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ON TUNNEL NUMBER ONE LINKS WITH SURGERIES YIELDING THE 3-SPHERE

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Abstract

Gordon and Luecke showed that knots are determined by their complements. Therefore a non-trivial Dehn surgery on a non-trivial knot does not yield the 3-sphere. But the situation for links is different from that for knots. Berge constructed some examples of Dehn surgeries of 2-component links yielding the 3-sphere with interesting properties. By extending Berge's example, we construct infinitely many examples of tunnel number one links in the 3-sphere, such that their components are non-trivial, and that non-trivial Dehn surgeries on them yield the 3-sphere.

1. Introduction and results

Let S^3 be the 3-sphere. By a Dehn surgery or Dehn filling yielding S^3 and a Heegaard diagram for S^3, we mean a Dehn surgery or Dehn filling yielding a 3-manifold homeomorphic to S^3 and a Heegaard diagram for a 3-manifold homeomorphic to S^3 respectively throughout this paper.

Gordon and Luecke [5] showed that knots are determined by their complements. In other words a non-trivial Dehn surgery on a non-trivial knot in S^3 does not yield S^3. But the situation for links is different from that for knots. In fact, there is a link in S^3 which admits a non-trivial Dehn surgery yielding S^3. Here a non-trivial Dehn surgery means a Dehn surgery along a non-meridional slopes. If a link has a trivial component or has a non-separating essential annulus in its exterior, we can easily see that the link admits infinitely many such surgeries. These are called trivial examples. Non-trivial examples of links with such surgeries have been constructed. Berge [1] gave some examples of tunnel number one links. Kawauchi [8], [9] showed that we can construct infinitely many examples of hyperbolic links of any number of components by using imitation theory. Teragaito [13] gave an example of an n-component link of which tunnel number is $n-1$ for any $n \geq 2$. Classes of links without such surgeries are also known. See for example [10].

Let L be a knot or link in a closed, orientable 3-manifold N, and let M be the exterior of L in N. L is a *tunnel number one link* if M is homeomorphic to a handle-

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body H of genus two with a single 2-handle attached to H along a simple closed curve C in ∂H. Note that L is a knot if and only if C is a non-separating curve in ∂H, and L is a two component link if and only if C is a separating curve in ∂H.

Berge [1] determined whether such M can be embedded in S^3 or not, and found all such embeddings if they exist. He showed that if L is a tunnel number one link in S^3, whose exterior $E(L)$ does not contain any non-separating essential annulus and any Dehn filling on one of the boundary components of $E(L)$ does not yield the solid torus, then L has at most five non-trivial Dehn surgeries yielding S^3. Also, he described a Heegaard diagram for the exterior of a link with five non-trivial Dehn surgeries yielding S^3.

Let M be a 3-manifold whose boundary components are k tori T_i $(i = 1, \dots, k)$, Λ be a subset of $\{1, \dots, k\}$ and m_j, m'_j $(j \in \Lambda)$ be essential simple closed curves in T_j. The Dehn filling of M along $\bigcup_{j\in\Lambda} m_j$ is said to be equivalent to that of M along $\bigcup_{j\in\Lambda} m'_j$ if $\bigcup_{j\in\Lambda} m_j$ is isotopic to $\bigcup_{j\in\Lambda} m'_j$ in ∂M. Two Dehn surgeries of a link L in S^3 are said to be equivalent if their corresponding Dehn fillings of the exterior $E(L)$ of L are equivalent.

Theorem 1.1. *There is an infinite family of mutually distinct tunnel number one links $\{L_n\}_{n=1}^{\infty}$ in S^3 such that each L_n has exactly five non-trivial Dehn surgeries yielding S^3 up to equivalence.*

REMARK 1.1. Since L_n has only finite non-trivial Dehn surgeries yielding S^3, L_n has the following properties.
(1) L_n has no trivial component.
(2) The exterior $E(L_n)$ of L_n does not contain any non-separating essential annulus.
(3) Any Dehn filling on one of the boundary components of $E(L_n)$ does not yield the solid torus.
Because L_n must have infinitely many non-trivial Dehn surgeries yielding S^3 up to equivalence if one of (1), (2) and (3) does not hold. All tunnel number one links whose exteriors contain a non-separating essential annulus are determined by [3], and all these links have a trivial component.

Theorem 1.2. *There is an infinite family of mutually distinct pairs of tunnel number one links $\{L_n, L'_n\}_{n=1}^{\infty}$ in S^3 with the following properties.*
(1) *L_n has no trivial component.*
(2) *L'_n has a trivial component.*
(3) *$E(L_n)$ is homeomorphic to $E(L'_n)$.*

REMARK 1.2. Berge [1] gave an example of a pair of distinct hyperbolic links without trivial component whose exteriors are homeomorphic to each other. The examples of Theorem 1.2 are entirely different from Berge's one.

2. Basic facts and notions for proofs of Theorems

In this section we will prepare some basic facts and notations for proofs of Theorems 1.1 and 1.2.

Following [6] we will recall Heegaard diagrams for 3-manifolds. Let H_n be a handlebody of genus n. A set of simple closed curves $u_1, u_2, \ldots, u_n \subset \partial H_n$ is a *meridian system of* H_n, if there are disks $D_1, D_2, \ldots, D_n \subset H_n$ such that $D_i \cap \partial H = \partial D_i = u_i$ (for each $i \in \{1, 2, \ldots, n\}$), $D_i \cap D_j = \emptyset$ (if $i \neq j$), and $\mathrm{Cl}\bigl(H_n - \mathrm{N}\bigl(\bigcup_{i=1}^n D_i\bigr)\bigr)$ is homeomorphic to the 3-ball B^3. Here $\mathrm{Cl}(\cdot)$ means the closure and $\mathrm{N}(\cdot)$ means regular neighborhood.

Let $H \cup_F H'$ be Heegaard splitting of a closed orientable 3-manifold M and $\{u_1, u_2, \ldots, u_n\}$ (resp. $\{u'_1, u'_2, \ldots, u'_n\}$) be a meridian system of H (resp. H'). We call $D = (F; \{u_1, u_2, \ldots, u_n\}, \{u'_1, u'_2, \ldots, u'_n\})$ a *Heegaard diagram* of genus n. This definition is extended to the definition of Heegaard diagrams for non-closed compact orientable 3-manifolds (H, H' are compression bodies), by choosing collections of core curves of 2-handles for each of compression bodies H, H'. A Heegaard diagram $D = (F; \{u_1, u_2, \ldots, u_n\}, \{u'_1, u'_2, \ldots, u'_n\})$ is said to be *normalized* if $\bigl(\bigcup_{i=1}^n u_i\bigr) \cap \bigl(\bigcup_{j=1}^n u'_j\bigr)$ contains no isotopically removable point. When $\bigl(\bigcup_{i=1}^n u_i\bigr) \cup \bigl(\bigcup_{j=1}^n u'_j\bigr)$ is connected, D is called a *connected diagram*. For normalized Heegaard diagram $D = (F; \{u_1, u_2, \ldots, u_n\}, \{u'_1, u'_2, \ldots, u'_n\})$, a simple arc w in F is called a *wave* if w satisfies the following conditions:

(1) there is a meridian $u \in \{u_1, u_2, \ldots, u_n\} \cup \{u'_1, u'_2, \ldots, u'_n\}$ satisfying $w \cap \Bigl(\bigl(\bigcup_{i=1}^n u_i\bigr) \cup \bigl(\bigcup_{j=1}^n u'_j\bigr)\Bigr) = w \cap u = \partial w$,

(2) a small neighborhood $\mathrm{N}(\partial w; w)$ of ∂w in w is the same side of u, that is, the closure of one component of $\mathrm{N}(u) - u$ contains $\mathrm{N}(\partial w; w)$,

(3) each component of $u - \partial w$ intersects $\{u_1, u_2, \ldots, u_n\} \cup \{u'_1, u'_2, \ldots, u'_n\} - \{u\}$.

The wave w is said to be *associated with* u specifying the meridian which w attaches. Note that any non-connected, normalized Heegaard diagram of genus two for S^3 is the standard one $D_0 = (F; \{u_1, u_2\}, \{u'_1, u'_2\})$, where D_0 is normalized and satisfies $u_i \cap u'_j = \{\text{a point}\}$ if $i = j$ and $u_i \cap u'_j = \emptyset$ if $i \neq j$ for $i, j \in \{1, 2\}$.

Theorem 2.1 (Homma, Ochiai and Takahashi [6]). *Any connected normalized Heegaard diagram of genus two for S^3 has a wave.*

For a survey of the proof, see [4].

Birman and Hilden [2] showed that every 3-manifold with a Heegaard splitting of genus two is two-sheeted cyclic branched cover of S^3 branched over a knot or link in S^3, see Takahashi [12] for alternative proof. By a solution of the Smith conjecture [11], we obtain the following well known theorem.

Theorem 2.2. *Let N be a closed, connected, simply connected* 3*-manifold with a Heegaard splitting of genus two. Then N is homeomorphic to S^3.*

By loop theorem and Schoenflies theorem, we obtain the following well known theorem.

Theorem 2.3. *Let M be a* 3*-manifold homeomorphic to the exterior of a knot. Then M is homeomorphic to the solid torus if and only if the fundamental group $\pi_1(M)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$.*

The following is the Dehn filling version theorem of Gordon–Luecke [5].

Theorem 2.4. *Let M be a* 3*-manifold homeomorphic to the exterior of a non-trivial knot in S^3. Then the Dehn filling of M yielding S^3 is unique up to equivalence.*

3. Proofs of Theorems

In this section, we will prove Theorems 1.1 and 1.2 by using Heegaard diagrams.

3.1. Definitions, Key Lemma and Basic Lemma. Let M be a handlebody H of genus two with a single 2-handle attached to H along a separating simple closed curve C in ∂H, and $\{u_1, u_2\}$ be a meridian system of H. Then $D = (\partial H; \{u_1, u_2\}, C)$ is a Heegaard diagram for M. Some definitions in the section 2 for a Heegaard diagram for a closed orientable 3-manifold can be extended to that for such a Heegaard diagram D. The Heegaard diagram $D = (\partial H; \{u_1, u_2\}, C)$ is said to be *normalized* if $(u_2 \cup u_2) \cap C$ contains no isotopically removable point. For normalized Heegaard diagram $D = (\partial H; \{u_1, u_2\}, C)$, a simple arc w in ∂H is called a *wave associated with C* if w satisfies the following conditions:
(1) $w \cap (u_1 \cup u_2 \cup C) = w \cap C = \partial w$,
(2) a small neighborhood $\mathrm{N}(\partial w; w)$ of ∂w in w is the same side of C, that is, the closure of one component of $\mathrm{N}(C) - C$ contains $\mathrm{N}(\partial w; w)$,
(3) each component of $C - \partial w$ intersects $u_1 \cup u_2$.

For Heegaard diagram $D = (\partial H; \{u_1, u_2\}, C)$ of genus two, by cutting ∂H open along u_1 and u_2, we obtain the 2-sphere with four disks (we name these A, a, B and b, where disks A, a are obtained by cutting ∂H open along u_1 and disks B, b are obtained by cutting ∂H open along u_2). Throughout this section, we consider such diagrams.

Key Lemma 3.1.1. *Let M be a handlebody H of genus two with a single* 2*-handle attached to H along a separating simple closed curve C in ∂H. Let $D = (\partial H; \{u_1, u_2\}, C)$ be a Heegaard diagram for M where $\{u_1, u_2\}$ is a meridian system of H. Suppose that D is of the type as shown in* Fig. 1 *below, where each arc represents a family of arcs parallel to it, the labels c, d, e, f for arcs indicate the numbers of arcs in each family respectively, and w_{ix} $(i \in \{1, 2, 3\}$, $x \in \{l, r\})$ is a wave associated with C in D. Let m_{ix} $(i \in \{1, 2, 3\}$, $x \in \{l, r\})$ be the simple closed curves $w_{ix} \cup \alpha_{ix}$ where α_{ix} is a component of $C - w_{ix}$. If c, d, e, $f \geq 1$, then for any Dehn filling of M yielding S^3 (if it exists),*

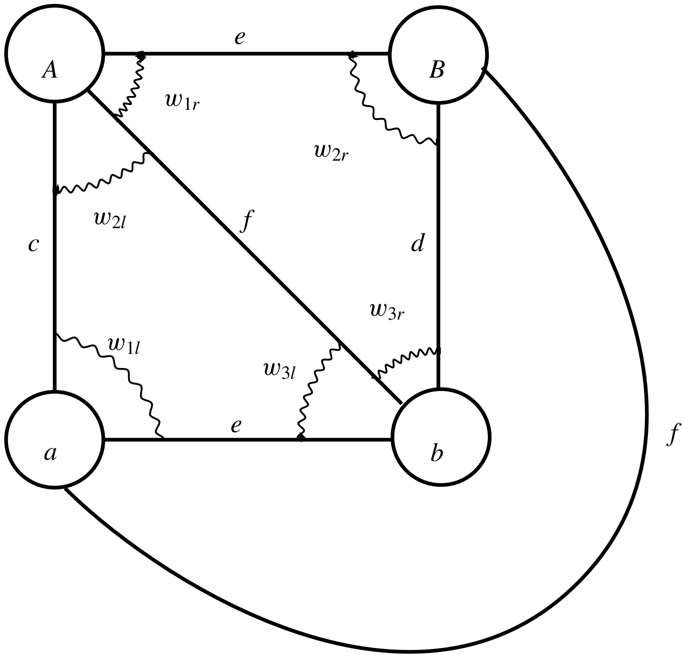


Fig. 1.

one of the two simple closed curves m_l, m_r *in* $\partial H - C$ *corresponding to this Dehn filling coincides with one of* $m_{1l}, m_{2l}, m_{3l}, m_{1r}, m_{2r}$ *and* m_{3r} *up to isotopy on the closure of one component of* $\partial H - C$.

Proof. Our proof is based on the idea of Berge [1]. We may assume that (1) $(m_l \cup m_r) \cap (u_1 \cup u_2)$ contains no isotopically removable point by isotopy keeping $(m_l \cup m_r) \cap C = \emptyset$. The triplet $D' = (\partial H; \{u_1, u_2\}, \{m_l, m_r\})$ is a Heegaard diagram for S^3. Suppose that D' is the standard Heegaard diagram for S^3. Then, without loss of generality, we may assume (2) $u_1 \cap m_l = \{\text{a point}\}$, $u_2 \cap m_r = \{\text{a point}\}$, $u_1 \cap m_r = \emptyset$, $u_2 \cap m_l = \emptyset$. By (2), $(m_l \cup m_r) \cap C = \emptyset$ and Fig. 1, C must contain c simple closed curves parallel to m_l and d simple closed curves parallel to m_r. This is contradicts connectivity of C because of $c + d \geq 2$. Therefore D' is not the standard Heegaard diagram for S^3 and so D' is connected. Then, by Theorem 2.1, D' has a wave w associated with m_x ($x = l$ or r) or u_j ($j = 1$ or 2). We may assume that (3) $w \cap C$ has no isotopically removable point by isotopy if necessary.

CASE 1. Suppose that w is a wave associated with m_x ($x = l$ or r). Then we have $w \cap C \neq \emptyset$ because, if $w \cap C = \emptyset$, w must be an arc as shown in Fig. 2 of D' obtained by cutting ∂H along m_l and m_r, where one of the two components of $m_x - \partial w$ does not intersect $u_1 \cup u_2$, and so is not a wave associated with m_x ($x = l$ or r). By $\partial w \subset m_x \subset \partial H - C$, ∂w is contained in one of two components of $\partial H - C$

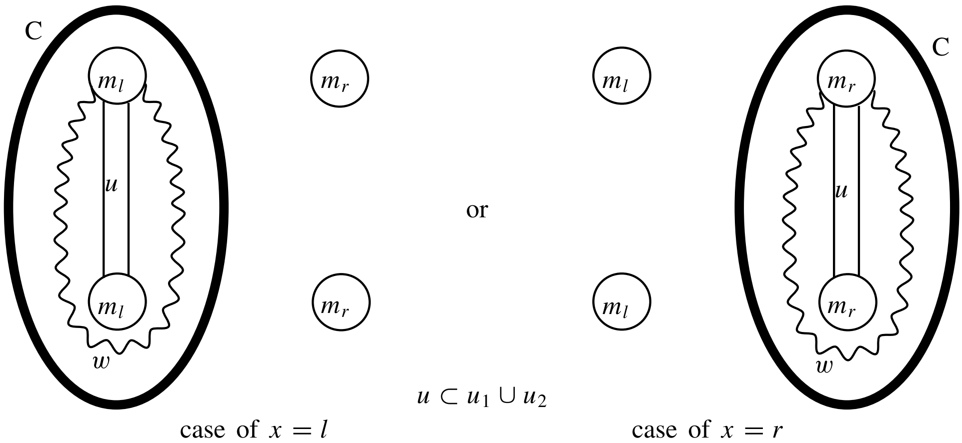


Fig. 2.

and so, by $w \cap C \neq \emptyset$, w contains a subarc w_C such that (4) $w_C \cap C = \partial w_C$, $w_C \cap (u_1 \cup u_2 \cup m_l \cup m_r) = \emptyset$, and $w_C - \partial w_C \subset F_y'$ where F_y' is the component of $\partial H - C$ containing m_y ($y \neq x$, $y = l$ or r). Then, by (3) and (4), w_C is a wave associated with C. Let β_{w_C} be any one of the two components β_1, β_2 of $C - \partial w_C$ and F_y be the closure of F_y'. (Note that two simple closed curves $w_C \cup \beta_1$ and $w_C \cup \beta_2$ are isotopic in F_y.) Then by (3), $w_C \cup \beta_{w_C}$ is an essential simple closed curve in F_y and so by $(w_C \cup \beta_{w_C}) \cap m_y \subset (w \cup C) \cap m_y = (w \cap m_y) \cup (C \cap m_y) = \emptyset$, $w_C \cup \beta_{w_C}$ is isotopic to m_y in F_y. By (3), w_C is isotopic (in F_y) to one wave w_{iy} ($i \in \{1, 2, 3\}$) keeping $(w_C - \partial w_C) \cap C = \emptyset$ and ∂w_C in C, and so $w_C \cup \beta_{w_C}$ is isotopic to m_{iy} in ∂H. Therefore, we have $m_y = m_{iy}$ up to isotopy in F_y.

CASE 2. Suppose that w is a wave associated with u_j ($j = 1$ or 2). We may consider Fig. 1 a graph in a 2-sphere Σ. For a wave w in Σ, there is a simple closed curve $u \in \{\partial A, \partial a, \partial B, \partial b\}$ satisfying (5) $w \cap (\partial A \cup \partial a \cup \partial B \cup \partial b) = w \cap u = \partial w$. Let $\tilde{u}$ be a component of $u - \partial w$. Then the simple closed curve $w \cup \tilde{u}$ in Σ separates $u' \cup u'' \cup u'''$ into $u' \cup u''$ and u''' where $\{u, u'\} = \{\partial A, \partial a\}$ or $\{\partial B, \partial b\}$, and $\{u'', u'''\} = \{\partial A, \partial a, \partial B, \partial b\} - \{u, u'\}$. Since the number of subarcs of C in Fig. 1 connecting $u' \cup u''$ and u''' is one of the integers $d + f$, $d + e$, $c + f$ and $c + e$, w intersects C at more than two points because of $d + f, d + e, c + f, c + e \geq 2$. Then w contains a subarc w_C such that (6) $w_C \cap C = \partial w_C$ and $w_C \cap (u_1 \cup u_2 \cup m_l \cup m_r) = \emptyset$. By (3) and (6), w_C is a wave associated with C. Choose m_y ($y = l$ or r) such that both m_y and $w_C - \partial w_C$ are contained in the same component F_y' of $\partial H - C$. Let F_y be the closure of F_y'. By (3) and (6), w_C is isotopic (in F_y) to one wave w_{iy} ($i \in \{1, 2, 3\}$) keeping $(w_C - \partial w_C) \cap C = \emptyset$ and ∂w_C in C. By the same argument in Case 1, we have $m_y = m_{iy}$ up to isotopy in F_y. $\square$

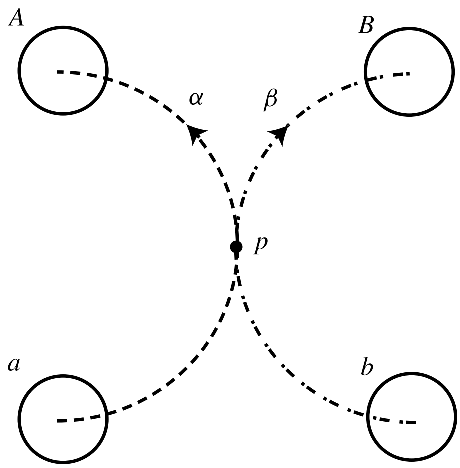


Fig. 3.

Let α and β be p-based closed curves in $H - \partial H$ as shown in Fig. 3, where p is base point. Let $[\alpha]_p$ (resp. $[\beta]_p$) be the p-base homotopy class of α (resp. β) in H. The fundamental group $\pi_1(H)$ $(= \pi_1(H, p))$ of H is the free group generated by $[\alpha]_p$ and $[\beta]_p$. Put $A = [\alpha]_p$, $B = [\beta]_p$ and $a = A^{-1}$, $b = B^{-1}$ in order to get a word expression for an element of $\pi_1(H)$ easily. For any (oriented) closed curve γ in H, its free homotopy class $[\gamma]$ in H contains a p-based closed curve $\gamma_p \in [\gamma]$. And $[\gamma_p]_p$ $(\in \pi_1(H) = \langle A, B\rangle)$ is represented by a word $W(\gamma)$ defined for a closed curve γ is unique up to conjugation in $\pi_1(H) = \langle A, B\rangle$. If an oriented simple closed curve γ in ∂H has finite transversal intersections with $u_1 \cup u_2$ and a starting point $q \in \gamma - (u_1 \cup u_2)$ is given, then we can obtain a word $W(\gamma)$ uniquely by reading the intersection $\gamma \cap (u_1 \cup u_2)$ along γ starting from q. This is a well-known algorithm to get $W(\gamma)$.

Let W_i $(i = 1, \dots, m)$ be a word in the alphabets A_j $(j = 1, \dots, n)$ and e be the unit element of the free group $\langle A_1, \dots, A_n\rangle$. Let $\mathrm{N}(W_1, \dots, W_m)$ be the smallest normal subgroup of $\langle A_1, \dots, A_n\rangle$ containing $\{W_1, \dots, W_m\}$. The factor group $\langle A_1, \dots, A_n\rangle/\mathrm{N}(W_1, \dots, W_m)$ is denoted by $\langle A_1 \cdots A_n \mid W_1 = e, \dots, W_m = e\rangle$. Note that $\langle A_1 \cdots A_n \mid W_1 = e, \dots, W_m = e\rangle \equiv \langle A_1 \cdots A_n \mid W_1' = e, \dots, W_m' = e\rangle$ holds if W_i is conjugate to W_i' in $\langle A_1, \dots, A_n\rangle$ for each $i = 1, \dots, m$. For two groups G_1 and G_2, $G_1 \equiv G_2$ means that G_1 is isomorphic to G_2. By van Kampen's theorem, the next lemma holds.

Basic Lemma 3.1.2. *Let M and C be the same ones in* Key Lemma 3.1.1. *Let m_l (resp. m_r) be an essential simple closed curve in one component (resp. the other component) of $\partial H - C$. Let $M(m_l)$ (resp. $M(m_r)$) be a* 3*-manifold obtained by Dehn filling of M along m_l (resp. m_r) as a meridian and $M(m_l, m_r)$ be the one along m_l and m_r as meridians. Then the followings hold.*

(1) $\pi_1(M) \equiv \langle A, B \mid W(C) = e \rangle$.
(2) $\pi_1(M(m_l)) \equiv \langle A, B \mid W(m_l) = e \rangle$, $\pi_1(M(m_r)) \equiv \langle A, B \mid W(m_r) = e \rangle$.
(3) $\pi_1(M(m_l, m_r)) \equiv \langle A, B \mid W(m_l) = e, W(m_r) = e \rangle$.

REMARK 3.1. Since

$$\begin{aligned} W_1^{-1} W(m_l) W_1 W_2^{-1} W(m_l)^{-1} W_2 &= W(C) \\ = W_1'^{-1} W(m_r) W_1' W_2'^{-1} W(m_r)^{-1} W_2' \end{aligned}$$

holds for certain words $W_1, W_2, W_1', W_2' \in \langle A, B \rangle = \pi_1(H)$, $W(m_l) = e$ (resp. $W(m_r) = e$) implies $W(C) = e$ and,

$$\mathrm{N}(W(C), W(m_l)) = \mathrm{N}(W(m_l)), \quad \mathrm{N}(W(C), W(m_r)) = \mathrm{N}(W(m_r))$$

and

$$\mathrm{N}(W(C), W(m_l), W(m_r)) = \mathrm{N}(W(m_l), W(m_r))$$

hold. And so,

$$\begin{aligned} \langle A, B \mid W(C) = e, W(m_l) = e \rangle &\equiv \langle A, B \mid W(m_l) = e \rangle, \\ \langle A, B \mid W(C) = e, W(m_r) = e \rangle &\equiv \langle A, B \mid W(m_r) = e \rangle \end{aligned}$$

and

$$\langle A, B \mid W(C) = e, W(m_l) = e, W(m_r) = e \rangle \equiv \langle A, B \mid W(m_l) = e, W(m_r) = e \rangle$$

hold.

3.2. Proof of Theorem 1.1. Let D_n be a Heegaard diagram $(\partial H; \{u_1, u_2\}, C_n)$ shown by Fig. 4 below where n is a positive integer. Note that D_n is a special case of D in Key Lemma 3.1.1. Throughout this subsection, we assume $D = D_n$, M_n, C_n mean M, C in Key Lemma 3.1.1 respectively and w_{ix}, m_{ix} $(i \in \{1, 2, 3\}, x \in \{l, r\})$ mean the ones in Key Lemma 3.1.1 in the case of $D = D_n$ respectively.

REMARK 3.2. A Heegaard diagram D_1 is an example given by Berge [1].

Lemma 3.2.1. *If C_n, m_{1l}, m_{2l}, m_{3l}, m_{1r}, m_{2r} and m_{3r} are oriented as shown in* Fig. 4 *respectively, then there exists a starting point on each of these simple closed curves respectively such that the following equations hold.*
(1)

$$\begin{aligned} W(C_n) = {} & B(abAB)^{(n-1)} a (BAba)^{(n-1)} BA (BabA)^{(n-1)} BB (AbaB)^{(n-1)} AA (BabA)^{(n-1)} \\ & \times B (AbaB)^{(n-1)} Ab (ABab)^{(n-1)} A (baBA)^{(n-1)} ba (bABa)^{(n-1)} bb (aBAb)^{(n-1)} \\ & \times aa (bABa)^{(n-1)} b (aBAb)^{(n-1)} a. \end{aligned}$$

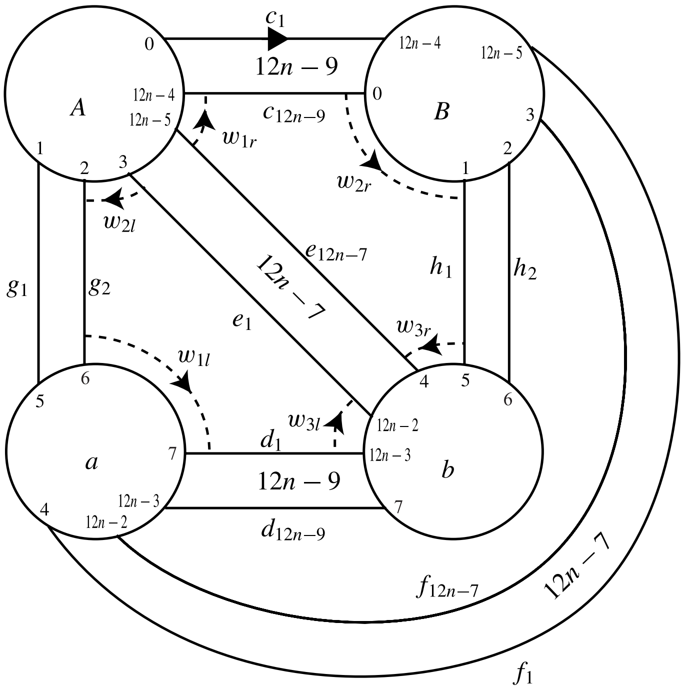


Fig. 4.

(2) $W(m_{1l}) = b(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABa)^{(n-1)}bb(aBAb)^{(n-1)}a.$
(3) $W(m_{2l}) = a(bABa)^{(n-1)}b(aBAb)^{(n-1)}aB(abAB)^{(n-1)}a(BAba)^{(n-1)}B.$
(4) $W(m_{3l}) = A(BabA)^{(n-1)}BB(AbaB)^{(n-1)}AA(BabA)^{(n-1)}B(AbaB)^{(n-1)}A.$
(5) $W(m_{1r}) = (BAba)^{(n-1)}BA(BabA)^{(n-1)}BB(AbaB)^{(n-1)}AA(BabA)^{(n-1)}B.$
(6) $W(m_{2r}) = b(aBAb)^{(n-1)}aa(bABa)^{(n-1)}b(aBAb)^{(n-1)}aB(abAB)^{(n-1)}a.$
(7) $W(m_{3r}) = (AbaB)^{(n-1)}Ab(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABa)^{(n-1)}b.$

Proof. Let c_i, d_i, e_i, f_i, g_i and h_i be subarcs of C_n respectively as shown in Fig. 4. Then C_n can be represented by connecting theses subarcs as

$$c_1 \prod_{i=0}^{n-2}(d_{12i+2}e_{12i+6}c_{12i+8}f_{12i+12})d_{12n-10}\prod_{i=0}^{n-2}(c_{12(n-i)-9}e_{12(n-i)-11}d_{12(n-i)-15}f_{12(n-i)-17}),$$

$$c_3e_1 \prod_{i=0}^{n-2}(f_{12i+2}d_{12i+4}e_{12i+8}c_{12i+10})f_{12n-10}h_2\prod_{i=0}^{n-2}(e_{12(n-i)-9}d_{12(n-i)-13}f_{12(n-i)-15}c_{12(n-i)-19}),$$

$$e_3g_1 \prod_{i=0}^{n-2}(f_{12i+4}d_{12i+6}e_{12i+10}c_{12i+12})f_{12n-8}\prod_{i=0}^{n-2}(e_{12(n-i)-7}d_{12(n-i)-11}f_{12(n-i)-13}c_{12(n-i)-17}),$$

$$e_5 d_1 \prod_{i=0}^{n-2}(c_{12i+2} f_{12i+6} d_{12i+8} e_{12i+12}) c_{12n-10} \prod_{i=0}^{n-2}(d_{12(n-i)-9} f_{12(n-i)-11} c_{12(n-i)-15} e_{12(n-i)-17}),$$

$$d_3 f_1 \prod_{i=0}^{n-2}(e_{12i+2} c_{12i+4} f_{12i+8} d_{12i+10}) e_{12n-10} h_1 \prod_{i=0}^{n-2}(f_{12(n-i)-9} c_{12(n-i)-13} e_{12(n-i)-15} d_{12(n-i)-19}),$$

$$f_3 g_2 \prod_{i=0}^{n-2}(e_{12i+4} c_{12i+6} f_{12i+10} d_{12i+12}) e_{12n-8} \prod_{i=0}^{n-2}(f_{12(n-i)-7} c_{12(n-i)-11} e_{12(n-i)-13} d_{12(n-i)-17}) f_5.$$

Take a starting point on $c_1 - \partial c_1$ for C_n and a starting point on $w_{ix} - \partial w_{ix}$ ($i \in \{1,2,3\}$, $x \in \{l, r\}$) for m_{ix} respectively. $\square$

For two words $W, W' \in \langle A, B\rangle = \pi_1(H)$, $W \equiv W'$ means that W is conjugate to W' in $\langle A, B\rangle = \pi_1(H)$.

Lemma 3.2.2. *Let $\varphi\colon \langle A, B\rangle \to \langle A, B\rangle$ (resp. $\varphi'\colon \langle A, B\rangle \to \langle A, B\rangle$) be an isomorphism defined by $\varphi(A) = B$ and $\varphi(B) = A$ (resp. $\varphi'(A) = A$ and $\varphi'(B) = ab$) and $\varphi''\colon \langle A, B\rangle \to \langle A, B\rangle$ be the composed isomorphism $\varphi' \circ \varphi$ (and so $\varphi''(A) = ab$ and $\varphi''(B) = A$). Then the followings hold.*

(1) $\varphi(W(m_{1l})) \equiv W(m_{2r})$, $\varphi(W(m_{2l})) \equiv W(m_{3r})$, $\varphi(W(m_{3l})) \equiv W(m_{1r})$,
$\varphi(W(m_{2r})(W(m_{3r}))^2) \equiv W(m_{1l})(W(m_{2l}))^2$,
$\varphi(W(m_{1r})(W(m_{2r}))^2) \equiv W(m_{3l})(W(m_{1l}))^2$,
$\varphi(W(m_{3r})(W(m_{1r}))^2) \equiv W(m_{2l})(W(m_{3l}))^2$.

(2) $\varphi'(W(m_{1l})) \equiv W(m_{1r})$, $\varphi'(W(m_{2r})(W(m_{3r}))^2) \equiv W(m_{2l})(W(m_{3l}))^2$.

(3) $\varphi''(W(m_{1l})) \equiv W(m_{2l})$, $\varphi''(W(m_{2r})(W(m_{3r}))^2) \equiv W(m_{1r})(W(m_{2r}))^2$.
And so the followings hold.

(4) $\langle A, B \mid W(m_{ix}) = e\rangle \equiv \langle A, B \mid W(m_{1l}) = e\rangle$ *for any $i \in \{1,2,3\}$ and any $x \in \{l, r\}$.*

(5) *Each of five groups $\langle A, B \mid W(m_{2l}) = e,\ W(m_{1r})(W(m_{2r}))^2 = e\rangle$,
$\langle A, B \mid W(m_{3l}) = e, W(m_{3r})(W(m_{1r}))^2 = e\rangle$, $\langle A, B \mid W(m_{1r}) = e, W(m_{2l})(W(m_{3l}))^2 = e\rangle$,
$\langle A, B \mid W(m_{2r}) = e, W(m_{1l})(W(m_{2l}))^2 = e\rangle$, $\langle A, B \mid W(m_{3r}) = e, W(m_{3l})(W(m_{1l}))^2 = e\rangle$
is isomorphic to
$\langle A, B \mid W(m_{1l}) = e,\ W(m_{2r})(W(m_{3r}))^2 = e\rangle$.*

Proof. We define words $W_{1l}, W_{2l}, W_{3l}, W_{1r}, W_{2r}, W_{3r} \in \langle A, B\rangle$ as follows.

$$W_{1l} = b(ABab)^{(n-1)}A(baBA)^{(n-1)}b,$$

$$W_{2l} = a(bABa)^{(n-1)}b(aBAb)^{(n-1)}a,$$

$$W_{3l} = A(BabA)^{(n-1)}BB(AbaB)^{(n-1)}A,$$

$$W_{1r} = (BAba)^{(n-1)}BA(BabA)^{(n-1)}B,$$

$$W_{2r} = b(aBAb)^{(n-1)}aa(bABa)^{(n-1)}b,$$

$$W_{3r} = (AbaB)^{(n-1)}Ab(ABab)^{(n-1)}A.$$

Then we can check the followings.

(1)

$$\begin{aligned}
\varphi(W(m_{1l})) &= W_{2r}^{-1}W(m_{2r})W_{2r},\\
\varphi(W(m_{2l})) &= W_{3r}^{-1}W(m_{3r})W_{3r},\\
\varphi(W(m_{3l})) &= W_{1r}^{-1}W(m_{1r})W_{1r},\\
\varphi(W(m_{2r})(W(m_{3r}))^2) &= W_{1l}^{-1}W(m_{1l})(W(m_{2l}))^2W_{1l},\\
\varphi(W(m_{1r})(W(m_{2r}))^2) &= W_{3l}^{-1}W(m_{3l})(W(m_{1l}))^2W_{3l},\\
\varphi(W(m_{3r})(W(m_{1r}))^2) &= W_{2l}^{-1}W(m_{2l})(W(m_{3l}))^2W_{2l}.
\end{aligned}$$

(2)

$$\begin{aligned}
\varphi'(W(m_{1l})) &= W_{1r}^{-1}W(m_{1r})W_{1r},\\
\varphi'(W(m_{2r})(W(m_{3r}))^2) &= W_{2l}^{-1}W(m_{2l})(W(m_{3l}))^2W_{2l}.
\end{aligned}$$

(3)

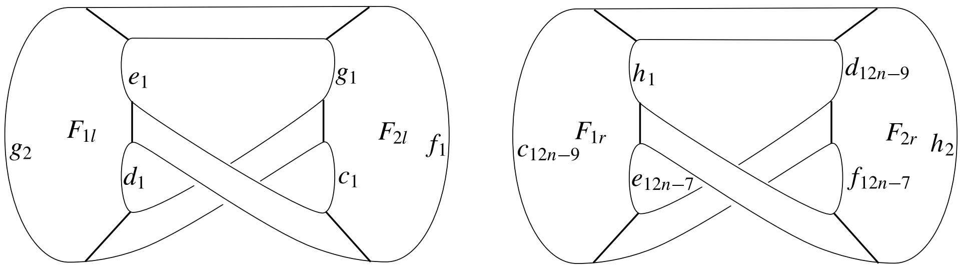
$$\begin{aligned}
\varphi''(W(m_{1l})) &= aW(m_{2l})A,\\
\varphi''(W(m_{2r})(W(m_{3r}))^2) &= aW(m_{1r})(W(m_{2r}))^2A.
\end{aligned}$$

□

Lemma 3.2.3. *If m_{ix} ($i \in \{1,2,3\}$, $x \in \{l,r\}$) is oriented as shown in* Fig. 4, *then the followings hold.*
1) *For each m_{ix} ($i \in \{1,2,3\}$, $x \in \{l,r\}$), there exists an oriented simple closed curve $\tilde{m}_{ix}$ in the component F'_x ($x \in \{l,r\}$) of $\partial H - C_n$ intersecting m_{ix} such that $W(\tilde{m}_{ix}) = W(m_{ix})$ and $\tilde{m}_{ix}$ is isotopic to m_{ix}.*
2) *For each word $W(m_{ix})(W(m_{jx}))^2$ ($(i,j) \in \{(1,2),(2,3),(3,1)\}$, $x \in \{l,r\}$), there exists an oriented simple closed curve $\tilde{m}_{ijjx}$ in the component of $\partial H - C_n$ intersecting $m_{ix} \cup m_{jx}$ such that $W(\tilde{m}_{ijjx}) = W(m_{ix})(W(m_{jx}))^2$.*

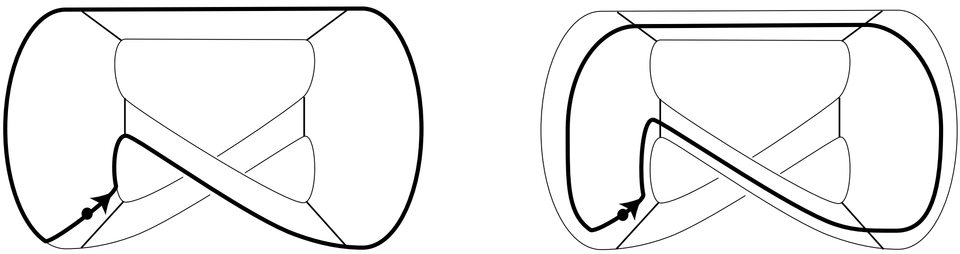
Proof. Let F_x be the closure of F'_x and F_{jx} ($j \in \{1,2,\dots,24n-15\}$) be the closure of each component of $F_x - (u_1 \cup u_2)$ such that $F_{1l} \supset e_1 \cup g_2 \cup d_1$, $F_{2l} \supset c_1 \cup f_1 \cup g_1$, $F_{1r} \supset c_{12n-9} \cup h_1 \cup e_{12n-7}$ and $F_{2r} \supset f_{12n-7} \cup d_{12n-9} \cup h_2$. Note that $F_{jx} \cap (u_1 \cup u_2)$ has three arc-components for $j \in \{1,2\}$, $x \in \{l,r\}$ or two arc-components for other case. Then $\bigcup_{j=3}^{24n-15} F_{jx}$ consists of three bands connecting F_{1x} and F_{2x} for each $x \in \{l,r\}$. By $F_x = \bigcup_{j=1}^{24n-15} F_{jx}$ and the subarc-expression of C_n in the proof of Lemma 3.2.1, F_x can be shown as in Fig. 5 up to homeomorphism. Note that i) $\partial F_x = C_n$ and ii) all subarcs of $u_1 \cup u_2$ in F_x are in three bands obtained by connecting F_{jx} ($j \in \{3,\dots,24n-15\}$) along two subarcs or one subarc of $u_1 \cup u_2$.

1) By deforming m_{ix} isotopically, we obtain a simple closed curve $\tilde{m}_{ix}$ in $F_x - \partial F_x = F_x - C_n$ such that $\tilde{m}_{ix} \cup F_{jx}$ and $m_{ix} \cup F_{jx}$ are both empty or parallel (two) arcs



F_l F_r

Fig. 5.



$F_l \supset m_{1l}$ $F_l \supset \tilde{m}_{1l}$

Fig. 6.

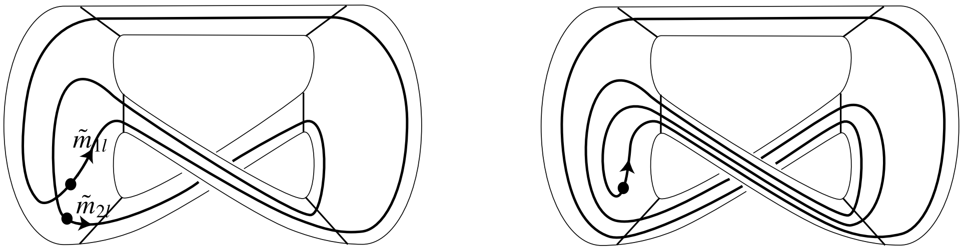
in F_{jx} for any $j \in \{1, 2, \ldots, 24n - 15\}$. Suppose that $\tilde{m}_{ix}$ has an orientation induced by that of m_{ix} and a starting point on an open arc $\tilde{m}_{ix} \cap (F_{1x} - \partial F_{1x})$. Then we have $W(\tilde{m}_{ix}) = W(m_{ix})$. (See Fig. 6 for the case of $i = 1$, $x = l$.)

2) Since waves w_{1x}, w_{2x}, w_{3x} ($x \in \{l, r\}$) are mutually disjoint and each component of $C_n - \partial w_{ix}$ contains one point of ∂w_{jx} for any $i \in \{1,2,3\}$ and any $j \in \{1,2,3\} - \{i\}$, we may assume that any two simple closed curves of $\tilde{m}_{1x}$, $\tilde{m}_{2x}$ and $\tilde{m}_{3x}$ ($x \in \{l, r\}$) in (1) intersect transversely at one point in $F_{1x} - \partial F_{1x}$. For $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$ and $x \in \{l, r\}$, let $\tilde{m}_{ijjx}$ be an oriented simple closed curve obtained from $\tilde{m}_{ix}$ with the orientation in (1) by applying the Dehn twist in F_x along $\tilde{m}_{jx}$ twice. Suppose that a starting point of $\tilde{m}_{ijjx}$ is the initial point of the oriented subarc $\tilde{m}_{ijjx} \cap \tilde{m}_{ix}$ of $\tilde{m}_{ix}$. Then we have $W(\tilde{m}_{ijjx}) = W(m_{ix})(W(m_{jx}))^2$. (See Fig. 7 for the case of $(i, j) = (1, 2)$ and $x = l$.) □

Lemma 3.2.4. 1) *Each of the six fundamental groups*

$$\pi_1(M_n(\tilde{m}_{1l}, \tilde{m}_{233r})), \quad \pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{122r})), \quad \pi_1(M_n(\tilde{m}_{3l}, \tilde{m}_{311r})),$$
$$\pi_1(M_n(\tilde{m}_{233l}, \tilde{m}_{1r})), \quad \pi_1(M_n(\tilde{m}_{122l}, \tilde{m}_{2r})), \quad \pi_1(M_n(\tilde{m}_{311l}, \tilde{m}_{3r}))$$

is trivial.



$F_l \supset \tilde{m}_{1l}, \tilde{m}_{2l}$ $\qquad$ $F_l \supset \tilde{m}_{122l}$

Fig. 7.

2) *Each* $\pi_1(M_n(\tilde{m}_{ix}))$ $(i \in \{1, 2, 3\},\ x \in \{l, r\})$ *is not isomorphic to the infinite cycle group* $\mathbb{Z}$.

Proof. 1) The presentations of the groups can be simplified by using “mutual substitutions” defined by Kaneto, see Definition 1 and Theorem 2 in [7]. Here we demonstrate how mutual substitutions can be applied.

$$\begin{aligned}
&\pi_1(M_n(\tilde{m}_{1l}, \tilde{m}_{233r}))\\
&\equiv \langle A, B \mid W(\tilde{m}_{1l}) = e,\ W(\tilde{m}_{233r}) = e\rangle \equiv \langle A, B \mid W(m_{1l}) = e,\ W(m_{2r})(W(m_{3r}))^2 = e\rangle\\
&\equiv \left\langle A, B \,\middle|\, \begin{array}{l} b(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABa)^{(n-1)}bb(aBAb)^{(n-1)}a = e,\\ b(aBAb)^{(n-1)}aa(bABa)^{(n-1)}b(baBA)^{(n-1)}ba(bABa)^{(n-1)}b(AbaB)^{(n-1)}A\times \\ b(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABa)^{(n-1)}b = e \end{array}\right\rangle\\
&\equiv \left\langle A, B \,\middle|\, \begin{array}{l} b(ABab)^{(n-1)}A(baBA)^{(n-1)}ba(bABa)^{(n-1)}bb(aBAb)^{(n-1)}a = e,\\ b(baBA)^{(n-1)}ba(bABa)^{(n-1)}b = e \end{array}\right\rangle\\
&\equiv \langle A, B \mid b = e,\ b(baBA)^{(n-1)}ba(bABa)^{(n-1)}b = e\rangle\\
&\equiv \langle A, B \mid b = e,\ a = e\rangle \equiv \{e\}.
\end{aligned}$$

By Basic Lemma 3.1.2 and Lemma 3.2.2 (5), we obtain the conclusion 1) of Lemma 3.2.4.

2) For a natural number N, let ξ_N be an N-th primitive root of unity, and put

$$\alpha_N = \begin{pmatrix} \xi_N & 0 \\ 0 & \xi_N^{-1} \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\rho\colon \langle A, B\rangle \to GL(2, \mathbb{C})$ be a homomorphism defined by $\rho(A) = \alpha_N$ and $\rho(B) = \beta$.

Since $\rho(W(\tilde{m}_{1l})) = \rho(W(m_{1l})) = \alpha_N^{-8n+5} = \begin{pmatrix} \xi_N^{-8n+5} & 0 \\ 0 & \xi_N^{8n-5} \end{pmatrix}$, by putting $N = 8n-5$, $\rho(W(\tilde{m}_{1l})) = e$, and so ρ keeps the relation $W(\tilde{m}_{1l}) = e$. Then we obtain the induced homomorphism $\tilde{\rho}\colon \langle A, B \mid W(\tilde{m}_{1l}) = e\rangle = \langle A, B\rangle/\mathrm{N}(W(\tilde{m}_{1l})) \to GL(2, \mathbb{C})$, so that $\tilde{\rho}(A) = \alpha_{8n-5}$, $\tilde{\rho}(B) = \beta$, here $AN(W(\tilde{m}_{1l}))$, $BN(W(\tilde{m}_{1l})) \in \langle A, B\rangle/\mathrm{N}(W(\tilde{m}_{1l}))$ are denoted by A, B respectively for convenience. Since two elements α_{8n-5} and β in $GL(2, \mathbb{C})$ are non-commutative, $\langle A, B \mid W(\tilde{m}_{1l}) = e\rangle$ is not isomorphic to $\mathbb{Z}$. By Lemma 3.2.2, each group $\langle A, B \mid W(\tilde{m}_{ix}) = e\rangle$ ($i \in \{1, 2, 3\}$, $x \in \{l, r\}$) is not isomorphic to $\mathbb{Z}$. By Basic Lemma 3.1.2, each group $\pi_1(M_n(\tilde{m}_{ix}))$ ($i \in \{1,2,3\}$, $x \in \{l,r\}$) is not isomorphic to $\mathbb{Z}$. $\square$

Lemma 3.2.5. *If $n \neq n'$, then $\pi_1(M_n)$ is not isomorphic to $\pi_1(M_{n'})$.*

Proof. Let ξ_N, α_N, β and ρ be same ones in the proof of Lemma 3.2.4. Since $\rho(W(C_n)) = \alpha_N^{16n-10} = \begin{pmatrix} \xi_N^{16n-10} & 0 \\ 0 & \xi_N^{-16n+10} \end{pmatrix}$, by putting $N = 16n-10$, $\rho(W(C_n)) = e$, and so ρ keeps the relation $W(C_n) = e$. Then we obtain the induced homomorphism $\hat{\rho}\colon \langle A, B \mid W(C_n) = e\rangle = \langle A, B\rangle/\mathrm{N}(W(C_n)) \to GL(2, \mathbb{C})$. Let G_N be the subgroup of $GL(2, \mathbb{C})$ generated by α_N and β. If $\pi_1(M_n)$ is isomorphic to $\pi_1(M_{n'})$ for $n' \leq n$, by Basic Lemma 3.1.2, there is a surjective homomorphism $\tau\colon \langle A, B \mid W(C_{n'}) = e\rangle = \langle A, B\rangle/\mathrm{N}(W(C_{n'})) \to G_{16n-10}$. Since τ is surjective, two elements $\tau(A)$, $\tau(B)$ are generators of G_{16n-10}. Any element of G_N is represented by α_N^k or $\alpha_N^k\beta$ for some integer k, because of $\beta^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta\alpha_N^k = \begin{pmatrix} 0 & \xi_N^{-k} \\ \xi_N^k & 0 \end{pmatrix} = \alpha_N^{-k}\beta$. Hence, any pair of two generators of G_N is represented by $\{\alpha_N^k, \alpha_N^l\beta\}$ or $\{\alpha_N^{k+l}\beta, \alpha_N^l\beta\}$, where k, l are integers, k and N are relatively prime. Note that ξ_N^k is also an N-th primitive root of unity if and only if k and N are relatively prime. Then there are following four cases for $\tau(A)$ and $\tau(B)$.

(1) If $\tau(A) = \alpha_{16n-10}^k$ and $\tau(B) = \alpha_{16n-10}^l\beta$, then $\tau(W(C_{n'})) = \alpha_{16n-10}^{(16n'-10)k}$.
(2) If $\tau(A) = \alpha_{16n-10}^l\beta$ and $\tau(B) = \alpha_{16n-10}^k$, then $\tau(W(C_{n'})) = \alpha_{16n-10}^{(16n'-10)k}$.
(3) If $\tau(A) = \alpha_{16n-10}^{k+l}\beta$ and $\tau(B) = \alpha_{16n-10}^l\beta$, then $\tau(W(C_{n'})) = \alpha_{16n-10}^{-(16n'-10)k}$.
(4) If $\tau(A) = \alpha_{16n-10}^l\beta$ and $\tau(B) = \alpha_{16n-10}^{k+l}\beta$, then $\tau(W(C_{n'})) = \alpha_{16n-10}^{(16n'-10)k}$.

Here k and $16n-10$ are relatively prime. On the other hand, since $W(C_{n'})$ represents a unit element of $\langle A, B \mid W(C_{n'}) = e\rangle = \langle A, B\rangle/\mathrm{N}(W(C_{n'}))$, $\tau(W(C_{n'})) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ holds. In each case of (1), (2), (3) and (4), $n = n'$ holds, because ξ_{16n-10}^k is a $(16n-10)$-th primitive root of unity and $16n'-10 \leq 16n-10$. $\square$

Here recall the definition of equivalence for Dehn fillings of a 3-manifold M with ∂M consisting of two tori. Two Dehn fillings of M yielding $M(m_1, m_2)$ and $M(m_1', m_2')$ respectively are said to be equivalent if $m_1 \cup m_2$ is isotopic to $m_1' \cup m_2'$ in ∂M.

Proposition 3.2.1. 1) *The Dehn fillings of* M_n *yielding* S^3 *are exactly the six ones yielding* $M_n(\tilde{m}_{1l}, \tilde{m}_{233r})$, $M_n(\tilde{m}_{2l}, \tilde{m}_{122r})$, $M_n(\tilde{m}_{3l}, \tilde{m}_{311r})$, $M_n(\tilde{m}_{233l}, \tilde{m}_{1r})$, $M_n(\tilde{m}_{122l}, \tilde{m}_{2r})$ *and* $M_n(\tilde{m}_{311l}, \tilde{m}_{3r})$ *respectively up to equivalence.*
2) *If* $n \neq n'$, *then* M_n *is not homeomorphic to* $M_{n'}$.

Proof. 1) By Lemma 3.2.4 1) and Theorem 2.2, each of Dehn fillings $M_n(\tilde{m}_{1l}, \tilde{m}_{233r})$, $M_n(\tilde{m}_{2l}, \tilde{m}_{122r})$, $M_n(\tilde{m}_{3l}, \tilde{m}_{311r})$, $M_n(\tilde{m}_{233l}, \tilde{m}_{1r})$, $M_n(\tilde{m}_{122l}, \tilde{m}_{2r})$ and $M_n(\tilde{m}_{311l}, \tilde{m}_{3r})$ of M_n is homeomorphic to S^3. We assume that a Dehn filling $M_n(m_l, m_r)$ of M_n yielding S^3. By Key Lemma 3.1.1, one of two simple closed curve m_l, m_r in $\partial H - C_n$ coincides with one of m_{1l}, m_{2l}, m_{3l}, m_{1r}, m_{2r} and m_{3r} up to isotopy on ∂M_n. Recall that $\tilde{m}_{ix}$ ($i \in \{1, 2, 3\}$, $x \in \{l, r\}$) is isotopic to m_{ix} in ∂M_n. By Lemma 3.2.4 2) and Theorem 2.3, each of Dehn fillings $M_n(\tilde{m}_{1l})$, $M_n(\tilde{m}_{2l})$, $M_n(\tilde{m}_{3l})$, $M_n(\tilde{m}_{1r})$, $M_n(\tilde{m}_{2r})$ and $M_n(\tilde{m}_{3r})$ of M_n is homeomorphic to the exterior of a non-trivial knot, because the exterior of the trivial knot is homeomorphic to the solid torus. Then, by Theorem 2.4, $m_l \cup m_r$ is isotopic to one of $\tilde{m}_{1l} \cup \tilde{m}_{233r}$, $\tilde{m}_{2l} \cup \tilde{m}_{122r}$, $\tilde{m}_{3l} \cup \tilde{m}_{311r}$, $\tilde{m}_{233l} \cup \tilde{m}_{1r}$, $\tilde{m}_{122l} \cup \tilde{m}_{2r}$ and $\tilde{m}_{311l} \cup \tilde{m}_{3r}$ on ∂M_n.

2) If $n \neq n'$, then, by Lemma 3.2.5, M_n is not homeomorphic to $M_{n'}$. □

By Proposition 3.2.1, there exists a homeomorphism h_n: $M_n(\tilde{m}_{1l}, \tilde{m}_{233r}) \to S^3$. The closure of $M_n(\tilde{m}_{1l}, \tilde{m}_{233r}) - M_n$ consists of two solid tori N_l, N_r such that $\partial N_l \supset \tilde{m}_{1l}$ and $\partial N_r \supset \tilde{m}_{233r}$. Then there are two homeomorphisms h_{nx}: $D^2 \times S^1 \to N_x$ ($x \in \{l, r\}$). Let $\tilde{K}_{nx}$ ($x \in \{l, r\}$) be the simple closed curve $h_{nx}(\mathbf{0} \times S^1)$ where $\mathbf{0}$ is the center of unit disk D^2. Let K_{nx} ($x \in \{l, r\}$) be the knot $h_n(\tilde{K}_{nx})$ in S^3 and L_n be the link $K_{nl} \cup K_{nr}$ in S^3. By the definitions of M_n and $M_n(\tilde{m}_{1l}, \tilde{m}_{233r})$, the link $\tilde{K}_{nl} \cup \tilde{K}_{nr}$ in $M_n(\tilde{m}_{1l}, \tilde{m}_{233r})$ is tunnel number one, and so the link $K_{nl} \cup K_{nr}$ in S^3 is tunnel number one.

In order to complete the proof of Theorem 1.1, we will show the next proposition.

Proposition 3.2.2. 1) *Each tunnel number one link* L_n *in* S^3 *has exactly five nontrivial Dehn surgeries yielding* S^3 *up to equivalence.*
2) *Two links* L, L' *are said to be equivalent if there is a homeomorphism* h: $S^3 \to S^3$ *satisfying* $h(L) = L'$. *If* $n \neq n'$, *then* L_n *is not equivalent to* $L_{n'}$.

Proof. By the definition of a link L_n, the exterior $E(L_n)$ of L_n is homeomorphic to M_n. Then we obtain 1) of Proposition 3.2.2 from 1) of Proposition 3.2.1. If $n \neq n'$, by 2) of Proposition 3.2.1, $E(L_n)$ is not homeomorphic to $E(L_{n'})$. Then L_n is not equivalent to $L_{n'}$. □

Theorem 1.1 follows from Proposition 3.2.2.

3.3. Proof of Theorem 1.2. Let D_n be a Heegaard diagram $(\partial H; \{u_1, u_2\}, C_n)$ shown by Fig. 8 below where n is a positive integer. Note that D_n is a special case of D in Key Lemma 3.1.1. Throughout this subsection, we assume $D = D_n$, M_n, C_n mean M, C in Key Lemma 3.1.1 respectively and w_{ix}, m_{ix} ($i \in \{1, 2, 3\}$, $x \in \{l, r\}$) mean the ones in Key Lemma 3.1.1 in the case of $D = D_n$ respectively.

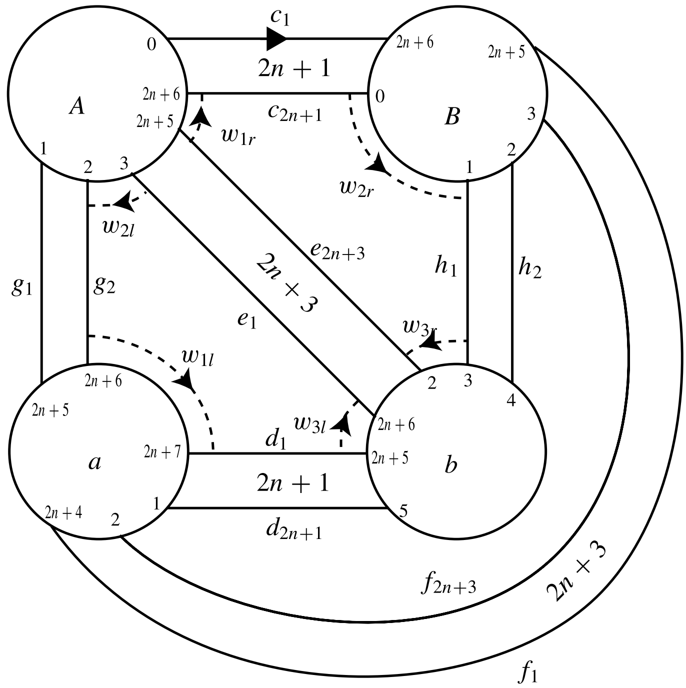


Fig. 8.

Lemma 3.3.1. *If C_n, m_{1l}, m_{2l}, m_{3l}, m_{1r}, m_{2r} and m_{3r} are oriented as shown in* Fig. 8 *respectively, then there exists a starting point on each of these simple closed curves respectively such that the following equations hold.*

(1) $W(C_n) = BABBAA(babA)^n babbaa(BABa)^n$.

(2) $W(m_{1l}) = babba$.

(3) $W(m_{2l}) = a(BABa)^n B$.

(4) $W(m_{3l}) = ABBAA(babA)^n$.

(5) $W(m_{1r}) = (BABa)^n BABB$.

(6) $W(m_{2r}) = baa$.

(7) $W(m_{3r}) = AA(babA)^n bab$.

Proof. Let c_i, d_i, e_i, f_i, g_i and h_i be subarcs of C_n respectively as shown in Fig. 8. Then C_n can be represented by connecting theses subarcs as

$$c_1 e_1 f_{2n+2} h_2 e_{2n+3} g_1 \prod_{i=0}^{n-1} (d_{2(n-i)+1} f_{2(n-i)+1} e_{2i+2} c_{2i+2}),$$

$$d_1 f_1 e_{2n+2} h_1 f_{2n+3} g_2 \prod_{i=0}^{n-1} (c_{2(n-i)+1} e_{2(n-i)+1} f_{2i+2} d_{2i+2}).$$

Take a starting point on $c_1 - \partial c_1$ for C_n and a starting point on $w_{ix} - \partial w_{ix}$ ($i \in \{1, 2, 3\}$, $x \in \{l, r\}$) for m_{ix} respectively. □

Lemma 3.3.2. *If m_{ix} ($i \in \{1,2,3\}$, $x \in \{l,r\}$) is oriented as shown in* Fig. 8, *then the followings hold.*
1) *For each m_{ix} ($i \in \{1,2,3\}$, $x \in \{l,r\}$), there exists an oriented simple closed curve $\tilde{m}_{ix}$ in the component F'_x ($x \in \{l,r\}$) of $\partial H - C_n$ intersecting m_{ix} such that $W(\tilde{m}_{ix}) = W(m_{ix})$ and $\tilde{m}_{ix}$ is isotopic to m_{ix}.*
2) *There exists an oriented simple closed curve $\tilde{m}_{21^{n+1}l}$ in the component of $\partial H - C_n$ intersecting $m_{2l} \cup m_{1l}$ such that $W(\tilde{m}_{21^{n+1}l}) = W(m_{2l})(W(m_{1l}))^{n+1}$.*

Lemma 3.3.2 can be proved by same argument in the proof of Lemma 3.2.3.

Lemma 3.3.3. 1) *Each of the two fundamental groups $\pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{3r}))$ and $\pi_1(M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r}))$ is trivial.*
2) *Each of the two fundamental groups $\pi_1(M_n(\tilde{m}_{2l}))$ and $\pi_1(M_n(\tilde{m}_{3r}))$ is not isomorphic to the infinite cycle group $\mathbb{Z}$.*
3) *The fundamental group $\pi_1(M_n(\tilde{m}_{2r}))$ is isomorphic to the infinite cycle group $\mathbb{Z}$.*

Proof. 1) We will check them by using mutual substitutions.

$$\begin{aligned}
&\pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{3r}))\\
&\equiv \langle A, B \mid W(\tilde{m}_{2l}) = e,\ W(\tilde{m}_{3r}) = e\rangle\\
&\equiv \langle A, B \mid W(m_{2l}) = e,\ W(m_{3r}) = e\rangle\\
&\equiv \langle A, B \mid a(BABa)^n B = e,\ AA(babA)^n bab = e\rangle\\
&\equiv \langle A, B \mid A(babA)^n b = e,\ AA(babA)^n bab = e\rangle\\
&\equiv \langle A, B \mid A(babA)^n b = e,\ b = e\rangle\\
&\equiv \langle A, B \mid A = e,\ b = e\rangle = \{e\}.
\end{aligned}$$

$$\begin{aligned}
&\pi_1(M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r}))\\
&\equiv \langle A, B \mid W(\tilde{m}_{21^{n+1}l}) = e,\ W(\tilde{m}_{2r}) = e\rangle\\
&\equiv \langle A, B \mid W(m_{2l})(W(m_{1l}))^{n+1} = e,\ W(m_{2r}) = e\rangle\\
&\equiv \langle A, B \mid a(BABa)^n abba(babba)^n = e,\ baa = e\rangle\\
&\equiv \langle A, B \mid (aBAB)^n aabba(babba)^n = e,\ aab = e\rangle\\
&\equiv \langle A, B \mid (aBAB)^{(n-1)} aabba(babba)^{(n-1)} = e,\ aab = e\rangle\\
&\cdots\\
&\equiv \langle A, B \mid aabba = e,\ aab = e\rangle\\
&\equiv \langle A, B \mid ba = e,\ aab = e\rangle \equiv \langle A, B \mid ba = e,\ a = e\rangle\\
&\equiv \langle A, B \mid b = e,\ a = e\rangle \equiv \{e\}.
\end{aligned}$$

2) Let ξ_N, α_N, β and ρ be same ones in the proof of Lemma 3.2.4 and $\sigma\colon \langle A,\, B\rangle \to GL(2,\,\mathbb{C})$ be a homomorphism defined by $\sigma(A) = \alpha_{N'}\beta,\ \sigma(B) = \beta$. Since $\sigma(W(\tilde{m}_{2l})) = \sigma(W(m_{2l})) = \alpha_{N'}^{2n+1}$ and $\rho(W(\tilde{m}_{3r})) = \rho(W(m_{3r})) = \alpha_N^{2n+3}$, by putting $N' = 2n+1,\ N = 2n+3,\ \sigma(W(\tilde{m}_{2r})) = e,\ \rho(W(\tilde{m}_{3r})) = e$, and so σ (resp. ρ) keeps the relation $W(\tilde{m}_{2l}) = e$ (resp. $W(\tilde{m}_{3r}) = e$). Then we obtain the induced homomorphisms $\tilde{\sigma}\colon \langle A,\, B \mid W(\tilde{m}_{2l}) = e\rangle = \langle A, B\rangle/\mathrm{N}(W(\tilde{m}_{2l})) \to GL(2,\mathbb{C})$ and $\tilde{\rho}\colon \langle A, B \mid W(\tilde{m}_{3r}) = e\rangle = \langle A, B\rangle/\mathrm{N}(W(\tilde{m}_{3r})) \to GL(2,\,\mathbb{C})$. Since two elements $\alpha_{2n+1}\beta$ and β (resp. α_{2n+3} and β) in $GL(2,\,\mathbb{C})$ are non-commutative, $\langle A,\, B \mid W(\tilde{m}_{2l}) = e\rangle$ (resp. $\langle A,\, B \mid W(\tilde{m}_{3r}) = e\rangle$) is not isomorphic to $\mathbb{Z}$. By Basic Lemma 3.1.2, $\pi_1(M_n(\tilde{m}_{2l}))$ (resp. $\pi_1(M_n(\tilde{m}_{3r}))$) is not isomorphic to $\mathbb{Z}$.

3) By changing generators A and B of free group $\langle A,\, B\rangle$ into A and $B' := W(\tilde{m}_{2r}) = W(m_{2r}) = baa$, we can check the following.

$$\pi_1(M_n(\tilde{m}_{2r})) \equiv \langle A,\, B \mid W(\tilde{m}_{2r}) = e\rangle \equiv \langle A,\, B' \mid B' = e\rangle \equiv \langle A \mid\ - \rangle \equiv \mathbb{Z}. \qquad \square$$

Lemma 3.3.4. *If $n \neq n'$, then $\pi_1(M_n)$ is not isomorphic to $\pi_1(M_{n'})$.*

Proof. Let ξ_N, α_N, β and ρ be same ones in the proof of Lemma 3.2.4 and G_N be same one in the proof of Lemma 3.2.5. Since $\rho(W(C_n)) = \alpha_N^{-4n-6} = \begin{pmatrix} \xi_N^{-4n-6} & 0 \\ 0 & \xi_N^{4n+6} \end{pmatrix}$, by putting $N = 4n+6,\ \rho(W(C_n)) = e$, and so ρ keeps the relation $W(C_n) = e$. Then we obtain the induced homomorphism $\hat{\rho}\colon \langle A,\, B \mid W(C_n) = e\rangle = \langle A,\, B\rangle/\mathrm{N}(W(C_n)) \to GL(2,\mathbb{C})$. If $\pi_1(M_n)$ is isomorphic to $\pi_1(M_{n'})$ for $n' \leq n$, by Basic Lemma 3.1.2, there is a surjective homomorphism $\tau\colon \langle A,\, B \mid W(C_{n'}) = e\rangle = \langle A,\, B\rangle/\mathrm{N}(W(C_{n'})) \to G_{4n+6}$. Since τ is surjective, two elements $\tau(A)$, $\tau(B)$ are generators of G_{4n+6}. By same argument in the proof of Lemma 3.2.5, there are following four cases for $\tau(A)$ and $\tau(B)$.
(1) If $\tau(A) = \alpha_{4n+6}^k$ and $\tau(B) = \alpha_{4n+6}^l\beta$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{(4n'+6)k}$.
(2) If $\tau(A) = \alpha_{4n+6}^l\beta$ and $\tau(B) = \alpha_{4n+6}^k$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{2k}$.
(3) If $\tau(A) = \alpha_{4n+6}^{k+l}\beta$ and $\tau(B) = \alpha_{4n+6}^l\beta$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{-(4n'+2)k}$.
(4) If $\tau(A) = \alpha_{4n+6}^l\beta$ and $\tau(B) = \alpha_{4n+6}^{k+l}\beta$, then $\tau(W(C_{n'})) = \alpha_{4n+6}^{(4n'+2)k}$.
Here k and $4n+6$ are relatively prime. On the other hand, since $W(C_{n'})$ represents a unit element of $\langle A,\, B \mid W(C_{n'}) = e\rangle = \langle A,\, B\rangle/\mathrm{N}(W(C_{n'}))$, $\tau(W(C_{n'})) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ holds. In the case (1), $n = n'$ holds, because ξ_{4n+6}^k is a $(4n+6)$-th primitive root of unity and $4n'+6 \leq 4n+6$. The other cases (2), (3) and (4) do not happen, because ξ_{4n+6}^k is a $(4n+6)$-th primitive root of unity and $2,\ 4n'+2 < 4n+6$. $\square$

Proposition 3.3.1. 1) *Each of the two Dehn fillings $M_n(\tilde{m}_{2l}, \tilde{m}_{3r})$ and $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r})$ is homeomorphic to S^3.*
2) *Each of the two Dehn fillings $M_n(\tilde{m}_{2l})$ and $M_n(\tilde{m}_{3r})$ is not homeomorphic to the solid torus.*
3) *A Dehn filling $M_n(\tilde{m}_{2r})$ is homeomorphic to the solid torus.*
4) *If $n \neq n'$, then M_n is not homeomorphic to $M_{n'}$.*

Proof. 1) By Lemma 3.3.3 1), each of the fundamental groups $\pi_1(M_n(\tilde{m}_{2l}, \tilde{m}_{3r}))$ and $\pi_1(M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r}))$ is trivial. Then, by Theorem 2.4, each of $M_n(\tilde{m}_{2l}, \tilde{m}_{3r})$ and $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r})$ is homeomorphic to S^3.

2) By Lemma 3.3.3 2) and Theorem 2.3, each of $M_n(\tilde{m}_{2l})$ and $M_n(\tilde{m}_{3r})$ is not homeomorphic to the solid torus.

3) A Dehn filling $M_n(\tilde{m}_{2r})$ is a submanifold of a Dehn filling of $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r})$. By 1), $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r})$ is homeomorphic to S^3. By Lemma 3.3.3 3), $\pi_1(M_n(\tilde{m}_{2r}))$ is isomorphic to $\mathbb{Z}$. Then, by Theorem 2.3, $M_n(\tilde{m}_{2r})$ is homeomorphic to the solid torus.

4) If $n \neq n'$, then, by Lemma 3.3.4, M_n is not homeomorphic to $M_{n'}$. □

By Proposition 3.3.1, there exists two homeomorphisms $h_n\colon M_n(\tilde{m}_{2l}, \tilde{m}_{3r}) \to S^3$ and $h'_n\colon M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r}) \to S^3$. The closure of $M_n(\tilde{m}_{2l}, \tilde{m}_{3r}) - M_n$ (resp. $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r}) - M_n$) consists of two solid tori N_l, N_r (resp. N'_l, N'_r) such that $\partial N_l \supset \tilde{m}_{2l}$ and $\partial N_r \supset \tilde{m}_{3r}$ (resp. $\partial N'_l \supset \tilde{m}_{21^{n+1}l}$ and $\partial N'_r \supset \tilde{m}_{2r}$). Then there are four homeomorphisms $h_{nx}\colon D^2 \times S^1 \to N_x$ $(x \in \{l, r\})$ and $h'_{nx}\colon D^2 \times S^1 \to N'_x$ $(x \in \{l, r\})$. Let $\tilde{K}_{nx}$ (resp. $\tilde{K}'_{nx}$) $(x \in \{l, r\})$ be the simple closed curve $h_{nx}(\mathbf{0} \times S^1)$ (resp. $h'_{nx}(\mathbf{0} \times S^1)$) where $\mathbf{0}$ is the center of unit disk D^2. Let K_{nx} (resp. K'_{nx}) $(x \in \{l, r\})$ be the knot $h_n(\tilde{K}_{nx})$ (resp. $h_n(\tilde{K}'_{nx})$) in S^3 and L_n (resp. L'_n) be the link $K_{nl} \cup K_{nr}$ (resp. $K'_{nl} \cup K'_{nr}$) in S^3. By the definitions of M_n, $M_n(\tilde{m}_{2l}, \tilde{m}_{3r})$ and $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}_{2r})$, each of the two links $\tilde{K}_{nl} \cup \tilde{K}_{nr}$ in $M_n(\tilde{m}_{2l}, \tilde{m}_{3r})$ and $\tilde{K}'_{nl} \cup \tilde{K}'_{nr}$ in $M_n(\tilde{m}_{21^{n+1}l}, \tilde{m}l_{3r})$ is tunnel number one, and so each of the two links $K_{nl} \cup K_{nr}$ and $K'_{nl} \cup K'_{nr}$ in S^3 is tunnel number one.

In order to complete the proof of Theorem 1.2 we will show the next proposition.

Proposition 3.3.2. 1) *L_n has no trivial component.*
2) *L'_n has a trivial component.*
3) *$E(L_n)$ is homeomorphic to $E(L'_n)$.*
4) *If $n \neq n'$, then L_n is not equivalent to $L_{n'}$.*

Proof. 1) By the definition of L_n, $E(K_l)$ (resp. $E(K_r)$) is homeomorphic to $M_n(\tilde{m}_{3r})$ (resp. $M_n(\tilde{m}_{2l})$). By Proposition 3.3.1, each of $E(K_l)$, $E(K_r)$ is not homeomorphic to the solid torus. Hence each of K_l, K_r is not a trivial knot.

2) By the definition of L'_n, $E(K'_l)$ is homeomorphic to $M_n(\tilde{m}_{2r})$. By Proposition 3.3.1, $E(K_l)$ is homeomorphic to the solid torus. Hence K_l is a trivial knot.

3) By the definition of L_n and L'_n, each of the exteriors $E(L_n)$, $E(L'_n)$ is homeomorphic to M_n. Hence $E(L_n)$ is homeomorphic to $E(L'_n)$.

4) If $n \neq n'$, then, by Proposition 3.3.1, M_n is not homeomorphic to $M_{n'}$, and so $E(L_n)$ is not homeomorphic to $E(L_{n'})$. Hence L_n is not equivalent to $L_{n'}$. □

Theorem 1.2 follows from Proposition 3.3.2.

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