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Osaka J. Math.
47 (2010), 177–187

THE MAXIMAL THURSTON–BENNEQUIN NUMBER OF A DOUBLED KNOT

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(Received May 7, 2008, revised October 1, 2008)

Abstract

We show that an upper bound for the maximal Thurston–Bennequin number of any double of a knot K given by the Kauffman polynomial is sharp if the bound is sharp for K. In particular, we give formulas for the maximal Thurston–Bennequin numbers of positive doubles of torus knots and two-bridge knots.

1. Introduction

A *contact structure* on 3-space $\mathbf{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbf{R}\}$ is a global differential 1-form ξ such that $\xi \wedge d\xi \neq 0$ everywhere on $\mathbf{R}^3$. We say that a contact structure on $\mathbf{R}^3$ is *standard* if it is given by a differential 1-form $dz - y\,dx$. The 3-space endowed with a contact structure $dz - y\,dx$ is called the *standard contact* 3-*space*. A *Legendrian link* is a smooth embedding of disjoint circles in the standard contact 3-space such that its tangent vector lies in the contact 2-plane, which is the kernel of the standard contact structure, at each point. The *front diagram* of a Legendrian link is its projection onto the (x, z)-plane. Generically, the only singularities of a front diagram are cusps and transverse double points [19]. We assume that all front diagrams are generic. For example, Fig. 1 (a) shows a generic front diagram of a Legendrian knot which is ambient isotopic to the figure eight knot. We obtain a link diagram of the same topological type from a front diagram by rounding the cusps and making the strand with smaller slope overcross at each double point. For example, we obtain a diagram of the figure eight knot as in Fig. 1 (b). For an oriented front diagram F of a Legendrian link, let $c(F)$ and $w(F)$ be the number of left cusps of F and the writhe of a link diagram obtained from F as above. The Thurston–Bennequin number is defined as $\mathbf{tb}(F) = w(F) - c(F)$. A *Legendrian isotopy* between Legendrian links J_0 and J_1 is an ambient isotopy between J_0 and J_1 with each level Legendrian. The Thurston–Bennequin number is known to be a Legendrian isotopy invariant of Legendrian links. For an oriented link L, we denote by $\mathbf{TB}(L)$ the maximal value of $\mathbf{tb}$ over all Legendrian link which are ambient isotopic to L. The integer $\mathbf{TB}(L)$ is called the *maximal Thurston–Bennequin number* of L. Let L be a link and D a diagram of L. The *Kauffman polynomial* $F_{(a,z)}(L) \in \mathbf{Z}[a^{\pm}, z^{\pm}]$ is defined as $a^{-w(D)} \wedge_{(a,z)} (D)$, where $\wedge_{(a,z)}(D)$

2000 Mathematics Subject Classification. Primary 57M25; Secondary 57M27.

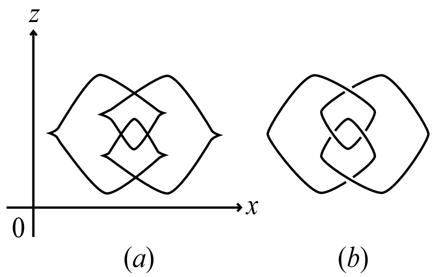

Fig. 1.

is a regular isotopy invariant with properties as follows.

(i) $\wedge_{(a,z)}(\bigcirc) = 1$;

(ii) $\wedge_{(a,z)}\big(\ \big) = a\,\wedge_{(a,z)}\big(\ \big)$ and $\wedge_{(a,z)}\big(\ \big) = a^{-1}\,\wedge_{(a,z)}\big(\ \big)$;

(iii) $\wedge_{(a,z)}\big(\ \big) - \wedge_{(a,z)}\big(\ \big) = z\Big(\wedge_{(a,z)}\big(\ \big) - \wedge_{(a,z)}\big(\ \big)\Big)$.

Let $f \in \mathbb{Z}[x^{\pm}, y^{\pm}]$ be a Laurent polynomial and write $f = \sum_i f_i(y)x^i$ where $f_i(y)$ are polynomials in $y^{\pm 1}$. We denote the largest (resp. the smallest) exponent of x in f by $\text{max-deg}_x f$ (resp. $\text{min-deg}_x f$). In the late of 1990's, an upper bound for the maximal Thurston–Bennequin number in terms of the Kauffman polynomial was given by Fuchs and Tabachnikov [6], [20] as follows.[1]

Theorem 1.1 (Fuchs and Tabachnikov [6], [20]). *Let K be a link in $\mathbb{R}^3$. Then* $\text{min-deg}_a F_{(a^{-1},z)}(L) - 1 \geq \mathbf{TB}(L)$.

We call the upper bound of the inequality in Theorem 1.1 the *Kauffman bound* on the maximal Thurston–Bennequin number. Then we consider the following problem.

PROBLEM. Which links have the sharpness for the Kauffman bound?

It is known that the Kauffman bound is sharp for any positive link and any alternating link [3], [10], [11], [21], [22], and recently T. Kálmán has shown that the bound is sharp for all $+$*adequate links* [8]. All positive links and alternating links are $+$adequate. Let $D_p^+(K)$ (resp. $D_p^-(K)$) a p-twisted positive (resp. negative) double of a knot K. (We shall give definitions in Section 3.) In this paper, we show the following.

Theorem 1.2. (1) *If* $\mathbf{TB}(K) \geq p$, *then the Kauffman bound is sharp for* $D_p^{\pm}(K)$ *and we have* $\mathbf{TB}(D_p^+(K)) = 1$ *and* $\mathbf{TB}(D_p^-(K)) = -3$;

[1] We take $\text{min-deg}_a F_{(a^{-1},z)}(L)$ instead of $-\text{min-deg}_a F_{(a,z)}(L)$ because it is a question of Stoimenow in Section 5 for which we shall a partial answer.

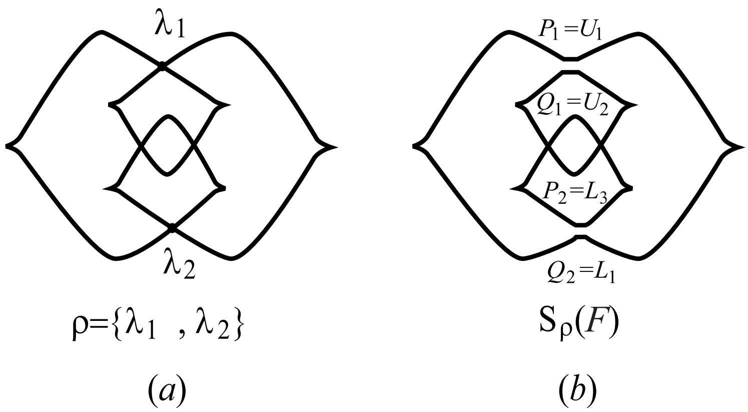

Fig. 2.

(2) *If K is a knot for which the Kauffman bound is sharp and* $\mathbf{TB}(K) < p$, *then the Kauffman bound is sharp for* $D_p^{\pm}(K)$ *and we have* $\mathbf{TB}(D_p^+(K)) = 1 - 2p + 2\,\mathbf{TB}(K)$ *and* $\mathbf{TB}(D_p^-(K)) = -2 - 2p + 2\,\mathbf{TB}(K)$.

Corollary 1.3. *If K is a knot for which the Kauffman bound is sharp, then the Kauffman bound for $D_p^{\pm}(K)$ is sharp for any integer p.*

REMARK. In general, the Kauffman bound is not necessarily sharp. For example, many negative torus knots do not have the sharpness as mentioned in [11].

This paper is organized as follows. In Section 2, we shall introduce results of D. Rutherford which will be used to prove Theorem 1.2 in Section 3. In Section 4, we shall give formulas for the maximal Thurston–Bennequin numbers of positive doubles of torus knots and two-bridge knots. In Section 5, we shall discuss a problem of A. Stoimenow.

2. Existence of rulings

In this section, we recall a work of D. Rutherford [13]. First we give the definition of a ruling for a front diagram of a Legendrian link. By planar isotopy, we assume that all singularities of a front diagram F have different x-coordinates. Give a subset $\rho = \{\lambda_1, \dots, \lambda_n\}$ of the set of crossings of F, with the x-coordinate of λ_i denoted x_i so that $x_i < x_{i+1}$, let $S_\rho(F)$ denote the front diagram obtained from F by resolving all crossings in ρ to parallel horizontal lines (see Fig. 2). The set ρ is called a *ruling* if
(i) every component T_j of $S_\rho(F)$ (as a Legendrian link) consists of two horizontal strands having one left cusp and no self-crossings. The upper is denoted U_j, and the lower L_j,

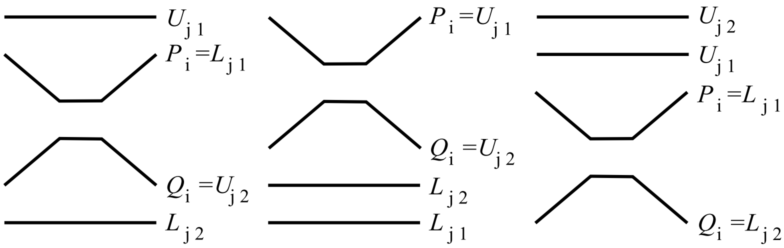

Fig. 3. Normality condition

(ii) for each i, the strands of $S_\rho(F)$ meeting where λ_i was in F belong to different components. Call the upper of these strands P_i and the lower Q_i,
(iii) one of the following normality conditions (Fig. 3) holds for each i: for some j_1, j_2,
(a) $P_i = L_{j_1}$ and $Q_i = U_{j_2}$;
(b) $P_i = U_{j_1}$ and $Q_i = U_{j_2}$, with the z-coordinate of L_{j_1} less than the z-coordinate of L_{j_2} at $x = x_i$;
(c) $P_i = L_{j_1}$ and $Q_i = L_{j_2}$, with the z-coordinate of U_{j_1} less than the z-coordinate of U_{j_2} at $x = x_i$.

REMARK. The set $\rho = \{\lambda_1, \lambda_2\}$ in Fig. 2 (a) is a ruling. See Fig. 2 (b).

D. Rutherford has shown the following result. (See Lemma 2.2 and Theorem 3.1 in [13].)

Theorem 2.1 (Rutherford [13]). (1) *A Legendrian link L has a front diagram with a ruling if and only if the Kauffman bound for the maximal Thurston–Bennequin number of L is sharp.*
(2) *If F is a front diagram with a ruling for a Legendrian link L, then* $\mathbf{tb}(F) = \mathbf{TB}(L)$.

REMARK. Theorem 2.1 gave an affirmative answer to a conjecture of D. Fuchs [4]. As mentioned in [13], the existence of a ruling of a front diagram of a Legendrian link is equivalent to the existence of an augmentation on the Legendrian contact DGA, defined by Chekanov [1] and Eliashberg [2]. (See [5] and [17].) D. Fuchs studied the existence of an augmentation of a doubled knot in [4].

3. Proof of Theorem

Take an embedding of an annulus A in $\mathbf{R}^3$. We denote the core curve of A by K. When we orient two boundary curves of the annulus so as to run around the annulus in the same direction, we denote the linking number of the boundary curves by p. Then add a clasp to the boundary curves as shown in Fig. 4. If we add a clasp (a)

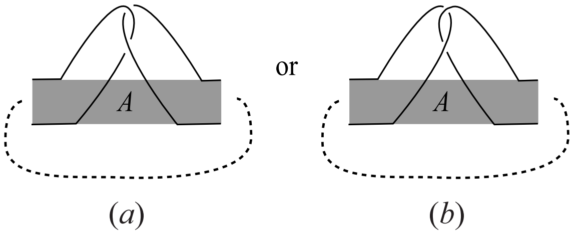

Fig. 4.

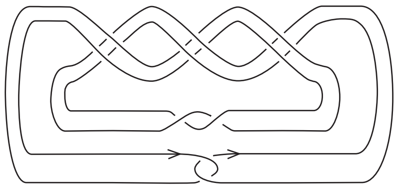

Fig. 5.

(resp. (b)), then we call the resultant knot a *p-twisted positive double* (*resp. negative*) *double of K*.

REMARK. For example, see [14] for the definition of a p-twisted positive double of a knot. The 2-twisted positive double of the positive trefoil knot is described in Fig. 5. We may define a p-twisted negative double of a knot as the mirror image of a $(-p)$-twisted positive double of the mirror image of the knot. To prove Theorem 1.2, we consider a front diagram for a double of a Legendrian knot obtained by doubling a front as follows. First take a front diagram F of an arbitrary Legendrian knot. Then we take a "double" of F as shown is Fig. 6. (Shift a copy of F slightly down.) Next we insert "full-twists" in a part of F which consists of a subarc and its copy as shown in Fig. 7. Finally we make a "clasp" at one portion of the obtained front diagram as shown in Fig. 8. If we insert a clasp (a), (b) or (c) in Fig. 8, then the resultant front diagram is denoted as $F^{+}_{m,n}$, $F^{-}_{m,n}$ or $F^{--}_{m,n}$ respectively. Notice that $F^{+}_{m,n}$ is a front diagram for a Legendrian representative of a positive double of a knot, and $F^{-}_{m,n}$ and $F^{--}_{m,n}$ are front diagrams for Legendrian representatives of negative doubles of a knot.

Proposition 3.1. *Let K be a knot in $\mathbf{R}^3$ and F a front diagram for a Legendrian representative of K. Then, for any integer p with $p \le \mathbf{tb}(F)$, $F^{+}_{0,\mathbf{tb}(F)-p}$ (resp. $F^{-}_{0,\mathbf{tb}(F)-p}$) is a front diagram with a ruling for a Legendrian representative of $D^{+}_{p}(K)$ (resp. $D^{-}_{p}(K)$), $\mathbf{tb}(F^{+}_{0,\mathbf{tb}(F)-p}) = 1$ and $\mathbf{tb}(F^{-}_{0,\mathbf{tb}(F)-p}) = -3$.*

Proof. By direct calculation, we have $p = m - n + \mathbf{tb}(F)$ for $F^{+}_{m,n}$ and $F^{-}_{m,n}$ as Legendrian representatives of $D^{+}_{p}(K)$ and $D^{-}_{p}(K)$, $\mathbf{tb}(F^{+}_{m,n}) = 1 - 2m$ and $\mathbf{tb}(F_{m,n}) =$

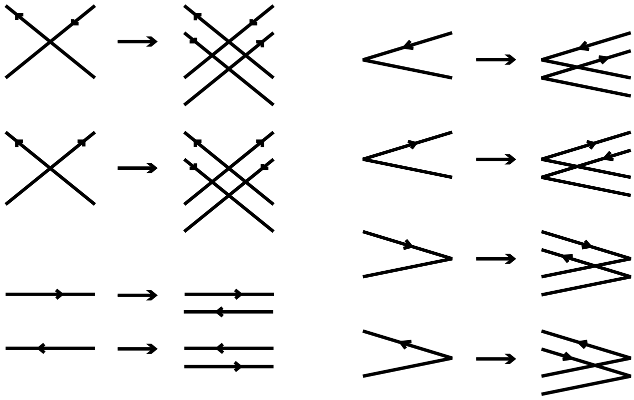

Fig. 6.

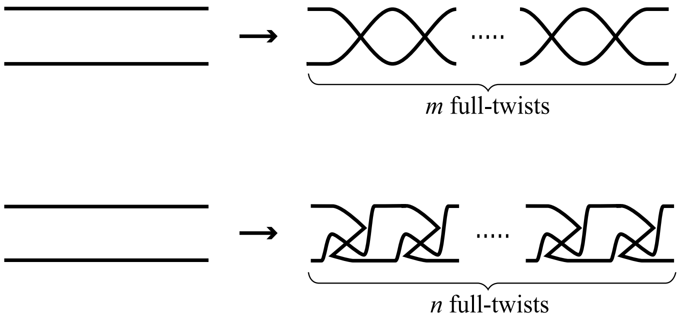

Fig. 7.

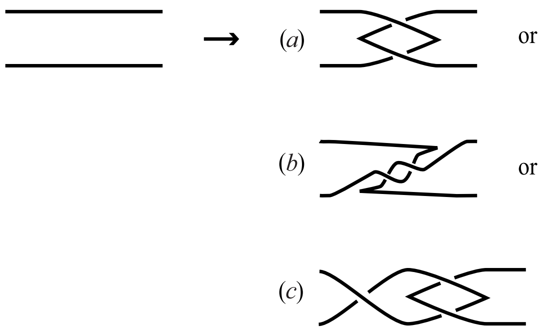

Fig. 8.

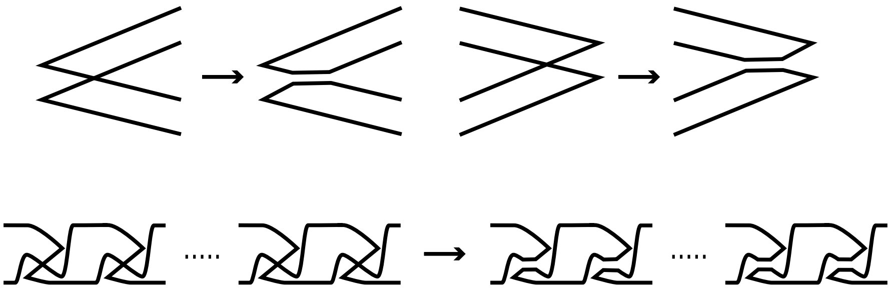

Fig. 9.

$-3-2m$. By assumption, we may assume that $m=0$. Thus we have $n=\mathbf{tb}(F)-p$, $\mathbf{tb}(F^{+}_{0,\mathbf{tb}(F)-p})=1$ and $\mathbf{tb}(F^{-}_{0,\mathbf{tb}(F)-p})=-3$. We know that $F^{+}_{0,\mathbf{tb}(F)-p}$ and $F^{-}_{0,\mathbf{tb}(F)-p}$ are front diagrams with rulings by considering resolutions of crossings as in Fig. 9. (We do not need to consider resolutions of crossings of clasps and of crossings near each crossing of F.) □

Proposition 3.2. *Let K be a knot in $\mathbf{R}^3$. If F is a front with a ruling for a Legendrian representative of K, then, for any integer p with $p>\mathbf{tb}(F)$, $F^{+}_{p-\mathbf{tb}(F),0}$ (resp. $F^{--}_{p-\mathbf{tb}(F)-1,0}$) is a front diagram with a ruling for a Legendrian representative of $D^{+}_{p}(K)$ (resp. $D^{-}_{p}(K)$), $\mathbf{tb}(F^{+}_{p-\mathbf{tb}(F),0})=1-2p+2\,\mathbf{tb}(F)$ and $\mathbf{tb}(F^{--}_{p-\mathbf{tb}(F),0})=-2-2p+2\,\mathbf{tb}(F)$.*

Proof. As in the proof of Proposition 3.1, we have $p=m-n+\mathbf{tb}(F)$ (resp. $p=m-n+\mathbf{tb}(F)+1$) for $F^{+}_{m,n}$ (resp. $F^{--}_{m,n}$), and $\mathbf{tb}(F^{+}_{m,n})=1-2m$ and $\mathbf{tb}(F^{--}_{m,n})=-4-2m$. By assumption, we may assume that $n=0$, and hence we have front diagrams $F^{+}_{p-\mathbf{tb}(F),0}$ and $F^{--}_{p-\mathbf{tb}(F)-1,0}$ such that $\mathbf{tb}(F^{+}_{p-\mathbf{tb}(F),0})=1-2p+2\,\mathbf{tb}(F)$ and $\mathbf{tb}(F^{--}_{p-\mathbf{tb}(F),0})=-2-2p+2\,\mathbf{tb}(F)$. By assumption that F has a ruling, we know that $F^{+}_{p-\mathbf{tb}(F),0}$ and $F^{--}_{p-\mathbf{tb}(F)-1,0}$ are front diagrams with rulings by considering resolutions of each crossing in the rulings of F and crossings near clasps as in Fig. 10. □

Proof of Theorem 1.2. Theorem 1.2 follows from Theorem 2.1, Proposition 3.1 and Proposition 3.2. □

4. Examples

In this section, we give formulas for positive doubles[2] of *torus knots* and *two-bridge knots* (cf. [9].)

[2]We are interested in a knot with nonnegative maximal Thurston–Bennequin number since it is not slice [15] [16].

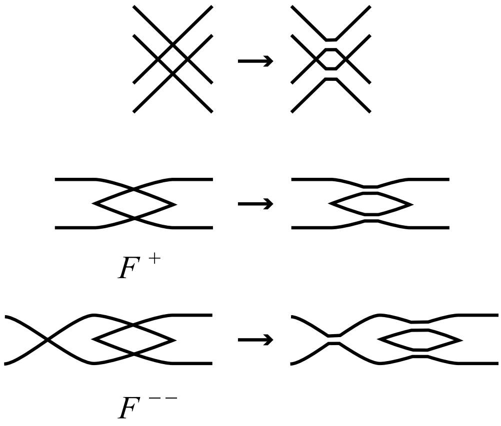

Fig. 10.

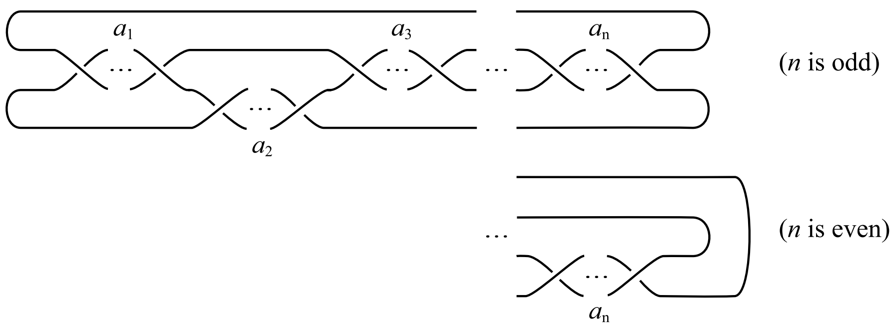

Fig. 11.

Proposition 4.1. *Let $T_{m,n}$ be the (m, n)-torus knot for positive integers m and n.*
(1) *If $mn - m - n \geq p$, then $\mathbf{TB}(D_p^+(T_{m,n})) = 1$;*
(2) *If $mn - m - n \leq p$, then $\mathbf{TB}(D_p^+(T_{m,n})) = 1 - 2p + 2(mn - m - n)$.*

Proof. As we will show in Remark of Section 5, we know that $\mathbf{TB}(T(m, n)) = (m-1)(n-1) - 1$. Therefore the result follows from Theorem 1.2. $\square$

A *two-bridge link* $T(a_1, a_2, \dots, a_n)$ is defined by a link diagram as in Fig. 11, where a_i denotes $|a_i|$ ($\neq 0$) crossing points with sign $\epsilon_i = a_i/|a_i| = \pm 1$. For a two-bridge knot, we have the following.

Proposition 4.2. *Let m be a positive integer. Then*

$$\mathbf{TB}(D_p^+(T(a_1,\dots,a_{2m})))$$
$$=\begin{cases} 1, & \text{if } \displaystyle\sum_{i=1}^{2m} a_i-\sum_{j=1}^{m}|a_{2j}|-1\geq p,\\ 1-2p+2\left\{\displaystyle\sum_{i=1}^{2m} a_i-\sum_{j=1}^{m}|a_{2j}|-1\right\}, & \text{if } \displaystyle\sum_{i=1}^{2m} a_i-\sum_{j=1}^{m}|a_{2j}|-1\leq p,\end{cases}$$

$$\mathbf{TB}(D_p^+(T(a_1,\dots,a_{2m+1})))$$
$$=\begin{cases} 1, & \text{if } \displaystyle\sum_{i=1}^{2m+1} a_i-\sum_{j=1}^{m}|a_{2j}|-2\geq p,\\ 1-2p+2\left\{\displaystyle\sum_{i=1}^{2m+1} a_i-\sum_{j=1}^{m}|a_{2j}|-2\right\}, & \text{if } \displaystyle\sum_{i=1}^{2m+1} a_i-\sum_{j=1}^{m}|a_{2j}|-2\leq p.\end{cases}$$

Proof. By a result of [21], $\mathbf{TB}(T(a_1,\dots,a_{2m}))=\sum_{i=1}^{2m}a_i-\sum_{j=1}^{m}|a_{2j}|-1$ and $\mathbf{TB}(T(a_1,\dots,a_{2m+1}))=\sum_{i=1}^{2m+1}a_i-\sum_{j=1}^{m}|a_{2j}|-2$. Thus we obtain the result by Theorem 1.2. □

5. A problem

Let K and L be a knot and a link in $\mathbb{R}^3$. A *Seifert surface* for L is a compact oriented surface none of whose components are closed and whose boundary is L. We define $\chi(L)$ to be the maximal Euler characteristic of all Seifert surfaces for L. We define $u(K)$ as the minimum number of crossing changes required to unknot K. The integer $u(K)$ is called the *unknotting number* of K. In [18], A. Stoimenow gave the following question.

QUESTION (Stoimenow [18]). Does $\text{min-deg}_a\, F_{(a^{-1},z)}(L)\leq 1-\chi(L)$ hold for any link L? Does $\text{min-deg}_a\, F_{(a^{-1},z)}(K)\leq 2u(K)$ hold for any knot K?

We can give a partial answer to this problem by using the following proposition.

Proposition 5.1. *Let K be a knot for which the Kauffman bound is sharp. Then we have* $\text{min-deg}_a\, F_{(a^{-1},z)}(K)\leq 1-\chi(K)$ *and* $\text{min-deg}_a\, F_{(a^{-1},z)}(K)\leq 2u(K)$.

Proof. By assumption, we have $\mathbf{TB}(K)=\text{min-deg}_a F_{(a^{-1},z)}(K)-1$. By a result of L. Rudolph in [15] and [16], we know that $\mathbf{TB}(K)\leq 2g_s(K)-1$, where $g_s(K)$ is the slice genus of K. Thus $\text{min-deg}_a\, F_{(a^{-1},z)}(K)\leq 2g_s(K)$. On the other hand, $2g_s(K)\leq 1-\chi(K)$ and $g_s(K)\leq u(K)$. Therefore, we have $\text{min-deg}_a\, F_{(a^{-1},z)}(K)\leq 1-\chi(K)$ and $\text{min-deg}_a\, F_{(a^{-1},z)}(K)\leq 2u(K)$. □

REMARK. Let L be a negative link that is a link which admit a diagram with all negative crossings. Then max-deg$_a F_{(a^{-1},z)}(L^*) - 1 \geq$ min-deg$_a$ $F_{(a^{-1},z)}(L^*) - 1 \geq 0$ by a result in [21] concerning a positive link that is a link which admit a diagram with all positive crossings. (Here, L^* is the mirror image of L.) By a formula $F_{(a^{-1},z)}(L^*) = F_{(a,z)}(L)$ [7] we have max-deg $F_{(a^{-1},z)}(L^*) =$ max-deg $F_{(a,z)}(L) = -$ min-deg $F_{(a^{-1},z)}(L)$, we know that min-deg $F_{(a^{-1},z)}(L) \leq -1$. On the other hand, if $K = T(p, q)$, then the inequalities of the above question are sharp. In fact, by a result in [21] and a result of Rasmussen [12] for a positive knot, min-deg$_a$ $F_{(a^{-1},z)}(T(p, q)) - 1 = \mathbf{TB}(T(p, q)) = s(T(p, q)) - 1 = 2u(T(p, q)) - 1 = 2(p - 1)(q - 1) - 1$, where s is the Rasmussen's s invariant in [12]. It is well-known that $1 - \chi(T(p, q)) = 2(p - 1)(q - 1)$. Thus we have min-deg$_a$ $F_{(a^{-1},z)}(T(p, q)) - 1 = -\chi(T(p, q))$.

ACKNOWLEDGEMENTS. The author would like to thank Prof. Akio Kawauchi for his helpful comments. This work was supported by the 21st century COE program at Osaka City University Advanced Mathematical Institute, and by a Korea Research Foundation grant funded by the Korean Goverment (MOEHRD) (KRF-2007-412-J02302).

References

[1] Y. Chekanov: *Differential algebra of Legendrian links*, Invent. Math. **150** (2002), 441–483.

[2] Y. Eliashberg: *Invariants in contact topology*; in Proceedings of the International Congress of Mathematicians, II (Berlin, 1998), Doc. Math., 1998, 327–338

[3] J. Epstein and D. Fuchs: *On the invariants of Legendrian mirror torus links*; in Symplectic and Contact Topology: Interactions and Perspectives (Toronto, ON/Montreal, QC, 2001), Fields Inst. Commun. **35**, Amer. Math. Soc., Providence, RI, 2003, 103–115.

[4] D. Fuchs: *Chekanov–Eliashberg invariant of Legendrian knots*: *existence of augmentations*, J. Geom. Phys. **47** (2003), 43–65.

[5] D. Fuchs and T. Ishkhanov: *Invariants of Legendrian knots and decompositions of front diagrams*, Mosc. Math. J. **4** (2004), 707–717, 783.

[6] D. Fuchs and S. Tabachnikov: *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997), 1025–1053.

[7] L.H. Kauffman: *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[8] T. Kálmán: *Maximal Thurston–Bennequin number of +adequate links*, arXiv:math/0610659.

[9] A. Kawauchi: A Survey of Knot Theory, Translated and revised from the 1990 Japanese original by the author, Birkhäuser, Basel, 1996.

[10] L. Ng: *Maximal Thurston–Bennequin number of two-bridge links*, Algebr. Geom. Topol. **1** (2001), 427–443.

[11] L. Ng: *A Legendrian Thurston–Bennequin bound from Khovanov homology*, Algebr. Geom. Topol. **5** (2005), 1637–1653.

[12] J. Rasmussen: *Khovanov homology and the slice genus*, arXiv:math.GT/0402131, to appear in Invent. Math.

[13] D. Rutherford: *Thurston–Bennequin number, Kauffman polynomial, and ruling invariants of a Legendrian link*: *the Fuchs conjecture and beyond*, Int. Math. Res. Not. (2006), Art. ID 78591, 15pp.

[14] L. Rudolph: *An obstruction to sliceness via contact geometry and "classical" gauge theory*, Invent. Math. **119** (1995), 155–163.

[15] L. Rudolph: *The slice genus and the Thurston–Bennequin invariant of a knot,* Proc. Amer. Math. Soc. **125** (1997), 3049–3050.
[16] L. Rudolph: *An obstruction to sliceness via contact geometry and "classical" gauge theory*, Invent. Math. **119** (1995), 155–163.
[17] J.M. Sabloff: *Augmentations and rulings of Legendrian knots*, Int. Math. Res. Not. (2005), 1157–1180.
[18] A. Stoimenow: *Some inequalities between knot invariants,* Internat. J. Math. **13** (2002), 373–393.
[19] J. Świątkowski: *On the isotopy of Legendrian knots,* Ann. Global Anal. Geom. **10** (1992), 195–207.
[20] S. Tabachnikov: *Estimates for the Bennequin number of Legendrian links from state models for knot polynomials*, Math. Res. Lett. **4** (1997), 143–156.
[21] T. Tanaka: *Maximal Bennequin numbers and Kauffman polynomials of positive links*, Proc. Amer. Math. Soc. **127** (1999), 3427–3432.
[22] T. Tanaka: *Maximal Thurston–Bennequin numbers of alternating links*, Topology Appl. **153** (2006), 2476–2483.

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