t))\, dt \le \int_{-\infty}^0 \mathbb{P}(\log X < t)\, dt \le \int_0^1 f_1(s) \log s \, ds.$$

From the assumption $\lim_{t\to\infty} t(\log t)^{2+\varepsilon} f_2(t) = 0$, $\lim_{t\to 0^+} t(\log t)^{2+\varepsilon} f_1(t) = 0$, we have

$$\int_0^\infty \mathbb{P}(\log X > t)\, dt < \infty, \quad \int_{-\infty}^0 \mathbb{P}(\log X < t)\, dt < \infty.$$

Therefore, $\log X$ is integrable and

$$E[\log X] = \int_0^\infty \mathbb{P}(\log X > t)\, dt - \int_{-\infty}^0 \mathbb{P}(\log X < t)\, dt,$$

which concludes

$$-\int_0^\infty f_1(s) \log s \, ds \le E[\log X] \le -\int_0^\infty f_2(s) \log s \, ds. \qquad \square$$

Assume that $C\varepsilon(B) < 1$. Then, we have the following theorem on the expectation of $\log \tau_B$:

Theorem 7. *For ψ-mixing or ϕ-mixing with summable ϕ processes, there exists a constant C' such that for all B with $0 < \varepsilon(B) < 1/C$*

$$|E[\log(\tau_B \xi_B \mathbb{P}(B))] + \gamma| < -C\varepsilon(B) \log(C\varepsilon(B)) + C'\varepsilon(B).$$

Proof. Let $c_B = C\varepsilon(B)$ and $p_B = \mathbb{P}(B)$ for notational simplicity.

Then, (3) implies that for $t > 0$

$$\mathbb{P}(\tau_B \xi_B p_B > t) \le e^{-t}(1 + c_B(t \vee 1)).$$

From the assumption $c_B < 1$, we have $\log(1 + c_B) < 1$; therefore,

$$\mathbb{P}(\tau_B \xi_B p_B > t) \le \begin{cases} 1, & 0 \le t \le \log(1 + c_B), \\ e^{-t}(1 + c_B), & \log(1 + c_B) < t \le 1, \\ e^{-t}(1 + c_B t), & t > 1. \end{cases}$$

Therefore, from Lemma 6, we have

$$
\begin{aligned}
E[\log(\tau_B \xi_B p_B)] &\le \int_{\log(1+c_B)}^{1} (1+c_B)e^{-t}\log t\, dt + \int_1^\infty (1-c_B+c_B t)e^{-t}\log t\, dt \\
&\le \int_{\log(1+c_B)}^{\infty} e^{-t}\log t\, dt + c_B\int_1^\infty (t-1)e^{-t}\log t\, dt \\
&= h(\log(1+c_B)) - \gamma + e^{-1}c_B < h(c_B) - \gamma + e^{-1}c_B.
\end{aligned}
$$

From (9), we have

$$
E[\log(\tau_B \xi_B p_B)] < -c_B \log c_B + c_B - \gamma + e^{-1}c_B.
$$

Since $\mathbb{P}(\tau_B = 1) = p_B$, Lemma 1 implies that $\mathbb{P}(\tau_B = k) \le p_B$ for all $k \in \mathbb{N}$. Therefore, for a real number $t > 0$

$$
\begin{aligned}
\mathbb{P}(\tau_B \xi_B p_B > t) &= 1 - \left(\mathbb{P}(\tau = 1) + \cdots + \mathbb{P}\left(\tau = \left\lfloor \frac{t}{\xi_B p_B} \right\rfloor\right)\right) \\
&\ge 1 - \left\lfloor \frac{t}{\xi_B p_B} \right\rfloor p_B \ge 1 - \frac{t}{\xi_B} \ge 1 - \frac{t}{\Xi_1}.
\end{aligned} \tag{10}
$$

Let t_0 be the positive real number that satisfies $1 - t_0/\Xi_1 = e^{-t_0}(1-c_B)$. Then, we have

$$
0 < t_0 < \frac{c_B}{{\Xi_1}^{-1} - 1 + c_B} < 1. \tag{11}
$$

Therefore, (3) and (10) imply that

$$
\mathbb{P}(\tau_B \xi_B p_B > t) \ge \begin{cases} 1 - \dfrac{t}{\Xi_1}, & 0 \le t \le t_0, \\ e^{-t}(1 - c_B(t \vee 1)), & t > t_0. \end{cases}
$$

Since $\int_1^\infty (t-1)e^{-t}\log t\, dt = e^{-1}$, from Lemma 6, we have

$$
\begin{aligned}
E[\log(\tau_B \xi_B p_B)] &\ge {\Xi_1}^{-1}\int_0^{t_0} \log t\, dt + \int_{t_0}^1 (1-c_B)e^{-t}\log t\, dt \\
&\quad + \int_1^\infty (1+c_B-c_B t)e^{-t}\log t\, dt \\
&> {\Xi_1}^{-1}(t_0\log t_0 - t_0) + \int_{t_0}^\infty e^{-t}\log t\, dt - e^{-1}c_B.
\end{aligned}
$$

Therefore, from (9), we have

$$
\begin{aligned}
E[\log(\tau_B \xi_B p_B)] &\ge {\Xi_1}^{-1}(t_0\log t_0 - t_0) + h(t_0) - \gamma - e^{-1}c_B \\
&> ({\Xi_1}^{-1} - 1)t_0\log t_0 - {\Xi_1}^{-1}t_0 - \gamma - e^{-1}c_B.
\end{aligned}
$$

From (11), we have

$$\begin{aligned}E[\log(\tau_B\xi_B p_B)] &> \frac{(\Xi_1{}^{-1}-1)c_B}{\Xi_1{}^{-1}-1+c_B}\log\left(\frac{c_B}{\Xi_1{}^{-1}-1+c_B}\right)\\ &\quad-\frac{\Xi_1{}^{-1}c_B}{\Xi_1{}^{-1}-1+c_B}-\gamma-e^{-1}c_B\\ &> c_B\log c_B-c_B\log\Xi_1{}^{-1}-\frac{c_B}{1-\Xi_1}-\gamma-e^{-1}c_B\\ &> c_B\log c_B-\left(\frac{1}{1-\Xi_1}+e^{-1}-\log\Xi_1\right)c_B-\gamma. \qquad \square\end{aligned}$$

Now, we have the following theorem for determining the expectation $\log\tau_B$ on $[B]$.

Theorem 8. *For ψ-mixing or ϕ-mixing with summable ϕ processes, if n is sufficiently large, then for each n-block B, we have*

$$E_B[\log(\tau_B\xi_B\mathbb{P}(B))]+\gamma\xi_B < -2\xi_B\sqrt{C\varepsilon(B)}\log(C\varepsilon(B))+\xi_B\sqrt{C\varepsilon(B)}$$

and

$$\begin{aligned}&E_B[\log(\tau_B\xi_B\mathbb{P}(B))]+\gamma\xi_B\\ &> (1-\xi_B)\log\mathbb{P}(B)+2\xi_B\sqrt{C\varepsilon(B)}\log\mathbb{P}(B)+\log(\xi_B(1-2\sqrt{C\varepsilon(B)})).\end{aligned}$$

Proof. For a simple calculation, we assume that $C\varepsilon(B) < 1/25$. Let $c_B = C\varepsilon(B)$ and $p_B = \mathbb{P}(B)$ for notational simplicity.

First, consider the upper bound of $E_B[\log(\tau_B\xi_B p_B)]$.

Let $t_0 = \log(1+2\sqrt{c_B(1+c_B)}+2c_B)$. Note that

$$0 < \rho(B) = \frac{2\sqrt{c_B}}{\sqrt{1+c_B}+\sqrt{c_B}} = 2\sqrt{c_B(1+c_B)}-2c_B < 1.$$

Then, we have

$$\begin{aligned}e^{\rho(B)} &< 1+\rho(B)+\frac{3}{4}\rho(B)^2 = 1+2\sqrt{c_B(1+c_B)}+c_B-\frac{6c_B\sqrt{c_B}}{\sqrt{c_B}+\sqrt{1+c_B}}\\ &< 1+2\sqrt{c_B(1+c_B)}+2c_B = e^{t_0}\\ &< 1+2\sqrt{c_B}\left(1+\frac{c_B}{2}\right)+2c_B = 1+2\sqrt{c_B}+\frac{(2\sqrt{c_B})^2}{2}+\frac{(2\sqrt{c_B})^3}{8} < e^{2\sqrt{c_B}},\end{aligned}$$

which implies that

$$\rho(B) < t_0 < 2\sqrt{c_B} < 1. \tag{12}$$

From Theorem 4, if $\xi_B \le 1$, then from (12)

$$\mathbb{P}_B(\tau_B \xi_B p_B > t) \le \begin{cases} 1, & 0 \le t < t_0, \\ \xi_B e^{-t}(1 + 2\sqrt{c_B(1+c_B)} + 2c_B), & t_0 \le t \le 1, \\ \xi_B e^{-t}(1 + 2\sqrt{c_B t(1+c_B t)} + 2c_B t), & t > 1. \end{cases}$$

Therefore, from Lemma 6, for $\xi_B \le 1$, we have

$$\begin{aligned} & E_B[\log(\tau_B \xi_B p_B)] \\ & < (1-\xi_B)\log t_0 + \xi_B(1 + 2\sqrt{c_B(1+c_B)} + 2c_B)\int_{t_0}^1 e^{-t}\log t\, dt \\ & \quad + \xi_B \int_1^\infty \left(1 + 2\sqrt{c_B t(1+c_B t)} - \frac{\sqrt{c_B}(1+2c_B t)}{\sqrt{t(1+c_B t)}} + 2c_B(t-1)\right) e^{-t}\log t\, dt \\ & < (1-\xi_B)\log t_0 + \xi_B \int_{t_0}^\infty e^{-t}\log t\, dt + 2\xi_B\sqrt{c_B}\int_1^\infty (\sqrt{t} + 2\sqrt{c_B}t)e^{-t}\log t\, dt \\ & < (1-\xi_B)\log t_0 + \xi_B(h(t_0) - \gamma) + \frac{14}{5}\xi_B\sqrt{c_B}\int_1^\infty t e^{-t}\log t\, dt. \end{aligned} \tag{13}$$

When $\xi_B > 1$, from Theorem 4 and (12), we have

$$\mathbb{P}_B(\tau_B \xi_B p_B > t) \le \begin{cases} 1, & 0 \le t \le t_0 + \log \xi_B, \\ \xi_B e^{-t}(1 + 2\sqrt{c_B(1+c_B)} + 2c_B), & t_0 + \log \xi_B < t \le 1, \\ \xi_B e^{-t}(1 + 2\sqrt{c_B t(1+c_B t)} + 2c_B t), & t > 1. \end{cases}$$

Note that from the assumption $c_B < 1/25$ and Corollary 5, we have

$$t_0 + \log \xi_B \le 2\sqrt{c_B} - \log(1 - 2\sqrt{c_B}) < \frac{2}{5} - \log\frac{3}{5} = 0.91082\cdots < 1. \tag{14}$$

Similarly, for $\xi_B > 1$, we have

$$E_B[\log(\tau_B \xi_B p_B)] < \xi_B \int_{t_0 + \log \xi_B}^\infty e^{-t}\log t\, dt + \frac{14}{5}\xi_B\sqrt{c_B}\int_1^\infty t e^{-t}\log t\, dt. \tag{15}$$

Let $D_0 := (14/5)\int_1^\infty e^{-t} t \log t\, dt = 1.644336\cdots$. Then, from (9) and (13), for $\xi_B \le 1$, we have

$$E_B[\log(\tau_B \xi_B p_B)] + \gamma\xi_B < (1-\xi_B)\log t_0 + \xi_B(-t_0\log t_0 + t_0) + \xi_B\sqrt{c_B}D_0.$$

Since $-x\log x + x$ is increasing for $0 < x < 1$, we have, from (12), for $\xi_B \le 1$

$$\begin{aligned} & E_B[\log(\tau_B \xi_B p_B)] + \gamma\xi_B \\ & < (1-\xi_B)\log(2\sqrt{c_B}) - \xi_B\sqrt{c_B}\log c_B + \xi_B\sqrt{c_B}(2 - 2\log 2 + D_0). \end{aligned}$$

For $\xi_B > 1$ by (9), (14), and (15), we have

$$
\begin{aligned}
&E_B[\log(\tau_B \xi_B p_B)] + \gamma \xi_B \\
&< \xi_B h(t_0 + \log \xi_B) + \xi_B \sqrt{c_B} D_0 \\
&< \xi_B \left(-(4\sqrt{c_B} + 3c_B) \log(4\sqrt{c_B}) + \frac{23}{5}\sqrt{c_B} + \sqrt{c_B} D_0 \right) \\
&= -\xi_B \sqrt{c_B} \left(2 \log c_B + 4 \log 4 + 3\sqrt{c_B} \log(4\sqrt{c_B}) - \frac{23}{5} - D_0 \right) \\
&< -\xi_B \sqrt{c_B} \left(2 \log c_B + 4 \log 4 - \frac{3}{4} e^{-1} - \frac{23}{5} - D_0 \right) \\
&< -\xi_B \sqrt{c_B} (2 \log c_B - 1) < -2\xi_B \sqrt{c_B} \log c_B + \xi_B \sqrt{c_B}.
\end{aligned}
$$

Now, we estimate the lower bound. Since $\tau_B \geq 1$, from Theorem 4, we have

$$
\mathbb{P}_B\left(\tau_B > \frac{t}{\xi_B p_B} \right) \geq \begin{cases} \xi_B e^{-t}(1 - 2\sqrt{c_B(t \vee 1)}), & t > \xi_B p_B, \\ 1, & 0 < t \leq \xi_B p_B. \end{cases}
$$

From Corollary 5, $\xi_B(1 - 2\sqrt{c_B}) \leq 1$; therefore, from Lemma 6, we have

$$
\begin{aligned}
&E_B[\log(\tau_B \xi_B p_B)] \\
&\geq (1 - \xi_B(1 - 2\sqrt{c_B})) \log(\xi_B p_B) + \int_0^1 \xi_B e^{-t}(1 - 2\sqrt{c_B}) \log t dt \\
&\quad + \int_1^\infty \xi_B e^{-t} \left(1 - 2\sqrt{c_B t} \left(1 - \frac{1}{2t} \right) \right) \log t \, dt \\
&> (1 - \xi_B(1 - 2\sqrt{c_B})) \log(\xi_B p_B) + \xi_B \int_0^\infty e^{-t} \log t \, dt \\
&\quad - 2\xi_B \sqrt{c_B} \int_0^1 e^{-t} \log t \, dt - 2\xi_B \sqrt{c_B} \int_1^\infty e^{-t} \sqrt{t} \log t \, dt \\
&> (1 - \xi_B(1 - 2\sqrt{c_B})) \log p_B + \log \xi_B - (1 - 2\sqrt{c_B}) \xi_B \log \xi_B - \gamma \xi_B,
\end{aligned}
$$

where the last inequality is from the fact that $\int_0^1 e^{-t} \log t \, dt + \int_1^\infty e^{-t} \sqrt{t} \log t \, dt < 0$. Since $\xi_B \log \xi_B \leq -\log(1 - 2\sqrt{c_B})/(1 - 2\sqrt{c_B})$, we have

$$
E_B[\log(\tau_B \xi_B p_B)] > (1 - \xi_B) \log p_B + 2\xi_B \sqrt{c_B} \log p_B + \log(\xi_B(1 - 2\sqrt{c_B})) - \gamma \xi_B
$$

which completes the proof. We note

$$
E_B[\log(\tau_B p_B)] > (1 - \xi_B) \log p_B + 2\xi_B \sqrt{c_B} \log p_B + \log(1 - 2\sqrt{c_B}) - \gamma \xi_B. \tag{16}
$$

□

4. Convergence of mean

For each $s \in \mathbb{N}$, Let $\mathcal{B}_n(s)$ be the set of $B \in \mathcal{A}^n$, which recurs before time n/s, i.e., $B = b_1 \cdots b_k b_1 \cdots b_k \cdots b_1 \cdots b_l$, where $1 \le l \le k$ for some $k < n/s$. Then, from [1], it can be noted that for any ϕ-mixing, there exists $s \in \mathbb{N}$ and two positive constants C_1 and c_1 such that

$$\mathbb{P}(\{x : x_1^n \in \mathcal{B}_n(s)\}) \le C_1 e^{-c_1 n}. \tag{17}$$

Also refer to [5] and [20].

In [1], Abadi shows that for exponentially ϕ-mixing processes, if $B \in \mathcal{A}^n \setminus \mathcal{B}_n(s)$, then

$$\sup_{t>0} \left| \mathbb{P}\left(\tau_B > \frac{t}{\mathbb{P}(B)} \right) - e^{-t} \right| < C_2 e^{-c_2 n}, \tag{18}$$

where C_2 and c_2 are constants. Combining (18) with (3), for exponentially ϕ-mixing processes, if $B \in \mathcal{A}^n \setminus \mathcal{B}_n(s)$, then

$$|\xi_B - 1| < C_3 e^{-c_3 n} \quad \text{and} \quad |\log \xi_B| < C_3 e^{-c_3 n}, \tag{19}$$

where C_3 and c_3 are constants.

Now we have the theorem on the convergence of the mean of the waiting time.

Theorem 9. *In the case of exponentially ϕ-mixing processes, we have*

$$\lim_{n\to\infty} E_{X\times Y}[\log(W_n(x, y)P_n(x))] = -\gamma \quad \textit{exponentially},$$

where $E_{X\times Y}$ is the expectation with respect to (x, y) in the product measure $\mathbb{P} \times \mathbb{P}$ and for almost every x

$$\lim_{n\to\infty} E_Y[\log(W_n(x, y)P_n(x))] = -\gamma \quad \textit{exponentially},$$

where E_Y is the expectation with respect to y.

Proof. From (4) and Theorem 7 we have

$$|E_Y[\log(W_n(x, y)\mathbb{P}(x_1^n))] - (-\gamma)| < |\log \xi_{x_1^n}| - CC_0 e^{-\Gamma n} \log(CC_0 e^{-\Gamma n}) + C' C_0 e^{-\Gamma n}.$$

By (19), for x with $x_1^n \in \mathcal{B}_n(s)$, we have

$$|E_Y[\log(W_n(x, y)\mathbb{P}(x_1^n))] - (-\gamma)| < C_3 e^{-c_3 n} - CC_0 e^{-\Gamma n} \log(CC_0 e^{-\Gamma n}) + C' C_0 e^{-\Gamma n}.$$

The Borel–Cantelli lemma with (17) implies that, for almost every x, $x_1^n \in \mathcal{B}_n(s)$ finitely many n's and

$$\lim_{n\to\infty} E_Y[\log(W_n(x, y)P_n(x))] = -\gamma \quad \text{exponentially}.$$

Also we have

$$\begin{aligned}&|E_{X\times Y}[\log(W_n(x,y)\mathbb{P}(x_1^n))]-(-\gamma)|\\&< E_X|\log \xi_{x_1^n}|-CC_0e^{-\Gamma n}\log(CC_0e^{-\Gamma n})+C'C_0e^{-\Gamma n}.\end{aligned}$$

Since $\xi_{x_1^n}$ is uniformly bounded, from (17) and (19), we have

$$\lim_{n\to\infty} E_{X\times Y}[\log(W_n(x,y)P_n(x))]=-\gamma \quad \text{exponentially.}$$ □

From Theorem 8, we have the following theorem.

Theorem 10. *In the case of exponentially ϕ-mixing processes, we have*

$$\lim_{n\to\infty} E[\log(R_n(x)P_n(x))]=-\gamma$$

exponentially.

Proof. From (4) and Theorem 8, we have for sufficiently large n

$$\begin{aligned}E_B[\log(\tau_B\mathbb{P}(B))]&< -\gamma\xi_B-3\xi_B\sqrt{C\varepsilon(B)}\log(C\varepsilon(B))-\log\xi_B\\&< -\gamma+3\Xi_2\sqrt{CC_0}\Gamma n e^{-\Gamma n/2}-\log\xi_B+\gamma(1-\xi_B),\end{aligned}$$

and from (17) and (19), we have for sufficiently large n

$$\begin{aligned}E[\log(R_nP_n)]&=\sum_{B\in\mathcal{B}_n(s)}E_B[\log(\tau_B\mathbb{P}(B))]\mathbb{P}(B)+\sum_{B\in\mathcal{A}^n\setminus\mathcal{B}_n(s)}E_B[\log(\tau_B\mathbb{P}(B))]\mathbb{P}(B)\\&\le -\gamma+3\Xi_2\sqrt{CC_0}\Gamma n e^{-\Gamma n/2}+\sum_{B\in\mathcal{B}_n(s)}(-\log\xi_B+\gamma(1-\xi_B))\mathbb{P}(B)\\&\quad+\sum_{B\in\mathcal{A}^n\setminus\mathcal{B}_n(s)}(-\log\xi_B+\gamma(1-\xi_B))\mathbb{P}(B)\\&\le -\gamma+3\Xi_2\sqrt{CC_0}\Gamma n e^{-\Gamma n/2}+(-\log\Xi_1+\gamma(1-\Xi_1))\mathbb{P}(x_1^n\in\mathcal{B}_n(s))\\&\quad+(1+\gamma)C_3e^{-c_3n}\mathbb{P}(x_1^n\in\mathcal{A}^n\setminus\mathcal{B}_n(s))\\&< -\gamma+3\Xi_2\sqrt{CC_0}\Gamma n e^{-\Gamma n/2}+(-\log\Xi_1+\gamma(1-\Xi_1))C_1e^{-c_1n}\\&\quad+(1+\gamma)C_3e^{-c_3n}.\end{aligned}$$

Therefore, we have the upper bound

$$\limsup_{n\to\infty} E[\log(R_nP_n)]\le-\gamma.$$

Now we consider the lower bound. From (4) and (16), we have for sufficiently large n

$$\begin{aligned} E_B[\log(\tau_B \mathbb{P}(B))] &> -\gamma\xi_B + \log(1 - 2\sqrt{C\varepsilon(B)}) \\ &\quad + (1-\xi_B)\log \mathbb{P}(B) + 2\xi_B\sqrt{C\varepsilon(B)}\log \mathbb{P}(B) \\ &> -\gamma + \log(1 - 2\sqrt{CC_0}e^{-\Gamma n/2}) \\ &\quad + (1 - \xi_B + 2\Xi_2\sqrt{CC_0}e^{-\Gamma n/2})\log \mathbb{P}(B) - \gamma(\xi_B - 1) \end{aligned}$$

and from (19), we have for sufficiently large n

$$\begin{aligned} &E[\log(R_n P_n)] \\ &= \sum_{B\in\mathcal{B}_n(s)} E_B[\log(\tau_B\mathbb{P}(B))]\mathbb{P}(B) + \sum_{B\in\mathcal{A}^n\setminus\mathcal{B}_n(s)} E_B[\log(\tau_B\mathbb{P}(B))]\mathbb{P}(B) \\ &\geq -\gamma + \log(1 - 2\sqrt{CC_0}e^{-\Gamma n/2}) \\ &\quad + \sum_{B\in\mathcal{B}_n}((1 - \Xi_1 + 2\Xi_2\sqrt{CC_0}e^{-\Gamma n/2})\log\mathbb{P}(B) - \gamma(\Xi_2 - 1))\mathbb{P}(B) \\ &\quad + \sum_{B\in\mathcal{A}^n\setminus\mathcal{B}_n}((C_3e^{-c_3 n} + 2\Xi_2\sqrt{CC_0}e^{-\Gamma n/2})\log\mathbb{P}(B) - \gamma C_3 e^{-c_3 n})\mathbb{P}(B) \\ &\geq -\gamma + \log(1 - 2\sqrt{CC_0}e^{-\Gamma n/2}) \\ &\quad + (1 - \Xi_1 + 2\Xi_2\sqrt{CC_0}e^{-\Gamma n/2})\sum_{B\in\mathcal{B}_n}\mathbb{P}(B)\log\mathbb{P}(B) - \gamma(\Xi_2 - 1)\mathbb{P}(\mathcal{B}_n(s)) \\ &\quad + (C_3e^{-c_3 n} + 2\Xi_2\sqrt{CC_0}e^{-\Gamma n/2})\sum_{B\in\mathcal{A}^n\setminus\mathcal{B}_n}\mathbb{P}(B)\log\mathbb{P}(B) - \gamma C_3 e^{-c_3 n}. \end{aligned}$$

Here, we have

$$\sum_{B\in\mathcal{A}^n\setminus\mathcal{B}_n(s)}\mathbb{P}(B)\log\mathbb{P}(B) \geq \sum_{B\in\mathcal{A}^n}\mathbb{P}(B)\log\mathbb{P}(B) \geq -n\log|\mathcal{A}|$$

and from (17), we have

$$\begin{aligned} \sum_{B\in\mathcal{B}_n(s)}\mathbb{P}(B)\log\mathbb{P}(B) &\geq \mathbb{P}(x_1^n\in\mathcal{B}_n(s))\log\frac{\mathbb{P}(x_1^n\in\mathcal{B}_n(s))}{|\mathcal{B}_n(s)|} \\ &\geq C_1e^{-c_1 n}\Big(\log C_1e^{-c_1 n} - \frac{n}{s}\log|\mathcal{A}|\Big). \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
E[\log(R_n P_n)] \ge -\gamma + \log(1 - 2\sqrt{CC_0} e^{-\Gamma n/2}) - \gamma(\Xi_2 - 1) C_1 e^{-c_1 n} - \gamma C_3 e^{-c_3 n} \\
+ (1 - \Xi_1 + 2\Xi_2 \sqrt{CC_0} e^{-\Gamma n/2}) C_1 e^{-c_1 n} \left(\log C_1 e^{-c_1 n} - \frac{n}{s} \log|\mathcal{A}| \right) \\
- (C_3 e^{-c_3 n} + 2\Xi_2 \sqrt{CC_0} e^{-\Gamma n/2}) n \log|\mathcal{A}|,
\end{aligned}
$$

which implies that

$$\liminf_{n\to\infty} E[\log(R_n P_n)] \ge -\gamma. \qquad \square$$

Similarly, we can show that

$$\lim_{n\to\infty} \mathrm{Var}_x[\log(R_n(x) P_n(x))] = \lim_{n\to\infty} \mathrm{Var}_y[\log(W_n(x,y) P_n(x))] = \frac{\pi^2}{6},$$

where Var_x and Var_y are the variance over x-variable and y-variable, respectively. For the nonoverlapping return time and hitting time consult [3].

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