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NON-CENTRAL FIXED POINT FREE SYMMETRIES OF BISYMMETRIC RIEMANN SURFACES

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Abstract

We study pairs of symmetries of a Riemann surface of genus $g \geq 2$, whose product has order $n > 2$, assuming that one of them is fixed point free. We start our considerations by giving some bounds for the number of ovals of a symmetry with fixed points and showing their attainment, later we take into account the number of points fixed by the product of the symmetries and we study some of its properties. Finally we deal the problem of finding the maximal possible power of 2 which can be realized as the order of their product.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. By a *symmetry* of X we mean an antiholomorphic involution σ of X. By the classical result of Harnack the set of fixed points of σ consists of at most $g+1$ disjoint simple closed curves, which are called *ovals*. If σ has $g+1-q$ ovals then we shall call it an $(M-q)$-*symmetry*, according to Natanzon's terminology from [12]. Furthermore, σ is called *separating* or *non-separating* if $X \setminus \mathrm{Fix}(\sigma)$ has two or one connected component respectively.

The study of symmetries of Riemann surfaces is important due to the categorical equivalence under which a compact, connected Riemann surface X corresponds to a smooth, complex, projective and irreducible algebraic curve $\mathcal{C}_X$. Furthermore, a Riemann surface X admits a symmetry σ if and only if the corresponding curve $\mathcal{C}_X$ has a real form $\mathcal{C}_X(\sigma)$ and two such symmetries give rise to the real forms non-isomorphic over the reals $\mathbb{R}$, if and only if they are not conjugate in the group $\mathrm{Aut}^{\pm}(X)$ of all, including antiholomorphic, automorphisms of X. Finally, the set $\mathrm{Fix}(\sigma)$ is homeomorphic to a smooth projective model of the corresponding real form $\mathcal{C}_X(\sigma)$ and in this paper we focus our attention on curves having two real forms one of which has no $\mathbb{R}$-rational points. The latter are known in the literature as the *purely imaginary curves* and they correspond to fixed point free symmetries of Riemann surfaces, an example of such a curve is the one given by $x^n + y^n = -1$ for n even.

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The aim of this paper is to solve some of the problems brought up by Bujalance, Costa and Singerman in [2] and Natanzon in [12], which were studied in [5, 9, 10], for the case of at least one of the symmetries being fixed point free. Here we fill the gaps existing in the literature of the topic, showing this way some differences and similarities to the case of both symmetries having fixed points. For our considerations we use important results given by Izquierdo and Singerman in [6] and following their terminology, we shall say that a Riemann surface of genus g admits the pair $(0, t)_n$, if it admits a pair of symmetries σ, τ where σ is fixed point free, τ has t ovals and $\sigma\tau$ has order n.

The results we give in the first parts of the paper complete the studies of some problems appearing in [5] and [9]. First of all, we give upper bounds for the number t of ovals of symmetry τ, in terms of the genus g of the surface and the order n of the product $\sigma\tau$. We also show attainment of these bounds for infinitely many values of g. Later we take into account the number m of points fixed by the product $\sigma\tau$. As in [5], we use theorem of Macbeath from [11] to give more specific bounds for t and we show their attainment. We also study properties of the number m of points fixed by the product $\sigma\tau$, showing that the upper bound for m can be realized for an orientation preserving automorphism being the product of our symmetries. Furthermore, we show some other values for m, which are attained and, in contrast to the case of pairs of symmetries with fixed points, we see that the number m of points fixed by the product has to be even. In the last part of the paper we study how far from being commutative can be a pair of symmetries one of which is a fixed point free symmetry, i.e. we find sharp upper and lower bound for the maximal order of the product of the symmetries in question, given the numbers of their ovals.

2. Preliminaries

We shall prove our results using theory of non-euclidean crystallographic groups (*NEC groups* in short), by which we mean discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$ including those that reverse orientation. The algebraic structure of such group Λ is coded in the signature:

$$s(\Lambda) = (h; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}), \tag{1}$$

where the brackets $(n_{i1}, \dots, n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i *proper periods* and finally h the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators*:

$$x_1, \dots, x_r, e_i, c_{ij}, \ 1 \le i \le k, \ 0 \le j \le s_i \ \text{and} \ a_1, b_1, \dots, a_h, b_h$$

if the sign is $+$ or $d_1, \dots, d_h$ otherwise, and relators:

$$x_i^{m_i},\ i=1,\dots,r,\quad c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i=1,\dots,k,\ j=1,\dots,s_i$$

and

$$x_1\cdots x_r e_1\cdots e_k a_1 b_1 a_1^{-1} b_1^{-1}\cdots a_h b_h a_h^{-1} b_h^{-1} \quad \text{or} \quad x_1\cdots x_r e_1\cdots e_k d_1^2\cdots d_h^2,$$

according to whether the sign is $+$ or $-$. The elements x_i are elliptic transformations, a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. Reflections c_{ij-1}, c_{ij} are said to be *consecutive*. Every element of finite order in Λ is conjugate either to a canonical reflection or to a power of some canonical elliptic element x_i, or to a power of the product of two consecutive canonical reflections.

Now an abstract group with such presentation can be realized as an NEC group Λ if and only if the value

$$2\pi\left(\varepsilon h+k-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon=2$ or 1 according to the sign being $+$ or $-$, is positive. By [13] this value turns out to be the hyperbolic area $\mu(\Lambda)$ of an arbitrary fundamental region for such group and we have the following Hurwitz–Riemann formula

$$[\Lambda:\Lambda']=\mu(\Lambda')/\mu(\Lambda)$$

for a subgroup Λ' of finite index in an NEC group Λ.

Now NEC groups having no orientation reversing elements are classical Fuchsian groups. They have signatures $(g;+;[m_1,\dots,m_r];\{-\})$, which shall be abbreviated as $(g;m_1,\dots,m_r)$. Given an NEC group Λ, the subgroup Λ^+ of Λ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* Λ and for a group with signature (1) it has, by [14], signature

$$(\varepsilon h+k-1; m_1, m_1,\dots,m_r,m_r,n_{11},\dots,n_{ks_k}). \tag{2}$$

A torsion free Fuchsian group Γ is called a *surface group* and it has signature $(g;-)$. In such case $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g and conversely, each compact Riemann surface can be represented as such orbit space for some Γ. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G=\Lambda/\Gamma$ for some NEC group Λ.

Let $C(G,g)$ denote the centralizer of an element g in the group G. The following result from [4] is crucial for the paper

Theorem 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta\colon \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry τ of X equals*

$$\sum [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to τ.

For a symmetry τ we shall denote by $\|\tau\|$ the number of its ovals. The index $w_i = [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|\theta(c_i)\|$.

3. Ovals of non-commuting pairs of symmetries

The starting point for this paper are the results of Bujalance, Costa and Singerman from [2] (see also Natanzon in [12]) and their generalizations, given in [9]. Recall, that the upper bound for the total number of ovals of a pair of symmetries with fixed points, whose product has order n, on a Riemann surface of genus g depends on the parity of n and is equal to $4g/n+2$ for n even and $2(g-1)/n+4$ for n odd. Moreover, these bounds are attained whenever $4g/n$ or $2(g-1)/n$ are integers. If not, then we can study the natural bounds, given by integer part, i.e. $[4g/n] + 2$ for n even and $[2(g-1)/n] + 4$ for n odd. In [9] it was shown that this new bound is attained for n even for infinitely many values of g. For odd n there is a better bound $[2(g-1)/n]+3$, which is attained. Here, we shall give the analogous bound for the number of ovals of symmetry τ. Obviously we shall assume that $n > 2$ is even, as otherwise both the symmetries would be fixed point free as conjugate ones. As we shall see, also in this case we get two different bounds, depending on the parity of $n/2$. Moreover, there are more conditions necessary for their attainment, as one has to be more careful about the epimorphism $\theta\colon \Lambda \to \mathrm{D}_n$ defining the surface and the action.

We shall use the lemma below, which follows easily from Theorem 2.1

Lemma 3.1 (see also Theorem 2 in [2]). *Let n be even and $G = \mathrm{D}_n = \Lambda/\Gamma$ be the group of automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ generated by two non-central symmetries: fixed point free symmetry σ and a symmetry τ and let $C = (n_1, \ldots, n_s)$ be a period cycle of Λ. Then reflections corresponding to C contribute to $\|\tau\|$ at most t ovals in total, where t is the number of even link periods if $s \geq 1$ and some n_i is even and at most 2 ovals in total for the remaining cases.*

Proof. Let $\theta\colon \Lambda \to G$ be the canonical epimorphism. The centralizer of any non-central involution in D_n has order 4. Since $c_i \in C(\Lambda, c_i)$, we have that $w_i \leq 2$. If c belongs to two odd link periods then we can assume that c does not contribute to $\|\tau\|$, while if c belongs to an even link period n' and cc' has order n' then $(cc')^{n'/2} \in$

$C(\Lambda, c)$. Now $\theta((cc')^{n'/2}c) \neq 1$ since $\ker\theta$ is a Fuchsian group and therefore we see that $\theta(C(\Lambda, c))$ has order 4. □

We also need the following two results from [6], which give some restrictions on the possible combinations of g, n and t.

Theorem 3.2. *If a Riemann surface X of even genus g admits a pair $(0, t)_n$ with $t > 0$, then $n \equiv 2 \bmod 4$ and t is odd.*

Theorem 3.3. *If a Riemann surface of genus g admits a pair $(0, t)_n$ and $n \equiv 0 \bmod 4$, then g is odd.*

Now we can give the upper bound for the number of ovals t for a Riemann surface X of genus g, admitting the pair $(0, t)_n$

Theorem 3.4. *Let X be a Riemann surface of genus g, admitting a pair $(0, t)_n$ where $n > 2$ is an even integer. Then, the following conditions hold*:
(1) *If $n \equiv 0 \bmod 4$, $n > 4$ then $t \leq 4(g-1)/n$ and the bound is attained when $4(g-1)/n$ is even or $n = 8$;*
(2) *If $n = 4$, then $t \leq g$ and the bound is attained for every odd g;*
(3) *If $n \equiv 2 \bmod 4$, then $t \leq 2(g-1)/n + 2$ and the bound is attained if $2(g-1)/n$ is an integer.*

Proof. Let first $n \equiv 0 \bmod 4$, $n > 4$ and let $G = \langle \sigma, \tau \rangle = D_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(-)^l, C_1, \ldots, C_k\}),$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$, $s_i \geq 1$ and let $s = s_1 + \cdots + s_k$. By the Hurwitz–Riemann formula we get $\mu(\Lambda) = 2\pi(g-1)/n$ and by Lemma 3.1, as $2l + s \geq t$, we have

$$\begin{aligned} \frac{2\pi(g-1)}{n} &= \mu(\Lambda) \\ &\geq 2\pi\left(\varepsilon h - 2 + k + l + \frac{r}{2} + \frac{s}{4}\right) \\ &\geq 2\pi\left(\varepsilon h - 2 + k + \frac{l}{2} + \frac{r}{2} + \frac{t}{4}\right) \end{aligned}$$

which gives $t \leq 4(g-1)/n$ whenever $\varepsilon h + k + l/2 + r/2 \geq 2$. So we only have to consider the following cases:
(a) $h = 1$, $\mathrm{sgn}(\Lambda) = -$, $k = r = 0$, $l = 1$,
(b) $h = r = k = 0$ and $l = 2$ or $l = 3$,
(c) $h = r = 0$, $k = l = 1$,

(d) $h = k = 0$, $r = 1$ and $l = 1$ or $l = 2$,
(e) $h = l = 0$, $k = r = 1$.

Observe that in case (a) the condition $\mu(\Lambda) > 0$ does not hold and in cases (b)–(d) there is no epimorphism $\theta\colon \Lambda \to D_n$ giving a pair $(0, t)_n$. So we only have to deal with the case (e). For the epimorphism $\theta: \Lambda \to D_n$ to exist, the signature of Λ must be of the form

$$(0; +; [m]; \{(n_1, \ldots, n_s)\})$$

for some $m \geq 4$. Now if there is at least one odd link period then, by Lemma 3.1, we obtain $s \geq t + 1$ and

$$\begin{aligned}\frac{2\pi(g-1)}{n} &= \mu(\Lambda)\\ &\geq 2\pi\left(-2 + 1 + 1 - \frac{1}{4} + \frac{t}{4} + \frac{1}{3}\right)\\ &> 2\pi\left(\frac{t}{4} + \frac{1}{12}\right)\end{aligned}$$

which gives $< 4(g-1)/n$.

Now if all the link periods are even and at least two of them are greater than 2, then

$$\begin{aligned}\frac{2\pi(g-1)}{n} &= \mu(\Lambda)\\ &\geq 2\pi\left(-2 + 1 + 1 - \frac{1}{4} + \frac{t-2}{4} + \frac{3}{4}\right)\\ &= \frac{2\pi\cdot t}{4}\end{aligned} \tag{3}$$

and the theorem holds as $t \leq 4(g-1)/n$.

Let now only one of the link periods be greater than 4. In such case it is easy to see, that for the epimorphism to exist it must be that $n \geq m \geq 8$. Hence

$$\begin{aligned}\frac{2\pi(g-1)}{n} &= \mu(\Lambda)\\ &\geq 2\pi\left(-2 + 1 + 1 - \frac{1}{8} + \frac{t-1}{4} + \frac{3}{8}\right)\\ &= \frac{2\pi\cdot t}{4}\end{aligned}$$

which again gives $t \leq 4(g-1)/n$.

So we may assume that Λ has a signature of the form

$$(0; +; [m]; \{(2, \overset{s}{\ldots}, 2)\}). \tag{4}$$

But in such case the epimorphism $\theta\colon \Lambda \to \mathrm{D}_n$ cannot exist for $n > 4$, as it would have to send the consecutive canonical reflections alternatively to τ and $\tau(\sigma\tau)^{n/2}$. Hence $\theta(e)$ may be equal to 1, $(\sigma\tau)^{n/2}$ or $(\sigma\tau)^{\pm n/4}$, depending on the parity of s. Moreover, $\theta(e)$ and $\theta(x_1)$ have the same order, as the relation $x_1 e = 1$ holds in Λ. It follows that in such case there is no epimorphism onto D_n for $n > 4$, $n \equiv 0 \mod 4$.

To see that the bound above is attained whenever $4(g-1)/n$ is even, consider an NEC group Λ with signature $(1; -; [-]; \{(2, \overset{4(g-1)/n}{\ldots}, 2)\})$ and an epimorphism $\theta\colon \Lambda \to \mathrm{D}_n$ defined by $\theta(d) = \sigma$, $\theta(e) = 1$ and sending canonical reflections alternatively to τ and $\tau(\sigma\tau)^{n/2}$. In such case $\Gamma = \ker(\theta)$ is a Fuchsian surface group and $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus g admitting a pair $(0, 4(g-1)/n)_n$.

For $n = 8$, we only have to consider the case when $(g-1)/2$ is odd, as we treated the opposite case $(g-1)/2$ even above. Consider an NEC group Λ with signature $(0; +; [8]; \{(2, \overset{(g-1)/2-1}{\ldots}, 2, 4)\})$ and an epimorphism $\theta\colon \Lambda \to \mathrm{D}_n$ defined by $\theta(x) = \theta(e)^{-1} = \sigma\tau$ and sending canonical reflections to

$$\underbrace{\tau,\ \tau(\sigma\tau)^4,\ \tau,\ \ldots,\ \tau}_{(g-1)/2},\ \tau(\sigma\tau)^2.$$

Also here θ gives rise to the configuration of symmetries we looked for.

Let us now consider the case of $n = 4$. It follows from the proof of the previous case, that $t \le g - 1$ for all the possible signatures of Λ except signature (4). In such case we define an epimorphism θ by putting $\theta(x) = \theta(e)^{-1} = \sigma\tau$ and sending consecutive canonical reflections alternatively to τ and $\tau(\sigma\tau)^2$, starting with the former and finishing with the latter. This leads to the Riemann surface $X = \mathcal{H}/\ker\theta$ of odd genus g, admitting the pair $(0, g)_4$.

Let finally $n \equiv 2 \mod 4$ and let $G = \langle \sigma, \tau \rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(-)^l, C_1, \ldots, C_k\}),$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$, $s_i \ge 1$. Observe, that all the link periods are odd, as symmetry σ is fixed point free and so images under θ of all the canonical reflections are conjugate to τ. By the Hurwitz–Riemann formula we get $\mu(\Lambda) = 2\pi(g-1)/n$ and by Lemma 3.1, as $2(k + l) \ge t$, we have

$$\begin{aligned} \frac{2\pi(g-1)}{n} &= \mu(\Lambda) \\ &\ge 2\pi\left(\varepsilon h - 2 + k + l + \frac{r}{2} + \frac{s}{4}\right) \\ &\ge 2\pi\left(\varepsilon h - 2 + \frac{r}{2} + \frac{t}{2}\right) \end{aligned}$$

which gives $t \le 2(g-1)/n + 2$ if $\varepsilon h + r/2 \ge 1$. So let $h = 0$, $r = 1$. In such case we have $\theta(e_i) \ne 1$ for some period cycle C_i. Observe that, by [6] (see in particular Cases 2 (i), 3 (i) on pp. 7–8, and Note on p. 9), any connecting generator mapped nontrivially forces the total possible amount of ovals to be diminished by 1. It follows that $2(k+l) \geq t+1$ with

$$\begin{aligned}\frac{2\pi(g-1)}{n} &\geq 2\pi\left(-2+1-\frac{1}{2}+\frac{t+1}{2}\right)\\ &\geq 2\pi\left(\frac{t}{2}-1\right)\end{aligned}$$

hence $t \leq 2(g-1)/n+2$.

So we only have to consider the case $h=r=0$. If so, there are at least two period cycles, say C_1, C_2 in the signature of Λ, with $\theta(e_1), \theta(e_2)$ being non-trivial. It follows, as above, that $2(k+l) \geq t+2$ and so

$$\begin{aligned}\frac{2\pi(g-1)}{n} &\geq 2\pi\left(-2+\frac{t+2}{2}\right)\\ &= 2\pi\left(\frac{t}{2}-1\right)\end{aligned}$$

which gives $t \leq 2(g-1)/n+2$.

To see that the bound is attained for $2(g-1)/n$ being even, consider an NEC group with signature

$$(1; -; [-]; \{(-)^{(g-1)/n+1}\})$$

and an epimorphism $\theta\colon \Lambda \to \mathrm{D}_n = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^n \rangle$ defined by $\theta(e_i)=1$, $\theta(c_{i0})=\tau$, $\theta(d_1)=\sigma$. In such case $X=\mathcal{H}/\ker(\theta)$ is a Riemann surface of genus g, admitting the pair $(0, 2(g-1)/n+2)_n$.

Now if $2(g-1)/n$ is odd, then we take an NEC group Λ with signature

$$(0; +; [2]; \{(-)^{(g-1)/n+3/2}\})$$

and an epimorphism onto D_n, defined by $\theta(c_{10})=\sigma\tau\sigma$, $\theta(e_1)=\theta(x_1)=(\sigma\tau)^{n/2}$, $\theta(c_{i0})=\tau$ and $\theta(e_i)=1$ for $i \geq 2$. Here we also obtained a configuration we were looking for. □

Now we shall prove that there are infinitely many values of g for which $n \equiv 0 \bmod 4$ does not divide $4(g-1)$ and the natural bound $[4(g-1)/n]$ is attained. On the other hand, for $n \equiv 2 \bmod 4$ and n not dividing $2(g-1)$ the bound $[2(g-1)/n]+2$ is not attained, but there is a new bound $[2(g-1)/n]+1$ which is attained for infinitely many values of g. These results are analogous to the case of symmetries with fixed points (see [9] for more details).

Theorem 3.5. *For arbitrary even* $n > 8$, $n \equiv 0 \bmod 4$ *there are infinitely many values of* g *for which* n *does not divide* $4(g-1)$ *and there exists Riemann surface of genus* g *admitting a pair* $(0, [4(g-1)/n])_n$.

Proof. Let Λ be an NEC group with signature

$$\left(0; +; [n]; \left\{\left(2, \overset{2u}{\ldots}, 2, \frac{n}{2}\right)\right\}\right)$$

for some nonnegative integer u and consider an epimorphism $\theta\colon \Lambda \to \mathrm{D}_n = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^n \rangle$ defined by $\theta(e_1) = \theta(x_1)^{-1} = \tau\sigma$, $\theta(c_{10}) = \tau$ and which sends first $2u+1$ reflections corresponding to the unique nonempty period cycle alternatively to τ and $\tau(\sigma\tau)^{n/2}$ and $\theta(c_{12u+1}) = \tau(\sigma\tau)^2$. Then, by Lemma 3.1, θ defines the configuration $(0, 2u+1)_n$ of two symmetries on a Riemann surface of genus $g = nu/2 + n/2 - 1$. Now observe that $4(g-1)/n = 2u + 2 - 8/n$ and so, as $n > 8$, we have $[4(g-1)/n] = 2u + 1$. Hence we have constructed the configuration of symmetries requested in the theorem indeed. □

Theorem 3.6. *Let* X *be a Riemann surface of genus* g, *admitting a pair* $(0, t)_n$, *where* $n > 2$ *is an even integer such that* $n \equiv 2 \bmod 4$ *and* n *does not divide* $2(g-1)$. *Then* $t \leq [2(g-1)/n] + 1$ *and there are infinitely many values of* g *for which this bound is attained.*

Proof. Let n, g be as in the theorem and let $G = \langle \sigma, \tau \rangle = \mathrm{D}_n$. Here $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(-)^l, C_1, \ldots, C_k\}),$$

where as usual $C_i = (n_{i1}, \ldots, n_{is_i})$, $s_i \geq 1$. Again, like in the proof of part (3) of Theorem 3.4, all the link periods are odd. As n does not divide $g-1$, it follows that there are proper or link periods in the signature of Λ. Let us assume first that $h = 0$. Here, for the epimorphism $\theta\colon \Lambda \to \mathrm{D}_n$ to exist there are several possibilities. First of all, assume that there are no link periods and all the connecting generators are mapped to 1. Then for the epimorphism to exist, there are two even proper periods, which are greater than 2 as $(g-1)/n$ is not an integer. Therefore

$$\begin{aligned}
\frac{2\pi(g-1)}{n} &= \mu(\Lambda) \\
&\geq 2\pi\left(-2 + 2 - \frac{2}{4} + k + l\right) \\
&\geq 2\pi\left(\frac{t}{2} - \frac{1}{2}\right)
\end{aligned}$$

as $2(k+l) \geq t$, by Lemma 3.1, and so $t \leq 2(g-1)/n+1$. If there are no link periods but there is some connecting generator e_i mapped nontrivially, then again there are at least two even proper periods greater than 2, as $(g-1)/n$ is not an integer. By [6], $2(k+l) \geq t+1$ and we have $2\pi(g-1)/n = \mu(\Lambda) \geq 2\pi(-2+2-2/4+(t+1)/2) \geq 2\pi \cdot t/2$, resulting with $t \leq 2(g-1)/n$. If there are at least two link periods, then for the epimorphism to exist either there are also at least two proper periods or one proper period and one connecting generator mapped nontrivially, or else at least two of the generators e_i are mapped nontrivially. In all of the cases we obtain $2\pi(g-1)/n = \mu(\Lambda) \geq 2\pi(-2+t/2+2/2+2/3) \geq 2\pi(t/2-1/3)$, giving $t \leq 2(g-1)/n+2/3$. Finally, if there is only one link period, then the connecting generator corresponding to the nonempty period cycle is mapped nontrivially and there is also an even proper period greater than 2, resulting with $2\pi(g-1)/n = \mu(\Lambda) \geq 2\pi(-2+3/4+(t+1)/2+1/3) \geq 2\pi(t/2-5/12)$ and so $t \leq 2(g-1)/n+10/12$.

Assume now that $h \geq 1$. If there are at least two proper periods, then $2\pi(g-1)/n = \mu(\Lambda) \geq 2\pi(1-2+2/2+t/2) \geq 2\pi \cdot t/2$ and so $t \leq 2(g-1)/n$. If there is only one proper period, then there is also at least one connecting generator e_i mapped nontrivially and, by [6] again, $2(k+l) \geq t+1$ giving $t \leq 2(g-1)/n$ as above. Finally, if there are no proper periods, then there is a nonempty period cycle and if there are at least two link periods, we have $2\pi(g-1)/n = \mu(\Lambda) \geq 2\pi(1-2+t/2+2/3) \geq 2\pi(t/2-1/3)$ and so $t \leq 2(g-1)/n+2/3$. Now if there is only one link period, then the connecting generator of the nonempty period cycle is mapped nontrivially and the image is not $(\sigma\tau)^{n/2}$. Hence the epimorphism cannot exist.

Now it is enough to observe, that all the values we obtained in resulting inequalities, are smaller than $2(g-1)/n+2$ and the difference is at least 1. Therefore there are also smaller than $[2(g-1)/n]+2$, which finishes the first part of the proof. We have showed that $t \leq [2(g-1)/n]+1$.

For the attainment, let us consider an NEC group Λ with signature

$$\left(0; +; [-]; \left\{\left(\frac{n}{2}\right)^2, (-)^u\right\}\right)$$

for some nonnegative integer u and an epimorphism defined by $\theta(c_{10}) = \theta(c_{21}) = \tau$, $\theta(c_{20}) = \theta(c_{11}) = \tau(\sigma\tau)^2$, $\theta(e_1) = \theta(e_2)^{-1} = \tau\sigma$ and $\theta(c_{i0}) = \tau$, $\theta(e_i) = 1$ for $i \geq 3$. Here $\mathcal{H}/\ker(\theta)$ gives rise to the Riemann surface of genus $g = nu+n-1$, which admits a pair $(0, 2u+2)_n$. Now $2(g-1)/n = 2u+2-4/n$, $n \geq 6$ and so $2u+2 = [2(g-1)/n]+1$ which means that indeed we constructed a configuration of symmetries in question. □

4. Points fixed by the product of symmetries

Now we shall state some results concerning the number m of points fixed by the product of two symmetries. The next result coming from [11] (see also [7]) is crucial for the paper

Proposition 4.1. *Let $\mathrm{Z}_n = \Delta/\Gamma$ be the cyclic group of orientation preserving automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ and let $x_1, x_2, \ldots, x_r$ be the set of canonical elliptic generators of a Fuchsian group Δ with periods $m_1, \ldots, m_r$ respectively. Then the number m of points of X fixed by an element $g \in \mathrm{Z}_n$ of order d is given by the formula*

$$m = n \sum \frac{1}{m_i},$$

where the sum is taken over those i for which d divides m_i.

Let X be a Riemann surface admitting a pair $(0, t)_n$ and let $G = \langle \sigma, \tau \rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature (1). The subgroup of Λ/Γ of orientation preserving automorphisms is generated by the product $\sigma\tau$ and is Λ^+/Γ. Observe now that in our case there are no link periods equal to n, as one of the symmetries is fixed point free. Therefore we have the following

Corollary 4.2. *The product of two symmetries σ and τ of a Riemann surface X has $2r$ fixed points, where r denotes the number of proper periods equal to n in the signature of Λ in the presentation $\langle \sigma, \tau \rangle = \Lambda/\Gamma$.*

Now we shall recall the upper bound for m, which was given in [8]:

Proposition 4.3. *The number of points fixed by an orientation preserving automorphism of order n of a Riemann surface of genus g, does not exceed $2(g+n-1)/(n-1)$ and the bound is attained whenever $n-1$ divides $2g$.*

In the sequel we shall denote the maximal possible number $2(g+n-1)/(n-1)$ of fixed points by M. It is not hard to prove, that the bound above can also be realized for the product of two symmetries having fixed points whenever $n-1$ divides $2g$. Now we shall give some conditions under which an integer $m \leq M = (2g+n-1)/(n-1)$ can be realized as the number of points fixed by the product of two symmetries, one of them being fixed point free. Observe that in particular taking $a = 0$ in part (2) of the next result gives us the necessary and sufficient condition for the maximal possible number M of fixed points to be attained

Theorem 4.4. *The following conditions hold*:
(1) *If there exists a Riemann surface of genus g, having a pair of symmetries, at least one without ovals, whose product has order n and has $M = 2(g+n-1)/(n-1)$ fixed points then $n, g \geq 2$ are integers such that $n-1$ divides g and n, g have different parity*;
(2) *Conversely, let g, n be integers such that M is attained as the number of points fixed by the product of two symmetries whose product has order n and one of them is*

fixed point free. Let also $1 < u < n$ be a proper divisor of n and $a \geq 0$ an integer. Then, if n is even any value $0 \leq m = M - 4ua$ is also attained and if n is odd, then all the values $0 \leq m = M - 2ua$ are attained.

Proof. We shall prove the necessity first. Let us assume that X is a Riemann surface, having a pair of symmetries σ, τ, such that at least one of them is fixed point free, their product has order n and has $M = 2(g+n-1)/(n-1)$ fixed points. Let $G = \langle \sigma, \tau \rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [n, \overset{M/2}{\ldots}, n, m_1, \ldots, m_r]; \{(-)^l, C_1, \ldots, C_k\}),$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$, $s_i \geq 1$. By the Hurwitz–Riemann formula we get $\mu(\Lambda) = 2\pi(g-1)/n$ and so

$$\begin{aligned}\frac{2\pi(g-1)}{n} &= \mu(\Lambda)\\ &= 2\pi\left(\varepsilon h - 2 + k + l + \frac{g}{n} + 1 - \frac{1}{n} + R_1 + R_2\right),\end{aligned}$$

where

$$R_1 = \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) \quad \text{and} \quad R_2 = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=0}^{s_i}\left(1 - \frac{1}{n_{ij}}\right).$$

It follows that $r = k = 0$ and $\varepsilon h + l = 1$, as otherwise there would be no epimorphism $\theta\colon \Lambda \to G$. By Corollary 4.2, M is even and so $n-1$ divides g. Now if n is even, $M/2$ is even and so $M = 2g/(n-1)+2$ is divisible by 4, which means that $g/(n-1)$ is odd. It follows that if n is even, then g is odd. Now if n is odd then g must be even, as $n-1$ divides g.

For the proof of (2), let first n be even. Consider an NEC group Λ with signature

$$\left(2(u-1)a; -; \left[\frac{n}{u}, \overset{2a}{\ldots}, \frac{n}{u}, n, \overset{M/2-2ua}{\ldots}, n\right]; \{(-)\}\right)$$

and an epimorphism $\theta\colon \Lambda \to \mathrm{D}_n = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^n \rangle$ defined by $\theta(e_1) = 1$, $\theta(c_{10}) = \tau$, $\theta(d_i) = \sigma$ and sending the canonical elliptic generators x_i, $i = 1, \ldots, 2a$ alternatively to $(\sigma\tau)^u$ and $(\tau\sigma)^u$ and generators x_i, $i = 2a+1, \ldots, M/2 - 2ua + 2a$ alternatively to $\sigma\tau$ and $\tau\sigma$. As before, $\mathcal{H}/\ker(\theta)$ gives rise to the Riemann surface of genus g, which admits a pair $(0,2)_n$ and the product of the symmetries has $M - 4ua$ fixed points.

Let now n be odd. If 4 divides M, then we take an NEC group Λ with signature

$$\left((u-1)a + 1; -; \left[\frac{n}{u}, \overset{a}{\ldots}, \frac{n}{u}, n, \overset{M/2-ua}{\ldots}, n\right]; \{-\}\right)$$

and an epimorphism $\theta\colon \Lambda \to \mathrm{D}_n = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^n \rangle$ defined as in the previous case if a is even. If a is odd we take θ such that $\theta(d_i) = \sigma$ and $\theta(x_1) = \theta(x_2) = (\sigma\tau)^{2u}$, $\theta(x_3) = (\tau\sigma)^{4u}$ and θ sends the canonical elliptic generators x_i, $i = 4, \dots, a$ alternatively to $(\sigma\tau)^u$ and $(\tau\sigma)^u$, $\theta(x_{a+1}) = \theta(x_{a+2}) = (\sigma\tau)^2$, $\theta(x_{a+3}) = (\tau\sigma)^4$ and generators x_i, $i = a+4, \dots, M/2 - ua + a$ alternatively to $\sigma\tau$ and $\tau\sigma$. In both cases we obtain a Riemann surface of genus g, admitting a pair $(0, 0)_n$ with the product having $m = M - 2ua$ fixed points.

Let now $M \equiv 2 \mod 4$. Consider an NEC group with the signature as in the previous case. If a is even, the epimorphism defined similarly as above for $M \equiv 0 \mod 4$ and a odd, gives rise to the configuration we looked for. If a is odd, then we take the epimorphism defined similarly as in the case $M \equiv 0 \mod 4$, a even. As before, $\mathcal{H}/\ker(\theta)$ gives rise to the Riemann surface of genus g, which admits a pair $(0, 0)_n$ and the product of the symmetries has $M - 2ua$ fixed points. $\square$

Now we shall take into account the number m of points fixed by the product $\sigma\tau$ and give bounds for the number of ovals of symmetry τ. Similarly to the case of two symmetries with fixed points (see also [5]), we obtain bounds for the number of ovals in terms of g, n, and m and show their attainment for some series of g, n, m. Observe, that we can assume that n is even, as otherwise both symmetries would be fixed point free as conjugate ones.

Theorem 4.5. *Let X be a Riemann surface, admitting a pair $(0, t)_n$, with the product having m fixed points. Then the following conditions hold*:
(1) *If $n \equiv 2 \mod 4$, then $t \le ((2g + m - 2)/n) + 4 - m$ and this bound is attained whenever g is odd, $m \ge 4$ and n divides $g + (m/2) - 1$;*
(2) *If $n \equiv 0 \mod 4$ and $m > M - 2$ then $t \le 2$ and this bound is attained for any odd g with $m = M$;*
(3) *If $n \equiv 0 \mod 4$, $n > 4$ and $m \le M - 2$, then $t \le ((4g + 2m - 4)/n) + 4 - 2m$ and this bound is attained if n divides $2g + m - 2$;*
(4) *If $n = 4$ and $m \le M - 2$, then $t \le g + 3 - (3m/2)$ and this bound is attained for any odd g.*

Proof. Let first $n \equiv 2 \mod 4$ and let as before $G = \langle \sigma, \tau \rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [n, \overset{m/2}{\dots}, n, m_1, \dots, m_r]; \{(-)^l, C_1, \dots, C_k\}), \tag{5}$$

where $C_i = (n_{i1}, \dots, n_{is_i})$, $s_i \ge 1$ and $s = s_1 + \dots + s_k$. Observe, as in the last case of the proof of Theorem 3.4, that all the link periods are odd. By the Hurwitz–Riemann formula we get $\mu(\Lambda) = 2\pi(g-1)/n$ and by Lemma 3.1, as $2(l+k) \geq t$, we have

$$\begin{aligned} \frac{2\pi(g-1)}{n} &= \mu(\Lambda) \\ &\geq 2\pi\left(\varepsilon h - 2 + k + l + \frac{m}{2}\cdot\left(1-\frac{1}{n}\right) + \frac{s}{4}\right) \\ &\geq 2\pi\left(-2 + \frac{m}{2}\cdot\left(1-\frac{1}{n}\right) + \frac{t}{2}\right) \end{aligned} \tag{6}$$

which gives $t \leq 2(g-1)/n + 4 - m(1-1/n) \leq (2g+m-2)/n + 4 - m$.

Now, for the attainment, consider g, n and $m \geq 4$ such that g is odd and n divides $g + m/2 - 1$. Observe that in such case $m/2$ is even. Let Λ be an NEC group with signature

$$(0; +; [n, \overset{m/2}{\ldots}, n]; \{(-)^l\}),$$

where $l = (g + m/2 - 1)/n + 2 - m/2$ and let $\theta\colon \Lambda \to \mathrm{D}_n = \langle\sigma, \tau\rangle$ be an epimorphism sending all the canonical reflections to τ, all the generators e_i to 1, and the canonical elliptic generators x_i alternatively to $\sigma\tau$ and $\tau\sigma$. Then $X = \mathcal{H}/\ker\theta$ is a Riemann surface of genus g, which admits a pair $(0, (2g+m-2)/n + 4 - m)_n$ with the product having m fixed points. Observe also, that the bound cannot be attained for $m = 0$, as in such case we would have $t = 2(g-1)/n + 4 > 2(g-1)/n + 2$, a contradiction.

Let now $n \equiv 0 \mod 4$ and $m > M-2$. Let, as above $G = \langle\sigma, \tau\rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature (5). If there are no even link periods in the signature of Λ, then as above we get $t \leq 2g/n + 2 - (m-2)(1-1/n)$.

If $k > 0$ and there are even link periods, then by Lemma 3.1, $2l + s \geq t$ and so

$$\begin{aligned} \frac{2\pi(g-1)}{n} &= \mu(\Lambda) \\ &\geq 2\pi\left(\varepsilon h - 2 + k + \frac{l}{2} + \frac{m}{2}\cdot\left(1-\frac{1}{n}\right) + \frac{s}{4}\right) \\ &\geq 2\pi\left(-1 + \frac{m}{2}\cdot\left(1-\frac{1}{n}\right) + \frac{t}{4}\right) \end{aligned} \tag{7}$$

which gives $t \leq 4(g-1)/n + 4 - 2m(1-1/n) = (4g+2m-4)/n + 4 - 2m$. Observe now, that for $m > M-2$ we get that $4g/n - 2(m-2)(1-1/n) < 2g/n + 2 - (m-2)(1-1/n) < 4$. So we may assume that there are no even link periods. If $\theta(e_i) \neq 1$ for some empty period cycle C_i, then this cycle contributes with 1 oval by Theorem 2.1 and so $2(l+k) \geq t+1$. Hence

$$\begin{aligned}\frac{2\pi(g-1)}{n} &= \mu(\Lambda) \\ &\geq 2\pi\left(\varepsilon h - 2 + k + l + \frac{m}{2}\cdot\left(1-\frac{1}{n}\right) + t + \frac{1}{2}\right) \\ &\geq 2\pi\left(-\frac{3}{2} + \frac{m}{2}\cdot\left(1-\frac{1}{n}\right) + \frac{t}{2}\right)\end{aligned}$$

which gives $t \leq 3/2$. Similarly, if there are nonempty period cycles we get $\mu(\Lambda) \geq 2\pi(-2 + t/2 + m/2\cdot(1-1/n) + 1/3)$ which in turn gives $t < 5/3 - 2/n < 2$.

Therefore we may assume that all the period cycles are empty and for all i, $\theta(e_i) = 1$, which means that all period cycles contribute with two ovals. Hence, as $t < 4$, it must be that $t \leq 2$. For the attainment it is enough to consider the signature and an epimorphism from the proof of Theorem 4.4, case when n is even, $a = 0$. Observe also that the bound is attained with $m = M$.

We shall treat the bound for cases (3), (4) together. As $m \leq M-2$, we get that $2g/n + 2 - (m-2)(1-1/n) \leq 4g/n - 2(m-2)(1-1/n)$ and so by the previous parts of the proof $t \leq (4g+2m-4)/n + 4 - 2m$, which gives $t \leq g + 3 - 3m/2$ for $n = 4$. Now for the proof of the attainment, consider an NEC group Λ with signature

$$(0;+;[n,\overset{m/2}{\ldots},n];\{(2,\overset{s}{\ldots},2)\}),$$

where $s = (4g+2m-4)/n + 4 - 2m$ is even and let $\theta\colon \Lambda \to \mathrm{D}_n = \langle\sigma,\tau\rangle$ be an epimorphism sending the consecutive canonical reflections alternatively to τ and $\tau(\sigma\tau)^{n/2}$, canonical elliptic generators alternatively to $\sigma\tau$ and $\tau\sigma$. If 4 divides m, then we take $\theta(e_1) = 1$ and if $n = 4$ and $m/2$ is odd, we take $\theta(e_1) = \sigma\tau$. In both cases we obtain a Riemann surface of genus g, admitting a pair $(0,t)_n$, with the product having m fixed points and t being maximal. □

5. Order of the product of the symmetries

In this section we investigate the case of non-conjugate and noncommuting pairs of symmetries. Our aim is to find sharp upper and lower bound for the maximal order of the product of the symmetries in question, given the numbers of their ovals. Now conjugate symmetries have the same topological properties, in particular they have the same numbers of ovals, so without loss of generality we shall restrict ourselves to non-conjugate symmetries. By the Sylow theorem we may assume that in fact these symmetries generate a dihedral 2-group as all Sylow 2-groups are conjugate. Observe, that Theorem 3.4 implies in particular restrictions for the order of the product of symmetries in question. Here we define a function

$$\mu_g\colon \{0,\ldots,g+1\} \to \mathbb{N},$$

where $\mu_g(q) = n$ if and only if 2^n is the biggest power of 2 being realized as the order of the product of a pair of $(M-q)$- and fixed point free symmetries on a Riemann surface of genus g. As we consider only pairs of non-commuting symmetries generating a 2-group, then by Theorem 3.3 we can assume that g is odd. Observe however, that this result does not give any conditions on the number of ovals of the second symmetry. In particular, it can also be a fixed point free symmetry. We shall study this case in the last part of the paper.

5.1. Pairs of symmetries with one being fixed point free. Similarly to the case of symmetries with fixed points, which was studied in [10], we shall use an upper bound for the number of ovals of one of the symmetries in question, i.e. $g+1-q$, in terms of the genus of the surface and the order of the product of the symmetries. By Theorem 3.4, the following corollary holds true, which allows us to give an upper bound for $\mu_g(q)$ in the next theorem.

Corollary 5.1. *Let X be a Riemann surface of genus g, having pair of $(M-q)$- and fixed point free symmetries σ, τ, whose product has order 2^n with $n \geq 2$. Then the following conditions hold*:
(1) *If $n > 2$ then $q \geq g+1-(g-1)/2^{n-2}$ and the bound is attained when $(g-1)/2^{n-2}$ is even or $n = 3$;*
(2) *If $n = 2$, then $q \geq 1$ and the bound is attained for any odd g.*

Theorem 5.2. *The order of the product of an $(M-q)$- and a fixed point free symmetry on a Riemann surface of genus g, satisfying*

$$g+1-\frac{g-1}{2^{n-2}} \leq q < g+1-\frac{g-1}{2^{n-1}} \tag{8}$$

for some $n > 2$, does not exceed 2^n and so $\mu_g(q) \leq n$. If the parameters g, q satisfy

$$1 \leq q < \frac{g+3}{2} \tag{9}$$

then the order does not exceed $2^2 = 4$ and so $\mu_g(q) \leq 2$.

Proof. Observe that in the first case the number of ovals of $(M-q)$-symmetry satisfies $g+1-q > g/2^{n-1}$. Let d denote the order of the product of the symmetries and assume to a contrary that d is a power of 2 such that $d \geq 2^{n+1}$. Now by Corollary 5.1 we have

$$q \geq g+1-\frac{g-1}{2^{n-1}}$$

which leads to a contradiction, as in the same time $q < g+1-g/2^{n-1}$.

Let us now consider the second case, that is g, q satisfy (9). Assume to a contrary that the product of our symmetries is at least $2^3 = 8$. Then by Corollary 5.1 we have $q \geq (g+3)/2$ and on the other hand $q < (g+3)/2$, a contradiction. $\square$

It is more challenging to show that this bound is in fact attained.

Theorem 5.3. *Let g, q and n be integers such that 2^{n-1} divides $g-1$ and for $n > 2$ or $n = 2$ respectively* (8) *or* (9) *holds. Then $\mu_g(q) = n$, that is there exists Riemann surface of genus g having a pair of $(M-q)$- and fixed point free symmetries σ, τ with the product of order 2^n.*

Proof. Observe that with our assumption $g = 2^{n-1}a + 1$ for some $a \geq 1$ and $g + 1 - (g-1)/2^{n-2}$ is an even integer, so we can assume that $q = g + 1 - (g-1)/2^{n-2} + \alpha = g + 1 - 2a + \alpha$ where $\alpha \geq 0$ is an integer. Observe also that the last equality in turn gives $g + 1 - q = 2a - \alpha$ which is the number of ovals of one of the symmetries in question.

Let first $\alpha = 2\beta$. Consider an NEC group Λ with signature

$$(1; -; [2, \overset{\beta}{\ldots}, 2]; \{(2, \overset{g+1-q}{\ldots}, 2)\})$$

and an epimorphism $\theta \colon \Lambda \to \mathrm{D}_{2^n} = \langle \sigma, \tau \rangle$ defined as follows for the consecutive canonical reflections corresponding to the nonempty period cycle

$$\underbrace{\sigma, \sigma(\sigma\tau)^{2^{n-1}}, \sigma, \sigma(\sigma\tau)^{2^{n-1}}, \ldots, \sigma}$$

for which $\theta(d_1) = \tau, \theta(x_i) = (\sigma\tau)^{2^{n-1}}$, $\theta(e) = 1$ for β even. If β is odd we take $\theta(e) = (\sigma\tau)^{2^{n-1}}$. With such a definition, by Lemma 3.1, θ gives rise to the configuration of two $(M-q)$- and fixed point free symmetries on a Riemann surface of genus g, whose product has order 2^n.

Let now $\alpha = 2\beta + 1$ with $n > 2$. Observe, that it follows that $q \leq g - 2$ as q is odd and (8) holds. Indeed, in such case $q < g + 1 - a \leq g$ as $a \geq 1$. Now take an NEC group Λ with signature

$$(1; -; [2, \overset{\beta}{\ldots}, 2]; \{(2, \overset{g-1-q}{\ldots}, 2, 4, 4)\})$$

and an epimorphism $\theta \colon \Lambda \to \mathrm{D}_{2^n} = \langle \sigma, \tau \rangle$ for which $\theta(d_1) = \tau$, $\theta(x_i) = (\sigma\tau)^{2^{n-1}}$ for $i = 1, \ldots, \beta$ and mapping the canonical reflections corresponding to the nonempty period cycle respectively to

$$\underbrace{\sigma, \sigma(\sigma\tau)^{2^{n-1}}, \sigma, \sigma(\sigma\tau)^{2^{n-1}}, \ldots, \sigma(\sigma\tau)^{2^{n-1}}}_{g-q}, \sigma(\sigma\tau)^{2^{n-2}}, \sigma$$

with $\theta(e) = 1$ for β even and $\theta(e) = (\sigma\tau)^{2^{n-1}}$ otherwise. Again, in both cases, θ defines the configuration of two $(M-q)$- and fixed point free symmetries on a Riemann surface of genus g, whose product has order 2^n.

Let now $n = 2$. Consider an NEC group Λ with signature

$$(0; +; [2, \overset{\gamma}{\ldots}, 2, 4]; \{(2, \overset{g+1-q}{\ldots}, 2)\})$$

where $\gamma = (q-3)/2 + 1$. Define an epimorphism $\theta\colon \Lambda \to G$ by taking $\theta(x_i) = (\sigma\tau)^2$ for $i = 1, \ldots, \gamma$, $\theta(x_{\gamma+1}) = \sigma\tau$ and which sends the consecutive canonical reflections alternatively to σ and $\tau\sigma\tau$. Furthermore, take $\theta(e) = \sigma\tau$ for γ odd and $\theta(e) = \tau\sigma$ otherwise. This definition leads to the configurations of symmetries in question. $\square$

The proof of the previous theorem gives us some information on a lower bound for $\mu_g(q)$ for $g = 2^u a + 1$. In such case $\mu_g(q) \geq u + 1$, provided that $q \geq g + 1 - (g-1)/2^{u-1}$ if $u > 1$ and $q \geq 1$ for $u = 1$. In particular taking $u = 1$ leads to the following corollary, which was also presented in [6].

Corollary 5.4. *For any odd g and $q \geq 1$ there exists Riemann surface of genus g, having a pair of non-commuting $(M-q)$- and fixed point free symmetries.*

5.2. Pairs of fixed point free symmetries. In this part we study the case of Riemann surfaces of genus g having two fixed point free symmetries σ, τ with the product of order 2^n, $n \geq 2$. By Theorem 3.3, we may assume that g is odd. So let $g = 2^u a + 1$ for some odd a and $u \geq 1$. First of all, we shall give the upper bound for $\mu_g(g+1)$.

Theorem 5.5. *Let $g, n \geq 2$ be integers such that g is odd and*

$$2^n - 1 \leq g < 2^{n+1} - 1.$$

Then $\mu_g(g+1) \leq n$ and this bound is attained if and only if $g = 2^{n+1} - 1 - 2^{n-l}$ and $0 \leq l < n$ or $g = 2^{n+1} + 1 - 2^{n+1-l}$ for $1 \leq l < n$.

Proof. Let n, g be integers holding conditions of the theorem and assume that we have a Riemann surface of genus g, having a pair of fixed point free symmetries with the product of order 2^v with $v > n$. Now let $G = \langle\sigma, \tau\rangle = D_{2^v} = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; -; [m_1, \ldots, m_r]; \{-\}). \tag{10}$$

Note, that there are no period cycles, as our symmetries have no ovals. Moreover, for the epimorphism $\theta\colon \Lambda \to \langle\sigma, \tau\rangle = D_{2^v}$ to exist it must be that $h > 0$ and sign is $-$.

By the Hurwitz–Riemann formula, we know that $(g-1)/2^v = \mu(\Lambda)/2\pi$. Now, as $g < 2^{n+1}-1$ and $v \geq n+1$, we see that

$$\frac{\mu(\Lambda)}{2\pi} = \frac{g-1}{2^v} < 1 - \frac{1}{2^{v-1}}. \tag{11}$$

Now, as $v > n \geq 2$ we see that there are proper periods in the signature of Λ and there are at least two as the relation $\theta(x_1 \cdots x_r) = 1$ must hold. Hence $r \geq 2$. Now if $h \geq 2$ we see that $\mu(\Lambda)/2\pi \geq 1$ and (11) does not hold. But if $h = 1$, then for the epimorphism to exist, there must a proper period equal 2^v, as otherwise the image of θ does not give D_{2^v}. Furthermore again, there are at least two such periods for the relation $\theta(x_1 \cdots x_r) = 1$ to hold. This gives $\mu(\Lambda)/2\pi \geq 1 - 1/2^{v-1}$ and also in this case we obtain the contradiction followed by the assumption $v > n$.

Note, that we only used the fact that $g < 2^{n+1}-1$. In the same way one shows, that for $g < 2^n - 1$ the order of the product of two fixed point free symmetries is strictly smaller than 2^n. Hence the bound for $\mu_g(g+1)$ can be attained only for $g \geq 2^n - 1$.

Now we shall prove the necessary and sufficient condition for the attainment of the bound. For, we shall find exact values of g for which $2^n - 1 \leq g < 2^{n+1}-1$ and $\mu_g(g+1) = n$. As before let $G = \langle \sigma, \tau \rangle = D_{2^n} = \Lambda/\Gamma$ for some Fuchsian surface group Γ and an NEC group Λ with signature (10). First of all, as $g < 2^{n+1}-1$ we see, again by the Hurwitz–Riemann formula, that $\mu(\Lambda)/2\pi < 2 - 1/2^{n-1}$. Observe now that if $h > 4$, $h = 3$, $r > 0$ or $h = 2, r \geq 4$ then $\mu(\Lambda)/2\pi > 2 - 1/2^{n-1}$ which is not our case here.

Thus let first $h = 2$ and $r = 3$. If two of the periods are equal 2, then the epimorphism $\theta : \Lambda \to D_{2^n}$ does not exist as the relation $\theta(x_1x_2x_3) = 1$ cannot hold in G. Hence at least two of them are greater than two which leads to a contradiction as in such case $\mu(\Lambda)/2\pi \geq 1/2 + 3/4 + 3/4 = 2$.

Now if $h = 2$, $r = 2$ then the proper periods are equal for the relation $\theta(x_1x_2) = 1$ to hold and must be strictly smaller than 2^n as otherwise $\mu(\Lambda)/2\pi = 2 - 1/2^{n-1}$, a contradiction.

Let finally $h = 1$. Observe that for our epimorphism θ to exist, there must be at least two proper periods equal 2^n. Now if there are more than three periods in total, then $\mu(\Lambda)/2\pi \geq 2 - 1/2^{n-1}$, which contradicts the assumption. Therefore the total number of periods for $h = 1$ is at most three.

Note that, as we have observed before, the case of $r = 1$ is impossible for we would have $\theta(x_1) = 1$ in such situation, which contradicts the fact that $\ker\theta$ is Fuchsian surface group.

Therefore the only possibilities for the signature are:

(1) $h = 1$, $r \leq 3$ and two of the periods are equal 2^n;

(2) $h = 2$, $r = 2$ and the periods are strictly smaller than 2^n;

(3) $h = 3$, $r = 0$.

For the case (1) and $r = 3$ we obtain an NEC group Λ with the signature

$$(1; -; [2^n, 2^n, 2^l]; \{-\}),$$

where $1 \le l < n$. By the Hurwitz–Riemann formula, we have the equality $(g-1)/2^n = \mu(\Lambda)/2\pi$ and as $\mu(\Lambda) = 2\pi(2 - 1/2^{n-1} - 1/2^l)$ we see that $g = 2^{n+1} - 1 - 2^{n-l}$. To see that there actually exists an epimorphism for such a group Λ define $\theta \colon \Lambda \to \langle \sigma, \tau \rangle = \mathrm{D}_{2^n}$ as $\theta(d) = \sigma$ for the only glide reflection, $\theta(x_1) = \sigma\tau$, $\theta(x_2) = (\sigma\tau)^{2^{n-l}-1}$ and $\theta(x_3) = (\tau\sigma)^{2^{n-l}}$. This definition leads to the configuration of two fixed point free symmetries with the product of order 2^n on the Riemann surface of genus $g = 2^{n+1} - 1 - 2^{n-l}$ for $1 \le l < n$.

Now for $r = 2$ we get the signature of an NEC group Λ being of the form

$$(1; -; [2^n, 2^n]; \{-\}).$$

Hence by the Hurwitz–Riemann formula again, $g = 2^n - 1$. For the sufficiency take an epimorphism θ to be defined as $\theta(d) = \sigma$, $\theta(x_1) = \theta(x_2)^{-1} = \sigma\tau$. We obtain the configuration of symmetries we looked for, on a Riemann surface of genus $g = 2^n - 1 = 2^{n+1} - 1 - 2^{n-l}$ for $l = 0$.

Given the conditions for the case (2) we have the signature of the form

$$(2; -; [2^l, 2^l]; \{-\})$$

with $1 \le l < n$ and for the necessary condition observe that $\mu(\Lambda) = 2 - 1/2^{l-1}$ and so by the Hurwitz–Riemann formula $g = 2^{n+1} + 1 - 2^{n+1-l}$. To see that this condition is also sufficient one takes an epimorphism θ defined as $\theta(d_1) = \sigma$, $\theta(d_2) = \tau$ and $\theta(x_1) = \theta(x_2)^{-1} = (\sigma\tau)^{2^{n-l}}$. This definition provides the configuration of fixed point free symmetries with the product of order 2^n on a Riemann surface of genus $g = 2^{n+1} + 1 - 2^{n+1-l}$ with $1 \le l < n$.

Finally let us consider the third case. Now our NEC group Λ has signature

$$(3; -; [-]; \{-\}).$$

By the Hurwitz–Riemann formula, $g = 2^n + 1$. As before, we can find the proper pair of symmetries by defining an epimorphism θ as $\theta(d_1) = \theta(d_2) = \sigma$, $\theta(d_3) = \tau$. The last case is equivalent to the previous one with $l = 1$. □

Now we shall give some results concerning a lower bound for $\mu_g(g+1)$, assuming that $g = 2^u a + 1$ for some $u \ge 1$ and odd a.

Theorem 5.6. *For $g = 2^u a + 1$ we have $\mu_g(g+1) \ge u$ and this bound is attained for arbitrary $u \ge 2$ and $a = 1$. Moreover, for any odd g there exists a Riemann surface of genus g, having a pair of non-commuting fixed point free symmetries.*

Proof. First we shall construct, for any g as in the theorem, a Riemann surface of genus g, having two fixed point free symmetries, whose product has order 2^u. Consider an NEC group Λ with signature

$$(a+2; -; [-]; \{-\})$$

and an epimorphism $\theta\colon \Lambda \to D_{2^u} = \langle \sigma, \tau \rangle$ defined by $\theta(d_1) = \sigma$, $\theta(d_2) = \cdots = \theta(d_{a+2}) = \tau$. This gives rise to the configuration in question, hence $\mu_g(g+1) \geq u$.

Now, for the attainment take $g = 2^u + 1$, $u \geq 2$ and let $n > u$ be arbitrary. Recall that from our previous considerations it follows that for $g < 2^n - 1$ the order of the product of the fixed point free symmetries is at most 2^{n-1}. Here in fact we have $g < 2^n - 1$ as $2^n - 2^u - 2 > 0$ for $n > u \geq 2$. Therefore for g of the form above, the order of the product of two fixed point free symmetries is at most 2^u. Together with the first part of the proof we get the equality $\mu_g(g+1) = u$.

We shall prove the last part of the theorem. Clearly g may be written in the form $g = 2b + 1$. If b is even then it is enough to take signature of an NEC group and epimorphism to be analogous to the one given in the first part of the proof. Now if $b = 2v + 1$, take an NEC group with signature

$$(v+1; -; [4, 4]; \{-\})$$

and an epimorphism $\theta\colon \Lambda \to D_4$ for which $\theta(d_i) = \sigma$, $\theta(x_1) = \theta(x_2)^{-1} = \sigma\tau$. □

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