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WELL-POSEDNESS AND ILL-POSEDNESS RESULTS FOR DISSIPATIVE BENJAMIN–ONO EQUATIONS

STÉPHANE VENTO

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Abstract

We study the Cauchy problem for the dissipative Benjamin–Ono equations $u_t + \mathcal{H}u_{xx} + |D|^\alpha u + uu_x = 0$ with $0 \le \alpha \le 2$. When $0 \le \alpha < 1$, we show the ill-posedness in $H^s(\mathbb{R})$, $s \in \mathbb{R}$, in the sense that the flow map $u_0 \mapsto u$ (if it exists) fails to be $\mathcal{C}^2$ at the origin. For $1 < \alpha \le 2$, we prove the global well-posedness in $H^s(\mathbb{R})$, $s > -\alpha/4$. It turns out that this index is optimal.

1. Introduction, main results and notations

1.1. Introduction. In this work we consider the Cauchy problem for the following dissipative Benjamin–Ono equations

$$\text{(dBO)} \qquad \begin{cases} u_t + \mathcal{H}u_{xx} + |D|^\alpha u + uu_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,\,\cdot\,) = u_0 \in H^s(\mathbb{R}), \end{cases}$$

with $0 \le \alpha \le 2$, and where $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H} f(x) = \frac{1}{\pi}\,\mathrm{pv}\left(\frac{1}{x} * f\right)(x) = \mathcal{F}^{-1}(-i\,\mathrm{sgn}(\xi)\,\hat{f}(\xi))(x),$$

and $|D|^\alpha$ is the Fourier multiplier with symbol $|\xi|^\alpha$.

When $\alpha = 0$, (dBO) is the ordinary Benjamin–Ono equation derived by Benjamin [2] and later by Ono [15] as a model for one-dimensional waves in deep water. The Cauchy problem for the Benjamin–Ono equation has been extensively studied these last years. It has been proved in [19] that (BO) is globally well-posed in $H^s(\mathbb{R})$ for $s \ge 3$, and then for $s \ge 3/2$ in [18] and [9]. In [21], Tao get the well-posedness of this equation for $s \ge 1$ by using a gauge transformation (which is a modified version of the Cole–Hopf transformation). Recently, combining a gauge transformation together with a Bourgain's method, Ionescu and Kenig [8] finally shown that one could go down to $L^2(\mathbb{R})$, and this seems to be, in some sense, optimal. It is worth noticing that all these results have been obtained by compactness methods. On the other hand, Molinet, Saut and Tzvetkov [13] proved that for all $s \in \mathbb{R}$, the flow map $u_0 \mapsto u$ is not

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of class $\mathcal{C}^2$ from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. Furthermore, building suitable families of approximate solutions, Koch and Tzvetkov proved in [10] that the flow map is actually not even uniformly continuous on bounded sets of $H^s(\mathbb{R})$, $s > 0$. As an important consequence of this, since a Picard iteration scheme would imply smooth dependance upon the initial data, we see that such a scheme cannot be used to get solutions in any space continuously embedded in $\mathcal{C}([0, T]; H^s(\mathbb{R}))$.

When $\alpha = 2$, (dBO) is the so-called Benjamin–Ono–Burgers equation

$$\text{(BOB)} \qquad u_t + (\mathcal{H} - 1)u_{xx} + uu_x = 0.$$

Edwin and Robert [6] have derived (BOB) by means of formal asymptotic expansions in order to describe wave motions by intense magnetic flux tube in the solar atmosphere. The dissipative effects in that context are due to heat conduction. (BOB) has been studied in many papers, see [4, 7, 23]. Working in Bourgain's spaces containing both dispersive and dissipative effects[1], Otani showed in [16] that (BOB) is globally well-posed in $H^s(\mathbb{R})$, $s > -1/2$. In this paper, we prove that this index is in fact critical since the flow map $u_0 \mapsto u$ is not of class $\mathcal{C}^3$ from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$, $s < -1/2$. As expected, since the dispersive and the dissipative operators are of the same order, this index coincides with the critical Sobolev space for the Burgers equation

$$u_t - u_{xx} + uu_x = 0,$$

see [5, 1]. This result is in a marked contrast with what occurs for the KdV-Burgers equation which is well-posed above $H^{-1}(\mathbb{R})$, see [11].

Now consider the general case $0 \leq \alpha \leq 2$. By running the approach of [11] combined with the smoothing relation obtained in [16], we can only get that the problem (dBO) is well-posed in $H^s(\mathbb{R})$ for $3/2 < \alpha \leq 2$ and $s > 1/2 - \alpha/2$. This was done by Otani in [17]. Here we improve this result by showing that (dBO) is globally well-posed in $H^s(\mathbb{R})$, for $1 < \alpha \leq 2$ and $s > -\alpha/4$. It is worth comparing (dBO) with the pure dissipative equation

$$\text{(1.1)} \qquad u_t + |D|^\alpha u + uu_x = 0.$$

In Appendix, we show that (1.1) with $1 < \alpha \leq 2$ is well-posed in $H^s(\mathbb{R})$ as soon as $s > 3/2 - \alpha$. The technics we use are very common in the context of semilinear parabolic problems and can be easily adapted to (dBO). In particular when $\alpha = 2$, this provides an alternative (and simpler) proof of our main result. When $\alpha < 2$, clearly we see that the dispersive part in (dBO) plays a key role in the low regularity of the solution.

We are going to perform a fixed point argument on the integral formulation of (dBO) in the weighted Sobolev space

$$\text{(1.2)} \qquad \|u\|_{X_\alpha^{b,s}} = \|\langle i(\tau - \xi|\xi|) + |\xi|^\alpha\rangle^b \langle\xi\rangle^s \mathcal{F}u(\tau, \xi)\|_{L^2(\mathbb{R}^2)}.$$

[1] Such spaces were first introduced by Molinet and Ribaud in [11] for the KdV-Burgers equation.

This will be achieved by deriving a bilinear estimate in these spaces. By Plancherel's theorem and duality, it reduces to estimating a weighted convolution of L^2 functions. In some regions where the dispersive effect is too weak to recover the lost derivative in the nonlinear term at low regularity ($s > -\alpha/4$), in particular when considering the high-high interactions, we are led to use a dyadic approach. In [20], Tao systematically studied some nonlinear dispersive equations like KdV, Schrödinger or wave equation by using such a dyadic decomposition and orthogonality. Following the spirit of Tao's works, we shall prove some estimates on dyadic blocks, which may be of independent interest. Indeed, we believe that they could certainly be used for other equations based on a Benjamin–Ono-type dispersion.

Next, we show that our well-posedness results turn out to be sharp. Adapting the arguments used in [13] to prove the ill-posedness of (BO), we find that the solution map $u_0 \mapsto u$ (if it exists) cannot be $\mathcal{C}^3$ at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ as soon as $s < -\alpha/4$. See also [3, 11, 12, 22] for situations where this method applies. Note that we need to prove the discontinuity of the third iterative term to obtain the condition $s < -\alpha/4$, whereas the second iterate is usually sufficient to get an optimal result. On the other hand, we prove using similar arguments, that in the case $0 \leq \alpha < 1$, the solution map fails to be $\mathcal{C}^2$ in any $H^s(\mathbb{R})$, $s \in \mathbb{R}$. This is mainly due to the fact that the operator $|D|^\alpha$ is too weak to counterbalance the lost derivative which appears in the nonlinear term $\partial_x u^2$.

1.2. Main results.

Let us now formally state our results.

Theorem 1.1. *Let $1 < \alpha \leq 2$ and $u_0 \in H^s(\mathbb{R})$ with $s > -\alpha/4$. Then for any $T > 0$, there exists a unique solution u of* (dBO) *in*

$$Z_T = \mathcal{C}([0, T]; H^s(\mathbb{R})) \cap X^{1/2,s}_{\alpha,T}.$$

Moreover, the map $u_0 \mapsto u$ is smooth from $H^s(\mathbb{R})$ to Z_T and u belongs to $\mathcal{C}((0, T], H^\infty(\mathbb{R}))$.

REMARK 1.1. The spaces $X^{b,s}_{\alpha,T}$ are restricted versions of $X^{b,s}_\alpha$ defined by the norm (1.2). See Section 1.3 for a precise definition.

REMARK 1.2. In [17], Otani studied a larger family of dispersive-dissipative equations taking the form

$$u_t - |D|^{1+a} u_x + |D|^\alpha u + u u_x = 0 \tag{1.3}$$

with $a \geq 0$ and $\alpha > 0$. He showed that (1.3) is globally well-posed in $H^s(\mathbb{R})$ provided $a + \alpha \leq 3$, $\alpha > (3 - a)/2$ and $s > -(a + \alpha - 1)/2$. If $a = 0$, it is clear that we get a better result, at least when $\alpha < 2$. It will be an interesting challenge to adapt our method of proofs to (1.3) in the case $a > 0$.

REMARK 1.3. Another interesting problem should be to consider the periodic dissipative BO equations

$$\begin{cases} u_t + \mathcal{H}u_{xx} + |D|^\alpha u + uu_x = 0, & t > 0,\ x \in \mathbb{T}, \\ u(0,\ \cdot\ ) = u_0 \in H^s(\mathbb{T}), \end{cases} \tag{1.4}$$

Recall that in [14], Molinet proved the global well-posedness of the periodic BO equation in $L^2(\mathbb{T})$. To our knowledge, equation (1.4) in the case $\alpha > 0$ has never been investigated.

Theorem 1.1 is sharp in the following sense.

Theorem 1.2. *Let $1 \le \alpha \le 2$ and $s < -\alpha/4$. There does not exist $T > 0$ such that the Cauchy problem* (dBO) *admits a unique local solution defined on the interval $[0, T]$ and such that the flow map $u_0 \mapsto u$ is of class $\mathcal{C}^3$ in a neighborhood of the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.*

In the case $0 \le \alpha < 1$, we have the following ill-posedness result.

Theorem 1.3. *Let $0 \le \alpha < 1$ and $s \in \mathbb{R}$. There does not exist $T > 0$ such that the Cauchy problem* (dBO) *admits a unique local solution defined on the interval $[0, T]$ and such that the flow map $u_0 \mapsto u$ is of class $\mathcal{C}^2$ in a neighborhood of the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.*

REMARK 1.4. At the end-point $\alpha = 1$, our proof of Theorem 1.3 fails. However, Theorem 1.2 provides the ill-posedness in $H^s(\mathbb{R})$, for $s < -1/4$. So, it is still not clear of what happens to (dBO) when $\alpha = 1$ and $s \ge -1/4$.

The structure of our paper is as follows. We introduce a few notation in the rest of this section. In Section 2, we recall some estimates related to the linear (dBO) equations. Next, we prove the crucial bilinear estimate in Section 3, which leads to the proof of Theorem 1.1 in Section 4. Section 5 is devoted to the ill-posedness results (Theorems 1.2 and 1.3). Finally, we briefly study the dissipative equation (1.1) in Appendix.

1.3. Notations. When writing $A \lesssim B$ (for A and B nonnegative), we mean that there exists $C > 0$ independent of A and B such that $A \le CB$. Similarly define $A \gtrsim B$ and $A \sim B$. If $A \subset \mathbb{R}^N$, $|A|$ denotes its Lebesgue measure and χ_A its characteristic function. For $f \in \mathcal{S}'(\mathbb{R}^N)$, we define its Fourier transform $\mathcal{F}(f)$ (or $\hat{f}$) by

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^N} e^{-i\langle x,\xi\rangle} f(x)\, dx.$$

The Lebesgue spaces are endowed with the norm

$$\|f\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |f(x)|^p \, dx\right)^{1/p}, \quad 1 \le p < \infty$$

with the usual modification for $p = \infty$. We also consider the space-time Lebesgue spaces $L^p_x L^q_t$ defined by

$$\|f\|_{L^p_x L^q_t} = \big\| \|f\|_{L^q_t(\mathbb{R})} \big\|_{L^p_x(\mathbb{R})}.$$

For $b, s \in \mathbb{R}$, we define the Sobolev spaces $H^s(\mathbb{R})$ and their space-time versions $H^{b,s}(\mathbb{R}^2)$ by the norms

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2},$$

$$\|u\|_{H^{b,s}} = \left(\int_{\mathbb{R}^2} \langle \tau \rangle^{2b} \langle \xi \rangle^{2s} |\hat{u}(\tau, \xi)|^2 \, d\tau \, d\xi\right)^{1/2},$$

with $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. Let $V(\cdot)$ be the free linear group associated to the linear Benjamin–Ono equation, i.e.

$$\forall t \in \mathbb{R}, \quad \mathcal{F}_x(V(t)\varphi)(\xi) = \exp(it\xi|\xi|)\hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'.$$

We will mainly work in the $X^{b,s}_\alpha$ space defined in (1.2), and in its restricted version $X^{b,s}_{\alpha,T}$, $T \ge 0$, equipped with the norm

$$\|u\|_{X^{b,s}_{\alpha,T}} = \inf_{w \in X^{b,s}_\alpha} \{\|w\|_{X^{b,s}_\alpha}, \ w(t) = u(t) \text{ on } [0, T]\}.$$

Note that since $\mathcal{F}(V(-t)u)(\tau, \xi) = \hat{u}(\tau + \xi|\xi|, \xi)$, we can re-express the norm of $X^{b,s}_\alpha$ as

$$\begin{aligned}
\|u\|_{X^{b,s}_\alpha} &= \|\langle i\tau + |\xi|^\alpha \rangle^b \langle \xi \rangle^s \hat{u}(\tau + \xi|\xi|, \xi)\|_{L^2(\mathbb{R}^2)} \\
&= \|\langle i\tau + |\xi|^\alpha \rangle^b \langle \xi \rangle^s \mathcal{F}(V(-t)u)(\tau, \xi)\|_{L^2(\mathbb{R}^2)} \\
&\sim \|V(-t)u\|_{H^{b,s}} + \|u\|_{L^2_t H^{s+\alpha b}_x}.
\end{aligned}$$

Finally, we denote by S_α the semigroup associated with the free evolution of (dBO),

$$\forall t \ge 0, \quad \mathcal{F}_x(S_\alpha(t)\varphi)(\xi) = \exp[it\xi|\xi| - |\xi|^\alpha t]\hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}',$$

and we extend S_α to a linear operator defined on the whole real axis by setting

$$\forall t \in \mathbb{R}, \quad \mathcal{F}_x(S_\alpha(t)\varphi)(\xi) = \exp[it\xi|\xi| - |\xi|^\alpha |t|]\hat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'. \tag{1.5}$$

2. Linear estimates

In this section, we collect together several linear estimates on the operators S_α introduced in (1.5) and L_α defined by

$$L_\alpha\colon\ f \mapsto \chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t S_\alpha(t-t')f(t')\,dt'.$$

Recall that (dBO) is equivalent to its integral formulation

$$u(t) = S_\alpha(t)u_0 - \frac{1}{2}\int_0^t S_\alpha(t-t')\partial_x(u^2(t'))\,dt'. \tag{2.1}$$

It will be convenient to replace the local-in-time integral equation (2.1) with a global-in-time truncated integral equation. Let ψ be a cutoff function such that

$$\psi \in \mathcal{C}_0^\infty(\mathbb{R}), \quad \operatorname{supp}\psi \subset [-2,2], \quad \psi \equiv 1 \text{ on } [-1,1],$$

and define $\psi_T(\,\cdot\,) = \psi(\,\cdot\,/T)$ for all $T > 0$. We can replace (2.1) on the time interval $[0,T]$, $T < 1$ by the equation

$$u(t) = \psi(t)\left[S_\alpha(t)u_0 - \frac{\chi_{\mathbb{R}_+}(t)}{2}\int_0^t S_\alpha(t-t')\partial_x(\psi_T^2(t')u^2(t'))\,dt'\right]. \tag{2.2}$$

Proofs of the results stated here can be obtained by a slight modification of the linear estimates derived in [11].

Lemma 2.1. *For all $s \in \mathbb{R}$ and all $\varphi \in H^s(\mathbb{R})$,*

$$\|\psi(t)S_\alpha(t)\varphi\|_{X_\alpha^{1/2,s}} \lesssim \|\varphi\|_{H^s}. \tag{2.3}$$

Lemma 2.2. *Let $s \in \mathbb{R}$. For all $0 < \delta < 1/2$ and all $v \in X_\alpha^{-1/2+\delta,s}$,*

$$\left\|\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t S_\alpha(t-t')v(t')\,dt'\right\|_{X_\alpha^{1/2,s}} \lesssim \|v\|_{X_\alpha^{-1/2+\delta,s}}. \tag{2.4}$$

To globalize our solution, we will need the next lemma.

Lemma 2.3. *Let $s \in \mathbb{R}$ and $\delta > 0$. Then for any $f \in X_\alpha^{-1/2+\delta,s}$,*

$$t \mapsto \int_0^t S_\alpha(t-t')f(t')\,dt' \in \mathcal{C}(\mathbb{R}_+; H^{s+\alpha\delta}).$$

Moreover, if (f_n) *is a sequence satisfying* $f_n \to 0$ *in* $X_\alpha^{-1/2+\delta,s}$*, then*

$$\left\| \int_0^t S_\alpha(t-t') f_n(t')\, dt' \right\|_{L^\infty(\mathbb{R}_+;H^{s+\alpha\delta})} \to 0.$$

3. Bilinear estimates

3.1. Dyadic blocks estimates. We introduce Tao's $[k;Z]$-multipliers theory [20] and derive the dyadic blocks estimates for the Benjamin–Ono equation.

Let Z be any abelian additive group with an invariant measure $d\eta$. For any integer $k \geq 2$ we define the hyperplane

$$\Gamma_k(Z) = \{(\eta_1, \ldots, \eta_k) \in Z^k : \eta_1 + \cdots + \eta_k = 0\}$$

which is endowed with the measure

$$\int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\eta_1, \ldots, \eta_{k-1}, -(\eta_1 + \cdots + \eta_{k-1}))\, d\eta_1 \cdots d\eta_{k-1}.$$

A $[k; Z]$-multiplier is defined to be any function $m : \Gamma_k(Z) \to \mathbb{C}$. The multiplier norm $\|m\|_{[k;Z]}$ is defined to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\eta) \prod_{j=1}^k f_j(\eta_j) \right| \leq \|m\|_{[k;Z]} \prod_{j=1}^k \|f_j\|_{L^2(Z)} \tag{3.1}$$

holds for all test functions $f_1, \ldots, f_k$ on Z. In other words,

$$\|m\|_{[k;Z]} = \sup_{\substack{f_j \in \mathcal{S}(Z) \\ \|f_j\|_{L^2(Z)} \leq 1}} \left| \int_{\Gamma_k(Z)} m(\eta) \prod_{j=1}^k f_j(\eta_j) \right|.$$

In his paper [20], Tao used the following notations. Capitalized variables N_j, L_j ($j = 1, \ldots, k$) are presumed to be dyadic, i.e. range over numbers of the form 2^l, $l \in \mathbb{Z}$. In this paper, we only consider the case $k = 3$, which corresponds to the quadratic nonlinearity in the equation. It will be convenient to define the quantities $N_{max} \geq N_{med} \geq N_{min}$ to be the maximum, median and minimum of N_1, N_2, N_3 respectively. Similarly, define $L_{max} \geq L_{med} \geq L_{min}$ whenever $L_1, L_2, L_3 > 0$. The quantities N_j will measure the magnitude of frequencies of our waves, while L_j measures how closely our waves approximate a free solution.

Here we consider $[3; \mathbb{R} \times \mathbb{R}]$-multipliers and we parameterize $\mathbb{R} \times \mathbb{R}$ by $\eta = (\tau, \xi)$ endowed with the Lebesgue measure $d\tau\, d\xi$. Define

$$h_0(\theta) = \theta|\theta|, \quad \lambda_j = \tau_j - h_0(\xi_j), \quad j = 1, 2, 3,$$

and the resonance function

$$h(\xi) = h_0(\xi_1) + h_0(\xi_2) + h_0(\xi_3), \quad \xi = (\xi_1, \xi_2, \xi_3).$$

By a dyadic decomposition of the variables ξ_j, λ_j, $h(\xi)$, we will be led to estimate

$$\|X_{N_1,N_2,N_3,H,L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \tag{3.2}$$

where

$$X_{N_1,N_2,N_3,H,L_1,L_2,L_3} = \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j|\sim N_j} \chi_{|\lambda_j|\sim L_j}. \tag{3.3}$$

From the identities

$$\xi_1 + \xi_2 + \xi_3 = 0 \tag{3.4}$$

and

$$\lambda_1 + \lambda_2 + \lambda_3 + h(\xi) = 0$$

on the support of the multiplier, we see that (3.3) vanishes unless

$$N_{max} \sim N_{med} \tag{3.5}$$

and

$$L_{max} \sim \max(H, L_{med}). \tag{3.6}$$

Lemma 3.1. *On the support of $X_{N_1,N_2,N_3,H,L_1,L_2,L_3}$, one has*

$$H \sim N_{max} N_{min}. \tag{3.7}$$

Proof. Recall that

$$h(\xi) = \xi_1|\xi_1| + \xi_2|\xi_2| + \xi_3|\xi_3|.$$

By symmetry, we can assume $|\xi_3| \sim N_{min}$. This forces by (3.4) $\xi_1\xi_2 < 0$. Suppose for example $\xi_1 > 0$ and $\xi_2 < 0$ (the other case being similar). Then if $\xi_3 > 0$,

$$h(\xi) = \xi_1^2 - \xi_2^2 + \xi_3^2 = \xi_1^2 - (\xi_1 + \xi_3)^2 + \xi_3^2 = -2\xi_1\xi_3$$

and in this case $|h(\xi)| \sim N_{max} N_{min}$. Now if $\xi_3 < 0$, then

$$h(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2 = (\xi_2 + \xi_3)^2 - \xi_2^2 - \xi_3^2 = 2\xi_2\xi_3$$

and it follows again that $|h(\xi)| \sim N_{max} N_{min}$. □

Let us now recall some lemmas proved in [20].

Lemma 3.2 (Comparison principle). *If m and M are $[k;Z]$-multipliers, and $|m(\xi)| \le M(\xi)$ for all $\xi \in \Gamma_k(Z)$, then $\|m\|_{[k;Z]} \le \|M\|_{[k;Z]}$.*

Lemma 3.3 (Tensor products). *Let Z_1, Z_2 be abelian groups, with $Z_1 \times Z_2$ parameterized by (ξ^1, ξ^2), and m_1, m_2 be $[k; Z_1]$ and $[k; Z_2]$ multipliers respectively. Define the tensor product $m_1 \otimes m_2$ to be the $[k; Z_1 \times Z_2]$ multiplier*

$$m_1 \otimes m_2((\xi_1^1, \xi_1^2), \dots, (\xi_k^1, \xi_k^2)) = m_1(\xi_1^1, \dots, \xi_k^1) m_2(\xi_1^2, \dots, \xi_k^2).$$

Then we have

$$\|m_1 \otimes m_2\|_{[k, Z_1 \times Z_2]} = \|m_1\|_{[k;Z_1]} \|m_2\|_{[k;Z_2]}.$$

Lemma 3.4. *For any function $m(\xi)$ from Z to $\mathbb{R}$, we have $\|m(\xi_1)\|_{[3;Z]} = \|m\|_{L^2}$.*

Lemma 3.5 (Box localization). *Suppose $(R+\eta)_{\eta \in \Sigma}$ is a box covering of Z (so Σ is a discrete subgroup of Z), and m is a $[k;Z]$-multiplier such that each $\mathrm{supp}_j(m)$ is contained in a box in this covering for all $1 \le j \le k-2$. Then*

$$\|m\|_{[k;Z]} \sim \sup_{\eta_{k-1}, \eta_k} \|m(\xi) \chi_{R+\eta_{k-1}}(\xi_{k-1}) \chi_{R+\eta_k}(\xi_k)\|_{[k;Z]}.$$

Lemma 3.6. *For any complex functions $m_1(\xi)$, $m_2(\xi)$ on Z we have*

$$\frac{\| |m_1|^2 * |m_2|^2 \|_{L^2}}{\| |m_1|^2 * |m_2|^2 \|_{L^1}^{1/2}} \le \|m_1(\xi_1) m_2(\xi_2)\|_{[3;Z]} \le \| |m_1|^2 * |m_2|^2 \|_{L^\infty}^{1/2}.$$

We are now ready to state the fundamental dyadic blocks estimates for the Benjamin–Ono equation.

Proposition 3.1. *Let $N_1, N_2, N_3, H, L_1, L_2, L_3 > 0$ satisfying* (3.5), (3.6), (3.7).

1. *In the high modulation case $L_{max} \sim L_{med} \gg H$, we have*

$$(3.2) \lesssim L_{min}^{1/2} N_{min}^{1/2}. \tag{3.8}$$

2. *In the low modulation case $L_{max} \sim H$,*
   (a) ((++) *coherence) if $N_{max} \sim N_{min}$, then*

$$(3.2) \lesssim L_{min}^{1/2} L_{med}^{1/4}, \tag{3.9}$$

(b) *((+−) coherence) if* $N_2 \sim N_3 \gg N_1$ *and* $H \sim L_1 \gtrsim L_2, L_3$, *we have for any* $\gamma > 0$

$$(3.2) \lesssim L_{min}^{1/2} \min\big(N_{min}^{1/2}, N_{max}^{1/2-1/2\gamma} N_{min}^{-1/2\gamma} L_{med}^{1/2\gamma}\big). \tag{3.10}$$

Similarly for permutations of the indexes $\{1, 2, 3\}$.
(c) *In all other cases, the multiplier* (3.3) *vanishes.*

Proof. First we consider the high modulation case $L_{max} \sim L_{med} \gg H$. Suppose for the moment that $L_1 \geq L_2 \geq L_3$ and $N_1 \geq N_2 \geq N_3$. By using the comparison principle (Lemma 3.2), we have

$$(3.2) \lesssim \| \chi_{|\xi_3|\sim N_3} \chi_{|\lambda_3|\sim L_3} \|_{[3;\mathbb{R}\times\mathbb{R}]}.$$

By Lemma 3.3 and Lemma 3.4,

$$(3.2) \lesssim \big\| \| \chi_{|\lambda_3|\sim L_3} \|_{[3;\mathbb{R}]} \chi_{|\xi_3|\sim N_3} \big\|_{[3;\mathbb{R}]} \lesssim L_3^{1/2} N_3^{1/2}.$$

It is clear from symmetry that (3.8) holds for any choice of L_j and N_j, $j = 1, 2, 3$.

Now we turn to the low modulation case $H \sim L_{max}$. Suppose for the moment that $N_1 \geq N_2 \geq N_3$. The ξ_3 variable is currently localized to the annulus $\{|\xi_3| \sim N_3\}$. By a finite partition of unity we can restrict it further to a ball $\{|\xi_3 - \xi_3^0| \ll N_3\}$ for some $|\xi_3^0| \sim N_3$. Then by box localization (Lemma 3.5) we may localize ξ_1, ξ_2 similarly to regions $\{|\xi_1 - \xi_1^0| \ll N_3\}$ and $\{|\xi_2 - \xi_2^0| \ll N_3\}$ where $|\xi_j^0| \sim N_j$. We may assume that $|\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_3$ since we have $\xi_1 + \xi_2 + \xi_3 = 0$. We summarize this symmetrically as

$$(3.2) \lesssim \Big\| \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j - \xi_j^0| \ll N_{min}} \chi_{|\lambda_j|\sim L_j} \Big\|_{[3;\mathbb{R}\times\mathbb{R}]}$$

for some ξ_j^0 satisfying

$$|\xi_j^0| \sim N_j \text{ for } j = 1, 2, 3; \quad |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min}.$$

Without loss of generality, we assume $L_1 \geq L_2 \geq L_3$. By Lemma 3.3, Lemma 3.2 and Lemma 3.6, we get

$$\begin{aligned}(3.2) &\lesssim \Big\| \chi_{|h(\xi)|\sim H} \prod_{j=2}^{3} \chi_{|\xi_j - \xi_j^0| \ll N_{min}} \chi_{|\lambda_j|\sim L_j} \Big\|_{[3;\mathbb{R}\times\mathbb{R}]} \\ &\lesssim |\{(\tau_2, \xi_2)\colon |\xi_2 - \xi_2^0| \ll N_{min},\ |\tau_2 - h_0(\xi_2)| \sim L_2, \\ &\qquad |\xi - \xi_2 - \xi_3^0| \ll N_{min},\ |\tau - \tau_2 - h_0(\xi - \xi_2)| \sim L_3\}|^{1/2}\end{aligned}$$

for some $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}$. For fixed ξ_2, the set of possible τ_2 ranges in an interval of length $O(L_3)$ and vanishes unless

$$h_0(\xi_2) + h_0(\xi - \xi_2) = \tau + O(L_2).$$

On the other hand, inequality $|\xi - \xi_2 - \xi_3^0| \ll N_{min}$ implies $|\xi + \xi_1^0| \ll N_{min}$, hence

$$(3.2) \lesssim L_3^{1/2} |\Omega_\xi|^{1/2}$$

for some ξ such that $|\xi + \xi_1^0| \ll N_{min}$ (in particular $|\xi| \sim N_1$) and with

$$\Omega_\xi = \{\xi_2 \colon |\xi_2 - \xi_2^0| \ll N_{min},\ h_0(\xi_2) + h_0(\xi - \xi_2) = \tau + O(L_2)\}.$$

Let us write $\Omega_\xi = \Omega_\xi^1 \cup \Omega_\xi^2$ with

$$\Omega_\xi^1 = \{\xi_2 \in \Omega_\xi : \xi_2(\xi - \xi_2) > 0\},$$
$$\Omega_\xi^2 = \{\xi_2 \in \Omega_\xi : \xi_2(\xi - \xi_2) < 0\}.$$

We only need to consider the three cases $N_1 \sim N_2 \sim N_3$, $N_2 \sim N_3 \gg N_1$ and $N_1 \sim N_2 \gg N_3$ (the case $N_1 \sim N_3 \gg N_2$ follows by symmetry).

Estimate of $|\Omega_\xi^1|$: In Ω_ξ^1 we can assume $\xi_2 > 0$ and $\xi - \xi_2 > 0$ (the other case being similar). Then we have

$$h_0(\xi_2) + h_0(\xi - \xi_2) = \xi_2^2 + (\xi - \xi_2)^2 = 2\left(\xi_2 - \frac{\xi}{2}\right)^2 + \frac{\xi^2}{2}$$

and thus

$$2\left(\xi_2 - \frac{\xi}{2}\right)^2 + \frac{\xi^2}{2} = \tau + O(L_2). \tag{3.11}$$

If $N_1 \sim N_2 \sim N_3$, we see from (3.11) that ξ_2 variable is contained in the union of two intervals of length $O(L_2^{1/2})$ at worst. Therefore $|\Omega_\xi^1| \lesssim L_2^{1/2}$ in this case. If $N_1 \sim N_2 \gg N_3$, then

$$\begin{aligned} \left|\left(\xi_2 - \frac{\xi}{2}\right) + \frac{\xi_1^0}{2}\right| &\le \left|\xi_2 - \xi_2^0 - \frac{\xi + \xi_1^0}{2} - \xi_3^0\right| + |\xi_1^0 + \xi_2^0 + \xi_3^0| \\ &\le |\xi_2 - \xi_2^0| + \frac{1}{2}|\xi + \xi_1^0| + |\xi_3^0| + |\xi_1^0 + \xi_2^0 + \xi_3^0| \\ &\lesssim N_3 \end{aligned}$$

and we get $|\xi_2 - \xi/2| \sim N_1$. From (3.11), we see that we must have $N_1^2 = O(L_2)$, which is in contradiction with $L_2 \lesssim L_1 \sim N_{max} N_{min}$. We deduce that the multiplier vanishes in this region. If $N_2 \sim N_3 \gg N_1$, then we obviously have $|\xi_2 - \xi/2| \sim N_2$ and, in the same way, the multiplier vanishes.

Estimate of $|\Omega^2_\xi|$: We can assume $\xi_2 > 0$ and $\xi - \xi_2 < 0$. It follows that

$$h_0(\xi_2) + h_0(\xi - \xi_2) = \xi_2^2 - (\xi - \xi_2)^2 = 2\xi\left(\xi_2 - \frac{\xi}{2}\right) = \tau + O(L_2). \tag{3.12}$$

If $N_1 \sim N_2 \sim N_3$, we see from (3.12) that ξ_2 variable is contained in the union of two intervals of length $O(N_1^{-1}L_2)$ at worst. But we have $L_2 \lesssim L_1 \sim N_1^2$ and thus $|\Omega^2_\xi| \lesssim L_2^{1/2}$ in this region. If $N_1 \sim N_2 \gg N_3$, we have $|\xi_2 - \xi/2| \sim N_1$ as previously and thus $N_1^2 = O(L_2)$, the multiplier vanishes. If $N_2 \sim N_3 \gg N_1$, then $|\xi_2 - \xi/2| \sim N_2$ and for any $\gamma > 0$, we have $|\xi_2 - \xi/2| \sim N_2^{1-\gamma}|\xi_2 - \xi/2|^\gamma$. Therefore we see from (3.12) that ξ_2 variable is contained in the union of two intervals of length $O(N_2^{1-1/\gamma}N_1^{-1/\gamma}L_2^{1/\gamma})$ at worst, and from $|\xi_2 - \xi_2^0| \ll N_{min}$ we see that $|\Omega^2_\xi| \lesssim N_{min}^{1/2}$, and (3.10) follows. □

3.2. Bilinear estimate. In this section we prove the following crucial bilinear estimate.

Theorem 3.1. *Let $1 < \alpha \le 2$ and $s > -\alpha/4$. For all $T > 0$, there exist $\delta, \nu > 0$ such that for all $u, v \in X_\alpha^{1/2,s}$ with compact support (in time) in $[-T, +T]$,*

$$\|\partial_x(uv)\|_{X_\alpha^{-1/2+\delta,s}} \lesssim T^\nu \|u\|_{X_\alpha^{1/2,s}} \|v\|_{X_\alpha^{1/2,s}}. \tag{3.13}$$

To get the required contraction factor T^ν in our estimates, the next lemma is very useful (see [17]).

Lemma 3.7. *Let $f \in L^2(\mathbb{R}^2)$ with compact support (in time) in $[-T, +T]$. For any $\theta > 0$, there exists $\nu = \nu(\theta) > 0$ such that*

$$\left\| \mathcal{F}^{-1}\left(\frac{\hat{f}(\tau, \xi)}{\langle \tau - \xi|\xi| \rangle^\theta} \right) \right\|_{L^2_{xt}} \lesssim T^\nu \|f\|_{L^2_{xt}}.$$

Proof of Theorem 3.1. By duality, Plancherel and Lemma 3.7, it suffices to show that

$$\left\| \frac{\xi_3 \langle \xi_3 \rangle^s \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s}}{\langle |\lambda_1| + |\xi_1|^\alpha \rangle^{1/2} \langle |\lambda_2| + |\xi_2|^\alpha \rangle^{1/2} \langle |\lambda_3| + |\xi_3|^\alpha \rangle^{1/2-\delta}} \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

By a dyadic decomposition of the variables ξ_j, λ_j, $h(\xi)$, we may assume $|\xi_j| \sim N_j$, $|\lambda_j| \sim L_j$ and $|h(\xi)| \sim H$. By the translation invariance of the $[k, Z]$-multiplier norm, we can always restrict our estimate on $L_j \gtrsim 1$ and $N_{max} \gtrsim 1$. The comparison principle and orthogonality reduce our estimate to show that

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_1,L_2,L_3\gtrsim 1} \frac{N_3\langle N_3\rangle^s\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}}{(L_1+\langle N_1\rangle^\alpha)^{1/2}(L_2+\langle N_2\rangle^\alpha)^{1/2}(L_3+\langle N_3\rangle^\alpha)^{1/2-\delta}} \times \|X_{N_1,N_2,N_3,L_{max},L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \tag{3.14}$$

and

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}}\ \sum_{H\ll L_{max}} \frac{N_3\langle N_3\rangle^s\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}}{(L_1+\langle N_1\rangle^\alpha)^{1/2}(L_2+\langle N_2\rangle^\alpha)^{1/2}(L_3+\langle N_3\rangle^\alpha)^{1/2-\delta}} \times \|X_{N_1,N_2,N_3,H,L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \tag{3.15}$$

are bounded, for all $N\gtrsim 1$.

We first show that (3.15) $\lesssim 1$. For $s>-1/2$, one has

$$N_3\langle N_3\rangle^s\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}\lesssim\langle N_{min}\rangle^{-s}N_{max}$$

and we get from (3.8),

$$\begin{aligned}(3.15) &\lesssim \sum_{N_{max}\sim N}\ \sum_{L_{max}\gg NN_{min}} \frac{\langle N_{min}\rangle^{-s}NL_{min}^{1/2}N_{min}^{1/2}}{L_{min}^{1/2}(L_{max}+N^\alpha)^{1/2-\delta}(L_{max}+\langle N_{min}\rangle^\alpha)^{1/2-\delta}L_{max}^\delta}\\ &\lesssim \sum_{N_{min}>0}\frac{N_{min}^{1/2}\langle N_{min}\rangle^{-s}N}{(NN_{min}+N^\alpha)^{1/2-\delta}(NN_{min}+\langle N_{min}\rangle^\alpha)^{1/2-\delta}}.\end{aligned}$$

When $N_{min}\lesssim 1$, we get

$$\begin{aligned}(3.15) &\lesssim \sum_{N_{min}\lesssim 1}\frac{N_{min}^{1/2}N}{N^{\alpha/2-\alpha\delta}(NN_{min})^{1/2-\delta}}\\ &\lesssim \sum_{N_{min}\lesssim 1} N_{min}^\delta N^{(1-\alpha)/2+\delta(\alpha+1)}\\ &\lesssim 1\end{aligned}$$

for $\delta\ll 1$ and $\alpha>1$. When $N_{min}\gtrsim 1$, then

$$\begin{aligned}(3.15) &\lesssim \sum_{N_{min}\gtrsim 1}\frac{N_{min}^{1/2-s}N}{(NN_{min})^{1/2-\delta-\varepsilon}N^{\alpha\varepsilon}(NN_{min})^{1/2-\delta}}\\ &\lesssim \sum_{N_{min}\gtrsim 1} N_{min}^{-1/2-s+2\delta+\varepsilon}N^{2\delta-\varepsilon(\alpha-1)}\\ &\lesssim 1\end{aligned}$$

for $\varepsilon = 2\delta/(\alpha - 1) > 0$, $\delta \ll 1$ and $s > -1/2$.

Now we show that (3.14) $\lesssim 1$. We first deal with the contribution where (3.9) holds. In this case $N_{min} \sim N_{max}$ and we get

$$\begin{aligned}(3.14) &\lesssim \sum_{L_{max}\sim N^2} \frac{N^{1-s} L_{min}^{1/2} L_{med}^{1/4}}{L_{min}^{1/2}(L_{med} + N^\alpha)^{1/2}(L_{max} + N^\alpha)^{1/2-2\delta} L_{max}^{\delta}}\\ &\lesssim \frac{N^{1-s}}{N^{\alpha/4} N^{1-4\delta}}\\ &\lesssim N^{-s-\alpha/4+4\delta} \lesssim 1\end{aligned}$$

for $s > -\alpha/4$ and $\delta \ll 1$.

Now we consider the contribution where (3.10) applies. By symmetry it suffices to treat the two cases

$$N_1 \sim N_2 \gg N_3,\ H \sim L_3 \gtrsim L_1, L_2,$$
$$N_2 \sim N_3 \gg N_1,\ H \sim L_1 \gtrsim L_2, L_3.$$

In the first case, estimate (3.10) applied with $\gamma = 1$ yields

$$(3.2) \lesssim L_{min}^{1/2} \min(N_3^{1/2}, N_3^{-1/2} L_{med}^{1/2}) \lesssim L_{min}^{1/2} N_3^{1/4} N_3^{-1/4} L_{med}^{1/4} \sim L_{min}^{1/2} L_{med}^{1/4}$$

and thus

$$\begin{aligned}(3.14) &\lesssim \sum_{N_3>0} \sum_{L_{max}\sim N N_3} \frac{N_3 \langle N_3\rangle^s N^{-2s} L_{min}^{1/2} L_{med}^{1/4}}{L_{min}^{1/2}(L_{med} + N^\alpha)^{1/2}(L_{max} + \langle N_{min}\rangle^\alpha)^{1/2-2\delta} L_{max}^{\delta}}\\ &\lesssim \sum_{N_3>0} \frac{N_3 \langle N_3\rangle^s N^{-2s}}{N^{\alpha/4}(N N_3)^{1/2-2\delta}}\\ &\lesssim \sum_{N_3>0} N_3^{1/2+2\delta} \langle N_3\rangle^s N^{-2s-\alpha/4-1/2+2\delta}.\end{aligned}$$

Since $-2s - \alpha/4 - 1/2 + 2\delta < 0$, we may write

$$\begin{aligned}(3.14) &\lesssim \sum_{N_3>0} N_3^{1/2+2\delta} \langle N_3\rangle^{-s-\alpha/4-1/2+2\delta}\\ &\lesssim \sum_{N_3\lesssim 1} N_3^{1/2+2\delta} + \sum_{N_3\gtrsim 1} N_3^{-s-\alpha/4+4\delta}\\ &\lesssim 1\end{aligned}$$

for $\delta \ll 1$ and $s > -\alpha/4$.

Finally consider the case $N_2 \sim N_3 \gg N_1$, $H \sim L_1 \gtrsim L_2, L_3$. Let $0 < \gamma \ll 1$. If we assume $N_{min}^{1/2} \lesssim N_{max}^{1/2-1/2\gamma} N_{min}^{-1/2\gamma} L_{med}^{1/2\gamma}$, i.e. $L_{med} \gtrsim N_{max}^{1-\gamma} N_{min}^{1+\gamma}$, then we get from (3.10) that

$$
\begin{aligned}
(3.14) &\lesssim \sum_{N_1>0} \sum_{L_{max}\sim NN_1} \frac{\langle N_1\rangle^{-s} N L_{min}^{1/2} N_1^{1/2}}{L_{min}^{1/2}(L_{med}+N^\alpha)^{1/2-\delta} L_{max}^{1/2-\delta} L_{max}^{\delta}} \\
&\lesssim \sum_{N_1>0} \frac{N_1^{1/2}\langle N_1\rangle^{-s} N}{(N^{1-\gamma}N_1^{1+\gamma}+N^\alpha)^{1/2-\delta}(NN_1)^{1/2-\delta}} \\
&\lesssim \sum_{N_1>0} \frac{N_1^{\delta}\langle N_1\rangle^{-s} N^{1/2+\delta}}{(N^{1-\gamma}N_1^{1+\gamma}+N^\alpha)^{1/2-\delta}}.
\end{aligned}
$$

If $N_1 \lesssim 1$, then

$$
(3.14) \lesssim \sum_{N_1\lesssim 1} N_1^{\delta} N^{(1-\alpha)/2+\delta(1+\alpha)} \lesssim 1
$$

for $\delta \ll 1$ and $\alpha > 1$. If $N_1 \gtrsim 1$, then

$$
\begin{aligned}
(3.14) &\lesssim \sum_{N_1\gtrsim 1} \frac{N_1^{-s+\delta} N^{1/2+\delta}}{(N^{1-\gamma}N_1^{1+\gamma})^{1/2-\delta-\varepsilon} N^{\alpha\varepsilon}} \\
&\lesssim \sum_{N_1\gtrsim 1} N_1^{-s-1/2+(1+\gamma)(\delta+\varepsilon)+\delta-\gamma/2} N^{\gamma(1/2-\delta)+2\delta-\varepsilon(\alpha-1+\gamma)} \\
&\lesssim 1
\end{aligned}
$$

for $\delta, \gamma \ll 1$, $s > -1/2$ and $\varepsilon = [2\delta + \gamma(1/2-\delta)]/(\alpha - 1 + \gamma) > 0$. If we assume $N_{min}^{1/2} \gtrsim N_{max}^{1/2-1/2\gamma} N_{min}^{-1/2\gamma} L_{med}^{1/2\gamma}$, i.e. $L_{med} \lesssim N_{max}^{1-\gamma} N_{min}^{1+\gamma}$, we get

$$
\begin{aligned}
(3.14) &\lesssim \sum_{N_1>0} \sum_{L_{max}\sim NN_1} \frac{\langle N_1\rangle^{-s} N L_{min}^{1/2} N^{1/2-1/2\gamma} N_1^{-1/2\gamma} L_{med}^{1/2\gamma}}{L_{min}^{1/2}(L_{med}+N^\alpha)^{1/2-\delta} L_{max}^{1/2-\delta} L_{max}^{\delta}} \\
&\lesssim \sum_{N_1>0} \sum_{L_{med}\lesssim N^{1-\gamma}N_1^{1+\gamma}} \frac{N_1^{-1/2\gamma-1/2+\delta}\langle N_1\rangle^{-s} N^{1-1/2\gamma+\delta} L_{med}^{1/2\gamma}}{(L_{med}+N^\alpha)^{1/2-\delta}}.
\end{aligned}
$$

When $N_1 \lesssim 1$, we have

$$
\begin{aligned}
(3.14) &\lesssim \sum_{N_1\lesssim 1} N_1^{-1/2\gamma-1/2+\delta} N^{1-1/2\gamma+\delta} N^{-\alpha/2+\alpha\delta} (N^{1-\gamma}N_1^{1+\gamma})^{1/2\gamma} \\
&\lesssim \sum_{N_1\lesssim 1} N_1^{\delta} N^{(1-\alpha)/2+\delta(1+\alpha)} \lesssim 1
\end{aligned}
$$

for $\delta \ll 1$ and $\alpha > 1$. When $N_1 \gtrsim 1$, then

$$\begin{aligned}(3.14) &\lesssim \sum_{N_1 \gtrsim 1} N_1^{-s-1/2-1/2\gamma+\delta} N^{1-1/2\gamma+\delta} (N^{1-\gamma} N_1^{1+\gamma})^{1/2\gamma-1/2+\delta+\varepsilon} N^{-\alpha\varepsilon} \\ &\lesssim \sum_{N_1 \gtrsim 1} N_1^{-s-1/2+(1+\gamma)(\delta+\varepsilon)+\delta-\gamma/2} N^{\gamma(1/2-\delta)+2\delta-\varepsilon(\alpha-1+\gamma)} \\ &\lesssim 1\end{aligned}$$

as previously. This completes the proof of Theorem 3.1. □

4. Proof of Theorem 1.1

In this section, we sketch the proof of Theorem 1.1 (see for instance [11] for the details).

Actually, local existence of a solution is a consequence of the following modified version of Theorem 3.1.

Proposition 4.1. *Given $s_c^+ > -\alpha/4$, there exist $\nu, \delta > 0$ such that for any $s \geq s_c^+$ and any $u, v \in X_\alpha^{1/2,s}$ with compact support in $[-T, +T]$,*

$$\|\partial_x(uv)\|_{X_\alpha^{-1/2+\delta,s}} \lesssim T^\nu \Big(\|u\|_{X_\alpha^{1/2,s_c^+}} \|v\|_{X_\alpha^{1/2,s}} + \|u\|_{X_\alpha^{1/2,s}} \|v\|_{X_\alpha^{1/2,s_c^+}} \Big). \tag{4.1}$$

Estimate (4.1) is obtained thanks to (3.13) and the triangle inequality

$$\forall s \geq s_c^+, \quad \langle\xi\rangle^s \leq \langle\xi\rangle^{s_c^+} \langle\xi_1\rangle^{s-s_c^+} + \langle\xi\rangle^{s_c^+} \langle\xi-\xi_1\rangle^{s-s_c^+}.$$

Let $u_0 \in H^s(\mathbb{R})$ with $s > -\alpha/4$. Define $F(u)$ as

$$F(u) = \psi(t)\bigg[S_\alpha(t)u_0 - \frac{\chi_{\mathbb{R}_+}(t)}{2} \int_0^t S_\alpha(t-t')\partial_x(\psi_T^2(t')u^2(t'))\,dt' \bigg].$$

We shall prove that for $T \ll 1$, F is contraction in a ball of the Banach space

$$Z = \Big\{ u \in X_\alpha^{1/2,s} : \|u\|_Z = \|u\|_{X_\alpha^{1/2,s_c^+}} + \gamma\|u\|_{X_\alpha^{1/2,s}} < +\infty \Big\},$$

where γ is defined for all nontrivial φ by

$$\gamma = \frac{\|\varphi\|_{H^{s_c^+}}}{\|\varphi\|_{H^s}}.$$

Combining (2.3), (2.4) as well as (4.1), it is easy to derive that

$$\|F(u)\|_Z \le C(\|u_0\|_{H^{s_c^+}} + \gamma\|u_0\|_{H^s}) + CT^\nu\|u\|_Z^2$$

and

$$\|F(u) - F(v)\|_Z \le CT^\nu\|u - v\|_Z\,\|u + v\|_Z$$

for some $C, \nu > 0$. Thus, taking $T = T\big(\|u_0\|_{H^{s_c^+}}\big)$ small enough, we deduce that F is contractive on the ball of radius $4C\|u_0\|_{H^{s_c^+}}$ in Z. This proves the existence of a solution u to $u = F(u)$ in $X^{1/2,s}_{\alpha,T}$.

Following similar arguments of [11], it is not too difficult to see that if $u_1, u_2 \in X^{1/2,s}_{\alpha,T}$ are solutions to (2.2) and $0 < \delta < T/2$, then there exists $\nu > 0$ such that

$$\|u_1 - u_2\|_{X^{1/2,s}_{\alpha,\delta}} \lesssim T^\nu\Big(\|u_1\|_{X^{1/2,s}_{\alpha,T}} + \|u_2\|_{X^{1/2,s}_{\alpha,T}}\Big)\|u_1 - u_2\|_{X^{1/2,s}_{\alpha,\delta}},$$

which leads to $u_1 \equiv u_2$ on $[0, \delta]$, and then on $[0, T]$ by iteration. This proves the uniqueness of the solution.

It is straightforward to check that $S_\alpha(\cdot)u_0 \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{R})) \cap \mathcal{C}(\mathbb{R}_+^*; H^\infty(\mathbb{R}))$. Then it follows from Theorem 3.1, Lemma 2.3 and the local existence of the solution that

$$u \in \mathcal{C}([0, T]; H^s(\mathbb{R})) \cap \mathcal{C}((0, T]; H^{s+\alpha\delta}(\mathbb{R}))$$

for some $T = T\big(\|u_0\|_{H^{s_c^+}}\big)$. By induction, we have $u \in \mathcal{C}((0, T]; H^\infty(\mathbb{R}))$. Taking the L^2-scalar product of (dBO) with u, we obtain that $t \mapsto \|u(t)\|_{H^{s_c^+}}$ is nonincreasing on $(0, T]$. Since the existence time of the solution depends only on the norm $\|u_0\|_{H^{s_c^+}}$, this implies that the solution can be extended globally in time.

5. Ill-posedness results

This section is devoted to the proof of Theorems 1.2 and 1.3. We adopt the notation $p(\xi) = \xi|\xi|$.

Assume that u is a solution to (dBO) such that the solution map $u_0 \mapsto u$ is of class $\mathcal{C}^k$ ($k = 2$ or $k = 3$) at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. The relation

$$F(u, \varphi) := u(t, \varphi) - S_\alpha(t)\varphi + \frac{1}{2}\int_0^t S_\alpha(t - t')\partial_x(u^2(t', \varphi))\,dt' \equiv 0$$

combined with implicit function theorem gives

$$u_1(t, x) := \frac{\partial u}{\partial\varphi}(t, x, 0)[h] = S_\alpha(t)h,$$
$$u_2(t, x) := \frac{\partial^2 u}{\partial\varphi^2}(t, x, 0)[h, h] = \int_0^t S_\alpha(t - t')\partial_x(u_1(t'))^2\,dt',$$

$$u_3(t,x) := \frac{\partial^3 u}{\partial \varphi^3}(t,x,0)[h,h,h] = \int_0^t S_\alpha(t-t')\partial_x(u_1(t')u_2(t'))\,dt',$$

etc.

Since the solution map is $\mathcal{C}^k$, we must have

$$\|u_k(t)\|_{H^s} \lesssim \|h\|_{H^s}^k, \quad \forall h \in H^s(\mathbb{R}). \tag{5.1}$$

In the sequel, we will show that (5.1) fails in the case $0 \le \alpha < 1$ and $k = 2$, and in the case $1 \le \alpha \le 2$, $k = 3$ and $s < -\alpha/4$.

5.1. The case $0 \le \alpha < 1$. It suffices to show the following lemma.

Lemma 5.1. *Let $0 \le \alpha < 1$ and $s \in \mathbb{R}$. There exists a sequence of functions $\{h_N\} \subset H^s(\mathbb{R})$ such that for all $T > 0$,*

$$\|h_N\|_{H^s} \lesssim 1,$$

and

$$\lim_{N\to\infty} \sup_{[0,T]} \left\| \int_0^t S_\alpha(t-t')\partial_x(S_\alpha(t')h_N)^2\,dt' \right\|_{H^s} = +\infty.$$

Proof. We define h_N by its Fourier transform[2]

$$\hat{h}_N(\xi) = \gamma^{-1/2}\chi_{I_1}(\xi) + \gamma^{-1/2}N^{-s}\chi_{I_2}(\xi)$$

with $I_1 = [\gamma/2, \gamma]$, $I_2 = [N, N+\gamma]$ and $N \gg 1$, $\gamma \ll N$ to be chosen later. Then it is clear that $\|h_N\|_{H^s} \sim 1$. Computing the Fourier transform of $u_2(t)$ leads to

$$\begin{aligned}
&\mathcal{F}_x(u_2(t))(\xi) \\
&= c\xi \int_0^t e^{i(t-t')p(\xi)}e^{-(t-t')|\xi|^\alpha}(e^{it'p(\xi)}e^{-t'|\xi|^\alpha}\hat{h}_N)^{*2}(\xi)\,dt' \\
&= c\xi e^{itp(\xi)}e^{-t|\xi|^\alpha}\int_{\mathbb{R}} \hat{h}_N(\xi_1)\hat{h}_N(\xi-\xi_1) \\
&\quad\times \int_0^t e^{it'(p(\xi_1)+p(\xi-\xi_1)-p(\xi))}e^{-t'(|\xi_1|^\alpha+|\xi-\xi_1|^\alpha-|\xi|^\alpha)}\,dt'\,d\xi_1 \\
&= c\xi e^{itp(\xi)}e^{-t|\xi|^\alpha}\int_{\mathbb{R}} \hat{h}_N(\xi_1)\hat{h}_N(\xi-\xi_1) \\
&\qquad\qquad\times \frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))}e^{-t(|\xi_1|^\alpha+|\xi-\xi_1|^\alpha-|\xi|^\alpha)}-1}{i(p(\xi_1)+p(\xi-\xi_1)-p(\xi))-(|\xi_1|^\alpha+|\xi-\xi_1|^\alpha-|\xi|^\alpha)}\,d\xi_1.
\end{aligned}$$

[2]As noticed in [13], h_N is not a real-valued function but the analysis works as well for $\Re e\, h_N$ instead of h_N.

Set

$$\chi(\xi, \xi_1) = i(p(\xi_1) + p(\xi - \xi_1) - p(\xi)) - (|\xi_1|^\alpha + |\xi - \xi_1|^\alpha - |\xi|^\alpha).$$

By support considerations, we have $\|u_2(t)\|_{H^s} \geq \|v_2(t)\|_{H^s}$ with

$$\mathcal{F}_x(v_2(t))(\xi) = cN^{-s}\gamma^{-1}\xi e^{itp(\xi)}e^{-t|\xi|^\alpha} \int_{K_\xi} \frac{e^{t\chi(\xi,\xi_1)} - 1}{\chi(\xi, \xi_1)}\, d\xi_1 \tag{5.2}$$

and

$$K_\xi = \{\xi_1 \colon \xi_1 \in I_1,\ \xi - \xi_1 \in I_2\} \cup \{\xi_1 \colon \xi_1 \in I_2,\ \xi - \xi_1 \in I_1\}.$$

We easily see that if $\xi_1 \in K_\xi$, then $\xi \in [N + \gamma/2, N + 2\gamma]$ and

$$p(\xi_1) + p(\xi - \xi_1) - p(\xi) = 2\xi_1(\xi_1 - \xi) \sim \gamma N,$$
$$|\xi_1|^\alpha + |\xi - \xi_1|^\alpha - |\xi|^\alpha \lesssim N^\alpha.$$

We deduce that for $\gamma = N^{\alpha-1} \ll N$, we have $|\chi(\xi, \xi_1)| \sim N^\alpha$. Now define

$$t_N = (N + 2\gamma)^{-\alpha-\varepsilon} \sim N^{-\alpha-\varepsilon}$$

so that $e^{-t_N|\xi|^\alpha} \gtrsim 1$. By a Taylor expansion of the exponential function,

$$\frac{e^{t_N\chi(\xi,\xi_1)} - 1}{\chi(\xi, \xi_1)} = t_N + R(t_N, \xi, \xi_1) \tag{5.3}$$

with

$$|R(t_N, \xi, \xi_1)| \lesssim \sum_{k\geq 2} \frac{t_N^k |\chi(\xi, \xi_1)|^{k-1}}{k!} \lesssim N^{-\alpha-2\varepsilon}.$$

Therefore the main contribution of (5.3) in (5.2) is given by t_N, and since $|K_\xi| \sim \gamma$, it follows that

$$\begin{aligned}|\mathcal{F}_x(v_2(t_N))(\xi)| &\gtrsim N^{-s+1}\gamma^{-1}e^{-(N+2\gamma)^{-\varepsilon}}\gamma N^{-\alpha-\varepsilon}\chi_{[N+\gamma/2,N+2\gamma]}(\xi)\\ &\gtrsim N^{-s+1-\alpha-\varepsilon}\chi_{[N+\gamma/2,N+2\gamma]}(\xi).\end{aligned}$$

We get the lower bound for the H^s-norm of $u_2(t_N)$

$$\|u_2(t_N)\|_{H^s} \gtrsim N^{-s+1-\alpha-\varepsilon}\left(\int_{N+\gamma/2}^{N+2\gamma}(1 + |\xi|^2)^s d\xi\right)^{1/2} \sim N^{1-\alpha-\varepsilon}\gamma^{1/2} \sim N^{(1-\alpha)/2-\varepsilon},$$

which leads to

$$\lim_{N\to\infty} \sup_{[0,T]}\|u_2(t)\|_{H^s} = +\infty$$

for $\varepsilon \ll 1$ and $\alpha < 1$, as desired. □

5.2. The case $1 \le \alpha \le 2$. Let $1 \le \alpha \le 2$ and $s < -\alpha/4$. As previously, it suffices to find a suitable sequence $\{h_N\}$ such that $\|h_N\|_{H^s} \lesssim 1$ and

$$\lim_{N\to\infty} \sup_{[0,T]} \|u_3(t)\|_{H^s} = +\infty.$$

With this purpose, we define the real-valued function h_N by

$$\hat{h}_N(\xi) = N^{-s}\gamma^{-1/2}(\chi_{I_N}(\xi) + \chi_{I_N}(-\xi)) \tag{5.4}$$

with $I_N = [N, N + 2\gamma]$, $N \gg 1$ and $\gamma \ll N$ to be chosen later. We have

$$\mathcal{F}_x(u_3(t))(\xi) = c\xi \int_0^t e^{i(t-t')p(\xi)} e^{-(t-t')|\xi|^\alpha} \mathcal{F}_x(S_\alpha(t')h_N) * \mathcal{F}_x(u_2(t'))(\xi)\, dt'$$

and

$$\mathcal{F}_x(S_\alpha(t')h_N) * \mathcal{F}_x(u_2(t'))(\xi) = c \int_{\mathbb{R}^2} \hat{h}_N(\xi_1)\hat{h}_N(\xi_2 - \xi_1)\hat{h}_N(\xi - \xi_2)\xi_2 \times e^{it'(p(\xi-\xi_2)+p(\xi_2))} e^{-t'(|\xi-\xi_2|^\alpha + |\xi_2|^\alpha)} \frac{e^{t'\chi(\xi_2,\xi_1)} - 1}{\chi(\xi_2, \xi_1)}\, d\xi_1\, d\xi_2.$$

Hence, we can write $u_3 = v_3 - w_3$ with

$$\begin{aligned}
&\mathcal{F}_x(v_3(t))(\xi)\\
&= c\xi e^{itp(\xi)} e^{-t|\xi|^\alpha} \int_{\mathbb{R}^2} \hat{h}_N(\xi_1)\hat{h}_N(\xi_2 - \xi_1)\hat{h}_N(\xi - \xi_2)\frac{\xi_2}{\chi(\xi_2, \xi_1)}\\
&\quad \times \int_0^t e^{it'(p(\xi_1)+p(\xi_2-\xi_1)+p(\xi-\xi_2)-p(\xi))} e^{-t(|\xi_1|^\alpha+|\xi_2-\xi_1|^\alpha+|\xi-\xi_2|^\alpha-|\xi|^\alpha)}\, dt'\, d\xi_1\, d\xi_2\\
&= c\xi e^{itp(\xi)} e^{-t|\xi|^\alpha} \int_{\mathbb{R}^2} \hat{h}_N(\xi_1)\, \hat{h}_N(\xi_2 - \xi_1)\, \hat{h}_N(\xi - \xi_2)\frac{\xi_2}{\chi(\xi_2, \xi_1)} \frac{e^{t\lambda(\xi,\xi_1,\xi_2)} - 1}{\lambda(\xi, \xi_1, \xi_2)}\, d\xi_1\, d\xi_2
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_x(w_3(t))(\xi)\\
&= c\xi e^{itp(\xi)} e^{-t|\xi|^\alpha} \int_{\mathbb{R}^2} \hat{h}_N(\xi_1)\hat{h}_N(\xi_2 - \xi_1)\hat{h}_N(\xi - \xi_2)\frac{\xi_2}{\chi(\xi_2, \xi_1)}\\
&\quad \times \int_0^t e^{t'\chi(\xi,\xi_2)} dt' d\xi_1 d\xi_2\\
&= c\xi e^{itp(\xi)} e^{-t|\xi|^\alpha} \int_{\mathbb{R}^2} \hat{h}_N(\xi_1)\hat{h}_N(\xi_2 - \xi_1)\hat{h}_N(\xi - \xi_2)\frac{\xi_2}{\chi(\xi_2, \xi_1)} \frac{e^{t\chi(\xi,\xi_2)} - 1}{\chi(\xi, \xi_2)}\, d\xi_1\, d\xi_2
\end{aligned}$$

where we set

$$\begin{aligned}
\lambda(\xi, \xi_1, \xi_2) = {}& i(p(\xi_1) + p(\xi_2 - \xi_1) + p(\xi - \xi_2) - p(\xi))\\
& - (|\xi_1|^\alpha + |\xi_2 - \xi_1|^\alpha + |\xi - \xi_2|^\alpha - |\xi|^\alpha).
\end{aligned}$$

Let $t_N = (N+4\gamma)^{-\alpha-\varepsilon}$ for some $0<\varepsilon\ll 1$. We get

$$|\mathcal{F}_x(v_3(t_N))(\xi)|\chi_{[N+3\gamma,N+4\gamma]}(\xi) \gtrsim N^{-3s+1}\gamma^{-3/2}\left|\int_{K_\xi} \frac{\xi_2}{\chi(\xi_2,\xi_1)}\frac{e^{t_N\lambda(\xi,\xi_1,\xi_2)}-1}{\lambda(\xi,\xi_1,\xi_2)}\,d\xi_1\,d\xi_2\right|$$

where $K_\xi = K_\xi^1\cup K_\xi^2\cup K_\xi^3$ and

$$\begin{aligned}
K_\xi^1 &= \{(\xi_1,\xi_2)\colon \xi_1\in I_N,\ \xi_2-\xi_1\in I_N,\ \xi-\xi_2\in -I_N\},\\
K_\xi^2 &= \{(\xi_1,\xi_2)\colon \xi_1\in I_N,\ \xi_2-\xi_1\in -I_N,\ \xi-\xi_2\in I_N\},\\
K_\xi^3 &= \{(\xi_1,\xi_2)\colon \xi_1\in -I_N,\ \xi_2-\xi_1\in I_N,\ \xi-\xi_2\in I_N\}.
\end{aligned}$$

If $\xi\in[N+3\gamma,N+4\gamma]$ and $(\xi_1,\xi_2)\in K_\xi$, we easily see that

$$\left|\frac{\xi_2}{\chi(\xi_2,\xi_1)}\right| \sim N^{-1}$$

and

$$\begin{aligned}
p(\xi_1)+p(\xi_2-\xi_1)+p(\xi-\xi_2)-p(\xi) &\sim \gamma^2,\\
|\xi_1|^\alpha+|\xi_2-\xi_1|^\alpha+|\xi-\xi_2|^\alpha-|\xi|^\alpha &\sim N^\alpha.
\end{aligned}$$

Thus we are led to choose $\gamma = N^{\alpha/2}\ll N$ for $N\gg 1$ so that $|\lambda(\xi,\xi_1,\xi_2)|\sim N^\alpha$. Then it follows that

$$\left|\frac{e^{t_N\lambda(\xi,\xi_1,\xi_2)}-1}{\lambda(\xi,\xi_1,\xi_2)}\right| = |t_N| + O(N^{-\alpha-2\varepsilon}).$$

Consequently,

$$\begin{aligned}
|\mathcal{F}_x(v_3(t_N))(\xi)|\chi_{[N+3\gamma,N+4\gamma]}(\xi) &\gtrsim N^{-3s+1}\gamma^{-3/2}N^{-1}\gamma^2N^{-\alpha-\varepsilon}\chi_{[N+3\gamma,N+4\gamma]}(\xi)\\
&\sim N^{-3s-\alpha-\varepsilon}\gamma^{1/2}\chi_{[N+3\gamma,N+4\gamma]}(\xi)\\
&\sim N^{-3s-3\alpha/4-\varepsilon}\chi_{[N+3\gamma,N+4\gamma]}(\xi),
\end{aligned}$$

since $|K_\xi|\sim\gamma^2$.

Concerning w_3, we verify that for $(\xi_1,\xi_2)\in K_\xi$, we have $|\chi(\xi,\xi_2)|\gtrsim \gamma N$ and then

$$\begin{aligned}
|\mathcal{F}_x(w_3(t_N))(\xi)|\chi_{[N+3\gamma,N+4\gamma]}(\xi) &\lesssim N^{-3s+1}\gamma^{-3/2}\gamma^2N^{-1}(\gamma N)^{-1}\chi_{[N+3\gamma,N+4\gamma]}(\xi)\\
&\sim N^{-3s-1}\gamma^{-1/2}\chi_{[N+3\gamma,N+4\gamma]}(\xi)\\
&\sim N^{-3s-1-\alpha/4}\chi_{[N+3\gamma,N+4\gamma]}(\xi).
\end{aligned}$$

Since $-3s-1-\alpha/4 < -3s-3\alpha/4-\varepsilon$ for $\alpha<2$, we deduce that the main contribution in the H^s-norm of u_3 is given by $\|v_3\|_{H^s}$, that is,

$$\|u_3(t_N)\|_{H^s} \gtrsim N^{-3s-3\alpha/4-\varepsilon}\gamma^{1/2}N^s \sim N^{-2s-\alpha/2-\varepsilon},$$

and we find the condition

$$-2s - \frac{\alpha}{2} > 0, \quad \text{i.e.} \quad s < -\frac{\alpha}{4}.$$

When $\alpha = 2$, the contributions of v_3 and w_3 are equivalent, and we must proceed with a bit more care, by considering directly the difference $u_3 = v_3 - w_3$. More precisely, for $\gamma = \varepsilon N \ll N$, we have

$$|\lambda(\xi, \xi_1, \xi_2)| \sim |\chi(\xi, \xi_2)| \sim N^2.$$

Noticing that

$$\lambda(\xi, \xi_1, \xi_2) - \chi(\xi, \xi_2) = \chi(\xi_2, \xi_1),$$

we deduce

$$\left| \frac{e^{t_N \lambda(\xi,\xi_1,\xi_2)} - 1}{\lambda(\xi, \xi_1, \xi_2)} - \frac{e^{t_N \chi(\xi,\xi_2)} - 1}{\chi(\xi, \xi_2)} \right| = t_N^2 |\chi(\xi_2, \xi_1)| + O(t_N^3 N^2 |\chi(\xi_2, \xi_1)|).$$

Setting again $t_N = N^{-2-\varepsilon}$, and since $|\xi_2| \sim N$, it follows that

$$|\mathcal{F}_x(u_3(t_N))(\xi)| \chi_{[N+3\gamma, N+4\gamma]} \gtrsim N^{-3s+1} \gamma^{-3/2} \gamma^2 N N^{-4-2\varepsilon} \chi_{[N+3\gamma, N+4\gamma]}(\xi)$$

and thus

$$\|u_3(t_N)\|_{H^s} \gtrsim N^{-2s-2-2\varepsilon} \gamma \sim N^{-2s-1-2\varepsilon},$$

which tends to infinity as soon as $-2s - 1 > 0$, i.e. $s < -1/2$.

6. Appendix

We prove here that the pure dissipative equation

$$u_t + |D|^\alpha u + u u_x = 0 \tag{6.1}$$

for $1 < \alpha \le 2$ is well-posed in $H^s(\mathbb{R})$, $s > s_\alpha$ where

$$s_\alpha = \frac{3}{2} - \alpha,$$

and that the solution map fails to be smooth when $s < s_\alpha$. The method of proof is classical and is based on the smoothing properties of the generalized heat kernel

$$G_\alpha(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-t|\xi|^\alpha} \, d\xi, \quad t > 0.$$

Theorem 6.1. *Let $1 < \alpha \le 2$, $s > s_\alpha$ and $u_0 \in H^s(\mathbb{R})$. Then there exist $T > 0$ and a unique solution $u \in \mathcal{C}([0, T]; H^s(\mathbb{R}))$ of (6.1) such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} < \infty \quad \text{if} \quad 1 < \alpha \le \frac{3}{2}, \tag{6.2}$$

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} + \sup_{t\in[0,T]} t^\beta \|u(t)\|_{L^{2/(\alpha-1)}} < \infty \quad \text{if} \quad \frac{3}{2} < \alpha \le 2 \tag{6.3}$$

where $\beta = -s/\alpha + (2-\alpha)/2\alpha$. The flow map $u_0 \mapsto u$ from $H^s(\mathbb{R})$ into the class defined by (6.2) and (6.3) is locally Lipschitz. Moreover, if $\|u_0\|_{H^s}$ is small enough, the solution can be extended to any time interval.

Proof. Observe that for any $p \in [1, \infty]$ and $\rho \ge 0$, we have

$$\| |D|^\rho G_\alpha(t)\|_{L^p} = c t^{-(1-1/p)/\alpha - \rho/\alpha}. \tag{6.4}$$

We use the Picard iteration theorem to show that the map F defined as

$$F(u) = G_\alpha(t) * u_0 - \frac{1}{2}\int_0^t G_\alpha(t-t') * \partial_x u^2(t')\, dt'$$

has a fixed point in some suitable Banach space.

We first consider the case $1 < \alpha \le 3/2$, and we choose $s_\alpha < s < 1/2$. Set $X_T = \mathcal{C}([0, T]; H^s(\mathbb{R}))$ endowed with the norm $\|u\|_{X_T} = \sup_{[0,T]}\|u(t)\|_{H^s}$. By Young inequality and (6.4), we have

$$\|G_\alpha(t) * u_0\|_{H^s} \lesssim \|G_\alpha(t)\|_{L^1}\|u_0\|_{H^s} \lesssim \|u_0\|_{H^s}. \tag{6.5}$$

Using the fractional Leibniz rule, we get

$$\begin{aligned}\int_0^t \|G_\alpha(t-t') * \partial_x u^2(t')\|_{H^s}\, dt' &\lesssim \int_0^t \|\partial_x G_\alpha(t-t')\|_{L^{(s+1/2)^{-1}}} \|\langle D\rangle^s u^2(t')\|_{L^{1/(1-s)}}\, dt' \\ &\lesssim \int_0^t (t-t')^{s/\alpha - 3/2\alpha} \|u(t')\|_{L^{(1/2-s)^{-1}}} \|u(t')\|_{H^s}\, dt'.\end{aligned}$$

Since $0 < s < 1/2$, we can take advantage of the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^{(1/2-s)^{-1}}(\mathbb{R})$. Since $s/\alpha - 3/2\alpha > -1$, we conclude

$$\int_0^t \|G_\alpha(t-t') * \partial_x u^2(t')\|_{H^s}\, dt' \lesssim T^\nu \|u\|_{X_T}^2 \tag{6.6}$$

with $\nu = 1 + s/\alpha - 3/2\alpha > 0$. Gathering (6.5) and (6.6) we infer

$$\|F(u)\|_{X_T} \lesssim \|u_0\|_{H^s} + T^\nu \|u\|_{X_T}^2$$

and in the same way,

$$\|F(u)-F(v)\|_{X_T} \lesssim T^{\nu}(\|u\|_{X_T}+\|v\|_{X_T})\|u-v\|_{X_T}.$$

This proves that for $T \ll 1$, F is contractive in a ball of X_T.

Now we solve (6.1) in the case $3/2<\alpha\le 2$ and $s_\alpha<s<0$. Define $Y_T = \mathcal{C}([0,T];H^s(\mathbb{R}))\cap\mathcal{C}^\beta([0,T];L^{2/(\alpha-1)}(\mathbb{R}))$ equipped with the norm

$$\|u\|_{Y_T}=\sup_{t\in[0,T]}\|u(t)\|_{H^s}+\sup_{t\in[0,T]}t^\beta\|u(t)\|_{L^{2/(\alpha-1)}}.$$

By Young inequality, we get

$$\|G_\alpha(t)*u_0\|_{L^{2/(\alpha-1)}}=\|\langle D\rangle^{-s}G_\alpha(t)*\langle D\rangle^s u_0\|_{L^{2/(\alpha-1)}}\lesssim\|\langle D\rangle^{-s}G_\alpha(t)\|_{L^{2/\alpha}}\|u_0\|_{H^s},$$

and it follows from (6.4) that

$$t^\beta\|\langle D\rangle^{-s}G_\alpha(t)\|_{L^{2/\alpha}}\lesssim t^\beta(t^{-(2-\alpha)/2\alpha}+t^{-(2-\alpha)/2\alpha+s/\alpha})\lesssim\langle T\rangle^{-s/\alpha}.$$

Now we deal with the nonlinear term. Using the Sobolev embedding $L^{(1/2-s)^{-1}}(\mathbb{R})\hookrightarrow H^s(\mathbb{R})$ valid for any $-1/2<s<0$, we obtain

$$\begin{aligned}\int_0^t\|G_\alpha(t-t')*\partial_x u^2(t')\|_{H^s}\,dt' &\lesssim \int_0^t\|\partial_x G_\alpha(t-t')\|_{L^{(5/2-s-\alpha)^{-1}}}\|u^2(t')\|_{L^{1/(\alpha-1)}}\,dt'\\ &\lesssim\int_0^t(t-t')^{-s/\alpha-1+1/2\alpha}t'^{-2\beta}t'^{2\beta}\|u(t')\|^2_{L^{2/(\alpha-1)}}\,dt'\\ &\lesssim T^\nu\|u\|^2_{Y_T}\end{aligned}$$

with $\nu=-s/\alpha+1/2\alpha-2\beta>0$. By similar calculations, we get

$$\begin{aligned}t^\beta\int_0^t\|G_\alpha(t-t')*\partial_x u^2(t')\|_{L^{2/(\alpha-1)}}\,dt' &\lesssim t^\beta\int_0^t\|\partial_x G_\alpha(t-t')\|_{L^{2/(3-\alpha)}}\|u^2(t')\|_{L^{1/(\alpha-1)}}\,dt'\\ &\lesssim t^\beta\int_0^t(t-t')^{-(\alpha+1)/2\alpha}t'^{-2\beta}\,dt'\|u\|^2_{Y_T}\\ &\lesssim T^\nu\|u\|^2_{Y_T}\end{aligned}$$

with $\nu=1-(\alpha+1)/2\alpha-\beta>0$. Finally, one has

$$\|F(u)\|_{Y_T}\lesssim\langle T\rangle^\nu\|u_0\|_{H^s}+T^\nu\|u\|^2_{Y_T}$$

and the claim follows. □

REMARK 6.1. Let $U_\alpha(t)=\mathcal{F}_\xi^{-1}(e^{it\xi|\xi|}e^{-t|\xi|^\alpha})$ be the fundamental solution of the linear (dBO) equation. Using that $|\mathcal{F}_xU_\alpha(t)|=|\mathcal{F}_xG_\alpha(t)|$ as well as the well-known estimate $\|f\|_{L^p} \lesssim \|\hat{f}\|_{L^{p'}}$, $p \geq 2$, $1/p + 1/p' = 1$, we easily check that Theorem 6.1 also holds for (dBO) equation.

Finally, we show that Theorem 6.1 is sharp.

Theorem 6.2. *Let $1 < \alpha \leq 2$ and $s < s_\alpha$. Then the solution map $u_0 \mapsto u$ associated with* (6.1) (*if it exists*) *is not of class $\mathcal{C}^2$ from $H^s(\mathbb{R})$ to $\mathcal{C}([0, T]; H^s(\mathbb{R}))$.*

Proof. The proof is similar to that of Theorems 1.2 and 1.3. Define h_N as in (5.4) and consider the high-high interactions in the convolution product $(e^{-t|\xi|^\alpha} h_N) * (e^{-t|\xi|^\alpha} h_N)$. We get that for $\xi \in [2N, 2N + 4\gamma]$, $\gamma = N^{1-\varepsilon}$ and $t_N \sim N^{-\alpha-\varepsilon}$,

$$|\mathcal{F}_x(u_2(t_N))(\xi)| \gtrsim N^{-2s-\alpha+1-\varepsilon} \chi_{[2N,2N+4\gamma]}(\xi)$$

where u_2 is defined by

$$u_2(t) = \int_0^t G_\alpha(t - t') * \partial_x (G_\alpha(t') * h_N)^2 \, dt'.$$

We conclude that

$$\|u_2(t_N)\|_{H^s} \gtrsim N^{-s-\alpha+1-\varepsilon} \gamma^{1/2} \gtrsim N^{-s+3/2-\alpha-3\varepsilon/2} \to +\infty$$

as soon as $s < 3/2 - \alpha$. □

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Université Paris-Est
Laboratoire d'Analyse et de Mathématiques Appliquées
5 bd. Descartes, Cité Descartes, Champs-Sur-Marne
77454 Marne-La-Vallée Cedex 2
France
e-mail: stephane.vento@univ-paris-est.fr