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# EXCEPTIONAL SURGERIES ON CERTAIN (1, 1)-KNOTS

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**Abstract**

We determine certain exceptional surgeries on a 3-parametric family of hyperbolic 1-bridge genus one knots ((1, 1)-knots, in short). In particular, we show that such knots admit two infinite series of lens space surgeries. Our work is related to a nice paper of Teragaito [16], since we represent his toroidal manifolds as 2-fold coverings of the 3-sphere branched over well-specified links.

## 1. Introduction

A *Dehn surgery* on a knot $K$ in the oriented 3-sphere $\mathbb{S}^3$ is a topological construction which yields a closed 3-manifold by removing an open tubular neighborhood of $K$ in $\mathbb{S}^3$, and glue a solid torus back. By gluing a solid torus back as it was, the surgery gives the 3-sphere again. Such a surgery is called *trivial*, and we will ignore it. A classical theorem of Wallace and Lickorish states that every pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries. For a hyperbolic knot in the 3-sphere, at most finitely many Dehn surgeries yield non-hyperbolic 3-manifolds by Thurston's hyperbolic surgery theorem. Such surgeries are said to be *exceptional*, and they have been studied extensively for many classes of knots. In particular, an exceptional surgery is called *toroidal* if the surgery manifold is toroidal, that is, it contains an incompressible torus. An exceptional surgery is called *Seifert-fibered* if the surgery manifold is a small Seifert-fibered space, that is, it has base space $\mathbb{S}^2$ and at most three singular fibers. One of the unsolved problems in Dehn surgery theory is to determine which knots in the 3-sphere admit Dehn surgeries yielding lens spaces. This problem is completely solved for torus knots and satellite knots. Also, there are hyperbolic knots with lens space surgeries. Such examples were first found by Fintushel and Stern in [5]. They proved that 18- and 19-surgeries on the $(-2, 3, 7)$-pretzel knot give the lens spaces $L(18, 5)$ and $L(19, 8)$, respectively. It was conjectured by Gordon that if a hyperbolic knot admits lens space surgery, then it is *doubly primitive* in the sense of Berge (unpublished manuscript). Here we recall this definition. Let $(V_1, V_2)$ be a genus two Heegaard splitting of $\mathbb{S}^3$ and $K$ a simple loop on $F = \partial V_1 = \partial V_2$. Then $K$ is called a *doubly primitive knot* if $K$ represents a free generator both of $\pi_1(V_1)$ and of $\pi_1(V_2)$. In this paper we study a 3-parametric family of hyperbolic 1-bridge genus one knots ((1, 1)-knots, in short), which includes the $(-2, 3, 2n-1)$-pretzel knots, a

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subclass of the Eudave-Muñoz knots [4], and certain knots considered by Teragaito in [16]. Then we determine certain exceptional surgeries on our (1, 1)-knots, and show that they admit two infinite series of lens space surgeries. We also obtain a presentation of the fundamental group of the constructed surgery manifolds. Finally, we describe covering properties of the toroidal manifolds obtained by Dehn surgery on the Teragaito knots [16].

## 2. The knots $K_{m,n,h}$

A knot $K$ in $\mathbb{S}^3$ is said to be a $(1,1)$-*knot* if $\mathbb{S}^3$ is a union of two solid tori $V_1$ and $V_2$ glued along their boundaries and if $K$ intersects each solid torus $V_i$ in a trivial arc $t_i$, $i = 1, 2$. It is known that every $(1,1)$-knot is a tunnel number one knot, and hence it is a 2-generator knot. Let $K_{m,n,h}$, $m \geq 1$, $h \geq 0$ and $n \geq m+2$, be the family of knots in the oriented 3-sphere $\mathbb{S}^3$ depicted in Fig. 1 (they were first considered in [15]). To make clear how strands run in the right-hand side in the figure, we have also depicted the knot $K_{2,4,2}$ in Fig. 2. If $n = m+2$, then $K_{m,n,h}$ is equivalent to the torus knot of type $((h+3)(n-1)-1, h+3)$. If $m = 1$ and $h = 0$, then $K_{m,n,h}$ is equivalent to the $(-2, 3, 2n-1)$-pretzel knot (in particular, the torus knot of type (5, 3) for $n = 3$). If $m = 2$ and $n = 6$, then $K_{m,n,h}$ is the knot $K_{\boldsymbol{n}}$, $\boldsymbol{n} = h+2$, with three consecutive toroidal Dehn surgeries, considered by Teragaito in [16]. If $m = 1$ and $n = 4$, then $K_{m,n,h}$ is equivalent to the Eudave-Muñoz knot $k(h+3, 1, 1, 0)$. In particular, for $h = 0$, the knot $k(3, 1, 1, 0)$ is the $(-2, 3, 7)$-pretzel knot. It is known that Eudave-Muñoz knots admit non-integral toroidal surgeries (see [4]).

For every $m$ such that $0 < m < n-2$, $K_{m,n,h}$ is a chiral strongly invertible hyperbolic (1, 1)-knot (see [15]). One can directly verify these facts for many values of the parameters by using the computer program SnapPea [19].

By the Wirtinger algorithm applied to the planar projection in Fig. 1, we get (see also [15]):

**Theorem 1.** *For every $m \geq 1$, $h \geq 0$ and $n \geq m+2$, the knot group of $K_{m,n,h}$ has the 2-generator presentation*

$$\pi(K_{m,n,h}) := \pi_1(\mathbb{S}^3 \setminus K_{m,n,h}) \cong \langle a, b\colon a(a^{h+2}b^{h+2})^{n-m-1}(a^{h+3}b^{h+3})^m = 1\rangle,$$

*where the path* $\mathbf{m} = ba$ *is a meridian of the knot.*

Let $E_{m,n,h} = E(K_{m,n,h})$ be the exterior of $K_{m,n,h}$ in $\mathbb{S}^3$, and choose a path

$$\boldsymbol{l}^* = a^{-(h+3)}(b^{-(h+2)}a^{-(h+2)})^{n-m-2}a^{-1}b^{-1}$$

as a longitudinal circle on $\partial E_{m,n,h}$. Of course, we have $[\boldsymbol{m}, \boldsymbol{l}^*] = 1$ by using the relation of

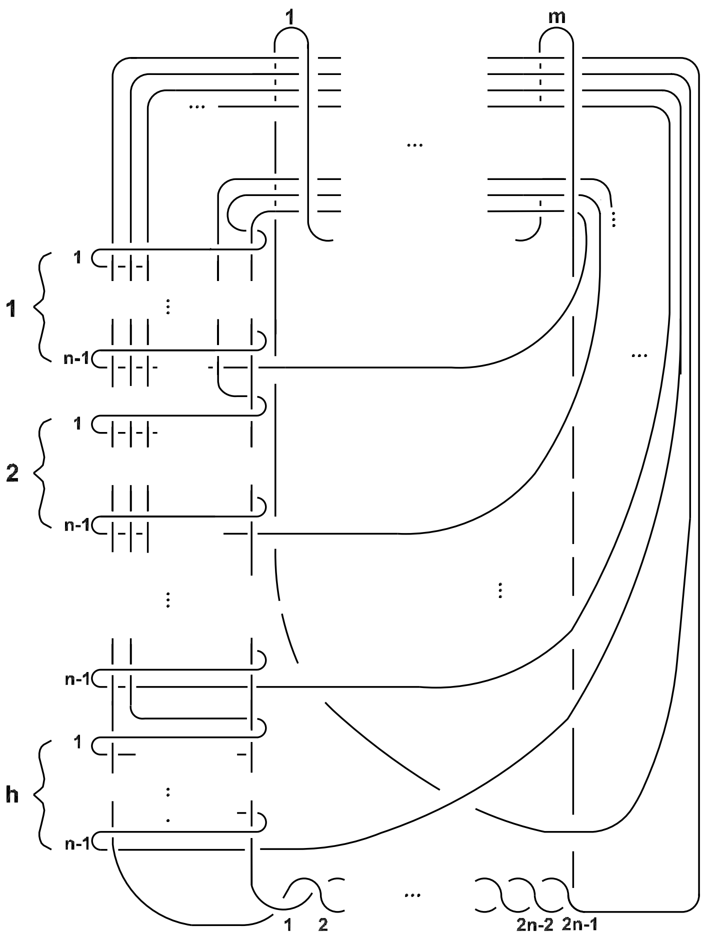


Fig. 1. The knot $K_{m,n,h}$, $m \geq 1$, $h \geq 0$, $n \geq m+2$.

the knot group. To find the null homologous longitude $\boldsymbol{l}$ in $E_{m,n,h}$, we form the expression

$$\boldsymbol{l} \sim \boldsymbol{l}^* + x\mathbf{m} = a^{-(h+3)}(b^{-(h+2)}a^{-(h+2)})^{n-m-2}a^{-1}b^{-1}(a^{-1}b^{-1})^{-x}$$

where $x \in \mathbb{Z}$ is not yet determined. Here the symbol $\sim$ means that two simple closed curves (in the exterior of $K_{m,n,h}$) are homologically equivalent. We show that the condition $\boldsymbol{l} \sim 0$ yields

$$x = -[(h+2)^2 n + (2h+5)m - h^2 - 3h - 3]$$

hence

$$\boldsymbol{l} \sim a^{-(h+3)}(b^{-(h+2)}a^{-(h+2)})^{n-m-2}(a^{-1}b^{-1})^{\xi},$$

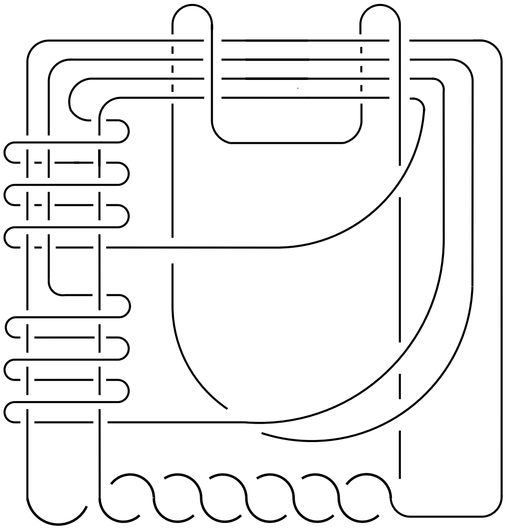

Fig. 2. The knot $K_{2,4,2}$.

where $\xi = (h+2)^2 n + (2h+5)m - h^2 - 3h - 2$. In fact, setting $\boldsymbol{l} \sim 0$ gives the equation

$$a(-h-4-(h+2)(n-m-2)+x) + b(-(h+2)(n-m-2)-1+x) = 0.$$

The relation of $\pi(K_{m,n,h})$, considered in its abelianization $H_1(E_{m,n,h})$, gives the equation

$$a(1+(h+2)(n-m-1)+(h+3)m) + b((h+2)(n-m-1)+(h+3)m) = 0.$$

Thus there exists an integer $\eta \in \mathbb{Z}$ such that

$$\begin{cases} \eta(1+(h+2)(n-m-1)+(h+3)m) = x-h-4-(h+2)(n-m-2), \\ \eta((h+2)(n-m-1)+(h+3)m) = -(h+2)(n-m-2)+x-1 \end{cases}$$

hence $\eta = -h-3$, that is, $x = -[(h+2)^2 n + (2h+5)m - h^2 - 3h - 3]$.

To obtain the surgery manifold $K_{m,n,h}(\gamma)$, $\gamma = p/q$, $\gamma \neq \infty$, we choose a simple closed curve $\phi_*(\boldsymbol{\mu}) = \mathbf{m}^p \boldsymbol{l}^q$, where $(\mathbf{m}, \boldsymbol{l})$ is the preferred frame, obtained above, $\boldsymbol{\mu}$ is the standard meridian of a solid torus $V = \mathbb{D}^2 \times \mathbb{S}^1$, and $\phi\colon \partial V \to \partial E_{m,n,h}$ is an attaching homeomorphism. Recall that a group presentation is said to be *balanced* if it has the same number of generators and relations.

Then we have

**Theorem 2.** *For every $m \geq 1$, $h \geq 0$ and $n \geq m+2$, the fundamental group of the surgery manifold $K_{m,n,h}(\gamma)$, $\gamma = p/q$, $\gamma \neq \infty$, has a balanced presentation with* *generators $a$ and $b$, and relations*

$$a(a^{h+2}b^{h+2})^{n-m-1}(a^{h+3}b^{h+3})^m = 1$$

*and*

$$(ba)^p[a^{-(h+3)}(b^{-(h+2)}a^{-(h+2)})^{n-m-2}(a^{-1}b^{-1})^{\xi}]^q = 1$$

*where*

$$\xi = (h+2)^2 n + (2h+5)m - h^2 - 3h - 2.$$

To complete the section, we determine the genus of our knots.

**Theorem 3.** *The genus of the knot $K_{m,n,h}$ is given by*

$$g(K_{m,n,h}) = (h+2)\left(\frac{(n-1)(h+1)}{2} + m\right).$$

Proof. Let us consider the planar picture of the knot $K_{m,n,h}$ in Fig. 1. Applying Seifert's algorithm to it, we can calculate the number $c$ of crossings and the number $s$ of Seifert circles in the sense of [12, Chapter 5]. We obtain $c = (h+1)(h+2)n + 2(h+2)m - h^2$ and $s = 3h + 3$. Now recall that the constructed Seifert surface $S$ has genus $g(S) = 1 - (1 + s - c)/2$ and $g(K_{m,n,h})$ is less than or equal to $g(S)$ (see [12, Exercise 10, p. 121]). To prove the reverse inequality, we use the free calculus of Fox and compute the degree of the Alexander polynomial $\Delta_{m,n,h}(t)$ of $K_{m,n,h}$ from the presentation of $\pi(K_{m,n,h})$ given in Theorem 1. Let $\pi^{ab} = \pi^{ab}(K_{m,n,h}) \cong \mathbb{Z}$ denote the abelianized group of $\pi = \pi(K_{m,n,h})$ and $\varphi\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi^{ab}] = \mathbb{Z}[t, t^{-1}]$ the abelianization map between the group rings. Setting $x = (ab)^{-1}$ and $y = a$ (with inverse relation $b = (xy)^{-1}$), $\pi^{ab}$ is freely generated by $x$ as $y = x^{\alpha}$, where $\alpha = (h+2)(n-m-1) + m(h+3)$. Furthermore, the relator of $\pi$ can be re-written as $R = y(y^{h+2}(y^{-1}x^{-1})^{h+2})^{n-m-1}(y^{h+3}(y^{-1}x^{-1})^{h+3})^m$. Then $\varphi(x) = t$, hence $\varphi(y) = t^{\alpha}$ and $\varphi(y^{-1}x^{-1}) = t^{-\alpha-1}$. Recall that the free derivatives of Fox satisfy the characteristic properties $\partial(uv)/\partial x = \partial u/\partial x + u\partial v/\partial x$, $\partial u^{-1}/\partial x = -u^{-1}\partial u/\partial x$ and $\partial u^n/\partial x = ((u^n - 1)/(u - 1))\partial u/\partial x$. Applying these to our case, we see that the Alexander polynomial $\Delta_{m,n,h}(t)$ is given by $\varphi(\partial R/\partial x)$ and its degree is equal to $(h+2)((h+1)(n-1) + 2m)$. Now recall that the degree of the Alexander polynomial of a knot $K$ in $\mathbb{S}^3$ cannot exceed $2g(K)$ (see [12, Exercise 10, p. 208]). This completes the proof. □

## 3. Exceptional surgeries on $K_{m,n,h}$

**Theorem 4.** *For $n = m + 3$, $m \geq 1$ and $h \geq 0$, we have the following results*:
i) *If $h = 0$ and $\gamma = 9m + 8$, then $K_{m,n,h}(\gamma)$ is the Seifert-fibered space defined by the invariants* $(O\ 0\ o\colon 0\ (2, 1)\ (5, 2)\ (m + 2, -1))$;

ii) *If* $\gamma = (h+3)^2m + 2h^2 + 9h + 9$, *then* $K_{m,n,h}(\gamma)$ *is the Seifert-fibered space defined by the invariants* $(O\,0\,o\colon 0\,(2,1)\,((h+3)m+1, h+3)\,(h+1,1))$. *In particular, if* $h = 0$, *we get the lens space* $L(9m+9, 3m+2)$;
iii) *If* $\gamma = (h+3)^2m + 2h^2 + 9h + 10$, *then* $K_{m,n,h}(\gamma)$ *is the Seifert-fibered space defined by the invariants* $(O\;0\;o\colon\;0\;(h+2,\;1)\;(2h+5,\;h+2)\;(m,\;1))$. *In particular, if* $m = 1$, *we get the lens space* $L(3h^2+15h+19, 3h+8)$.

Proof. i) By Theorem 2, $\pi_1 = \pi_1(K_{m,n,h}(\gamma))$, $\gamma = 9m+8$, has a balanced presentation with generators $a$ and $b$, and relations $a(a^2b^2)^2(a^3b^3)^m = 1$ and $baba^3b^2a^3 = 1$. Using the second relation, we can express the first relation as $(a^{-4}b^{-1})^m a^{-1}ba^{-1}b^{-1} = 1$. Setting $x = a^{-4}b^{-1}$ and $y = a$ (with inverse relation $b = x^{-1}y^{-4}$) yields a balanced presentation for $\pi_1$ with generators $x$ and $y$ and relations $x^{m+2} = (xy)^2$ and $(xy)^2y^3(xy)^2y^2 = 1$. From the first relation, we see that $x^{m+2}$ commutes with $xy$, hence it commutes with $y$. Then $\pi_1 \cong \langle x, y\colon x^{m+2} = (xy)^2, (xy)^4 = y^{-5}\rangle$. Let now $\Sigma$ be the Seifert-fibered manifold defined by the invariants $(O\;0\;o\colon\;0\;(2,\;1)\;(5,\;2)\;(m+2,\;-1))$, where $m > 1$ (for $m = 1$ the considered knot is the $(-2,3,7)$-pretzel knot, and the result is known). Since $1/2+1/5+1/(m+2) < 1$ for $m > 1$, $\Sigma$ is a large Seifert manifold in the sense of [8, p. 92], hence it is aspherical (for the classification of small Seifert manifolds we refer to Section 5.4 of [8, p. 99]). We recall by [8, p. 91] that $\pi_1(\Sigma) \cong \langle q_1, q_2, q_3, u\colon q_1^2u = 1, q_2^5u^2 = 1, q_3^{m+2} = u$, $q_1q_2q_3 = 1, [q_j, u] = 1, j = 1, 2, 3\rangle$. Eliminating the generators $u$ $(= q_3^{m+2})$ and $q_1$ $(= q_3^{-1}q_2^{-1})$ we get the presentation $\pi_1(\Sigma) \cong \langle q_2, q_3\colon q_3^{m+2} = (q_3q_2)^2, (q_3q_2)^2 = q_2^{-5}\rangle$, hence $\pi_1(\Sigma) \cong \pi_1(K_{m,n,h}(\gamma)) = \pi_1$. In particular, the element $x^{m+2}$ $(= q_3^{m+2} = u)$ generates an infinite cyclic group $\langle x^{m+2}\rangle$, which is the nontrivial center of $\pi_1$. Suppose $K_{m,n,h}(\gamma)$ is not prime. Since the knot has tunnel number one, the genus of $K_{m,n,h}(\gamma)$ is less than or equal to 2, and it can be decomposed in a connected sum $K_{m,n,h}(\gamma) = M_1 \# M_2$ where $M_i$ is prime for $i = 1, 2$, and $\pi_1 \cong \pi_1(M_1) * \pi_1(M_2)$. But a free product of non-trivial groups admits only a trivial center and here $\pi_1$ has a non-trivial one (generated by $u = x^{m+2}$). This implies that $\pi_1(M_1) = \pi_1$ and $\pi_1(M_2) \cong 0$. Hence $M_2$ is the 3-sphere. Now $K_{m,n,h}(\gamma) \cong M_1$ is prime, different from $\mathbb{S}^1 \times \mathbb{S}^2$ (because $\pi_1 \neq \mathbb{Z}$), then it is irreducible. By [2] the surgery manifold is Seifert-fibered, and hence it is homeomorphic to $\Sigma$ by [8, p. 134].

ii) By Theorem 2, $\pi_1 = \pi_1(K_{m,n,h}(\gamma))$, $\gamma = (h+3)^2m+2h^2+9h+9$, is presented by $\langle a, b\colon a(a^{h+2}b^{h+2})^2(a^{h+3}b^{h+3})^m = 1, b^{h+2}a^{h+3}ba^{h+3} = 1\rangle$. Setting $x = a^{h+3}b$ and $y = a$ (with inverse relation $b = y^{-h-3}x$), the second relation becomes $x^2(y^{-h-3}x)^{h+1} = 1$. So $y^{-h-3}x$ commutes with $x^2$, hence $y^{h+3}$ commutes with $x^2$. Using this fact, we see that the first relation of $\pi_1$ becomes $y(y^{h+2}(y^{-h-3}x)^{h+2})^2(y^{h+3}(y^{-h-3}x)^{h+3})^m = 1$, or, equivalently, $x^2y^{m(h+3)+1} = 1$. Then we have $\pi_1 \cong \langle x, y\colon y^{m(h+3)+1} = (y^{-h-3}x)^{h+1} = x^{-2}\rangle$. By [13, Theorems 2.1 and 3.1], our surgery manifold is the fibered tetrahedron manifold defined by the Seifert invariants $(O\;0\;o\colon\;-1\;(m(h+3)+1,\;-h-3)\;(2,\;-1)\;(h+1,\;h))$. But such a manifold is homeomorphic to that specified in (ii) by using standard modifications of the Seifert invariants (see [7, Chapter 4, p. 147]). If $h = 0$, then the surgery manifold is the Seifert-fibered space $(O\;0\;o\colon\;0\;(2,\;1)\;(3m+1,\;3)\;(1,\;1))$, which is homeomorphic to the Seifert manifold $(O\;0\;o\colon\;1\;(2,\;1)\;(3m+1,\;3))$. Recall now that a fibered space defined by the invariants $(O\,0\,o\colon b\,(\alpha_1,\beta_1)\,(\alpha_2,\beta_2))$ is the lens space $L(\xi,\eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1|$, $\eta = r\alpha_2 + s\beta_2$, and $r\alpha_1 - s(b\alpha_1 + \beta_1) = 1$. In our case, we have $(\alpha_1, \beta_1) = (2, 1)$, $(\alpha_2, \beta_2) = (3m+1, 3)$, $b = 1$, $r = s = -1$, hence $\xi = 9m+9$ and $\eta = -3m-4$. Since $(-3m-4)(3m+2) \equiv 1 \bmod (9m+9)$, the surgery manifold is the lens space $L(\xi, \eta) \cong L(9m+9, 3m+2)$.

iii) By Theorem 2, $\pi_1 = \pi_1(K_{m,n,h}(\gamma))$, $\gamma = (h+3)^2 m + 2h^2 + 9h + 10$, is presented by $\langle a, b\colon a(a^{h+2}b^{h+2})^2(a^{h+3}b^{h+3})^m = 1,\ a^{2h+5}b^{h+2} = 1\rangle$. Since $b^{h+2}$ is central in $\pi_1$, the first relation becomes $(a^{-(h+2)}b)^{m-1}a^{-(h+2)}b^{h+3} = 1$. Then we have $\pi_1 \cong \langle a, b\colon a^{2h+5}b^{h+2} = 1,\ (a^{-(h+2)}b)^{m-1}a^{-(h+2)}b^{h+3} = 1\rangle$. This presentation is geometric since it arises from a genus 2 Heegaard diagram of the manifold. Now we can apply Theorem 2.2 of [3] to conclude that our surgery manifold is the Seifert-fibered space defined by the invariants $(O\,0\,o\colon -1\,(h+2, h+3)\,(2h+5, h+2)\,(m, 1))$. But such a manifold is homeomorphic to that specified in (iii) by using standard modifications of the Seifert invariants (see [7, Chapter 4, p. 147]). If $m = 1$, then the surgery manifold is the Seifert-fibered space $(O\,0\,o\colon 0\,(h+2, 1)\,(2h+5, h+2)\,(1, 1))$, which is homeomorphic to $(O\,0\,o\colon 1\,(h+2, 1)\,(2h+5, h+2))$. As above, the surgery manifold is the lens space $L(\xi, \eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1|$, $\eta = r\alpha_2 + s\beta_2$, and $r\alpha_1 - s(b\alpha_1 + \beta_1) = 1$, with $(\alpha_1, \beta_1) = (h+2, 1)$, $(\alpha_2, \beta_2) = (2h+5, h+2)$ and $b = 1$. Then we have $\xi = 3h^2 + 15h + 19$, $r = s = -1$, and $\eta = -3h - 7$. Since $(-3h-7)(3h+8) \equiv 1 \bmod (3h^2 + 15h + 19)$, the surgery manifold is the lens space $L(\xi, \eta) \cong L(3h^2 + 15h + 19, 3h + 8)$. □

If $m = 1$ and $h = 0$ (hence $n = 4$) in Theorem 4, then $K_{m,n,h}$ is the $(-2, 3, 7)$-pretzel knot. So from (ii) and (iii) above we re-obtain the wellknown results that 18-surgery and 19-surgery on it give the lens spaces $L(18, 5)$ and $L(19, 8)$, respectively (see [5]). If $m = 1$ and $n = 4$, the knot $K_{m,n,h}$ is the Eudave-Muñoz knot $k(h+3, 1, 1, 0)$ (in particular, the $(-2, 3, 7)$-pretzel knot for $h = 0$), and its exceptional surgeries are known in the literature. For $h = 0$, there are exactly six nontrivial exceptional surgeries which correspond to the slopes 16, 17, 18, 19, 20, and $37/2$. The slopes 16, 20 and $37/2$ give toroidal manifolds; the slope 17 gives the Seifert-fibered space $(O\,0\,o\colon 0\,(2, 1)\,(5, 2)\,(3, -1))$; the slopes 18 and 19 give the lens spaces $L(18, 5)$ and $L(19, 8)$, respectively. For $h > 0$, there are four nontrivial exceptional surgeries which correspond to the slopes $3h^2 + 15h + 18$, $3h^2 + 15h + 19$, $3h^2 + 15h + 20$, and $3h^2 + 15h + 18 + (1/2)$. The slope $3h^2 + 15h + 18$ gives the Seifert-fibered space $(O\,0\,o\colon 0\,(2, 1)\,(h+4, h+3)\,(h+1, 1))$ (see (ii) in Theorem 4). The slope $3h^2 + 15h + 19$ gives the lens space $L(3h^2 + 15h + 19, 3h + 8)$ (see (iii) in Theorem 4). The slope $3h^2 + 15h + 20$ gives a toroidal manifold as shown in Theorem 5 below. Finally, $3h^2 + 15h + 18 + (1/2)$ is the non-integral toroidal slope for the knot $k(h+3, 1, 1, 0)$ which was determined in [4] (see also [17, Section 2, p. 3]).

**Theorem 5.** *For $n = m + 3$, $m \geq 1$ and $h \geq 0$, the surgery manifold $K_{m,n,h}(\gamma)$ is toroidal if $\gamma = (h+3)^2 m + 2h^2 + 9h + 11$. If further $h = 0$, then $K_{m,n,h}(\gamma)$ is also toroidal when $\gamma = 9m + 7$.*

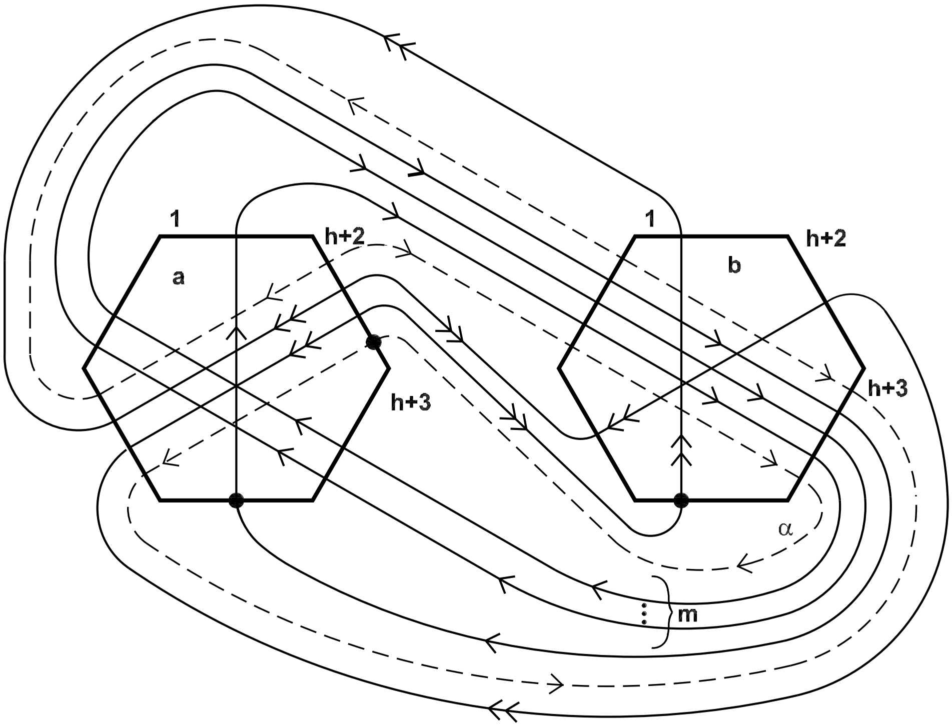


Fig. 3. An RR-system inducing the presentation of $\pi_1(M)$, where $M = K_{m,n,h}(\gamma)$, $\gamma = (h+3)^2m + 2h^2 + 9h + 11$ and $n = m + 3$.

Proof. The proofs are similar in both cases, so we shall illustrate that of the first statement. By Theorem 2, $\pi_1 = \pi_1(M)$, where $M = K_{m,n,h}(\gamma)$ and $\gamma = (h+3)^2m + 2h^2 + 9h + 11$, has a balanced presentation with generators $a$ and $b$ and relations $a(a^{h+2}b^{h+2})^2(a^{h+3}b^{h+3})^m = 1$ and $b = a^{h+2}b^{h+2}a^{h+2}$. From the second relation $b^{h+3} = (a^{h+2}b^{h+2})^2$, it follows that $b^{h+3}$ commutes with $a^{h+2}b^{h+2}$, hence $b^{h+3}$ commutes with $a^{h+2}$. Thus $G = \langle a^{h+2}, b^{h+3}\rangle$ is a finitely generated abelian subgroup of $\pi_1$. By Theorem 9.13 of [6], $G$ is isomorphic to one of $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Z}\oplus\mathbb{Z}$, or $\mathbb{Z}\oplus\mathbb{Z}_2$, where $p > 0$. By Theorem 9.12 of [6], if $G \cong \mathbb{Z}\oplus\mathbb{Z}_2$, then $M$ would be non-orientable, which is not the case. If $G \cong \mathbb{Z}$ or $\mathbb{Z}_p$, then there exists $\eta \in \mathbb{Z}$ (resp. $\xi \in \mathbb{Z}$) such that $b^{h+3} = a^{\eta(h+2)}$ (resp. $a^{h+2} = b^{\xi(h+3)}$). In these cases, the order of the abelianized group $\pi_1^{ab}$ is different to the integral surgery slope $\gamma = (h+3)^2m + 2h^2 + 9h + 11$, giving a contradiction. Thus $G = \langle a^{h+2}, b^{h+3}\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$. Furthermore, $\pi_1$ can be presented by $\langle a, b\colon ab^{h+3}(a^{h+3}b^{h+3})^m = 1, ba^{-(h+2)}b^{-(h+2)}a^{-(h+2)} = 1\rangle$. This presentation is geometric since it arises from a genus 2 Heegaard diagram of the manifold. To see this it suffices to draw an RR-system (*rail road system*) which induces the above presentation. See Fig. 3 (for the theory of RR-systems we refer to [9], [10] and [11]). The first (resp. second) relation arises from the simple closed curve whose orientation is given by one (resp. two) arrow(s), starting from the marked point in Fig. 3. The relation $[a^{h+2}, b^{h+3}] = 1$ arises from the dotted curve $\alpha$ on the Heegaard surface of the splitting (see Fig. 3), hence it is bicollared in $M$. Then we have a map $f: B^2 \to M$ such that for some neighborhood $A$ of $\partial B^2$ in $B^2$, the restricted map $f_{|A}: A \to M$ is an embedding, $f^{-1}(f(A)) = A$ and $f(\partial B^2) = \alpha$. Then $f_{|\partial B^2}: \partial B^2 \to M$, representing $\alpha$, extends to an embedding $g: B^2 \to M$ by Dehn's lemma (see [6, Lemma 4.1]). So $M$ contains an incompressible non-separating torus since $\alpha$ defines the relation $[a^{h+2}, b^{h+3}] = 1$ and $G = \langle a^{h+2}, b^{h+3}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. □

**Theorem 6.** *For $m = 1$, $n > 4$ and $h \geq 0$, we have the following results*:
i) *If $\gamma = (h+2)^2 n - h^2 - h + 2$, then $K_{m,n,h}(\gamma)$ is the Seifert-fibered space defined by the invariants $(O\ 0\ o\colon -1\ (2,1)\ (h+4, h+3)\ ((h+2)n - 3h - 7, h+2))$.*
ii) *If $\gamma = (h+2)^2 n - h^2 - h + 3$, then $K_{m,n,h}(\gamma)$ is the Seifert-fibered space defined by the invariants $(O\ 0\ o\colon -1\ (h+3, 1)\ (2h+5, h+2)\ (n-3, -1))$.*
iii) *If $\gamma = (h+2)^2 n - h^2 - h + 4$, then $K_{m,n,h}(\gamma)$ is toroidal.*

Proof. i) By Theorem 2, $\pi_1 = \pi_1(K_{m,n,h}(\gamma))$, $\gamma = (h+2)^2 n - h^2 - h + 2$, has a balanced presentation with generators $a$ and $b$ and relations $a(a^{h+2}b^{h+2})^{n-2}a^{h+3}b^{h+3} = 1$ and $a^{-(h+3)}(b^{-(h+2)}a^{-(h+2)})^{n-3}a^{-1}b^{-1} = 1$. Using the first relation, the second one becomes $a = b^{h+2}a^{h+3}b^{h+2}$. Then the first relation is equivalent to the relation $a^{h+4}(b^{-1})^{(h+2)n-3h-7} = 1$ by using standard Tietze transformations. So we have

$$\pi_1 \cong \left\langle a, b\colon a^{h+4}(b^{-1})^{(h+2)n-3h-7} = 1,\ ab^{-(h+2)}a(b^{-1})^{(h+2)n-2h-5} = 1\right\rangle.$$

This presentation is geometric since it arises from a genus 2 Heegaard diagram of the manifold. Now we can apply Theorem 2.2 of [3] to conclude that our surgery manifold is the fibered space defined by the Seifert invariants in (i).

ii) By Theorem 2, the group $\pi_1 = \pi_1(K_{m,n,h}(\gamma))$, $\gamma = (h+2)^2 n - h^2 - h + 3$, has a balanced presentation with generators $a$ and $b$, and relations $a(a^{h+2}b^{h+2})^{n-2}a^{h+3}b^{h+3} = 1$ and $(a^{h+2}b^{h+2})^{n-3}a^{h+3} = 1$. So $a^{h+3}$ is central in $\pi_1$. By using Tietze's transformations we get $\pi_1 \cong \langle a, b\colon a^{h+3}b^{2h+5} = 1, (ab^{h+3})^{2-n}ab^{-h-2} = 1\rangle$. This presentation is geometric since it arises from a genus 2 Heegaard diagram of the manifold. Now we can apply Theorem 2.2 of [3] to obtain the result.

iii) By Theorem 2, $\pi_1 = \pi_1(M)$, where $M = K_{m,n,h}(\gamma)$, $\gamma = (h+2)^2 n - h^2 - h + 4$, has a balanced presentation with generators $a$ and $b$ and relations $a(a^{h+2}b^{h+2})^{n-2}a^{h+3}b^{h+3} = 1$ and $(a^{h+2}b^{h+2})^{n-2} = b^{h+3}$. Using the second relation, the first relation becomes $ab^{h+3}a^{h+3}b^{h+3} = 1$, or, equivalently, $a^{h+2} = (a^{h+3}b^{h+3})^2$. This implies that $a^{h+2}$ commutes with $a^{h+3}b^{h+3}$, hence $a^{h+2}$ commutes with $b^{h+3}$. Reasoning as in Theorem 5, it follows that $M$ contains an incompressible non-separating torus whose fundamental group is given by $\langle a^{h+2}, b^{h+3}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. □

If $m = 2$ and $n = 6$, then $K_{m,n,h}$ is the Teragaito knot $K_{\boldsymbol{n}}$, with $\boldsymbol{n} = h + 2$, having three consecutive toroidal slopes $5h^2 + 25h + 31$, $5h^2 + 25h + 32$ and $5h^2 + 25h + 33$. For the remaining cases, we obtain the following result.

**Theorem 7.** *For either $m = 2$ and $n \geq 7$ or $m \geq 3$ and $n - m \geq 4$, the surgery manifold $K_{m,n,h}(\gamma)$ is toroidal if the slopes are either $\gamma_1 = (h+2)^2 n + (2h+5)m - h^2 - 3h - 2$ or $\gamma_2 = (h+2)^2 n + (2h+5)m - h^2 - 3h - 1$.*

Proof. If $m = 2$ and $n \geq 7$, then the surgery manifold $K_{m,n,h}(\gamma_1)$ has a fundamental group presented by

$$\pi_1 = \langle a, b \colon a(a^{h+2}b^{h+2})^{n-3}(a^{h+3}b^{h+3})^2 = 1,\ a^{h+3}(a^{h+2}b^{h+2})^{n-4} = 1\rangle.$$

By using the second relation, the first relation becomes $ba^{h+3}b^{h+3}a^{h+3} = 1$, hence

$$b^{-1} = a^{h+3}b^{h+2}ba^{h+3},$$

which is equivalent to $b^{-1} = a^{h+3}b^{h+2}a^{-(h+3)}b^{-(h+3)}$, i.e., $[a^{h+3}b^{h+2}] = 1$. Reasoning as in Theorem 5, we see that the surgery manifold contains an incompressible non-separating torus whose fundamental group is $\langle a^{h+3}, b^{h+2}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Under the same conditions above, the surgery manifold $K_{m,n,h}(\gamma_2)$ has a fundamental group presented by

$$\begin{aligned}\pi_1 &\cong \langle a, b \colon a(a^{h+2}b^{h+2})^{n-3}(a^{h+3}b^{h+3})^2 = 1,\ b^{h+3} = (a^{h+2}b^{h+2})^{n-3}\rangle \\ &\cong \langle a, b \colon ab^{h+3}(a^{h+3}b^{h+3})^2 = 1,\ b^{h+3} = (a^{h+2}b^{h+2})^{n-3}\rangle.\end{aligned}$$

It follows that $b^{h+3}$ commutes with $a^{h+2}b^{h+2}$, hence $b^{h+3}$ commutes with $a^{h+2}$. So the result is proved by considering the subgroup $\langle a^{h+2}, b^{h+3}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. If $m \geq 3$ and $n - m \geq 4$, then the surgery manifold $K_{m,n,h}(\gamma_1)$ has a fundamental group presented by

$$\pi_1 = \langle a, b \colon a(a^{h+2}b^{h+2})^{n-m-1}(a^{h+3}b^{h+3})^m = 1,\ a^{h+3} = (b^{-(h+2)}a^{-(h+2)})^{n-m-2}\rangle.$$

Then $a^{h+3}$ commutes with $a^{h+2}b^{h+2}$, hence $a^{h+3}$ commutes with $b^{h+2}$. We have again a subgroup $\langle a^{h+3}, b^{h+2}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Under the same conditions above, the fundamental group of $K_{m,n,h}(\gamma_2)$ is presented by

$$\langle a, b \colon a(a^{h+2}b^{h+2})^{n-m-1}(a^{h+3}b^{h+3})^m = 1,\ (b^{-(h+2)}a^{-(h+2)})^{n-m-1}b^{h+3} = 1\rangle.$$

It follows that $b^{h+3}$ commutes with $a^{h+2}$, and so there exists a subgroup $\langle b^{h+3}, a^{h+2}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. $\square$

By using SnapPea program [19] and taking in mind the known restrictions on the exceptional slopes, we have verified that there are no other (nontrivial) exceptional surgeries on $K_{m,n,h}$ for many fixed values of the parameters (besides those obtained above).

## 4. Representing the Teragaito manifolds as branched coverings

Let $M_1(h)$, $M_2(h)$, and $M_3(h)$ denote the closed toroidal 3-manifolds obtained by Dehn surgeries on $K_{m,n,h}$, $m = 2$ and $n = 6$, with slopes $5h^2+25h+31$, $5h^2+25h+32$, and $5h^2 + 25h + 33$, respectively. Such manifolds were first considered by Teragaito in [16]. Here we represent them by simple Heegaard diagrams of genus two. As a consequence, we prove that these manifolds are 2-fold cyclic coverings of the 3-sphere branched over well-specified links. From Theorem 2 (using suitable Tietze transformations), we get

**Theorem 8.** *For every non negative integer $h$, the fundamental groups $G_i \cong \pi_1(M_i(h))$ of the toroidal manifolds $M_i(h)$, for $i = 1,2,3$, admit the following 2-generator presentations*:

$$G_1 = \langle a, b\colon a = b^{h+2}a^{h+3}b^{h+3}a^{h+3}b^{h+2},\ b^{-1} = a^{h+3}b^{h+2}a^{h+2}b^{h+2}a^{h+3}\rangle,$$
$$G_2 = \langle a, b\colon b^{2h+5}a^{h+3}b^{h+3}a^{h+3} = 1,\ a^{2h+5}b^{h+2}a^{h+2}b^{h+2} = 1\rangle,$$

*and*

$$G_3 = \langle a, b\colon ab^{h+3}(a^{h+3}b^{h+3})^2 = 1,\ b^{-1}a^{h+2}(b^{h+2}a^{h+2})^2 = 1\rangle.$$

*In particular, these presentations are geometric, that is, they correspond to spines of the associated manifolds. In other words, the presentation $G_i$ is induced by a genus 2 Heegaard diagram representing $M_i(h)$, for $i = 1, 2, 3$ (see* Figs. 4, 5 *and* 6).

**Theorem 9.** *For every non negative integer $h$, the toroidal manifolds $M_1(h)$, $M_2(h)$, and $M_3(h)$ are the 2-fold cyclic coverings of the 3-sphere branched over the hyperbolic 3-bridge links $L_1$, $L_2$, and $L_3$ depicted in* Figs. 7, 8 and 9, *respectively. In particular, $L_1$ and $L_3$ are knots while $L_2$ is a 2-component link.*

Proof of Theorem 9. The Heegaard diagrams in Figs. 4, 5 and 6 admit two different symmetries of order two: one of them interchanges the circles $a$ and $-b$ and the other one is an orientation-preserving involution $\tau$ which fixes two symmetry axes on the circles $a$ and $-b$ and one axis on the circle obtained by the horizontal line plus infinity. Let us consider the planar graph in Fig. 4. The circle denoted by $a$ and its dual $-a$ have exactly $5h + 15$ vertices, while the circles $b$ and $-b$ have $5h + 12$ vertices. If the parameter $h$ is even, say $h = 2n$, then the fixed axis of $a$ connects the vertex $2h+7$ and the middle point between $2+n$ and $h+5+n$, the fixed axis of $-b$ connects the points labelled by $-\bar{1}$ and $-(2 + n)$, while the axis on the horizontal line joins the middle point between $5n + 4$ and $5n + 5$ with the vertex $15n + 13$. If $h = 2n + 1$ for a suitable $n$, then the involution $\tau$ fixes the axis of $a$ with endpoints $2h + 7$ and $\overline{h + 5 + n}$, the axis of $-b$ connecting $-\bar{1}$ and the middle point between $-(\overline{h + 5 + n})$ and $-(\overline{3 + n})$ and the horizontal axis which joins the vertex $5n + 7$ with the middle

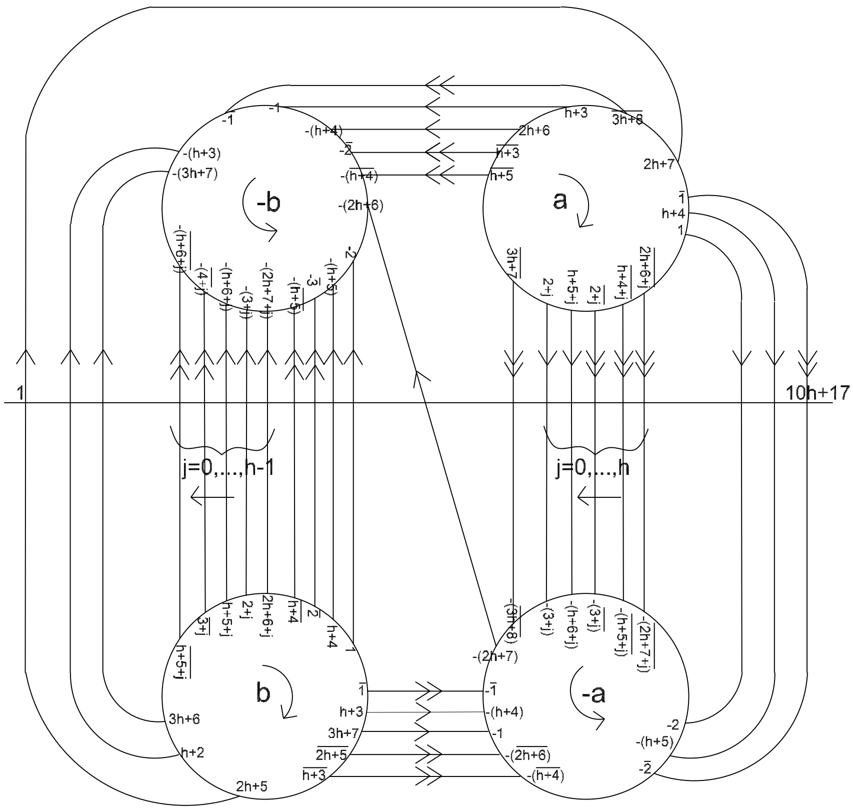


Fig. 4. A genus 2 Heegaard diagram of $M_1(h)$.

point between $15n+20$ and $15n+21$. The graph in Fig. 5 has $5h+13$ (resp. $5h+12$) vertices on the circle $a$ and $-a$ (resp. $b$ and $-b$). If $h=2n$, then the fixed axis of $a$ connects $\overline{h+3}$ and the middle point between $2+n$ and $h+5+n$, the fixed axis of $-b$ has endpoints $-(2h+6)$ and $-(2+n)$, while the fixed axis of the horizontal line plus infinity joins the middle point between $5n+4$ and $5n+5$ with the vertex $15n+13$. If $h=2n+1$, then the fixed axis of $a$ has endpoints on $\overline{h+3}$ and $\overline{2h+7+n}$, the fixed axis of $-b$ connects $-(2h+6)$ with the middle point between $-(\overline{h+4+n})$ and $\overline{2+n}$ and the fixed axis of the horizontal line plus infinity joins $5n+7$ with the middle point between the vertices $15n+20$ and $15n+21$. Finally, let us focus on the Heegaard diagram in Fig. 6, having exactly $5h+13$ (resp. $5h+14$) vertices on the circles $a$ and $-a$ (resp. $b$ and $-b$). If $h=2n$, then the fixed axes of the diagram connect $\bar{1}$ and the middle point between $\overline{3+n}$ and $\overline{h+6+n}$ on the circle $a$, the points $-(2h+5)$ and $-(\overline{h+5+n})$ on the circle $-b$ and the middle point between $5n+5$ and $5n+6$ with the vertex $15n+14$ on the horizontal line plus infinity. If $h=2n+1$, then the fixed axis of $a$ under the involution $\tau$ has endpoints $\bar{1}$ and $h+n+4$, the fixed axis of $-b$

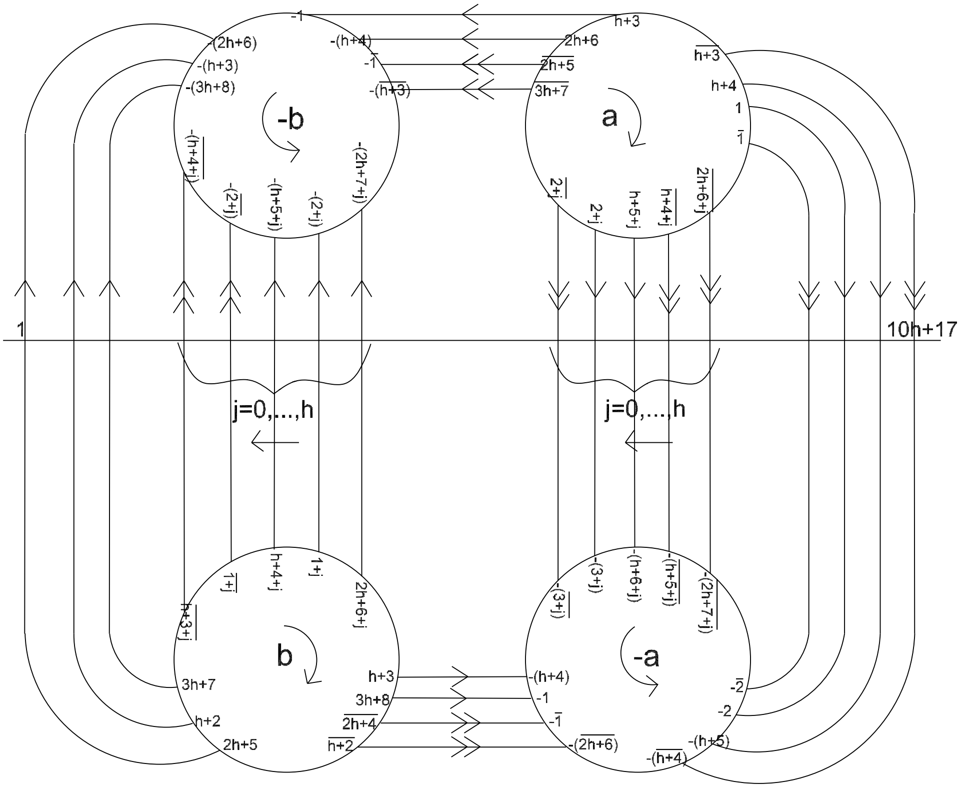


Fig. 5. A genus 2 Heegaard diagram of $M_2(h)$.

joins the vertex $-(2h+5)$ with the middle point between $-(h+n+4)$ and $-(n+2)$ and finally, the fixed axis of the horizontal line plus infinity connects the middle point between $5n+5$ and $5n+6$ with the point $15n+14$. As described in [1], [14] and [18], the fixed axes of each Heegaard diagram under the action of the involution $\tau$ become the bridges of a well-specified 3-bridge link or knot. This is the branch set of the related manifold represented as double branched covering of $\mathbb{S}^3$. By a sequence of Reidemeister moves, we can reduce the obtained branch sets in the simple forms shown in Figs. 7, 8 and 9. □

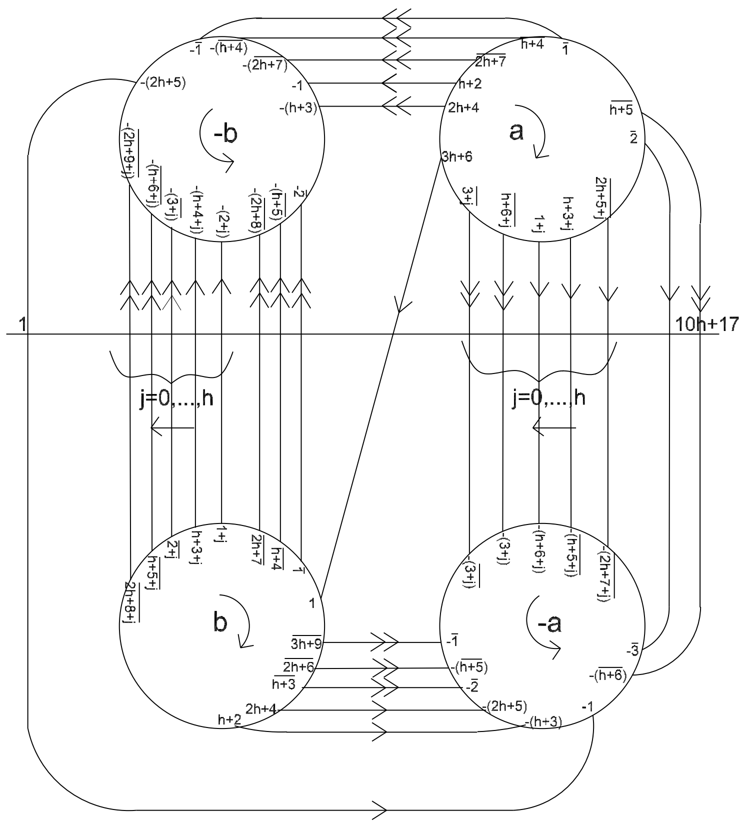

Fig. 6. A genus 2 Heegaard diagram of $M_3(h)$.

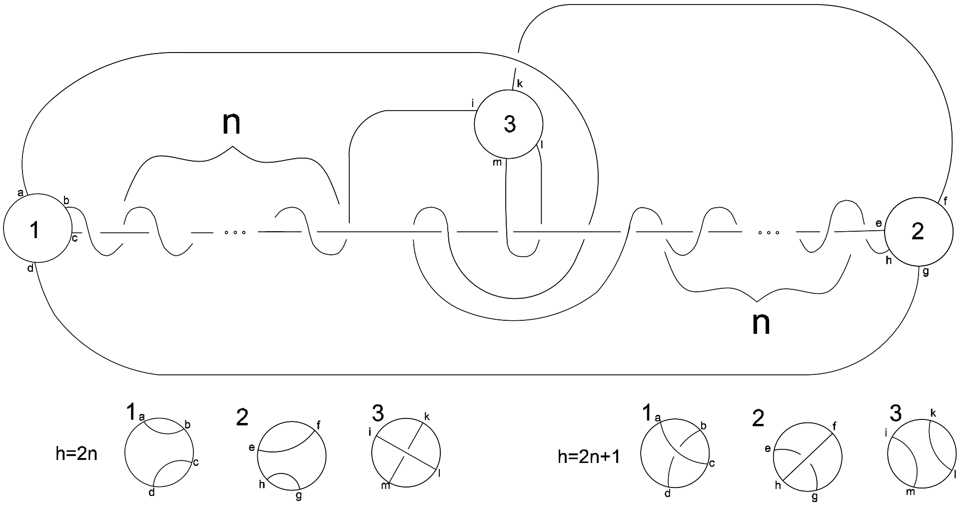


Fig. 7. The 3-bridge hyperbolic knot $L_1$.

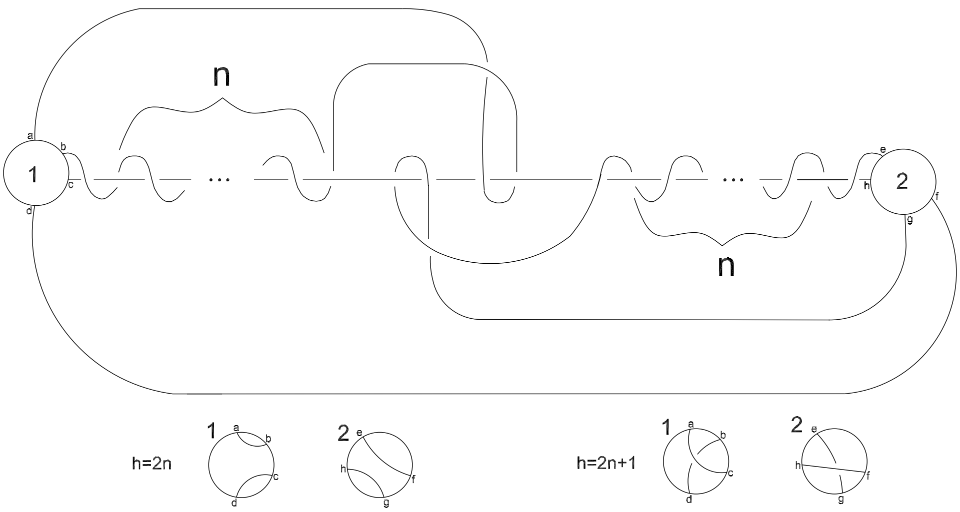


Fig. 8. The 3-bridge hyperbolic link $L_2$.

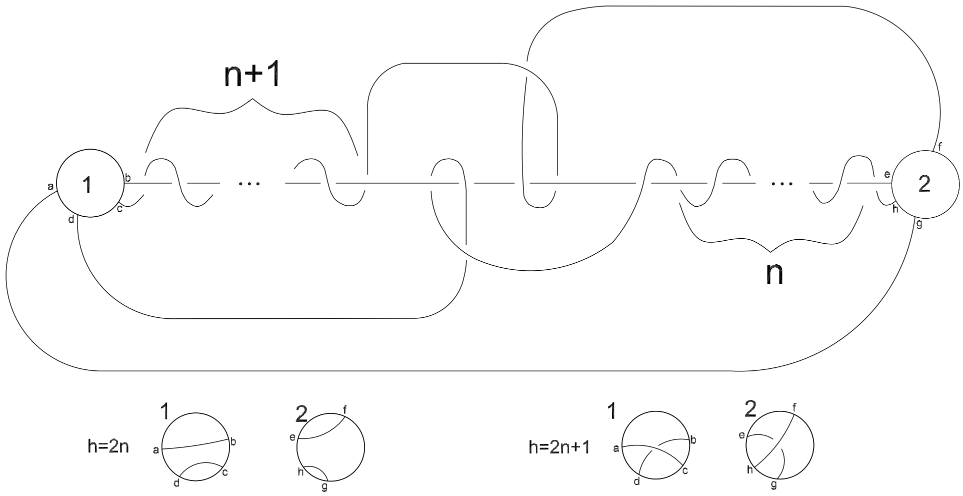


Fig. 9. The 3-bridge hyperbolic knot $L_3$.

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