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Osaka J. Math.
47 (2010), 215–236

THE SMITH SET OF THE GROUP $S_5 \times C_2 \times \cdots \times C_2$

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(Received July 2, 2008, revised October 24, 2008)

Abstract

In 1960, P.A. Smith raised an isomorphism problem. Is it true that the tangential G-modules at two fixed points of an arbitrary smooth G-action on a sphere with exactly two fixed points are isomorphic to each other? Given a finite group, the Smith set of the group means the subset of real representation ring consisting of all differences of Smith equivalent representations. Many researchers have studied the Smith equivalence for various finite groups. But the Smith sets for non-perfect groups were rarely determined. In particular, the Smith set for a non-gap group has not been determined unless it is trivial. We determine the Smith set for the non-gap group $G = S_5 \times C_2 \times \cdots \times C_2$.

1. Introduction

Throughout this paper, let G be a finite group. In 1960, P.A. Smith [30] raised the next problem.

SMITH ISOMORPHISM PROBLEM. Is it true that the tangential G-modules at two fixed points of an arbitrary smooth G-action on a sphere with exactly two fixed points are isomorphic to each other?

Following [25], two real G-modules V and W are called *Smith equivalent* if there exists a smooth action of G on a homotopy sphere S such that $S^G = \{x, y\}$ for two points x and y at which $T_x(S) \cong V$ and $T_y(S) \cong W$ as real G-modules.

Let $\mathrm{RO}(G)$ denote the real representation ring of G. Define the *Smith set* $\mathrm{Sm}(G)$ to be

$$\mathrm{Sm}(G) := \{[V] - [W] \in \mathrm{RO}(G) \mid V \text{ and } W \text{ are Smith equivalent}\}.$$

In general, we don't know whether $\mathrm{Sm}(G)$ is a subgroup of $\mathrm{RO}(G)$. The Smith isomorphism problem can be restated as follows.

SMITH ISOMORPHISM PROBLEM. Is it true that $\mathrm{Sm}(G) = 0$?

2000 Mathematics Subject Classification. Primary 55M35; Secondary 57S17, 57S25, 20C15.

It is easy to show that the answer is affirmative if G is a group such that each element has the order 1, 2 or 4. Important breakthroughs on the problem came in the following.
(1) M.F. Atiyah–R. Bott [1]: If $G = C_p$, a cyclic group of order p, where p an odd prime, then $\mathrm{Sm}(G) = 0$.
(2) J. Milnor [11]: If G is a compact group and the action semi-free, then $T_x(S) \cong T_y(S)$.
(3) C.U. Sanchez [28]: If G is a group with odd-prime-power order or G is a group with $|G| = pq$, where p and q are odd primes, then $\mathrm{Sm}(G) = 0$.
(4) T. Petrie [24], [26]: If G is an odd order finite abelian group with at least four non-cyclic Sylow subgroups, then $\mathrm{Sm}(G) \neq 0$.
(5) S.E. Cappell–J.L. Shaneson [2]: If G is a cyclic group of order $4m$ such that $m \geq 2$ then $\mathrm{Sm}(G) \neq 0$.
By the character theory, we have $\mathrm{Sm}(C_6) = 0$ and $\mathrm{Sm}(D_6) = 0$ where D_6 is a dihedral group of order 6. So, C_8 is the smallest group with $\mathrm{Sm}(G) \neq 0$. T. Petrie and his collaborators found various pairs of non-isomorphic Smith equivalent real G-modules, e.g. K.H. Dovermann–T. Petrie [3], K.H. Dovermann–D.Y. Suh [5].

In 1996, in the case where G is an Oliver group, E. Laitinen [10, Appendix] lighted the problem again with the next conjecture.

***A_G*-Conjecture.** *If G is an Oliver group with $a_G \geq 2$, then* $\mathrm{Sm}(G) \neq 0$.

After E. Laitinen–M. Morimoto [8], a finite group G is called an *Oliver group* if and only if G never admits a normal series

$$P \trianglelefteq H \trianglelefteq G$$

such that $|P|$ and $[G : H]$ are prime powers and H/P is a cyclic group. For an element $g \in G$, let (g) denote the conjugacy class of g in G. The union $(g)^{\pm} = (g) \cup (g^{-1})$ is called the *real conjugacy class* of g in G. Let a_G denote the number of the real conjugacy classes $(g)^{\pm}$ in G such that the order of g is not a prime power.

We have affirmative answers for the A_G-Conjecture in the following cases.

- E. Laitinen–K. Pawałowski [10]: G is a finite perfect group.
- E. Laitinen–K. Pawałowski [10]: $G \cong A_n$, $\mathrm{SL}(2, p)$ or $\mathrm{PSL}(2, q)$ where n is a natural number, and p and q are primes.
- K. Pawałowski–R. Solomon [21]: G is a finite Oliver group of odd order.
- K. Pawałowski–R. Solomon [21]: G is a finite Oliver group with a cyclic quotient of order pq for two distinct odd primes p and q.
- K. Pawałowski–R. Solomon [21]: G is a finite non-solvable gap group and $G \ncong \mathrm{P\Sigma L}(2, 27)$, where $\mathrm{P\Sigma L}(2, 27)$ is the splitting extension of $\mathrm{PSL}(2, 27)$ by the group $\mathrm{Aut}(\mathbb{F}_{27})$.
- M. Morimoto [13]: $G \cong \mathrm{P\Sigma L}(2, 27)$.

In 2006, M. Morimoto gave a counterexample to the A_G-Conjecture.

- M. Morimoto [14]: If $G = \mathrm{Aut}(A_6)$, then $a_G = 2$ and $\mathrm{Sm}(G) = 0$.

We refer to the articles [27], [4], [20], [6] for survey of related results. K. Pawałowski–T. Sumi claim $\mathrm{Sm}(G) \neq 0$ for many Oliver groups G such that $a_G \geq 2$ and G is not a gap group. Recent information of this topic is found in [22], [32] and [23].

For a prime p, let $G^{\{p\}}$ denote the smallest normal subgroup H of G such that $[G : H]$ is a power of p (possibly 1). Let G^{nil} denote the smallest normal subgroup H of G such that G/H is nilpotent. It is known that

$$G^{\mathrm{nil}} = \bigcap_p G^{\{p\}}.$$

We introduce notation for several families consisting of subgroups of G.

$$\mathcal{S}(G) := \{H \leq G\}.$$

$$\mathcal{P}(G) := \{P \in \mathcal{S}(G) \mid P \text{ is a } p\text{-subgroup}$$
$$\text{for some prime } p \text{ (possibly a trivial group)}\}.$$

$$\mathcal{L}(G) := \{L \in \mathcal{S}(G) \mid G^{\{p\}} \subseteq L \text{ for some prime } p\}.$$

$$\mathcal{G}^1(G) := \{H \in \mathcal{S}(G) \mid \exists P \trianglelefteq H \text{ and } H/P \text{ is cyclic for some } P \in \mathcal{P}(G)\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be families consisting of subgroups of G. A real G-module V is said to be *$\mathcal{X}$-free* if $V^H = 0$ for any $H \in \mathcal{X}$. If M is a subset of $\mathrm{RO}(G)$ then for the families $\mathcal{X}$, $\mathcal{Y}$, we define

$$M_{\mathcal{X}} := \{x = V - W \in M \mid \mathrm{Res}^G_H V \cong \mathrm{Res}^G_H W \text{ for all } H \in \mathcal{X}\},$$

$$M^{\mathcal{Y}} := \{x = V - W \in M \mid V \text{ and } W \text{ are } \mathcal{Y}\text{-free}\},$$

$$M^{\mathcal{Y}}_{\mathcal{X}} := M_{\mathcal{X}} \cap M^{\mathcal{Y}}.$$

Let $\mathcal{HP}(G)$ denote the set of all pairs (H, P) consisting of $H \in \mathcal{S}(G)$ and $P \in \mathcal{P}(H)$ such that $P \neq H$. A real G-module V is called a *gap module* if it satisfies $\dim V^P > 2 \dim V^H$ for all pairs $(H, P) \in \mathcal{HP}(G)$. A finite group G is called a *gap group* if G admits a $\mathcal{L}(G)$-free gap module. Let $V^{=H}$ denote the set consisting of all points $x \in V$ with isotropy subgroup $G_x = H$, and $\dim V^{=H}$ as the maximum of the dimension of all connected components of $V^{=H}$. A real G-module V is said to satisfy the *weak gap condition* if it satisfies the following.

(WG1) $\dim V^P \geq 2 \dim V^H$ for all pairs $(H, P) \in \mathcal{HP}(G)$.

(WG2) If $\dim V^P = 2 \dim V^H$ for a pair $(H, P) \in \mathcal{HP}(G)$, then $[H : P] = 2$.

(WG3) If $\dim V^P = 2 \dim V^H$ and $\dim V^P = 2 \dim V^K$ for pairs $(H, P), (K, P) \in \mathcal{HP}(G)$ respectively, then $\langle H, K \rangle$ belongs to $\mathcal{S}(G) \setminus \mathcal{L}(G)$.

(WG4) $\dim V^P \geq 5$ for all $P \in \mathcal{P}(G)$.

(WG5) $\dim V^{=H} \geq 2$ for all $H \in \mathcal{G}^1(G)$.

(WG6) If $\dim V^P = 2 \dim V^H$ for a pair $(H, P) \in \mathcal{HP}(G)$, then for all $g \in N_G(P) \cap N_G(H)$, the associated transformations $g \colon V^H \to V^H$ are orientation preserving.

Throughout this paper let X_2 be a finite group isomorphic to a direct product of groups isomorphic to C_2, namely $X_2 \cong C_2 \times \cdots \times C_2$ (n-fold) where C_2 is the cyclic group of order 2. Let S_5 be the symmetric group on the five letters, and A_5 be the alternating group on the five letters.

Many authors have studied the Smith equivalence for various finite groups. But the Smith sets $\mathrm{Sm}(G)$ were rarely determined. In particular, when G is a non-solvable, non-perfect group, the Smith set $\mathrm{Sm}(G)$ was not determined except the case $\mathrm{Sm}(G) = 0$. Most finite Oliver groups are gap group, while neither S_5 nor $\mathrm{Aut}(A_6)$ is a gap group. We have interested in the group S_5, because it is an Oliver group which is not a gap group, but it's subgroup A_5 is an Oliver and gap group. In fact $\mathrm{Sm}(S_5) = \mathrm{Sm}(A_5) = 0$ ([21, Example E4, E5]). But what about the case $S_5 \times X_2$ and $A_5 \times X_2$?

Theorem A. *If* $K = A_5 \times X_2$ *then* $\mathrm{Sm}(K) = \mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)} \cong \mathbb{Z}^{2(2^n-1)}$.

This theorem follows from the following 4 lemmas, and the rank of the Smith set follows from Lemma 6.1 and Proposition 6.2.

Lemma 1.1 (K. Pawałowski–R. Solomon). *If G is an Oliver, gap group, then* $\mathrm{Sm}(G) \supseteq \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$.

This result was given as [21, p. 850, Realization Theorem]. The next lemma is well known (see [10, Lemma 2.6]).

Lemma 1.2. *If G contains no elements of order* 8, *then* $\mathrm{Sm}(G) = \mathrm{Sm}(G)_{\mathcal{P}(G)}$.

Lemma 1.3. *If G/G^{nil} is isomorphic to a direct product of groups isomorphic to C_2, then* $\mathrm{Sm}(G)_{\mathcal{P}(G)} \subseteq \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$.

This lemma immediately follows from [14, Proposition 2.2].

Lemma 1.4. *If $K = A_5 \times X_2$ then the following hold.*

(1) *K is an Oliver, gap group.*

(2) *K does not contain an element of order* 8.

(3) $K^{\mathrm{nil}} = A_5$ *and* $K/K^{\mathrm{nil}} \cong X_2$.

The purpose of this paper is to show the next.

Theorem B. *If* $G = S_5 \times X_2$ *then* $\mathrm{Sm}(G) = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \cong \mathbb{Z}^{2^n-1}$.

For $G = S_5 \times X_2$, we can check the following.

(1) G is an Oliver, but not a gap group.

(2) G does not contain an element of order 8.

(3) $G^{\mathrm{nil}} = A_5$ ($\subseteq S_5$) and $G/G^{\mathrm{nil}} \cong C_2 \times X_2$.
To prove Theorem B, we need to obtain an extended result of Lemma 1.1. Thus we will prove the next lemma.

Lemma 1.5. *Let G be an Oliver group. For $x = V_0 - W_0 \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ such that V_0 and W_0 are $\mathcal{L}(G)$-free real G-modules, if there exists a real G-module U such that $V_0 \oplus U$ and $W_0 \oplus U$ are $\mathcal{L}(G)$-free and satisfy the weak gap condition, then $x \in \mathrm{Sm}(G)$.*

In addition, we will show

Lemma 1.6. *Let $G = S_5 \times X_2$. For each $x \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$, there exist real G-modules U, V and W such that $x = V - W$, and $V \oplus U$ and $W \oplus U$ are $\mathcal{L}(G)$-free and satisfy the weak gap condition.*

Hence Theorem B follows from Lemmas 1.2, 1.3, 1.5 and 1.6, and the rank of the Smith set follows from Lemma 6.1 and Proposition 6.2. A key to proving Lemma 1.6 is the next.

Lemma 1.7. *If $K = A_5 \times X_2$ and $G = S_5 \times X_2$ then*

$$\mathrm{Ind}_K^G\left(\mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}\right) = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}.$$

The organization of the paper is as follows. Section 2 is devoted to describing lemmas which are useful to construct smooth G-actions on spheres with non-isomorphic Smith equivalent tangential modules for a general Oliver group G, and we give a proof of Lemma 1.5. In Section 3 we exhibit results on the groups $K = A_5 \times C_2$ and $G = S_5 \times C_2$ obtained by concrete computation and show that $\mathrm{Sm}(K)$ and $\mathrm{Sm}(G)$ are isomorphic to $\mathbb{Z}^2$ and $\mathbb{Z}$, respectively. In Section 4 we observe the induction homomorphism $\mathrm{Ind}_K^G\colon \mathrm{RO}(K) \to \mathrm{RO}(G)$ and the restriction homomorphism $\mathrm{Res}_K^G\colon \mathrm{RO}(G) \to \mathrm{RO}(K)$, and prove Lemma 1.7. In Section 5 we introduce the notion of orientation triviality. Section 6 completes proofs of Theorems A and B.

2. Construction of non-isomorphic Smith equivalent G-modules

If G is not of prime power order, define a real G-module $V(G)$ by

$$V(G) := (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_p (\mathbb{R}[G/G^{\{p\}}] - \mathbb{R})$$

where p runs over the set of primes dividing $|G|$. Let $kV(G) = V(G) \oplus \cdots \oplus V(G)$ (k-fold). We recall some properties of $V(G)$.

Lemma 2.1 (E. Laitinen–M. Morimoto). *For any finite group G, the module $V(G)$ satisfies the following properties.*
(1) $\dim V(G)^P \geq 2 \dim V(G)^H$ *for all* $(H, P) \in \mathcal{HP}(G)$.
(2) *Suppose* $(H, P) \in \mathcal{HP}(G)$ *and* $P \in \mathcal{S}(G) \setminus \mathcal{L}(G)$. *Then* $\dim V(G)^P = 2 \dim V(G)^H$ *holds if and only if* $[H : P] = 2$, $[\langle H, G^{\{2\}}\rangle : \langle P, G^{\{2\}}\rangle] = 2$ *and* $\langle P, G^{\{p\}}\rangle = G$ *for all odd prime* p.

This Lemma was given as [8, Theorem 2.3]. Reader can refer to [8] for fundamental properties of $V(G)$.

Lemma 2.2. *Let G be an Oliver group, n an integer ≥ 1, and V and W real G-modules. Suppose the following* (1)–(3):
(1) *There exists a smooth G-action on a homotopy sphere Σ_1 with exactly one G-fixed point, x_1 say, such that the tangential G-module $T_{x_1}(\Sigma_1)$ at x_1 of Σ_1 is isomorphic to $V \oplus nV(G)$.*
(2) *There exists a smooth G-action on a homotopy sphere Σ_2 with exactly one G-fixed point, x_2 say, such that $T_{x_2}(\Sigma_2)$ is isomorphic to $W \oplus nV(G)$.*
(3) *There exists a smooth G-action on a disk Δ with exactly two G-fixed points, y_1 and y_2 say, such that $T_{y_1}(\Delta)$ and $T_{y_2}(\Delta)$ are isomorphic to $V \oplus nV(G)$ and $W \oplus nV(G)$ respectively.*
Then there exists a smooth G-action on a standard sphere Σ with exactly two G-fixed points, z_1 and z_2 say, such that $T_{z_1}(\Sigma)$ and $T_{z_2}(\Sigma)$ are isomorphic to $V \oplus nV(G)$ and $W \oplus nV(G)$ respectively. Hence the element $V - W$ of $\mathrm{RO}(G)$ *belongs to* $\mathrm{Sm}(G)$.

Proof. Let Σ_1, Σ_2 and Δ be spheres and a disk appearing in (1)–(3) above. Let Σ_3 denote the sphere obtained as the double of Δ, namely $\Sigma_3 = \Delta \cup \Delta'$, where Δ' is a copy of Δ. Then Σ_3^G consists of y_1, y_2, y_1' and y_2' such that $T_{y_1}(\Sigma_3) = T_{y_1'}(\Sigma_3) \cong V \oplus nV(G)$ and $T_{y_2}(\Sigma_3) = T_{y_2'}(\Sigma_3) \cong W \oplus nV(G)$. Let Σ_4 denote the G-connected sum of Σ_3 with Σ_1 and Σ_2 with respect to the pairs of points (y_1', x_1) and (y_2', x_2). Since $n \geq 1$, $\dim \Sigma_3^P \geq 2$ and Σ_3 contains (infinitely many) points of isotropy subgroup P for each Sylow subgroup of G. By the [9, Proposition 1.3], we can obtain the standard sphere Σ as the resulting manifold of iterated G-connected sum of Σ_3 with copies of $G \times_P \mathrm{Res}_P^G \Sigma_3$, where P runs over the set of all Sylow subgroups of G. □

Lemma 2.3 (M. Morimoto). *Let G be an Oliver group and V an $\mathcal{L}(G)$-free real G-module satisfying the weak gap condition. Then there exists a smooth G-action on a sphere Σ_1 with exactly one G-fixed point, x_1 say, such that $T_{x_1}(\Sigma_1)$ is isomorphic to V.*

Proof. By [18], Oliver group has a smooth fixed-point-free action on a disk. Thus we can construct a smooth action of G on a disk $D = D(V)$ with exactly one G-fixed point x_1. Taking the double of D, we obtain a smooth action of G on $\Sigma_1 = D \cup_{\partial D} D$

with $\Sigma_1^G = \{x_1, x_2\}$. Clearly $\Sigma_1 \cong_G S(\mathbb{R} \oplus V)$. We can check that the action of G on Σ_1 satisfies Conditions (1)–(5) of [16, Theorem 36]. Therefore we can delete x_2 from Σ_1^G. Namely there exists a smooth action of G on a sphere Σ_2 with exactly one G-fixed point. □

Lemma 2.4 (B. Oliver, M. Morimoto–K. Pawałowski). *Let G be an Oliver group and V_1 and W_1 $\mathcal{L}(G)$-free real G-modules such that $\mathrm{Res}_P^G V_1$ is isomorphic to $\mathrm{Res}_P^G W_1$ for all Sylow subgroups P. Then there exists an integer N such that for every $n \geq N$, the m-dimensional disk Δ, where $m = \dim V_1 + n \dim V(G)$, admits a smooth G-action with exactly two G-fixed points, y_1 and y_2 say, such that $T_{y_1}(\Delta)$ and $T_{y_2}(\Delta)$ are isomorphic to $V_1 \oplus nV(G)$ and $W_1 \oplus nV(G)$ respectively.*

This lemma follows from [15, Theorem 0.3] but crucial part of the proof was due to [19].

Proof of Lemma 1.5. Set $V_1 = V_0 \oplus U$ and $W_1 = W_0 \oplus U$. Clearly, V_1 and W_1 are $\mathcal{L}(G)$-free real G-modules such that $\mathrm{Res}_P^G V_1 \cong \mathrm{Res}_P^G W_1$ for all Sylow subgroups P. Apply Lemma 2.4 to V_1 and W_1, for finding an integer N such that for each $k \geq N$, putting $n = 2k$, there exists a smooth G-action on a disk Δ described in Lemma 2.4. Set $V = V_1 \oplus 2kV(G)$, and $W = W_1 \oplus 2kV(G)$, where $k \geq N$. Apply Lemma 2.3 to V for obtaining a smooth G-action on a sphere Σ_1 described in Lemma 2.4. Then $V_1 \oplus 2kV(G)$ and $W_1 \oplus 2kV(G)$ satisfy the weak gap condition. Obtain Σ_2 for W similarly to Σ_1 replacing V by W. Then by Lemma 2.2, we obtain a desired smooth G-action on the m-dimensional sphere Σ for arbitrary $k \geq N$. □

3. Computation of Sm($S_5 \times C_2$)

Proposition 3.1. *The following equalities hold for $G = S_5 \times C_2$ and $K = A_5 \times C_2$.*
(1) $\mathrm{Sm}(K) \cong \mathbb{Z}^2$ *and* $\mathrm{Sm}(G) \cong \mathbb{Z}$.
(2) $\mathrm{Ind}_K^G(\mathrm{Sm}(K)) = \mathrm{Sm}(G)$.
Here the map $\mathrm{Ind}_K^G \colon \mathrm{RO}(K) \to \mathrm{RO}(G)$ *is the induction homomorphism*:

$$[V] \mapsto [\mathbb{R}[G] \otimes_{\mathbb{R}[K]} V].$$

Proof. (1) By means of GAP [33], the irreducible complex characters of $K = A_5 \times C_2$ are as in Table 1. The notation in the table reads that, for example in the case "5b", the first letter 5 of "5b" indicates the order of an element belonging to the corresponding conjugacy class and the second letter b of "5b" is used to distinguish conjugacy classes.

Since A_5 is a simple group, it follows that $K^{\{2\}} = A_5$ and $K^{\{p\}} = K$ $(p \neq 2)$. Thus $K^{\mathrm{nil}} = A_5$, and $K/K^{\mathrm{nil}} \cong C_2$. Clearly K contains no elements of 8.

K is an Oliver group, because K is non-solvable. Let $\{\delta_i, 1 \leq i \leq 8\}$ be the $\mathbb{Z}$-basis of $\mathrm{RO}(K)$ such that the complification of δ_i is $\delta_{i\mathbb{C}}$. In fact, $4\delta_3 + 3\delta_5 + \delta_7 + 2\delta_8 +$

Table 1. The complex characters of $K = A_5 \times C_2$ where $A = -\omega - \omega^4 = (1-\sqrt{5})/2$, $\hat{A} = -\omega^2 - \omega^3 = (1+\sqrt{5})/2$, $\omega = \exp(2\pi\sqrt{-1})/5$.

	1a	2a	3a	6a	2b	2c	5a	10a	5b	10b
$\delta_{1\mathbb{C}}$	1	1	1	1	1	1	1	1	1	1
$\delta_{2\mathbb{C}}$	1	−1	1	−1	1	−1	1	−1	1	−1
$\delta_{3\mathbb{C}}$	3	3	0	0	−1	−1	A	A	$\hat{A}$	$\hat{A}$
$\delta_{4\mathbb{C}}$	3	3	0	0	−1	−1	$\hat{A}$	$\hat{A}$	A	A
$\delta_{5\mathbb{C}}$	3	−3	0	0	−1	1	A	$-A$	$\hat{A}$	$-\hat{A}$
$\delta_{6\mathbb{C}}$	3	−3	0	0	−1	1	$\hat{A}$	$-\hat{A}$	A	$-A$
$\delta_{7\mathbb{C}}$	4	4	1	1	0	0	−1	−1	−1	−1
$\delta_{8\mathbb{C}}$	4	−4	1	−1	0	0	−1	1	−1	1
$\delta_{9\mathbb{C}}$	5	5	−1	−1	1	1	0	0	0	0
$\delta_{10\mathbb{C}}$	5	−5	−1	1	1	−1	0	0	0	0

$\delta_9 + 3\delta_{10}$ is a $\mathcal{L}(K)$-free gap K-module, so K is a gap group. (The fact that K is a gap group was theoretically proved by T. Sumi [31, Proposition 3.3].) By Lemmas 1.1, 1.2, and 1.3, we get $\mathrm{Sm}(K) = \mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}$.

By a straightforward computation [33], a $\mathbb{Z}$-basis of $\mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}$ is $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$, where $\boldsymbol{x}_1 = \delta_3 - \delta_5 - 2\delta_7 + 2\delta_8 + \delta_9 - \delta_{10}$, $\boldsymbol{x}_2 = \delta_4 - \delta_6 - 2\delta_7 + 2\delta_8 + \delta_9 - \delta_{10}$.

(2) By means of GAP [33], the irreducible complex characters of $G = S_5 \times C_2$ are as in Table 2.

Since A_5 is a simple group, it follows that $G^{\{2\}} = A_5$ and $G^{\{p\}} = G$ ($p \neq 2$). Thus $G^{\mathrm{nil}} = A_5$, and $G/G^{\mathrm{nil}} \cong C_2 \times C_2$. Clearly G contains no elements of 8. G is an Oliver group, because G is non-solvable. By Lemmas 1.2 and 1.3, $\mathrm{Sm}(G) \subseteq \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$.

Let $\{\xi_i,\ 1 \le i \le 14\}$ be the $\mathbb{Z}$-basis of $\mathrm{RO}(G)$ such that the complification of ξ_i is $\xi_{i\mathbb{C}}$. By a straightforward computation [33], $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \cong \mathbb{Z}$. We take the $\mathbb{Z}$-basis element $\boldsymbol{y} = V - W$ of $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ such that $V = 2\xi_5 + 2\xi_7 + \xi_{10} + \xi_{12} + \xi_{14}$ and $W = 2\xi_6 + 2\xi_8 + \xi_9 + \xi_{11} + \xi_{13}$. Let $U = \xi_5 + 2\xi_6 + 2\xi_8 + 3\xi_{10} + 3\xi_{12}$. We can check that $V \oplus 2U$ and $W \oplus 2U$ satisfy the weak gap condition. By Lemma 1.5, we obtain $n\boldsymbol{y} \in \mathrm{Sm}(G)$ for any $n \in \mathbb{Z}$, thus $\{\boldsymbol{y}\}$ is a $\mathbb{Z}$-basis of $\mathrm{Sm}(G)$.

Since the equalities

$$\begin{aligned}
&\mathrm{Ind}_K^G\,\delta_1 = \xi_1 + \xi_4, && \mathrm{Ind}_K^G\,\delta_2 = \xi_2 + \xi_3,\\
&\mathrm{Ind}_K^G\,\delta_3 = \xi_{13}, && \mathrm{Ind}_K^G\,\delta_4 = \xi_{13},\\
&\mathrm{Ind}_K^G\,\delta_5 = \xi_{14}, && \mathrm{Ind}_K^G\,\delta_6 = \xi_{14},\\
&\mathrm{Ind}_K^G\,\delta_7 = \xi_5 + \xi_7, && \mathrm{Ind}_K^G\,\delta_8 = \xi_6 + \xi_8,\\
&\mathrm{Ind}_K^G\,\delta_9 = \xi_9 + \xi_{11}, && \mathrm{Ind}_K^G\,\delta_{10} = \xi_{10} + \xi_{12}
\end{aligned}$$

Table 2. The complex characters of $G = S_5 \times C_2$.

	1a	2a	2b	2c	3a	6a	2d	2e	4a	4b	6b	6c	5a	10a
$\xi_{1\mathbb{C}}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\xi_{2\mathbb{C}}$	1	−1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1
$\xi_{3\mathbb{C}}$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
$\xi_{4\mathbb{C}}$	1	1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1
$\xi_{5\mathbb{C}}$	4	4	−2	−2	1	1	0	0	0	0	1	1	−1	−1
$\xi_{6\mathbb{C}}$	4	−4	−2	2	1	−1	0	0	0	0	1	−1	−1	1
$\xi_{7\mathbb{C}}$	4	4	2	2	1	1	0	0	0	0	−1	−1	−1	−1
$\xi_{8\mathbb{C}}$	4	−4	2	−2	1	−1	0	0	0	0	−1	1	−1	1
$\xi_{9\mathbb{C}}$	5	5	1	1	−1	−1	1	1	−1	−1	1	1	0	0
$\xi_{10\mathbb{C}}$	5	−5	1	−1	−1	1	1	−1	−1	1	1	−1	0	0
$\xi_{11\mathbb{C}}$	5	5	−1	−1	−1	−1	1	1	1	1	−1	−1	0	0
$\xi_{12\mathbb{C}}$	5	−5	−1	1	−1	1	1	−1	1	−1	−1	1	0	0
$\xi_{13\mathbb{C}}$	6	6	0	0	0	0	−2	−2	0	0	0	0	1	1
$\xi_{14\mathbb{C}}$	6	−6	0	0	0	0	−2	2	0	0	0	0	1	−1

hold, we obtain $\mathrm{Ind}_K^G(\boldsymbol{x}_1) = \mathrm{Ind}_K^G(\boldsymbol{x}_2) = -\boldsymbol{y}$, which determines the induction map $\mathrm{Ind}_K^G \colon \mathrm{Sm}(K) \to \mathrm{Sm}(G)$. □

4. Induction and restriction

Let G be a finite group.

Lemma 4.1. *If $K \le G$, then*

$$\mathrm{Ind}_K^G\big(\mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}\big) \subseteq \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}.$$

Proof. By definition,

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)_{\mathcal{P}(G)} \cap \mathrm{RO}(G)^{\mathcal{L}(G)}.$$

So we will prove following (1) and (2).
(1) $\mathrm{Ind}_K^G(\mathrm{RO}(K)_{\mathcal{P}(K)}) \subseteq \mathrm{RO}(G)_{\mathcal{P}(G)}$.
(2) $\mathrm{Ind}_K^G(\mathrm{RO}(K)^{\mathcal{L}(K)}) \subseteq \mathrm{RO}(G)^{\mathcal{L}(G)}$.

(1) Let $x = V - W \in \mathrm{RO}(K)_{\mathcal{P}(K)}$ where V and W are real K-modules. It suffices to prove

$$\mathrm{Res}_P^G\big(\mathrm{Ind}_K^G V\big) \cong \mathrm{Res}_P^G\big(\mathrm{Ind}_K^G W\big)$$

for all $P \in \mathcal{P}(G)$. By the Mackey decomposition, we have

$$\operatorname{Res}_P^G\big(\operatorname{Ind}_K^G V\big) = \bigoplus_{PgK \in P\backslash G/K} \operatorname{Ind}_{P\cap gKg^{-1}}^P\big(g_* \operatorname{Res}_{K\cap g^{-1}Pg}^K V\big),$$
$$\operatorname{Res}_P^G\big(\operatorname{Ind}_K^G W\big) = \bigoplus_{PgK \in P\backslash G/K} \operatorname{Ind}_{P\cap gKg^{-1}}^P\big(g_* \operatorname{Res}_{K\cap g^{-1}Pg}^K W\big).$$

Since $V - W \in \mathrm{RO}(K)_{\mathcal{P}(K)}$, it follows that

$$\operatorname{Res}_{K\cap g^{-1}Pg}^K V \cong \operatorname{Res}_{K\cap g^{-1}Pg}^K W.$$

(2) Let $x = V - W \in \mathrm{RO}(K)^{\mathcal{L}(K)}$ where V and W are $\mathcal{L}(K)$-free real K-modules. By definition, $V^{K^{\{p\}}} = 0 = W^{K^{\{p\}}}$ for all primes p. By the Mackey decomposition, we have

$$\operatorname{Res}_{G^{\{p\}}}^G\big(\operatorname{Ind}_K^G V\big) = \bigoplus_{G^{\{p\}}gK \in G^{\{p\}}\backslash G/K} \operatorname{Ind}_{G^{\{p\}}\cap gKg^{-1}}^{G^{\{p\}}}\big(g_* \operatorname{Res}_{K\cap g^{-1}G^{\{p\}}g}^K V\big).$$

Clearly $[K : (K \cap G^{\{p\}})]$ is a p-power. Thus we have

$$\begin{aligned} V^{K\cap G^{\{p\}}} &= (V^{K^{\{p\}}})^{(K\cap G^{\{p\}})/K^{\{p\}}} \\ &= 0^{(K\cap G^{\{p\}})/K^{\{p\}}} \\ &= 0. \end{aligned}$$

Similarly, $W^{K\cap G^{\{p\}}} = 0$. Thus $\big(\operatorname{Ind}_K^G V\big)^{G^{\{p\}}} = 0 = \big(\operatorname{Ind}_K^G W\big)^{G^{\{p\}}}$. □

Lemma 4.2. *If $K \le G$, then*

$$\operatorname{Res}_K^G(\mathrm{RO}(G)_{\mathcal{P}(G)}) \subseteq \mathrm{RO}(K)_{\mathcal{P}(K)}.$$

Moreover, if $G^{\{p\}} = G$ $(p \ne 2)$ and $K \supseteq G^{\{2\}}$ then

$$\operatorname{Res}_K^G\big(\mathrm{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)}\big) \subseteq \mathrm{RO}(K)_{\mathcal{P}(K)}^{\mathcal{L}(K)}.$$

Proof. Let $V - W \in \mathrm{RO}(G)_{\mathcal{P}(G)}$ where V and W are real G-modules. So

$$\operatorname{Res}_P^G V \cong \operatorname{Res}_P^G W$$

for all $P \in \mathcal{P}(K) \subseteq \mathcal{P}(G)$. In general, $\operatorname{Res}_P^K\big(\operatorname{Res}_K^G V\big) = \operatorname{Res}_P^G V$. Therefore

$$\operatorname{Res}_P^K\big(\operatorname{Res}_K^G V\big) \cong \operatorname{Res}_P^K\big(\operatorname{Res}_K^G W\big).$$

Thus,

$$\operatorname{Res}_K^G\big(\mathrm{RO}(G)_{\mathcal{P}(G)}\big) \subseteq \mathrm{RO}(K)_{\mathcal{P}(K)}.$$

Suppose $G^{\{p\}} = G$ $(p \neq 2)$ and $K \supseteq G^{\{2\}}$. Since $G^{\{2\}} = K \cap G^{\{2\}} \trianglelefteq K$, we have $K^{\{2\}} \subseteq G^{\{2\}}$. Since $K^{\{2\}} \trianglelefteq K$ and $G^{\{2\}} \subseteq K$, we get $K^{\{2\}} \trianglelefteq G^{\{2\}}$. For all $g \in G$, we obtain

$$gK^{\{2\}}g^{-1} \subseteq gG^{\{2\}}g^{-1} = G^{\{2\}}.$$

Let $a \in G^{\{2\}}$. Then

$$a(gK^{\{2\}}g^{-1})a^{-1} = g(g^{-1}ag)K^{\{2\}}(g^{-1}ag)^{-1}g^{-1}.$$

Since $g^{-1}ag \in G^{\{2\}}$ and $K^{\{2\}} \trianglelefteq G^{\{2\}}$, we get $(g^{-1}ag)K^{\{2\}}(g^{-1}ag)^{-1} = K^{\{2\}}$. Thus

$$a(gK^{\{2\}}g^{-1})a^{-1} = gK^{\{2\}}g^{-1}.$$

That is $gK^{\{2\}}g^{-1} \trianglelefteq G^{\{2\}}$. Set

$$S = \bigcap_{g \in G} gK^{\{2\}}g^{-1}.$$

Clearly, $S \trianglelefteq G$. We know $G^{\{2\}}/K^{\{2\}}$ is a subgroup of $K/K^{\{2\}}$. Since $K/K^{\{2\}}$ is a 2-group, it follows that $G^{\{2\}}/K^{\{2\}}$ is a 2-group. It is easy to show that $G^{\{2\}}/S$ is a 2-group. Since $G/G^{\{2\}} \cong (G/S)/(G^{\{2\}}/S)$, G/S is a 2-group. Therefore $S = G^{\{2\}} = K^{\{2\}}$.

Let $U_1 - U_2 \in \mathrm{RO}(G)^{\mathcal{L}(G)}$ where U_1 and U_2 are $\mathcal{L}(G)$-free real G-modules. We obtain

$$\left(\mathrm{Res}^G_K U_1\right)^{K^{\{2\}}} = U_1^{G^{\{2\}}} = 0$$

and

$$\left(\mathrm{Res}^G_K U_2\right)^{K^{\{2\}}} = U_2^{G^{\{2\}}} = 0. \qquad \square$$

Let $G = S_5 \times X_2$ and $K = A_5 \times X_2$ where $X_2 = C_2 \times \cdots \times C_2$.

Proof of Lemma 1.7. The conjugacy classes of the maximal elementary subgroups of G not belonging to K are represented by $E_1 = D_8 \times X_2$ and $E_2 = C_6 \times X_2$. As $E_2 \subset H_2 = D_{12} \times X_2$, by Brauer's theorem [29, p. 78]

$$\mathrm{RO}(G) = \mathrm{Ind}^G_{E_1} \mathrm{RO}(E_1) + \mathrm{Ind}^G_{H_2} \mathrm{RO}(H_2) + \mathrm{Ind}^G_K \mathrm{RO}(K).$$

Thus we have

$$1_{\mathbb{R}} = \mathrm{Ind}^G_{E_1} t + \mathrm{Ind}^G_{H_2} u + \mathrm{Ind}^G_K v$$

for some $t \in \mathrm{RO}(E_1)$, $u \in \mathrm{RO}(H_2)$ and $v \in \mathrm{RO}(K)$. Let x be an arbitrary element of $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. Then we have

$$x = \mathrm{Ind}^G_{E_1}\left(t \cdot \mathrm{Res}^G_{E_1} x\right) + \mathrm{Ind}^G_{H_2}\left(u \cdot \mathrm{Res}^G_{H_2} x\right) + \mathrm{Ind}^G_K\left(v \cdot \mathrm{Res}^G_K x\right).$$

Since E_1 is a 2-group, $\mathrm{Res}^G_{E_1} x = 0$ and hence

$$x = \mathrm{Ind}^G_{H_2}\big(u \cdot \mathrm{Res}^G_{H_2} x\big) + \mathrm{Ind}^G_K\big(v \cdot \mathrm{Res}^G_K x\big).$$

Let $H \le G$ and $a \in \mathrm{RO}(H)$. Then $\mathcal{P}(H)$ is a subset of $\mathcal{P}(G)$. Thus for $P \in \mathcal{P}(H)$, we have

$$\begin{aligned}\mathrm{Res}^H_P\big(a \cdot \mathrm{Res}^G_H x\big) &= \mathrm{Res}^H_P(a)\big(\mathrm{Res}^H_P\big(\mathrm{Res}^G_H x\big)\big)\\ &= \mathrm{Res}^H_P(a)\big(\mathrm{Res}^G_P x\big)\\ &= \mathrm{Res}^H_P(a) \cdot 0\\ &= 0.\end{aligned}$$

Namely $a \cdot \mathrm{Res}^G_H x \in \mathrm{RO}(H)_{\mathcal{P}(H)}$.

Suppose $A_5 \le H \le G$. Write $a = U_1 - U_2$ and $x = V_1 - V_2$ with real H-modules U_1 and U_2 and $\mathcal{L}(G)$-free real G-modules V_1 and V_2. Then note $V_1^{A_5} = V_2^{A_5} = 0$ and

$$\begin{aligned}a \cdot \mathrm{Res}^G_H x = &\big\{\big(U_1 \otimes \mathrm{Res}^G_H V_1\big) \oplus \big(U_2 \otimes \mathrm{Res}^G_H V_2\big)\big\}\\ &- \big\{\big(U_1 \otimes \mathrm{Res}^G_H V_2\big) \oplus \big(U_2 \otimes \mathrm{Res}^G_H V_1\big)\big\}.\end{aligned}$$

Let $W_1 = \big(U_1 \otimes \mathrm{Res}^G_H V_1\big) \oplus \big(U_2 \otimes \mathrm{Res}^G_H V_2\big)$ and $W_2 = \big(U_1 \otimes \mathrm{Res}^G_H V_2\big) \oplus \big(U_2 \otimes \mathrm{Res}^G_H V_1\big)$. Since A_5 does not have subgroups with index 2, we have

$$\begin{aligned}W_1^{A_5} &= \big(U_1 \otimes \mathrm{Res}^G_H V_1\big)^{A_5} \oplus \big(U_2 \otimes \mathrm{Res}^G_H V_2\big)^{A_5}\\ &= \big(U_1^{A_5} \otimes \big(\mathrm{Res}^G_H V_1\big)^{A_5}\big) \oplus \big(U_2^{A_5} \otimes \big(\mathrm{Res}^G_H V_2\big)^{A_5}\big)\\ &= \big(U_1^{A_5} \otimes 0\big) \oplus \big(U_2^{A_5} \otimes 0\big)\\ &= 0.\end{aligned}$$

Similarly, $W_2^{A_5} = 0$. Therefore, $a \cdot \mathrm{Res}^G_H x \in \mathrm{RO}(H)^{\mathcal{L}(H)}_{\mathcal{P}(H)}$. Consequently,

$$\begin{aligned}x &= \mathrm{Ind}^G_{H_2}\big(u \cdot \mathrm{Res}^G_{H_2} x\big) + \mathrm{Ind}^G_K\big(v \cdot \mathrm{Res}^G_K x\big)\\ &= \mathrm{Ind}^G_{H_2} x_1 + \mathrm{Ind}^G_K x_2\end{aligned}$$

with $x_1 = u \cdot \mathrm{Res}^G_{H_2} x \in \mathrm{RO}(H_2)_{\mathcal{P}(H_2)}$ and $x_2 = v \cdot \mathrm{Res}^G_K x \in \mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}$.

In order to show $\mathrm{Ind}^G_{H_2} x_1 = 0$, we regard

$$\mathrm{RO}(H_2) = \mathrm{RO}(D_{12}) \otimes_{\mathbb{Z}} \mathrm{RO}(X_2)$$

in a canonical way. For each $T \le X_2$ with $[X_2 : T] \le 2$, there is a unique 1-dimensional real X_2-representation ξ_T such that the kernel of ξ_T is T. The set

$$\{\xi_T \mid T \le X_2,\ [X_2 : T] \le 2\}$$

is a $\mathbb{Z}$-basis of $\mathrm{RO}(X_2)$. Thus we can regard

$$\mathrm{RO}(H_2) = \mathrm{RO}(D_{12})\xi_{T_1} \oplus \mathrm{RO}(D_{12})\xi_{T_2} \oplus \cdots \oplus \mathrm{RO}(D_{12})\xi_{T_{2^n}}.$$

We can write x_1 above in the form

$$x_1 = \sum_{i=1}^{2^n} u_{T_i} \cdot \xi_{T_i}$$

with $u_{T_i} \in \mathrm{RO}(D_{12})$. Since $x \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ and $\mathrm{Ind}_K^G x_2 \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$, we get $\mathrm{Ind}_{H_2}^G x_1 \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. Since $x_1 \in \mathrm{RO}(H_2)_{\mathcal{P}(H_2)}$, it must hold that

$$u_{T_i} \in \mathrm{RO}(D_{12})_{\mathcal{P}_2(D_{12})}$$

for each i and

$$\sum_{i=1}^{2^n} u_{T_i} \in \mathrm{RO}(D_{12})_{\mathcal{P}_3(D_{12})}$$

where $\mathcal{P}_p(H) := \{P \le H \mid P \text{ is a } p\text{-group}\}$.

Regard $D_{12} = \langle a, b, c\rangle$ with $a = (1,2,3)$, $b = (1,2)$, $c = (4,5)$. By a straightforward computation [33], we can check that $\{U_1 - U_2, U_3 - U_4\}$ is a basis of $\mathrm{RO}(D_{12})_{\mathcal{P}_2(D_{12})}$, where U_i are real D_{12}-modules of dimension 2 with action:

$$\begin{aligned}
&U_1\colon a \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, && b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\\
&U_2\colon a \mapsto \begin{bmatrix} \cos\frac{2}{3}\pi & -\sin\frac{2}{3}\pi \\ \sin\frac{2}{3}\pi & \cos\frac{2}{3}\pi \end{bmatrix}, && b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\\
&U_3\colon a \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, && b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\\
&U_4\colon a \mapsto \begin{bmatrix} \cos\frac{2}{3}\pi & -\sin\frac{2}{3}\pi \\ \sin\frac{2}{3}\pi & \cos\frac{2}{3}\pi \end{bmatrix}, && b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad c \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\left(\mathrm{Ind}_{D_{12}}^{S_5}(U_1 - U_2)\right)^{S_5} &= \mathbb{R}, \quad \left(\mathrm{Ind}_{D_{12}}^{S_5}(U_3 - U_4)\right)^{S_5} = 0,\\
\left(\mathrm{Ind}_{D_{12}}^{S_5}(U_1 - U_2)\right)^{A_5} &= \mathbb{R}, \quad \left(\mathrm{Ind}_{D_{12}}^{S_5}(U_3 - U_4)\right)^{A_5} = \mathbb{R}.
\end{aligned}$$

Then we can write $\operatorname{Ind}_{H_2}^G x_1$ in the form

$$\operatorname{Ind}_{H_2}^G x_1 = \sum_{i=1}^{2^n} \left\{ m_{T_i} \operatorname{Ind}_{D_{12}}^{S_5}(U_1 - U_2) + n_{T_i} \operatorname{Ind}_{D_{12}}^{S_5}(U_3 - U_4) \right\} \cdot \xi_{T_i}.$$

Note

$$\begin{aligned}\left(\operatorname{Ind}_{H_2}^G x_1\right)^G &= \left\{ m_{X_2} \left(\operatorname{Ind}_{D_{12}}^{S_5}(U_1 - U_2)\right)^{S_5} + n_{X_2} \left(\operatorname{Ind}_{D_{12}}^{S_5}(U_3 - U_4)\right)^{S_5} \right\} \cdot \xi_{X_2} \\ &= m_{X_2} \mathbb{R} \cdot \xi_{X_2} \\ &= 0.\end{aligned}$$

This shows $m_{X_2} = 0$. Next note

$$\begin{aligned}\left(\operatorname{Ind}_{H_2}^G x_1\right)^{S_5 \times T_i} &= \left\{ m_{T_i} \left(\operatorname{Ind}_{D_{12}}^{S_5}(U_1 - U_2)\right)^{S_5} + n_{T_i} \left(\operatorname{Ind}_{D_{12}}^{S_5}(U_3 - U_4)\right)^{S_5} \right\} \cdot \xi_{T_i} \\ &\quad + n_{X_2} \left(\operatorname{Ind}_{D_{12}}^{S_5}(U_3 - U_4)\right)^{S_5} \cdot \xi_{X_2} \\ &= m_{T_i} \mathbb{R} \cdot \xi_{T_i} \\ &= 0.\end{aligned}$$

This shows $m_{T_i} = 0$. Therefore we get the equality

$$\operatorname{Ind}_{H_2}^G x_1 = \sum_{i=1}^{2^n} n_{T_i} \operatorname{Ind}_{D_{12}}^{S_5}(U_3 - U_4) \cdot \xi_{T_i}.$$

The equalities

$$\begin{aligned}\left(\operatorname{Ind}_{H_2}^G x_1\right)^{A_5 \times X_2} &= n_{X_2} \left(\operatorname{Ind}_{D_{12}}^{S_5}(U_3 - U_4)\right)^{A_5} \cdot \xi_{X_2} \\ &= n_{X_2} \mathbb{R} \cdot \xi_{X_2} \\ &= 0\end{aligned}$$

conclude $n_{X_2} = 0$. Similarly, we can show $n_{T_i} = 0$. Thus we have established $\operatorname{Ind}_{H_2}^G x_1 = 0$. □

5. Orientation triviality

We use the following notation.

$$\mathcal{HP}(G, 2) := \{(H, P) \in \mathcal{HP}(G) \mid [H : P] = 2\},$$

$$\mathcal{HP}(G, 2)_0 := \{(H, P) \in \mathcal{HP}(G, 2) \mid [\langle H, G^{\{2\}} \rangle : \langle P, G^{\{2\}} \rangle] = 2 \text{ and } \langle P, G^{\{q\}} \rangle = G \text{ for any odd prime } q\},$$

$$\mathcal{A}(G) := \{(H, g) \in \mathcal{S}(G) \times G \mid g \in N_G(H),\ \exists P \triangleleft H \text{ satisfying } (H, P) \in \mathcal{HP}(G, 2)\},$$

$$\mathcal{B}(G) := \{(H, g) \in \mathcal{S}(G) \times G \mid g \in N_G(H),\ \exists P \triangleleft H \text{ satisfying } (H, P) \in \mathcal{HP}(G, 2)_0\}.$$

For each element $x = V - W \in \mathrm{RO}(G)$, we define a map

$$\psi\colon \mathcal{A}(G) \times \mathrm{RO}(G) \to \mathbb{Z}_2$$

by

$$\psi((H, g), x) = \mathrm{Ori}(g, V^H) - \mathrm{Ori}(g, W^H)$$

where

$$\mathrm{Ori}(g, V^H) = \begin{cases} 0 & \text{if} \quad g\colon V^H \to V^H \quad \text{is orientation preserving,} \\ 1 & \text{if} \quad g\colon V^H \to V^H \quad \text{is orientation reversing.} \end{cases}$$

The value $\psi((H, g), x)$ is also written as $\mathrm{Ori}(g, x^H)$.

Lemma 5.1. *For a real G-module V and $(H,g) \in \mathcal{A}(G)$,* $\mathrm{Ori}(g, V^H) = \dim V^{\langle H, g^2\rangle} - \dim V^{\langle H, g\rangle} \pmod 2$.

For a subset $\mathcal{C} \subset \mathcal{A}(G)$, $x \in \mathrm{RO}(G)$ is called *orientation trivial on $\mathcal{C}$* if $\mathrm{Ori}(g, x^H) = 0$ for all $(H, g) \in \mathcal{C}$.

In the following, we always invoke the next hypothesis.

HYPOTHESIS 5.2. Let K be a gap subgroup of G of index 2.

Let U be a K-module and set $V = \mathrm{Ind}_K^G U$. If $H \subseteq K$ then we have

$$\begin{aligned}\mathrm{Res}_H^G V &= \bigoplus_{HgK \in H\backslash G/K} \mathrm{Ind}_{H\cap g^{-1}Kg}^H g_*\big(\mathrm{Res}_{K\cap gHg^{-1}}^K U\big) \\ &= \big(\mathrm{Res}_H^K U\big) \oplus g_*\big(\mathrm{Res}_{K\cap gHg^{-1}}^K U\big)\end{aligned}$$

where g is an arbitrary element in $G \setminus K$.

Lemma 5.3. *Let U be a gap K-module. Then for $V = \mathrm{Ind}_K^G U$ and $(H, P) \in \mathcal{HP}(K)$, the inequality* $\dim V^P > 2\dim V^H$ *holds.*

Proof. By the formula above, we get $\dim V^H = \dim U^H + \dim U^{gHg^{-1}}$ and $\dim V^P = \dim U^P + \dim U^{gPg^{-1}}$. Since U is a gap K-module, we have $\dim U^P > 2\dim U^H$ and $\dim U^{gPg^{-1}} > 2\dim U^{gHg^{-1}}$. These imply $\dim V^P > 2\dim V^H$. □

Lemma 5.4. *Let $V_0 - W_0 \in \mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}$ and U_0 a gap K-module. Then $V_1 = V_0 \oplus (\dim V_0 + 1)U_0$ and $W_1 = W_0 \oplus (\dim V_0 + 1)U_0$ are gap K-modules. For $V = \mathrm{Ind}_K^G V_1$, $W = \mathrm{Ind}_K^G W_1$ and $U = 2(\dim V_1 + 1)V(G)$, the real G-modules $V \oplus U$ and $W \oplus U$ fulfill the gap condition for any $(H, P) \in \mathcal{HP}(G)$ whenever $H \leq K$ or $(H, P) \notin \mathcal{HP}(G, 2)_0$.*

Proof. Let $(H, P) \in \mathcal{HP}(G)$. First observe the computation

$$\begin{aligned}\dim V_1^P - 2\dim V_1^H &= \dim V_0^P - 2\dim V_0^H + (\dim V_0 + 1)(\dim U_0^P - 2\dim U_0^H)\\ &\geq \dim V_0^P - 2\dim V_0^H + (\dim V_0 + 1)\\ &\geq (\dim V_0 + 1) - \dim V_0^H\\ &> 0.\end{aligned}$$

Thus V_1 is a gap K-module. Similarly, W_1 is a gap K-module. By Lemma 5.3 V and W fulfill the gap condition for the pair (H, P) whenever $H \leq K$.

Now assume $(H, P) \notin \mathcal{HP}(G, 2)_0$. By Lemma 2.1 (2), the inequality $\dim V(G)^P > 2\dim V(G)^H$ holds. Thus we get

$$\begin{aligned}\dim(V \oplus U)^P - 2\dim(V \oplus U)^H &= \dim\big(\mathrm{Ind}_K^G V_1\big)^P - 2\dim\big(\mathrm{Ind}_K^G V_1\big)^H\\ &\quad + 2(\dim V_1 + 1)(\dim V(G)^P - 2\dim V(G)^H)\\ &\geq 2(\dim V_1 + 1) - \dim\big(\mathrm{Ind}_K^G V_1\big)^H\\ &\geq 2(\dim V_1 + 1) - \dim \mathrm{Ind}_K^G V_1\\ &> 0.\end{aligned}$$

This shows that $V \oplus U$ fulfills the gap condition for the pair (H, P). Similarly, $W \oplus U$ fulfills the gap condition for the pair (H, P). □

To apply Morimoto's surgery result for $y \in \mathrm{Ind}_K^G\big(\mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}\big)$, we need to show that y is orientation trivial on the set

$$\widetilde{\mathcal{B}(G)}_2 := \{(H, g) \in \mathcal{B}(G) \mid \mathrm{Ord}(g) = 2^l \text{ for some } l \in \mathbb{N} \text{ and } H \not\subseteq K\}.$$

In Proposition 3.1, we checked the orientation triviality holds for the group $G = S_5 \times C_2$. In order to show that the orientation triviality holds for $G = S_5 \times X_2$ with $X_2 = C_2 \times \cdots \times C_2$ (n-fold) such that $n \geq 2$, we introduce the notation

$$\widetilde{\mathcal{B}(G)}_{2_{\mathrm{even}}} := \{(H, g) \in \widetilde{\mathcal{B}(G)}_2 \mid |H| = 2^k \text{ for some } k \in \mathbb{N}\}$$

and

$$\widetilde{\mathcal{B}(G)}_{2_{\mathrm{odd}}} := \widetilde{\mathcal{B}(G)}_2 \setminus \widetilde{\mathcal{B}(G)}_{2_{\mathrm{even}}}.$$

We can prove the following two lemmas without difficulties.

Lemma 5.5. *Let $G = S_5 \times X_2$ and $a = (\sigma, b) \in G$ with $\sigma \in S_5 \setminus A_5$ and $b \in X_2$. Then there exists an isomorphism $\varphi\colon G \to G$ such that*
(1) $\varphi(\sigma) = a$,

(2) $\varphi(x) = x$ *for all* $x \in A_5 \cup X_2$, *and*
(3) $\varphi \circ \varphi = \mathrm{id}_G$.

Lemma 5.6. *Let* $G = S_5 \times X_2$. *Then the implication*

$$\widetilde{\mathcal{B}(G)}_{2_{\text{even}}} \subset \bigcup_{\substack{Y \leq G \\ Y:\, 2\text{-group}}} \mathcal{A}(Y)$$

holds.

Then we have the next lemma.

Lemma 5.7. *The implication*

$$\widetilde{\mathcal{B}(G)}_{2_{\text{odd}}} \subset \bigcup_{\substack{T \leq G \\ T \cong S_5 \times C_2}} \mathcal{A}(T)$$

holds.

Proof. Let $(H, g) \in \widetilde{\mathcal{B}(G)}_{2_{\text{odd}}}$. By definition, we get $H \not\subseteq A_5 = G^{\{2\}}$ as well as $H \not\subseteq K$. It is easy to show the following.
(1) $|H| = 2p$ for $p = 3$ or 5.
(2) H has a unique (normal) Sylow p-subgroup $P = \langle u \rangle$ such that the order of u is p.
(3) P is a unique (normal) Sylow p-subgroup of $L = \langle H, g \rangle$ $(\subset G)$.
(4) $P \subset A_5$.

Since $p = 3$ or 5, H is isomorphic to C_{2p} or D_{2p}. Thus, we can take $a \in H \setminus A_5$ of order 2. Write

$$a = (\sigma, b)$$

and

$$g = (\tau, c)$$

with $\sigma, \tau \in S_5$ and $b, c \in X_2$. Since $H \not\subseteq K$, $\sigma \notin A_5$. In addition, since the order of g is a power of 2 by definition, the order is 2 or 4. There exists an isomorphism $\varphi \colon G \to G$ such that $\varphi(H) \subset S_5$ and $\varphi|_{X_2} = \mathrm{id}_{X_2}$. Then $\varphi(L) = \langle \varphi(H), \varphi(g) \rangle$ is a subgroup of $S_5 \times \langle c \rangle$. Thus (L, g) belongs to $\mathcal{A}(T)$ for some $T \leq G$ such that $T \cong S_5 \times C_2$. $\square$

Lemma 5.8. *Let* $G = S_5 \times X_2$ *and* $K = A_5 \times X_2$. *For an arbitrary element* $x \in \mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}$, $y = \mathrm{Ind}_K^G\, x$ *is orientation trivial on* $\widetilde{\mathcal{B}(G)}_2$.

Proof. By Lemmas 5.6 and 5.7, the implication

$$\widetilde{\mathcal{B}(G)}_2 \subset \bigcup_{\substack{T \leq G \\ T \cong S_5 \times C_2}} \mathcal{A}(T) \cup \bigcup_{\substack{Y \leq G \\ Y:\, \text{2-group}}} \mathcal{A}(Y)$$

holds. Clearly, y is orientation trivial on $\mathcal{A}(Y)$ because Y is a 2-group. In the proof of Proposition 3.1, we saw that for the basis element $y = V - W$ of $\mathrm{RO}(T)^{\mathcal{L}(T)}_{\mathcal{P}(T)}$, $V \oplus 2U$ and $W \oplus 2U$ satisfy the weak gap condition. Thus each element of $\mathrm{RO}(T)^{\mathcal{L}(T)}_{\mathcal{P}(T)}$ is orientation trivial on $\mathcal{A}(T)$. □

6. Completion of proofs of Theorems A and B

In this section, we proceed as follows. Firstly, we give proofs of Lemmas 1.4 and 1.6. Secondly, for $G = S_5 \times X_2$ and $A_5 \times X_2$, we compute the rank of the Smith set of G.

Proof of Lemma 1.4. Let $K = A_5 \times X_2$. Since A_5 is a simple group, it follows that $K^{\{2\}} = A_5$ and $K^{\{p\}} = K$ $(p \neq 2)$. Thus $K^{\mathrm{nil}} = A_5$, and $K/K^{\mathrm{nil}} \cong X_2$. Clearly K contains no elements of 8. K is an Oliver group, because K is non-solvable. Clearly $\mathcal{P}(K) \cap \mathcal{L}(K) = \emptyset$. Since $A_5 \times C_2$ is a gap group (see the proof of Proposition 3.1), by [17, Theorem 0.4], it follows that K is a gap group. □

Proof of Lemma 1.6. For arbitrary $x \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$, there exists an element $y \in \mathrm{RO}(K)^{\mathcal{L}(K)}_{\mathcal{P}(K)}$ such that $x = \mathrm{Ind}_K^G y$. Let $y = V_0 - W_0$ such that V_0 and W_0 are $\mathcal{L}(K)$-free real K-modules, and U_0 $\mathcal{L}(K)$-free gap K-module. Then $V_1 = V_0 \oplus (\dim V_0 + 1)U_0$ and $W_1 = W_0 \oplus (\dim V_0 + 1)U_0$ are $\mathcal{L}(K)$-free gap K-modules. Set $V = \mathrm{Ind}_K^G V_1$, $W = \mathrm{Ind}_K^G W_1$ and $U = \max\{6, 2(\dim V_1 + 1)\}V(G)$.

For subgroups H, K of G and a real G-module X,

$$\begin{aligned}\mathrm{Res}^G_H\big(\mathrm{Ind}^G_K X\big) &= \bigoplus_{HgK \in H\backslash G/K} \mathrm{Ind}^H_{H\cap gKg^{-1}}\big(g_* \mathrm{Res}^K_{K\cap g^{-1}Hg} X\big) \\ &= \begin{cases} \mathrm{Res}^K_H X \oplus g_* \mathrm{Res}^K_{g^{-1}Hg} X & \text{if} \quad H \leq K \quad (\text{here } g \in S_5 \setminus A_5), \\ \mathrm{Ind}^H_{H\cap K}\big(\mathrm{Res}^K_{H\cap K} X\big) & \text{if} \quad H \not\leq K. \end{cases}\end{aligned}$$

Hence

$$\dim\big(\mathrm{Ind}^G_K X\big)^H = \begin{cases} \dim X^H + \dim X^{g^{-1}Hg} & \text{if} \quad H \leq K \quad (\text{here } g \in S_5 \setminus A_5), \\ \dim X^{H\cap K} & \text{if} \quad H \not\leq K. \end{cases}$$

Let $(H, P) \in \mathcal{HP}(G, 2)$.

CASE $H \leq K$. By Lemma 5.4, $V \oplus U$ and $W \oplus U$ satisfy the gap condition for (H, P).

CASE $P \not\leq K$. We obtain

$$\dim V^P - 2\dim V^H = \dim V_1^{P\cap K} - 2\dim V_1^{H\cap K}.$$

Note $[H \cap K : P \cap K] = 2$, because $[P : P \cap K] = 2$ and $[H : H \cap K] = 2$. Thus

$$\dim V_1^{P\cap K} - 2\dim V_1^{H\cap K} > 0.$$

By Lemma 2.1 (1), $\dim U^P \geq 2\dim U^H$. Thus $V \oplus U$ satisfies the gap condition for (H, P). Similarly $W \oplus U$ satisfies the gap condition for (H, P).

CASE $P \leq K$, $H \not\leq K$. For an element $g \in H \setminus P$, we obtain

$$\dim V^P - 2\dim V^H = \dim V_1^P + \dim V_1^{g^{-1}Pg} - 2\dim V_1^{H\cap K}.$$

Since $P \triangleleft H$ and $H \cap K = P$, it follows that

$$\begin{aligned}\dim V_1^P + \dim V_1^{g^{-1}Pg} - 2\dim V_1^{H\cap K} &= 2\dim V_1^P - 2\dim V_1^P \\ &= 0.\end{aligned}$$

By Lemma 5.8, $V - W$ is orientation trivial on $\widetilde{\mathcal{B}(G)}_2$. Thus $V \oplus U$ satisfies (WG6). By [8, Corollary 3.5], $6V(G)$ satisfies (WG1)–(WG6). Hence $V \oplus U$ satisfies (WG1), (WG2), (WG4), (WG5). By [12, Theorem 2.5], $V \oplus U$ satisfies (WG3). Similarly $W \oplus U$ satisfies the weak gap condition. □

Let H be a normal subgroup of G. We denote by $b_{G,H}$ the number of real conjugacy classes $(gH)^{\pm}$ in G/H of cosets gH containing elements of G not of prime power order.

Lemma 6.1. *If $G^{\mathrm{nil}} = G^{\{p\}}$ for some prime p, then*

$$\mathrm{Rank}_{\mathbb{Z}}\big(\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}\big) = a_G - b_{G,G^{\mathrm{nil}}}.$$

Proof. By [21, p. 858, Subgroup Lemma], we have

$$\mathrm{RO}(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)} \subseteq \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subseteq \mathrm{RO}(G)^{\{G^{\{p\}}\}}_{\mathcal{P}(G)}.$$

Since $G^{\mathrm{nil}} = G^{\{p\}}$, it follows $\mathrm{RO}(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. By [21, p. 856, Second Rank Lemma],

$$\begin{aligned}\mathrm{Rank}_{\mathbb{Z}}\big(\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}\big) &= \mathrm{Rank}_{\mathbb{Z}}\big(\mathrm{RO}(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}\big) \\ &= a_G - b_{G,G^{\mathrm{nil}}}.\end{aligned}$$

□

Proposition 6.2. *Let $G = S_5 \times X_2$ and $K = A_5 \times X_2$ where $X_2 = C_2 \times \cdots \times C_2$ (n-folds). Then the following hold.*
(1) $a_G = 1 + 3(2^n - 1)$ *and* $b_{G,G^{\mathrm{nil}}} = 2^{n+1} - 1$.
(2) $a_K = 3(2^n - 1)$ *and* $b_{K,K^{\mathrm{nil}}} = 2^n - 1$.

The proof is straightforward.

ACKNOWLEDGMENT. I would like to express my deep and sincere gratitude to my supervisor, Professor M. Morimoto, for his detailed and constructive comments, and for his important support throughout this paper.

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