Arike, Y.
Osaka J. Math.
47 (2010), 535–557

A CONSTRUCTION OF SYMMETRIC LINEAR FUNCTIONS ON THE RESTRICTED QUANTUM GROUP $\bar{U}_q(sl_2)$

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(Received July 14, 2008, revised February 6, 2009)

Abstract

In this article we construct all the primitive idempotents of the restricted quantum group $\bar{U}_q(sl_2)$ and also describe $\bar{U}_q(sl_2)$ as the subalgebra of the direct sum of matrix algebras. By using this result we construct a basis of the space of symmetric linear functions of $\bar{U}_q(sl_2)$ and determine the decomposition of the integral of the dual of $\bar{U}_q(sl_2)$ twisted by the balancing element to the basis of the space of symmetric linear functions.

1. Introduction

The restricted quantum group $\bar{U}_q(sl_2)$ at roots of unity has been studied in various contexts. In [2] and [3] it is shown that the category of modules of the irrational vertex operator algebra $\mathcal{W}(p)$ is closely related with the category of finite-dimensional modules of $\bar{U}_q(sl_2)$ at $q = \exp(\pi\sqrt{-1}/p)$ for $p \geq 2$. More precisely, the Grothendieck group and the center of $\bar{U}_q(sl_2)$ are determined and it is also proved that the center and the space related with $\mathcal{W}(p)$, which is invariant under the canonical $SL_2(\mathbb{Z})$ action, are isomorphic to each other. Then it is also proved that, if $p = 2$, the category of finite-dimensional modules of $\bar{U}_q(sl_2)$ and the category of modules of $\mathcal{W}(p)$ are equivalent to each other. Furthermore we can expect that the equivalence of these categories holds for any $p \geq 2$.

Let $\mathrm{SLF}(\bar{U}_q(sl_2))$ be the space of all the symmetric linear functions on $\bar{U}_q(sl_2)$, that is, $\mathrm{SLF}(\bar{U}_q(sl_2))$ is the space over the complex number field $\mathbb{C}$ formed by all the $\mathbb{C}$-linear maps $\varphi\colon \bar{U}_q(sl_2) \to \mathbb{C}$ with $\varphi(ab) = \varphi(ba)$ for all $a, b \in \bar{U}_q(sl_2)$. In this paper we construct a basis of $\mathrm{SLF}(\bar{U}_q(sl_2))$. In order to construct a basis we determine a certain basis of $\bar{U}_q(sl_2)$ which corresponds to indecomposable projective modules. Since $\bar{U}_q(sl_2)$ is a finite-dimensional unimodular Hopf algebra and the square of the antipode is inner, we can see that $\mathrm{SLF}(\bar{U}_q(sl_2))$ is isomorphic to the center by [10]. For $\bar{U}_q(sl_2)$ the center is $(3p-1)$-dimensional (see [2]) so we see that $\mathrm{SLF}(\bar{U}_q(sl_2))$ is also $(3p-1)$-dimensional. It also follows from [10] that the linear functions given by the action of the balancing element of $\bar{U}_q(sl_2)$ to the left and right integrals of the dual Hopf algebra

2000 Mathematics Subject Classification. Primary 16W35, ; Secondary 17B37, 81R05.

of $\bar{U}_q(sl_2)$ are symmetric. We determine the decomposition of this linear function into the basis of $\mathrm{SLF}(\bar{U}_q(sl_2))$.

Our motivation to study $\mathrm{SLF}(\bar{U}_q(sl_2))$ comes from conformal field theories. For conformal field theory associated with a vertex operator algebra (VOA), the representation theory of VOAs plays an important role. In fact, the conformal field theory with the factorization property for any VOA with some regularity and some finiteness condition is established over the projective line in [9]. Note that such a VOA is rational. For an elliptic curve, under the rationality and the finiteness condition, the space of conformal blocks with a VOA V is finite-dimensional and its dimension coincides with the number of simple modules for V. Any conformal block is related with symmetric linear functions on Zhu's algebra $A_0(V)$ (cf. [7], [12]).

On the other hand, the theory for irrational VOAs is difficult because there could be logarithmic modules (see [7]). In this case, under the same finiteness condition, the number of simple modules is less than or equal to the dimension of the space of conformal blocks and we have to consider pseudo-trace functions on logarithmic modules.

There is an example of irrational conformal field theory which is called logarithmic conformal field theory. A typical example of irrational conformal field theory is a VOA $\mathcal{W}(p)$, whose conformal blocks involve logarithmic function of modulus q (recall that no logarithmic terms appear in rational cases). In this example it is difficult to determine the dimension of conformal blocks.

By the arguments given in [7] and [8], we can expect that the space of conformal blocks is isomorphic to the space of the symmetric linear functions on a finite dimensional algebra whose category of modules is equivalent to the category of V-modules (also see [6]). This naturally indicates that the space of conformal blocks associated to $\mathcal{W}(p)$ is closely related with $\mathrm{SLF}(\bar{U}_q(sl_2))$.

This paper is organized as follows. In Section 2 we recall the definitions and basic properties of symmetric linear functions and integrals of finite-dimensional Hopf algebras. We also introduce the relationship given in [10] between the space of symmetric linear functions on a finite-dimensional Hopf algebra and its center.

In Section 3 we recall the definition of $\bar{U}_q(sl_2)$. We describe left and right integrals of $\bar{U}_q(sl_2)$, which turn out to be equal to each other, hence $\bar{U}_q(sl_2)$ is unimodular. Left and right integrals of the dual of $\bar{U}_q(sl_2)$ are determined. In addition, we recall that the square of the antipode of $\bar{U}_q(sl_2)$ is inner. The above results can be found in [2].

In Section 4, by using the structure of projective modules of $\bar{U}_q(sl_2)$ given in [2], we construct all indecomposable left ideals of $\bar{U}_q(sl_2)$ which are isomorphic to projective modules and we also construct all primitive idempotents of $\bar{U}_q(sl_2)$.

In Section 5 we describe $\bar{U}_q(sl_2)$ as the subalgebra of the direct sum of matrix algebras. By using this result we construct a basis of $\mathrm{SLF}(\bar{U}_q(sl_2))$. Moreover we determine the decomposition of the linear functions given by the action of the balancing element to the left and right integrals into the basis.

2. Preliminaries

In this paper we will always work over the complex number field $\mathbb{C}$. For any vector space V we denote the space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by V^*.

2.1. Symmetric linear functions. Let A be a finite-dimensional associative algebra. A symmetric linear function φ on A is an element of A^* which satisfies $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. We denote the space of symmetric linear functions on A by $\mathrm{SLF}(A)$. If A is a finite-dimensional Hopf algebra, the space $\mathrm{SLF}(A)$ coincides with the space of cocommutative elements of A^* ([10]).

2.2. Integrals and the square of antipode of Hopf algebras. Let A be a finite-dimensional Hopf algebra with the coproduct Δ, the counit ε and the antipode S. Each element of the subspaces

$$\mathcal{L}_A = \{\Lambda \in A \mid a\Lambda = \varepsilon(a)\Lambda \text{ for all } a \in A\},$$
$$\mathcal{R}_A = \{\Lambda \in A \mid \Lambda a = \varepsilon(a)\Lambda \text{ for all } a \in A\},$$

is called a left integral and a right integral of A, respectively. Since A is finite-dimensional, the space $\mathcal{L}_A$ (respectively $\mathcal{R}_A$) is one-dimensional (cf. [1]). Similarly a left (respectively, right) integral of the dual Hopf algebra A^* is an element $\lambda \in A^*$ which satisfies $p\lambda = p(1)\lambda$ (respectively, $\lambda p = p(1)\lambda$) for all $p \in A^*$. Equivalently we can see

$$\mathcal{L}_{A^*} = \{\lambda \in A^* \mid (1 \otimes \lambda)\Delta(x) = \lambda(x) \text{ for all } x \in A\},$$
$$\mathcal{R}_{A^*} = \{\lambda \in A^* \mid (\lambda \otimes 1)\Delta(x) = \lambda(x) \text{ for all } x \in A\}.$$

If $\mathcal{L}_A = \mathcal{R}_A$ the Hopf algebra A is called unimodular.

Proposition 2.1 ([10]). *Let A be a finite-dimensional unimodular Hopf algebra with the antipode S. Suppose that λ is a left integral of A^* and that μ is a right integral of A^*. Then*
(1) $\lambda(ab) = \lambda(bS^2(a))$,
(2) $\mu(ab) = \mu(S^2(b)a)$.

The square of the antipode is called *inner* if there exists an invertible element t such that $S^2(x) = txt^{-1}$ for all $x \in A$.

Denote by $\rightharpoonup$ and $\leftharpoonup$ the left and right actions of A on A^*:

$$a \rightharpoonup p(b) = p(ba), \quad p \leftharpoonup a(b) = p(ab),$$

for $p \in A^*$ and $a, b \in A$.

Proposition 2.2 ([10]). *Let A be a finite-dimensional unimodular Hopf algebra with the antipode S. Suppose that λ is a left integral of A^* and that μ is a right integral of A^*. If S^2 is inner, the linear maps $f_l,\ f_r\colon Z(A) \to \mathrm{SLF}(A)$ defined by $f_l(c) = t^{-1}c \rightharpoonup \lambda$ and $f_r(c) = \mu \leftharpoonup tc$ are isomorphisms.*

3. The restricted quantum group $\bar{U}_q(sl_2)$

3.1. Definition. Let $p \geq 2$ be an integer and $q = \exp(\pi\sqrt{-1}/p)$ be a primitive $2p$-th root of unity. The restricted quantum group $\bar{U}_q(sl_2)$ is a Hopf algebra over $\mathbb{C}$ generated by E, F, K and K^{-1} with the relations

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$E^p = F^p = 0, \quad K^{2p} = 1,$$

as an algebra. We define the coproduct Δ, the counit ε and the antipode S by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$
$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1,$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.$$

Lemma 3.1. *The $2p^3$ elements $E^mF^nK^l$, where $0 \leq m, n \leq p-1$ and $0 \leq l \leq 2p-1$, form a basis of $\bar{U}_q(sl_2)$ as a vector space.*

For $m, n \in \mathbb{Z}$ we use the standard notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]! = [n][n-1]\cdots[2][1], \quad [0]! = 1,$$
$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[n]!\,[m-n]!} \quad \text{for} \quad n \geq 0 \quad \text{and} \quad m - n \geq 0.$$

We can write down the coproduct of the basis of $\bar{U}_q(sl_2)$ in Lemma 3.1 by using induction.

Lemma 3.2.

$$\Delta(E^mF^nK^l) = \sum_{r=0}^{m}\sum_{s=0}^{n} q^{r(m-r)+s(n-s)-2rs} \begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ s \end{bmatrix} \times E^rF^{n-s}K^{-s+l} \otimes E^{m-r}F^sK^{r+l}.$$

3.2. The integrals. The integral of $\bar{U}_q(sl_2)$ and the right integral of $\bar{U}_q(sl_2)^*$ is given in [2]. A basis of the space of left integrals is given by

$$E^{p-1}F^{p-1}\sum_{l=0}^{2p-1}K^l,$$

which also belongs to the space of right integrals. Therefore we can see that $\bar{U}_q(sl_2)$ is unimodular.

Define the elements in $\bar{U}_q(sl_2)^*$ by

$$\lambda(E^mF^nK^l) = \delta_{m,p-1}\delta_{n,p-1}\delta_{l,p-1},$$
$$\mu(E^mF^nK^l) = \delta_{m,p-1}\delta_{n,p-1}\delta_{l,p+1},$$

for $0 \le m, n \le p-1$ and $0 \le l \le 2p-1$.

Proposition 3.3. *Each of the spaces of left integrals and right integrals of the dual Hopf algebra of $\bar{U}_q(sl_2)$ is spanned by λ and μ respectively.*

Proof. It follows from Lemma 3.2. □

3.3. The square of the antipode. It is shown in [2] that there exists a ribbon Hopf algebra $\bar{D}$ which contains $\bar{U}_q(sl_2)$ as a Hopf subalgebra. It is also shown in [2] that the Drinfeld element and the ribbon element of $\bar{D}$ belong to $\bar{U}_q(sl_2)$. Thus the balancing element of $\bar{D}$ is in $\bar{U}_q(sl_2)$. The balancing element is given by $g = K^{p+1}$ in [2].

Proposition 3.4 ([2]). *The square of the antipode of $\bar{U}_q(sl_2)$ is inner, in particular, $S^2(x) = gxg^{-1}$ for all $x \in \bar{U}_q(sl_2)$ where $g = K^{p+1}$.*

3.4. Irreducible modules. Let $\{\mathcal{X}_s^\alpha \mid \alpha = \pm,\ 1 \le s \le p\}$ be the complete set of non-isomorphic irreducible modules of $\bar{U}_q(sl_2)$. The irreducible module $\mathcal{X}_s^\alpha$ has a basis formed by weight vectors $a_n^\alpha(s)$, $0 \le n \le s-1$ with the action of $\bar{U}_q(sl_2)$ defined by

$$Ka_n^\alpha(s) = \alpha q^{s-1-2n}a_n^\alpha(s),$$
$$Ea_n^\alpha(s) = \alpha[n][s-n]a_{n-1}^\alpha(s),$$
$$Fa_n^\alpha(s) = a_{n+1}^\alpha(s),$$

where $a_{-1}^\alpha(s) = a_s^\alpha(s) = 0$.

3.5. Casimir element. The Casimir element of $\bar{U}_q(sl_2)$ is given by

$$C = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} \in Z(\bar{U}_q(sl_2)). \tag{3.1}$$

Proposition 3.5 ([2]). *The minimal polynomial relation of the Casimir element is*

$$\Phi_p(x) = (x - \beta_0)(x - \beta_p) \prod_{s=1}^{p-1} (x - \beta_s)^2 \tag{3.2}$$

where $\beta_s = (q^s + q^{-s})/(q - q^{-1})^2$.

This relation gives the decomposition of $\bar{U}_q(sl_2)$ into its subalgebras by the Casimir element:

$$\bar{U}_q(sl_2) = \bigoplus_{s=0}^{p} Q_s, \tag{3.3}$$

where Q_s for $0 \le s \le p$ is a generalized eigenspace with eigenvalue β_s.

4. Idempotents of $\bar{U}_q(sl_2)$

In this section we construct primitive idempotents of $\bar{U}_q(sl_2)$ by referring to the structure of projective modules (see [2]). We construct all indecomposable left ideals generated by primitive idempotents in $\bar{U}_q(sl_2)$. It is well known that these ideals are indecomposable projective modules.

4.1. Indecomposable modules in $\bar{U}_q(sl_2)$. In order to construct projective modules in $\bar{U}_q(sl_2)$, first we construct the module whose socle is isomorphic to the irreducible module $\mathcal{X}_s^\alpha$.

We quote the useful lemma.

Lemma 4.1 ([4]). *For* $1 \le m \le p - 1$, *the following relations hold in* $\bar{U}_q(sl_2)$:

$$\begin{aligned}
[E, F^m] &= [m]F^{m-1}\frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}} \\
&= [m]\frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}F^{m-1}, \\
[E^m, F] &= [m]E^{m-1}\frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}} \\
&= [m]\frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}}E^{m-1}.
\end{aligned}$$

Then we have a generalization of this lemma (cf. [5], [11]).

Lemma 4.2.

$$[E^r, F^s] = \sum_{i=1}^{\min(r,s)} E^{r-i} F^{s-i} f_i^{r,s}(K),$$

where $f_i^{r,s}(z) \in \mathbb{C}[z, z^{-1}]$.

For $1 \le s \le p$, $1 \le t \le s$ and $\alpha = \pm$, we set

$$v^\alpha(s, t) = \sum_{l=0}^{2p-1} (\alpha q^{-(s-2t+1)})^l K^l.$$

Then we can see that $K v^\alpha(s, t) = \alpha q^{s-2t+1} v^\alpha(s, t)$.

Define

$$a_0^\alpha(s, t) = E^{p-1} F^{p-t} v^\alpha(s, t). \tag{4.1}$$

This is a highest weight vector of highest weight αq^{s-1}. Set $a_n^\alpha(s, t) = F^n a_0^\alpha(s, t)$. Then Lemma 4.1 shows

$$K a_n^\alpha(s, t) = \alpha q^{s-1-2n} a_n^\alpha(s, t), \tag{4.2}$$

$$E a_n^\alpha(s, t) = \alpha [n][s-n] a_{n-1}^\alpha(s, t), \tag{4.3}$$

$$F a_n^\alpha(s, t) = a_{n+1}^\alpha(s, t), \tag{4.4}$$

and $E a_s^\alpha(s, t) = 0$. By (4.3), we have $a_n^\alpha(s, t) \neq 0$ for $0 \le n \le s-1$. Let $\mathcal{X}_s^\alpha(t)$ be a space spanned by $a_n^\alpha(s, t)$, $0 \le n \le s-1$. It is clear that the space $\mathcal{X}_p^\alpha(t)$ is isomorphic to the irreducible module $\mathcal{X}_p^\alpha$ for $1 \le t \le p$. It is expected that $a_s^\alpha(s, t)$ is zero for $1 \le s \le p-1$ but it is hard to prove by direct calculation. In fact, this fact will be proved in the following argument.

We consider the element which is sent to $a_0^\alpha(s, t)$ by the action of F.

Lemma 4.3. *For* $1 \le s \le p-1$ *and* $1 \le t \le s$, *we have*

$$a_0^\alpha(s, t) = F \sum_{n=1}^{p-s} \mu_n^\alpha(s) E^{p-n} F^{p-t-n} v^\alpha(s, t)$$

where $\mu_n^\alpha(s) = \prod_{k=p-s-(n-1)}^{p-s-1} (-\alpha [k][p-s-k])$.

Proof. By Lemma 4.1,

$$
\begin{aligned}
&FE^{p-n}F^{p-t-n}v^{\alpha}(s,t)\\
&= E^{p-n}F^{p-t-n+1}v^{\alpha}(s,t)\\
&\quad - \frac{[p-n]}{q-q^{-1}}(q^{-(p-n-1)}K - q^{p-n-1}K^{-1})E^{p-n-1}F^{p-n-t}v^{\alpha}(s,t)\\
&= E^{p-n}F^{p-t-n+1}v^{\alpha}(s,t) + \alpha[n][p-s-n]E^{p-n-1}F^{p-n-t}v^{\alpha}(s,t).
\end{aligned}
$$

We can see

$$
\begin{aligned}
&F\sum_{n=1}^{p-s}\mu_n^{\alpha}(s)E^{p-n}F^{p-t-n}v^{\alpha}(s,t)\\
&= \sum_{n=1}^{p-s}\mu_n^{\alpha}(s)E^{p-n}F^{p-t-n+1}v^{\alpha}(s,t)\\
&\quad + \sum_{n=1}^{p-s}\mu_n^{\alpha}(s)\alpha[n][p-s-n]E^{p-n-1}F^{p-n-t}v^{\alpha}(s,t)\\
&= E^{p-1}F^{p-t}v^{\alpha}(s,t)\\
&\quad + \sum_{n=1}^{p-s-1}(\mu_{n+1}^{\alpha}(s) + \alpha[n][p-s-n]\mu_n^{\alpha}(s))E^{p-n-1}F^{p-n-t}v^{\alpha}(s,t).
\end{aligned}
$$

Since $\mu_n^{\alpha}(s) = \prod_{k=p-s-(n-1)}^{p-s-1}(-\alpha[k][p-s-k])$, the second term vanishes. □

Set

$$
x_k^{\alpha}(s,t) = \frac{E^{p-s-k-1}}{\prod_{i=k+1}^{p-s-1}(-\alpha[i][p-s-i])}\sum_{n=1}^{p-s}\mu_n^{\alpha}(s)E^{p-n}F^{p-t-n}v^{\alpha}(s,t). \tag{4.5}
$$

Then we can easily see that

$$
x_0^{\alpha}(s,t) = \frac{\mu_{p-s}^{\alpha}(s)E^{p-1}F^{s-t}v^{\alpha}(s,t)}{\prod_{i=1}^{p-s-1}(-\alpha[i][p-s-i])} = E^{p-1}F^{s-t}v^{\alpha}(s,t),
$$

so $x_k^{\alpha}(s,t)$ is non-zero and $x_0^{\alpha}(s,t)$ is a highest weight vector of highest weight $-\alpha q^{p-s-1}$. Then, by Lemma 4.1 and Lemma 4.3, we have

$$
Kx_k^{\alpha}(s,t) = -\alpha q^{p-s-1-2k}x_k^{\alpha}(s,t), \tag{4.6}
$$

$$
Ex_k^{\alpha}(s,t) = \begin{cases} -\alpha[k][p-s-k]x_{k-1}^{\alpha}(s,t), & 1 \le k \le p-s-1,\\ 0, & k=0,\end{cases} \tag{4.7}
$$

$$
Fx_k^{\alpha}(s,t) = \begin{cases} x_{k+1}^{\alpha}(s,t), & 0 \le k \le p-s-2,\\ a_0^{\alpha}(s,t), & k=p-s-1.\end{cases} \tag{4.8}
$$

By above relations we obtain

$$a_s^\alpha(s,t) = F^s a_0^\alpha(s,t) = F^{s+p-s} x_0^\alpha(s,t) = 0$$

for $1 \le s \le p-1$ and $1 \le t \le s$.

Proposition 4.4. *For $1 \le s \le p$ and $1 \le t \le s$, the space $\mathcal{X}_s^\alpha(t)$ spanned by the vectors of the form $a_n^\alpha(s,t)$, $0 \le n \le s-1$ is isomorphic to the irreducible module $\mathcal{X}_s^\alpha$.*

Set

$$b_n^\alpha(s,t) = F^n \sum_{n=1}^{p-s} \mu_n^\alpha(s) E^{p-n-1} F^{p-t-n} v^\alpha(s,t), \tag{4.9}$$

$$y_k^\alpha(s,t) = F^{s+k} b_0^\alpha(s,t) \tag{4.10}$$

for $0 \le n \le s-1$, $0 \le k \le p-s-1$, and $\alpha = \pm$. By Lemma 4.1 we have following relations:

$$K b_n^\alpha(s,t) = \alpha q^{s-1-2n} b_n^\alpha(s,t), \tag{4.11}$$

$$E b_n^\alpha(s,t) = \begin{cases} \alpha[n][s-n] b_{n-1}^\alpha(s,t) + a_{n-1}^\alpha(s,t), & 1 \le n \le s-1, \\ x_{p-s-1}^\alpha(s,t), & n = 0, \end{cases} \tag{4.12}$$

$$F b_n^\alpha(s,t) = \begin{cases} b_{n+1}^\alpha(s,t), & 0 \le n \le s-2, \\ y_0^\alpha(s,t), & n = s-1, \end{cases} \tag{4.13}$$

$$K y_k^\alpha(s,t) = -\alpha q^{p-s-1-2k} y_k^\alpha(s,t), \tag{4.14}$$

$$E y_k^\alpha(s,t) = \begin{cases} -\alpha[k][p-s-k] y_{k-1}^\alpha(s,t), & 1 \le k \le p-s-1, \\ a_{s-1}^\alpha(s,t), & k = 0, \end{cases} \tag{4.15}$$

$$F y_k^\alpha(s,t) = \begin{cases} y_{k+1}^\alpha(s,t), & 0 \le k \le p-s-2, \\ 0, & k = p-s-1. \end{cases} \tag{4.16}$$

By (4.15), we have $b_n^\alpha(s,t) \neq 0$ for $0 \le n \le s-1$ and $y_k^\alpha(s,t) \neq 0$ for $0 \le k \le p-s-1$. Let $\mathcal{P}_s^\alpha(t)$ be a space spanned by the elements of the form

$$b_n^\alpha(s,t),\ x_k^\alpha(s,t),\ y_k^\alpha(s,t),\ a_n^\alpha(s,t), \quad 0 \le n \le s-1,\ 0 \le k \le p-s-1,$$

for $1 \le s \le p-1$ and $1 \le t \le s$. By (4.2)–(4.4), (4.6)–(4.8) and (4.11)–(4.16), the $2p$-dimensional space $\mathcal{P}_s^\alpha(t)$ is an indecomposable left $\bar{U}_q(sl_2)$-module. Note that the modules $\mathcal{P}_s^\alpha(t)$ for $1 \le t \le s$ are isomorphic to each other so we set $\mathcal{P}_s^\alpha \cong \mathcal{P}_s^\alpha(t)$.

Proposition 4.5. *The $2p$-dimensional indecomposable left module $\mathcal{P}_s^{\alpha}$, $1 \le s \le p-1$ and $\alpha = \pm$, has a basis formed by weight vectors*

$$b_n^{\alpha}(s), \quad x_k^{\alpha}(s), \quad y_k^{\alpha}(s), \quad a_n^{\alpha}(s),$$

for $0 \le n \le s-1$ and $0 \le k \le p-s-1$ with left actions defined by

$$Kb_n^{\alpha}(s) = \alpha q^{s-1-2n} b_n^{\alpha}(s),$$

$$Eb_n^{\alpha}(s) = \begin{cases} \alpha[n][s-n]b_{n-1}^{\alpha}(s) + a_{n-1}^{\alpha}(s), & 1 \le n \le s-1, \\ x_{p-s-1}^{\alpha}(s), & n = 0, \end{cases}$$

$$Fb_n^{\alpha}(s) = \begin{cases} b_{n+1}^{\alpha}(s), & 0 \le n \le s-2, \\ y_0^{\alpha}(s), & n = s-1, \end{cases}$$

$$Kx_k^{\alpha}(s) = -\alpha q^{p-s-1-2k} x_k^{\alpha}(s),$$

$$Ex_k^{\alpha}(s) = \begin{cases} -\alpha[k][p-s-k]x_{k-1}^{\alpha}(s), & 1 \le k \le p-s-1, \\ 0, & k = 0, \end{cases}$$

$$Fx_k^{\alpha}(s) = \begin{cases} x_{k+1}^{\alpha}(s), & 0 \le k \le p-s-2, \\ a_0^{\alpha}(s), & k = p-s-1, \end{cases}$$

$$Ky_k^{\alpha}(s) = -\alpha q^{p-s-1-2k} y_k^{\alpha}(s),$$

$$Ey_k^{\alpha}(s) = \begin{cases} -\alpha[k][p-s-k]y_{k-1}^{\alpha}(s), & 1 \le k \le p-s-1, \\ a_{s-1}^{\alpha}(s), & k = 0, \end{cases}$$

$$Fy_k^{\alpha}(s) = \begin{cases} y_{k+1}^{\alpha}(s), & 0 \le k \le p-s-2, \\ 0, & k = p-s-1, \end{cases}$$

$$Ka_n^{\alpha}(s) = \alpha q^{s-1-2n} a_n^{\alpha}(s),$$

$$Ea_n^{\alpha}(s) = \begin{cases} \alpha[n][s-n]a_{n-1}^{\alpha}(s), & 1 \le n \le s-1, \\ 0, & n = 0, \end{cases}$$

$$Fa_n^{\alpha}(s) = \begin{cases} a_{n+1}^{\alpha}(s), & 0 \le n \le s-2, \\ 0, & n = s-1. \end{cases}$$

It is shown in [2] that the action $(C-\beta_s)^2$ vanishes on the modules $\mathcal{P}_s^+$ and $\mathcal{P}_{p-s}^-$ for $1 \le s \le p-1$. Hence there are inclusion maps $\mathcal{P}_s^+ \to Q_s$ and $\mathcal{P}_{p-s}^- \to Q_s$ for $1 \le s \le p-1$. It is also shown in [2] each of the actions of $(C-\beta_0)$ and $(C-\beta_p)$ vanishes on $\mathcal{X}_p^-$ and $\mathcal{X}_p^+$ respectively. Thus there are inclusion maps $\mathcal{X}_p^- \to Q_0$ and $\mathcal{X}_p^+ \to Q_p$.

4.2. Primitive idempotents of $\bar{U}_q(sl_2)$. First we show that the irreducible modules $\mathcal{X}_p^{\alpha}(t)$ contain primitive idempotens of $\bar{U}_q(sl_2)$.

Proposition 4.6. *Set*

$$e^{\alpha}(p,t) = \frac{1}{2p\prod_{i=1}^{p-1}(\alpha[i][p-i])} a^{\alpha}_{t-1}(p,t) \in \mathcal{X}^{\alpha}_{s}(t). \tag{4.17}$$

Then

$$e^{\alpha}(p,t_1)e^{\alpha}(p,t_2) = \begin{cases} e^{\alpha}(p,t_2), & t_1 = t_2, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.18}$$

In particular $e^{+}(p,t) \in Q_p$ and $e^{-}(p,t) \in Q_0$ for $1 \le t \le p$ are primitive idempotents.

Proof. It is clear that $e^{\alpha}(p,t)$ generates the irreducible module $\mathcal{X}^{\alpha}_{p}$ by the structure of irreducible modules. By the left action of K on the irreducible module,

$$v^{\alpha}(p,t_1)a^{\alpha}_{t_1-1}(p,t_2) = \begin{cases} 2pa^{\alpha}_{t_2-1}(p,t_2), & t_1 = t_2, \\ 0, & \text{otherwise}. \end{cases}$$

Thus we have

$$a^{\alpha}_{t-1}(p,t)^2 = 2pF^{t-1}E^{p-1}F^{p-t}a^{\alpha}_{t-1}(p,t) = 2p\prod_{i=1}^{p-1}(\alpha[i][p-i])a^{\alpha}_{t-1}(p,t). \qquad \square$$

To find other primitive idempotents of $\bar{U}_q(sl_2)$ the following lemma is useful.

Lemma 4.7. *Let φ_n be an element in $\bar{U}_q(sl_2)$ with weight q^{s-1-2n} for $0 \le n \le s-1$ and ψ_k be an element in $\bar{U}_q(sl_2)$ with weight $-q^{p-s-1-2k}$ for $0 \le k \le p-s-1$. Then*

$$v^{+}(s,t)\varphi_n = \begin{cases} 2p\varphi_n, & n = t-1, \\ 0, & \textit{otherwise}, \end{cases}$$
$$v^{+}(s,t)\psi_k = 0,$$
$$v^{-}(p-s,u)\varphi_n = 0,$$
$$v^{-}(p-s,u)\psi_k = \begin{cases} 2p\psi_k, & k = u-1, \\ 0, & \textit{otherwise}, \end{cases}$$

for $1 \le s \le p-1$, $1 \le t \le s$ and $1 \le u \le p-s$.

Proof. It follows from direct calculation. $\square$

For $1 \leq s \leq p-1$ and $1 \leq t \leq s$, set

$$e^{\alpha}(s,t) = \frac{1}{\gamma^{\alpha}(s)}\left(b^{\alpha}_{t-1}(s,t) - \frac{\delta^{\alpha}(s)}{\gamma^{\alpha}(s)} a^{\alpha}_{t-1}(s,t)\right) \in \mathcal{P}^{\alpha}_{s}(t) \tag{4.19}$$

where

$$\gamma^{\alpha}(s) = 2p \prod_{m=1}^{p-s-1}(-\alpha[m][p-s-m]) \prod_{i=1}^{s-1}(\alpha[i][s-i]), \tag{4.20}$$

$$\begin{aligned}\delta^{\alpha}(s) &= 2p \prod_{m=1}^{p-s-1}(-\alpha[m][p-s-m]) \sum_{j=1}^{s-1} \prod_{\substack{k=1\\k\neq j}}^{s-1}(\alpha[k][s-k]) \\ &\quad + 2p \prod_{i=1}^{s-1}(\alpha[i][s-i]) \sum_{n=1}^{p-s-1} \prod_{\substack{k=1\\k\neq n}}^{p-s-1}(-\alpha[k][p-s-k]).\end{aligned} \tag{4.21}$$

Note that $\gamma^{+}(s) = \gamma^{-}(p-s)$ and $\delta^{+}(s) = \delta^{-}(p-s)$. Then we see that $e^{\alpha}(s,t)$ generates the module $\mathcal{P}^{\alpha}_{s}$.

Using Lemma 4.7 and the action of $\bar{U}_q(sl_2)$, we have the following.

Proposition 4.8. *The elements $e^{+}(s,t)$ for $1 \leq t \leq s$ and $e^{-}(p-s,u)$ for $1 \leq u \leq p-s$ are mutually orthogonal primitive idempotents of Q_s for $1 \leq s \leq p-1$.*

Proof. Using Lemma 4.7, we have

$$\begin{aligned}&(e^{+}(s,t))^2 \\ &= \frac{F^{t-1}}{(\gamma^{+}(s))^2}\left(b^{+}_{0}(s,t) - \frac{\delta^{+}(s)}{\gamma^{+}(s)} a^{+}_{0}(s,t)\right)\left(b^{+}_{t-1}(s,t) - \frac{\delta^{+}(s)}{\gamma^{+}(s)} a^{+}_{t-1}(s,t)\right) \\ &= 2p\frac{F^{t-1}}{(\gamma^{+}(s))^2}\left(\sum_{i=1}^{p-s} \mu^{+}_{i}(s) E^{p-i-1}F^{p-t-i} - \frac{\delta^{+}(s)}{\gamma^{+}(s)} E^{p-1}F^{p-t}\right) \\ &\qquad \times \left(b^{+}_{t-1}(s,t) - \frac{\delta^{+}(s)}{\gamma^{+}(s)} a^{+}_{t-1}(s,t)\right).\end{aligned}$$

By the action of $\bar{U}_q(sl_2)$ on $\mathcal{P}_s^+$, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{p-s}\mu_i^+(s)E^{p-i-1}F^{p-t-i}\right)\left(b_{t-1}^+(s,t)-\frac{\delta^+(s)}{\gamma^+(s)}a_{t-1}^+(s,t)\right)\\
&=\mu_{p-s}^+(s)\prod_{i=1}^{s-1}([i][s-i])\left(b_0^+(s,t)-\frac{\delta^+(s)}{\gamma^+(s)}a_0^+(s,t)\right)\\
&\quad+\mu_{p-s}^+(s)\sum_{i=1}^{s-1}\prod_{\substack{j=1\\ j\neq i}}^{s-1}([j][s-j])a_0^+(s,t)\\
&\quad+\prod_{i=1}^{s-1}([i][s-i])\sum_{i=1}^{p-s-1}\mu_i^+(s)\prod_{j=1}^{p-s-1-i}(-[j][p-s-j])a_0^+(s,t)\\
&=\frac{\gamma^+(s)}{2p}\left(b_0^+(s,t)-\frac{\delta^+(s)}{\gamma^+(s)}a_0^+(s,t)\right)+\frac{\delta^+(s)}{2p}a_0^+(s,t),
\end{aligned}$$

and

$$\begin{aligned}
&E^{p-1}F^{p-t}\left(b_{t-1}^+(s,t)-\frac{\delta^+(s)}{\gamma^+(s)}a_{t-1}^+(s,t)\right)\\
&=\prod_{i=1}^{p-s-1}(-[i][p-s-i])\prod_{j=1}^{s-1}([j][s-j])a_0^+(s,t)=\frac{\gamma^+(s)}{2p}a_0^+(s,t),
\end{aligned}\tag{4.22}$$

since $\mu_i^+(s)=\prod_{j=p-s-(i-1)}^{p-s-1}(-[j][p-s-j])$. Hence we have

$$\begin{aligned}
(e^+(s,t))^2=2p\frac{F^{t-1}}{(\gamma^+(s))^2}\Bigg(&\frac{\gamma^+(s)}{2p}\left(b_0^+(s,t)-\frac{\delta^+(s)}{\gamma^+(s)}a_0^+(s,t)\right)\\
&+\frac{\delta^+(s)}{2p}a_0^+(s,t)-\frac{\delta^+(s)}{\gamma^+(s)}\frac{\gamma^+(s)}{2p}a_0^+(s,t)\Bigg)=e^+(s,t).
\end{aligned}$$

By Lemma 4.7 these idempotents are mutually orthogonal. □

Corollary 4.9. *The modules $\mathcal{X}_p^\alpha$ and $\mathcal{P}_s^\alpha$ for $\alpha=\pm$ and $1\leq s\leq p-1$ are indecomposable projective modules.*

It is clear that

$$\bar{U}_q(sl_2)\supseteq\bigoplus_{s=1}^{p-1}\bigoplus_{t=1}^{s}(\mathcal{P}_s^+(t)\oplus\mathcal{P}_s^-(t))\oplus\bigoplus_{t=1}^{p}(\mathcal{X}_p^+(t)\oplus\mathcal{X}_p^-(t)).\tag{4.23}$$

Then the dimension of right hand side is $2p^3$ so the above inclusion is an equality.

Theorem 4.10. *The following equality holds*:

$$\bar{U}_q(sl_2) = \bigoplus_{s=1}^{p-1}\bigoplus_{t=1}^{s}(\mathcal{P}_s^+(t) \oplus \mathcal{P}_s^-(t)) \oplus \bigoplus_{t=1}^{p}(\mathcal{X}_p^+(t) \oplus \mathcal{X}_p^-(t)).$$

5. Symmetric linear functions on $\bar{U}_q(sl_2)$

5.1. Basis and matrix representation of Q_s. Let A be a finite-dimensional associative $\mathbb{C}$-algebra and let $\{P_i \mid i \in I\}$ be the complete set of non-isomorphic projective A-modules. Then the map $\varphi\colon A \to \bigoplus_{i\in I}\mathrm{End}_{\mathbb{C}}(P_i)$ defined by $\varphi(a) = (\rho_i(a))_{i\in I}$ for $a \in A$ is an algebra monomorphism where ρ_i is the representation of A on P_i.

Recall that $\bar{U}_q(sl_2)$ has the decomposition into subalgebras:

$$\bar{U}_q(sl_2) = \bigoplus_{s=0}^{p} Q_s.$$

By Proposition 4.6, each of the subalgebras Q_0 and Q_p has unique projective irreducible module $\mathcal{X}_p^{\mp}$, respectively. By Proposition 4.8, the subalgebra Q_s for $1 \le s \le p-1$ has two projective modules $\mathcal{P}_s^+$ and $\mathcal{P}_{p-s}^-$. So we can define the algebra monomorphisms

$$\begin{aligned}
&\varphi_0\colon\ Q_0 \to \mathrm{End}_{\mathbb{C}}(\mathcal{X}_p^-),\\
&\varphi_p\colon\ Q_p \to \mathrm{End}_{\mathbb{C}}(\mathcal{X}_p^+),\\
&\varphi_s\colon\ Q_s \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}_s^+) \oplus \mathrm{End}_{\mathbb{C}}(\mathcal{P}_{p-s}^-),
\end{aligned}$$

for $1 \le s \le p-1$. In order to determine the images of above monomorphisms, we first give a certain basis of Q_s and their actions on projective modules.

For Q_0 we can choose a basis

$$A_n^-(p,t) = \frac{1}{2p\prod_{i=1}^{p-1}(-[i][p-i])}a_n^-(p,t), \quad 0 \le n \le p-1,\ 1 \le t \le p. \tag{5.1}$$

For Q_p we can choose a basis

$$A_n^+(p,t) = \frac{1}{2p\prod_{i=1}^{p-1}([i][p-i])}a_n^+(p,t), \quad 0 \le n \le p-1,\ 1 \le t \le p. \tag{5.2}$$

By the action of $\bar{U}_q(sl_2)$ on the irreducible module $\mathcal{X}_p^{\alpha}$ and Lemma 4.7, we have

$$A_m^{\alpha}(p,t)a_n^{\alpha}(p) = \begin{cases} a_m^{\alpha}(p), & n = t-1\\ 0, & \text{otherwise.}\end{cases} \tag{5.3}$$

By (5.3) we have the following theorem.

Theorem 5.1. *The subalgebra Q_s for $s = 0, p$ is isomorphic to $M_p(\mathbb{C})$. The isomorphism is given by $A^{\alpha}_{m-1}(p, t) \mapsto E_{m,t}$ for $1 \le m, t \le p$ where $E_{m,t}$ is a matrix unit of $M_p(\mathbb{C})$.*

For $1 \le s \le p-1$ the following elements define a basis of Q_s:

$$B^+_n(s, t) := \frac{1}{\gamma_s}\left(b^+_n(s, t) - \frac{\delta_s}{\gamma_s} a^+_n(s, t)\right), \tag{5.4}$$

$$X^+_k(s, t) := \frac{1}{\gamma_s} x^+_k(s, t) = \frac{E^{p-s-k}}{\prod_{i=k+1}^{p-s-1}(-[i][p-s-i])} B^+_0(s, t), \tag{5.5}$$

$$Y^+_k(s, t) := \frac{1}{\gamma_s} y^+_k(s, t) = F^{s+k} B^+_0(s, t), \tag{5.6}$$

$$A^+_n(s, t) := \frac{1}{\gamma_s} a^+_n(s, t) = F^{n+1} E B^+_0(s, t), \tag{5.7}$$

$$B^-_k(p-s, u) := \frac{1}{\gamma_s}\left(b^-_k(p-s, u) - \frac{\delta_s}{\gamma_s} a^-_k(p-s, u)\right), \tag{5.8}$$

$$X^-_n(p-s, u) := \frac{1}{\gamma_s} x^-_n(p-s, u) = \frac{E^{s-n}}{\prod_{i=n+1}^{s-1}([i][s-i])} B^-_0(p-s, u), \tag{5.9}$$

$$Y^-_n(p-s, u) := \frac{1}{\gamma_s} y^-_n(p-s, u) = F^{p-s+n} B^-_0(p-s, u), \tag{5.10}$$

$$A^-_k(p-s, u) := \frac{1}{\gamma_s} a^-_k(p-s, u) = F^{k+1} E B^-_0(p-s, u), \tag{5.11}$$

for $1 \le t \le s$, $1 \le u \le p-s$, $0 \le n \le s-1$ and $0 \le k \le p-s-1$ where

$$\gamma_s = \gamma^+(s) = \gamma^-(p-s),$$
$$\delta_s = \delta^+(s) = \delta^-(p-s).$$

By the action of $\bar{U}_q(sl_2)$ on projective modules $\mathcal{P}^+_s$ given in Proposition 4.5 and Lemma 4.7, the following hold:

$$B^+_m(s, t) b^+_n(s) = \begin{cases} b^+_m(s), & n = t-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.12}$$

$$B^+_m(s, t) x^+_k(s) = 0, \tag{5.13}$$

$$B^+_m(s, t) y^+_k(s) = 0, \tag{5.14}$$

$$B^+_m(s, t) a^+_n(s) = \begin{cases} a^+_m(s), & n = t-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.15}$$

Table 1. The actions of the basis of Q_s on $\mathcal{P}_s^+$.

$Q_s \backslash \mathcal{P}_s^+$	$b_{t-1}^+(s)$	$x_{u-1}^+(s)$	$y_{u-1}^+(s)$	$a_{t-1}^+(s)$
$B_n^+(s,t)$	$b_n^+(s)$	0	0	$a_n^+(s)$
$X_k^+(s,t)$	$x_k^+(s)$	0	0	0
$Y_k^+(s,t)$	$y_k^+(s)$	0	0	0
$A_n^+(s,t)$	$a_n^+(s)$	0	0	0
$B_k^-(p-s,u)$	0	$x_k^+(s)$	$y_k^+(s)$	0
$X_n^-(p-s,u)$	0	0	$a_n^+(s)$	0
$Y_n^-(p-s,u)$	0	$a_n^+(s)$	0	0
$A_k^-(p-s,u)$	0	0	0	0

Table 2. The actions of the basis of Q_s on $\mathcal{P}_{p-s}^-$.

$Q_s \backslash \mathcal{P}_{p-s}^-$	$b_{u-1}^-(p-s)$	$x_{t-1}^-(p-s)$	$y_{t-1}^-(p-s)$	$a_{u-1}^-(p-s)$
$B_n^+(s,t)$	0	$x_n^-(p-s)$	$y_n^-(p-s)$	0
$X_k^+(s,t)$	0	0	$a_k^-(p-s)$	0
$Y_k^+(s,t)$	0	$a_k^-(p-s)$	0	0
$A_n^+(s,t)$	0	0	0	0
$B_k^-(p-s,u)$	$b_k^-(p-s)$	0	0	$a_k^-(p-s)$
$X_n^-(p-s,u)$	$x_n^-(p-s)$	0	0	0
$Y_n^-(p-s,u)$	$y_n^-(p-s)$	0	0	0
$A_k^-(p-s,u)$	$a_k^-(p-s)$	0	0	0

$$B_k^-(p-s,u)b_n^+(s) = 0, \tag{5.16}$$

$$B_k^-(p-s,u)x_l^+(s) = \begin{cases} x_k^+(s), & l = u-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.17}$$

$$B_k^-(p-s,u)y_l^+(s) = \begin{cases} y_k^+(s), & l = u-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.18}$$

$$B_k^-(p-s,u)a_n^+(s) = 0, \tag{5.19}$$

for $0 \le m, n \le s-1$, $0 \le k, l \le p-s-1$, $1 \le t \le s$ and $1 \le u \le p-s$. By (5.4)–(5.11) and (5.12)–(5.19), we can determine the action of Q_s on $\mathcal{P}_s^+$. Similarly we can also determine the action of Q_s on $\mathcal{P}_{p-s}^-$. The actions of Q_s on $\mathcal{P}_s^+$ and $\mathcal{P}_{p-s}^-$ are given in Table 1 and Table 2. We note that the actions which do not appear in Table 1 or 2 are zero.

Let us expand $v \in Q_s$ by the basis of Q_s for $1 \le s \le p-1$:

$$
\begin{aligned}
v = \sum_{t=1}^{s} \Bigg\{ & \sum_{n=1}^{s} (\varphi^{+}_{n,t}(v) B^{+}_{n-1}(s,t) + \psi^{+}_{n,t}(v) A^{+}_{n-1}(s,t)) \\
& + \sum_{k=1}^{p-s} (\xi^{+}_{k,t}(v) X^{+}_{k-1}(s,t) + \zeta^{+}_{k,t}(v) Y^{+}_{k-1}(s,t)) \Bigg\} \\
+ \sum_{u=1}^{p-s} \Bigg\{ & \sum_{k=1}^{p-s} (\varphi^{-}_{k,u}(v) B^{-}_{k-1}(p-s,u) + \psi^{-}_{k,u}(v) A^{-}_{k-1}(p-s,u)) \\
& + \sum_{n=1}^{s} (\xi^{-}_{n,u}(v) X^{+}_{n-1}(p-s,u) + \zeta^{-}_{n,u}(v) Y^{-}_{n-1}(p-s,u)) \Bigg\}.
\end{aligned}
\tag{5.20}
$$

By Table 1, we have

$$
\begin{aligned}
v b^{+}_{m-1}(s) &= \sum_{n=1}^{s} (\varphi^{+}_{n,m}(v) b^{+}_{n-1}(s) + \psi^{+}_{n,m}(v) a^{+}_{n-1}(s)) \\
&\quad + \sum_{k=1}^{p-s} (\xi^{+}_{k,m}(v) x^{+}_{k-1}(s) + \zeta^{+}_{k,m}(v) y^{+}_{k-1}(s)), \\
v x^{+}_{l-1}(s) &= \sum_{k=1}^{p-s} \varphi^{-}_{k,l}(v) x^{+}_{k-1}(s) + \sum_{n=1}^{s} \zeta^{-}_{n,l}(v) a^{+}_{n-1}(s), \\
v y^{+}_{l-1}(s) &= \sum_{k=1}^{p-s} \varphi^{-}_{k,l}(v) y^{+}_{k-1}(s) + \sum_{n=1}^{s} \xi^{-}_{n,l}(v) a^{+}_{n-1}(s), \\
v a^{+}_{m-1}(s) &= \sum_{n=1}^{s} \varphi^{+}_{n,m}(v) a^{+}_{n-1}(s),
\end{aligned}
$$

for $1 \le m \le s$ and $1 \le l \le p-s$. Set

$$K_s(v) = [\varphi^{+}_{n,m}(v)] \in M_s(\mathbb{C}), \tag{5.21}$$

$$K_{p-s}(v) = [\varphi^{-}_{k,l}(v)] \in M_{p-s}(\mathbb{C}), \tag{5.22}$$

$$H_s(v) = [\psi^{+}_{n,m}(v)] \in M_s(\mathbb{C}), \tag{5.23}$$

$$A_{p-s,s}(v) = [\xi^{+}_{k,m}(v)] \in M_{p-s,s}(\mathbb{C}), \tag{5.24}$$

$$B_{p-s,s}(v) = [\zeta^{+}_{k,m}(v)] \in M_{p-s,s}(\mathbb{C}), \tag{5.25}$$

$$C_{s,p-s}(v) = [\xi^{-}_{n,l}(v)] \in M_{s,p-s}(\mathbb{C}), \tag{5.26}$$

$$D_{s,p-s}(v) = [\zeta^{-}_{n,l}(v)] \in M_{s,p-s}(\mathbb{C}), \tag{5.27}$$

$$\tilde{H}_{p-s}(v) = [\psi^{-}_{k,l}(v)] \in M_{p-s}(\mathbb{C}). \tag{5.28}$$

Let $\rho_s^+\colon Q_s \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}_s^+)$ and $\rho_s^-\colon Q_s \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}_{p-s}^-)$ be the representations. The linear map $\rho_s^+(v)$ can be expressed as follows with respect to the basis $b_n^+(s)$, $x_k^+(s)$, $y_k^+(s)$, $a_n^+(s)$:

$$\rho_s^+(v) = \begin{bmatrix} K_s(v) & 0 & 0 & 0 \\ A_{p-s,s}(v) & K_{p-s}(v) & 0 & 0 \\ B_{p-s,s}(v) & 0 & K_{p-s}(v) & 0 \\ H_s(v) & D_{s,p-s}(v) & C_{s,p-s}(v) & K_s(v) \end{bmatrix}.$$

By Table 2, the linear map $\rho_s^-(v)$ can be expressed as follows with respect to the basis $b_k^-(p-s)$, $x_n^-(p-s)$, $y_n^-(p-s)$, $a_k^-(p-s)$:

$$\rho_s^-(v) = \begin{bmatrix} K_{p-s}(v) & 0 & 0 & 0 \\ C_{s,p-s}(v) & K_s(v) & 0 & 0 \\ D_{s,p-s}(v) & 0 & K_s(v) & 0 \\ \tilde{H}_{p-s}(v) & B_{p-s,s}(v) & A_{p-s,s}(v) & K_{p-s}(v) \end{bmatrix}.$$

Thus we have the following theorem, which is one of our main results in this paper.

Theorem 5.2. *For $1 \le s \le p-1$, the image of φ_s is given by the subalgebra of $M_{2p}(\mathbb{C}) \oplus M_{2p}(\mathbb{C})$ as follows*:

$$\left(\begin{bmatrix} K_s & 0 & 0 & 0 \\ A_{p-s,s} & K_{p-s} & 0 & 0 \\ B_{p-s,s} & 0 & K_{p-s} & 0 \\ H_s & D_{s,p-s} & C_{s,p-s} & K_s \end{bmatrix}, \begin{bmatrix} K_{p-s} & 0 & 0 & 0 \\ C_{s,p-s} & K_s & 0 & 0 \\ D_{s,p-s} & 0 & K_s & 0 \\ \tilde{H}_{p-s} & B_{p-s,s} & A_{p-s,s} & K_{p-s} \end{bmatrix} \right)$$

where $K_s, H_s \in M_s(\mathbb{C})$, $K_{p-s}, \tilde{H}_{p-s} \in M_{p-s}(\mathbb{C})$, $A_{p-s,s}, B_{p-s,s} \in M_{p-s,s}(\mathbb{C})$ and $C_{s,p-s}, D_{s,p-s} \in M_{s,p-s}(\mathbb{C})$.

5.2. The center. To determine the space of symmetric linear functions on $\bar{U}_q(sl_2)$, it is important to describe the structure of the center of $\bar{U}_q(sl_2)$. The structure of the center of $\bar{U}_q(sl_2)$ is given in [2].

Proposition 5.3 ([2]). *The center of $\bar{U}_q(sl_2)$ is $(3p-1)$-dimensional.*

The subalgebra Q_s, $0 \le s \le p$, has a central idempotent e_s which acts on Q_s as an identity and annihilates $Q_{s'}$ for $s \ne s'$. For $1 \le s \le p-1$, Q_s also has two nilpotent elements $w_s^\pm$. The action of w_s^+ on Q_s is given by $w_s^+ B_n^+(s,t) = A_n^+(s,t)$ for $1 \le t \le s$ and $0 \le n \le s-1$. Similarly the action of w_s^- on Q_s is given by $w_s^- B_k^-(p-s,u) = A_k^-(p-s,u)$ for $1 \le u \le p-s$ and $0 \le k \le p-s-1$. For other elements of the basis of Q_s, the central element $w_s^\pm$ acts as zero.

5.3. Symmetric linear functions. Since $\bar{U}_q(sl_2)$ is unimodular and the square of the antipode is inner, the dimension of the space of symmetric linear functions on $\bar{U}_q(sl_2)$ is equal to the dimension of the center of $\bar{U}_q(sl_2)$. By Proposition 2.2 and Proposition 5.3, we have the dimension of the space of symmetric linear functions on $\bar{U}_q(sl_2)$.

Proposition 5.4. *The space of symmetric linear functions on $\bar{U}_q(sl_2)$ is $(3p-1)$-dimensional.*

We define linear functions T_0 on Q_0 and T_p on Q_p by

$$T_0(v) = \sum_{t=1}^{p} \psi^-_{t,t}(v), \quad \text{for} \quad v = \sum_{t=1}^{p}\sum_{n=1}^{p} \psi^-_{n,t}(v)A^-_{n-1}(p,t) \in Q_0,$$
$$T_p(w) = \sum_{t=1}^{p} \psi^+_{t,t}(w), \quad \text{for} \quad w = \sum_{t=1}^{p}\sum_{n=1}^{p} \psi^+_{n,t}(w)A^+_{n-1}(p,t) \in Q_p,$$

respectively. By Theorem 5.1, we may regard T_0 and T_p as the traces on $M_p(\mathbb{C})$.

Let us define linear functions on Q_s for $1 \le s \le p-1$

$$T_s^+(v) = \mathrm{tr}(K_s(v)),$$
$$T_s^-(v) = \mathrm{tr}(K_{p-s}(v)),$$
$$G_s(v) = \mathrm{tr}(H_s(v)) + \mathrm{tr}(\tilde{H}_{p-s}(v)),$$

for $v \in Q_s$. By (5.20)–(5.28), we note that

$$T_s^+(v) = \sum_{n=1}^{s} \varphi^+_{n,n}(v),$$
$$T_s^-(v) = \sum_{k=1}^{p-s} \varphi^-_{k,k}(v),$$
$$G_s(v) = \sum_{n=1}^{s} \psi^+_{n,n}(v) + \sum_{k=1}^{p-s} \psi^-_{k,k}(v).$$

Then we have the main theorem in this paper.

Theorem 5.5. *The linear functions*

$$T_0, \quad T_p, \quad T_s^{\pm}, \quad G_s,$$

for $1 \le s \le p-1$ form a basis of $\mathrm{SLF}(\bar{U}_q(sl_2))$.

Proof. It is sufficient to prove that the linear functions $T_s^{\pm}$ and G_s form a basis of $\mathrm{SLF}(Q_s)$ for $1 \le s \le p-1$.

We can easily see that linear functions $T_s^{\pm}$ are symmetric by Theorem 5.2.

By Theorem 5.2, we can see

$$\begin{aligned}H_s(vw) = {} & H_s(v)K_s(w) + D_{s,p-s}(v)A_{p-s,s}(w) \\ & + C_{s,p-s}(v)B_{p-s,s}(w) + K_s(v)H_s(w), \\ \tilde{H}_{p-s}(vw) = {} & \tilde{H}_{p-s}(v)K_{p-s}(w) + B_{p-s,s}(v)C_{s,p-s}(w) \\ & + A_{p-s,s}(v)D_{s,p-s}(w) + K_{p-s}(v)\tilde{H}_{p-s}(w),\end{aligned}$$

for $v, w \in Q_s$. Thus we have $G_s \in \mathrm{SLF}(Q_s)$.

Suppose

$$aT_s^+ + bT_s^- + cG_s = 0, \tag{5.29}$$

for $a, b, c \in \mathbb{C}$. By applying (5.29) to $B_{t-1}^+(s,t)$, $B_{u-1}^-(p-s,u)$ or $A_{t-1}^+(s,t)$, we have $a = b = c = 0$. $\square$

5.4. Integrals and symmetric linear functions. The linear functions $g^{-1} \rightharpoonup \lambda$ and $\mu \leftharpoonup g$ on $\bar{U}_q(sl_2)$ are in $\mathrm{SLF}(\bar{U}_q(sl_2))$ by Proposition 2.1. Note that

$$g^{-1} \rightharpoonup \lambda(E^m F^n K^l) = \mu \leftharpoonup g(E^m F^n K^l) = \delta_{m,p-1}\delta_{n,p-1}\delta_{l,0}. \tag{5.30}$$

Recall that the map $c \mapsto \mu \leftharpoonup gc$ gives an isomorphism from the center of $\bar{U}_q(sl_2)$ to the space of symmetric linear functions on $\bar{U}_q(sl_2)$. For the left integral λ, the map $c \mapsto g^{-1}c \rightharpoonup \lambda$ is also isomorphism. Since $g^{-1} \rightharpoonup \lambda = \mu \leftharpoonup g$, the linear function $g^{-1}c \rightharpoonup \lambda$ coincides with $\mu \leftharpoonup gc$ for any central element c. Thus we only consider the linear function $\mu \leftharpoonup gc$.

Proposition 5.6. *For $s = 0, p$, the following relations hold*:

$$\mu \leftharpoonup ge_s = \alpha_s T_s,$$

where $\alpha_0 = 1/\big(2p\prod_{i=1}^{p-1}(-[i][p-i])\big)$ *and* $\alpha_p = 1/\big(2p\prod_{i=1}^{p-1}([i][p-i])\big)$.

Proof. Recall that the central element e_p acts on Q_p as an identity and annihilates Q_s for $s \neq p$. Therefore $\mu \leftharpoonup ge_s$ is a linear function on Q_s. Recall that

$$A_n^+(p,t) = \frac{F^n}{2p\prod_{i=1}^{p-1}([i][p-i])}E^{p-1}F^{p-t}v^+(p,t).$$

Then, by Lemma 4.2, we see that the term $E^{p-1}F^{p-1}$ appears in $A^+_{t-1}(p,t)$ only. Thus we have

$$\mu \leftharpoonup ge_s(A^+_n(p,t)) = \begin{cases} \dfrac{1}{2p\prod_{i=1}^{p-1}([i][p-i])}, & n = t-1, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Proposition 5.7. *For* $1 \le s \le p-1$, *the following relations hold*:

$$\mu \leftharpoonup ge_s = \alpha_s(T^+_s + T^-_s) + \beta_s G_s,$$
$$\mu \leftharpoonup gw^+_s = \beta_s T^+_s, \quad \mu \leftharpoonup gw^-_s = \beta_s T^-_s,$$

where $\alpha_s = -\delta_s/(\gamma_s)^2$ *and* $\beta_s = 1/\gamma_s$.

Proof. Recall

$$A^+_n(s,t) = \frac{F^n}{\gamma_s}E^{p-1}F^{p-t}v^+(s,t),$$
$$B^+_n(s,t) = \frac{F^n}{\gamma_s}\left(\sum_{l=1}^{p-s}\mu^+_l(s)E^{p-l-1}F^{p-t-l} - \frac{\delta_s}{\gamma_s}E^{p-1}F^{p-t}\right)v^+(s,t).$$

By Lemma 4.2, we see that the term $E^{p-1}F^{p-1}$ appears in $A^+_{t-1}(s,t)$ and $B^+_{t-1}(s,t)$ only. Therefore we obtain

$$\mu \leftharpoonup ge_s(A^+_n(s,t)) = \begin{cases} \dfrac{1}{\gamma_s}, & n = t-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu \leftharpoonup ge_s(B^+_n(s,t)) = \begin{cases} -\dfrac{\delta_s}{(\gamma_s)^2}, & n = t-1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly we have

$$\mu \leftharpoonup ge_s(A^-_k(p-s,u)) = \begin{cases} \dfrac{1}{\gamma_s}, & k = u-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu \leftharpoonup ge_s(B^-_k(p-s,u)) = \begin{cases} -\dfrac{\delta_s}{(\gamma_s)^2}, & u = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have the first relation.

Recall that $w_s^+ B_n^+(s,t) = A_n^+(s,t)$ and w_s^+ annihilates other elements of the basis of $\bar{U}_q(sl_2)$. Hence we have

$$\mu \leftharpoonup g w_s^+(B_n^+(s,t)) = \begin{cases} \dfrac{1}{\gamma_s}, & n = t-1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly we have

$$\mu \leftharpoonup g w_s^-(B_k^-(p-s,u)) = \begin{cases} \dfrac{1}{\gamma_s}, & u = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

So we obtain the second and the third relations. $\square$

Now we can see that

$$\mu \leftharpoonup g = \mu \leftharpoonup \left(g \sum_{s=0}^{p} e_s \right).$$

By Proposition 5.6 and Proposition 5.7, we have

$$\mu \leftharpoonup g = \alpha_0 T_0 + \left(\sum_{s=1}^{p-1} \alpha_s (T_s^+ + T_s^-) + \beta_s G_s \right) + \alpha_s T_p.$$

We can see that

$$\begin{aligned} \beta_s &= \frac{1}{2p \prod_{i=1}^{s-1} [i][s-i] \prod_{i=1}^{p-s-1} (-[i][p-s-i])} \\ &= \frac{(-1)^{p-s-1}[s]^2}{2p([p-1]!)^2} \\ &= \frac{(-1)^{p-s-1}}{2p^3}[s]^2 \left(2 \sin \frac{\pi}{p} \right)^{2(p-1)}, \end{aligned}$$

since $[p-1]! = \prod_{l=1}^{p-1} \sin(l\pi/p)/\sin^{p-1}(\pi/p)$ and $\prod_{l=1}^{p-1} \sin(l\pi/p) = p/2^{p-1}$. In the same way, we have

$$\begin{aligned} \alpha_p &= \frac{1}{2p^3} \left(2 \sin \frac{\pi}{p} \right)^{2(p-1)}, \\ \alpha_0 &= \frac{(-1)^{p-1}}{2p^3} \left(2 \sin \frac{\pi}{p} \right)^{2(p-1)}. \end{aligned}$$

By (4.22), we see

$$\alpha_s^{\pm} = -\frac{\delta_s}{(\gamma_s)^2} = -\beta_s\left(\sum_{l=1}^{s-1}\frac{1}{[l][s-l]} - \sum_{l=1}^{p-s-1}\frac{1}{[l][p-s-l]}\right).$$

ACKNOWLEDGMENT. The author is grateful to Jun Murakami for many valuable comments and to Kiyokazu Nagatomo for discussions. He appreciates Etsuro Date, Yusuke Sakane and the referee for reading manuscript, pointing out some errors and useful comments.

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