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INVERSE PROBLEM OF A TIME-DEPENDENT GINGZBURG–LANDAU MODEL FOR SUPERCONDUCTIVITY WITH THE FINAL OVERDETERMINATION

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Abstract

In this paper we prove the existence, uniqueness and stability of solutions of an inverse problem to a time-dependent Ginzburg–Landau model for superconductivity with the final overdetermination.

1. Introduction

We consider an inverse problem of the following Ginzburg–Landau equations for superconductivity under the Coulomb gauge:

$$\eta\psi_t + i\eta k\phi\psi + \left(\frac{i}{k}\nabla + A\right)^2\psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

$$A_t + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\left\{\left(\frac{i}{k}\nabla\psi + \psi A\right)\bar{\psi}\right\} = \operatorname{curl} H, \tag{1.2}$$

$$\int_\Omega \phi\, dx = 0, \quad \operatorname{div} A = 0 \quad \text{in} \quad \Omega \times (0, T) \tag{1.3}$$

with boundary and initial conditions

$$\nabla\psi \cdot \nu = 0, \quad A \cdot \nu = 0, \quad \operatorname{curl} A \times \nu = H \times \nu \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.4}$$

$$\psi(\,\cdot\,, 0) = \psi_0, \quad A(\,\cdot\,, 0) = A_0 \quad \text{in} \quad \Omega \subset \mathbb{R}^3 \tag{1.5}$$

and the final overdetermination

$$A(x, T) = \chi(x). \tag{1.6}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with smooth boundary $\partial\Omega$, ν is the unit outward normal vector of $\partial\Omega$, $i = \sqrt{-1}$, $\bar{\psi}$ denotes the complex conjugate of ψ, $\operatorname{Re}\psi := (\psi + \bar{\psi})/2$. Also, ψ, A and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^3$-valued and

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$\mathbb{R}$-valued functions, and they are the order parameter, the magnetic potential and the electric potential, respectively. Moreover, $H(x)$ is the applied magnetic field which is to be determined, η, k are the Ginzburg–Landau positive constants and $|\psi|^2$ is the density of superconducting carriers. In (1.1)–(1.6), the unknown functions are ψ, A, ϕ and $H(x)$.

For given initial data $\psi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $A_0 \in H^1(\Omega)$, Chen, Elliott, Tang and Du [2, 3, 5, 15] proved the existence and uniqueness of global strong solutions to (1.1)–(1.5) in the case of Coulomb and Lorentz as well as temporal gauges. The regularity of solutions has been studied by Liang [10].

For given initial data $\psi_0 \in H^1(\Omega)$, $A_0 \in H^1(\Omega)$, Tang and Wang [16] studied the Coulomb gauge case and proved the existence and uniqueness of strong solutions to (1.1)–(1.5). Very recently, Fan and Jiang [6] proved the existence of global weak solutions when $(\psi_0, A_0) \in L^2(\Omega) \times L^2(\Omega)$ in the case of Coulomb gauge or Lorentz gauge, which answered an open problem in [16]. Zaouch [18] proved the existence of time-periodic solutions to (1.1)–(1.4). Phillips and Shin [12], Chen and Hoffmann [4] proved the existence and uniqueness of classical solutions to the non-isothermal models for superconductivity.

In this paper, we study the nonlinear inverse problem consisting of finding a set of the functions $\{\psi, A, \phi, H\}$ satisfying (1.1)–(1.6). This is an inverse problems with the final overdetermination. There are many studies on inverse problem for final overdetermination for parabolic equations and Navier–Stokes equations [1, 7, 8, 9, 13, 14].

Unless otherwise stated, we always assume
(H1) $\psi_0 \in H^2(\Omega)$, $|\psi_0| \le 1$ in Ω, $\nabla\psi_0 \cdot \nu = 0$ on $\partial\Omega$,
(H2) $A_0 \in H^2(\Omega)$, $\operatorname{div} A_0 = 0$ in Ω, $A_0 \cdot \nu = 0$, $\operatorname{curl} A_0 \times \nu = H \times \nu$ on $\partial\Omega$,
(H3) $\chi \in H^2(\Omega)$, $\operatorname{div} \chi = 0$ in Ω, $\chi \cdot \nu = 0$, $\operatorname{curl} \chi \times \nu = H \times \nu$ on $\partial\Omega$,
through this paper.

We first give the existence and uniqueness result to the direct problem (1.1)–(1.5).

Theorem 1.1. *Let* $\psi_0 \in H^1(\Omega)$, $|\psi_0| \le 1$ *in* Ω, $A_0 \in H^1(\Omega)$, $\operatorname{div} A_0 = 0$ *in* Ω, $A_0 \cdot \nu = 0$ *on* $\partial\Omega$, $H \in L^2(\Omega)$. *Then there exists a unique solution* (ψ, A, ϕ) *to* (1.1)–(1.5) *which satisfies*

$$\begin{cases} \psi \in L^\infty((0,T); H^1(\Omega)) \cap L^2((0,T); H^2(\Omega)), \\ \psi_t \in L^2(\Omega \times ((0,T))), \\ |\psi| \le 1 \quad \text{in} \quad \Omega \times (0,T), \end{cases}$$

$$\begin{cases} A \in L^\infty((0,T); H^1(\Omega)) \cap L^2((0,T); H^2(\Omega)), \\ A_t \in L^2(\Omega \times ((0,T))), \\ \operatorname{div} A = 0 \quad \text{in} \quad \Omega \times (0,T) \end{cases}$$

and $\phi \in L^\infty((0,T); H^1(\Omega))$.

Moreover, let (H1) *and* (H2) *be satisfied and* curl $H \in L^2(\Omega)$, *then*

$$\psi \in L^\infty((0, T); H^2(\Omega)), \quad \psi_t \in L^\infty((0, T); L^2(\Omega)),$$
$$A \in L^\infty((0, T); H^2(\Omega)), \quad A_t \in L^\infty((0, T); L^2(\Omega)),$$
$$\phi \in L^\infty((0, T); H^2(\Omega)).$$

REMARK 1. The existence and uniqueness part has been proved in [1]. The regularity result has been proved in [10].

In order to determine $H(x)$ uniquely, we further assume:

$$\operatorname{div} H = 0 \quad \text{in} \quad \Omega, \quad H \cdot \nu = 0 \quad \text{on} \quad \partial\Omega. \tag{1.7}$$

Let $\xi \in L^2(\Omega)$ be such that

$$\operatorname{div} \xi = 0 \quad \text{a.e.} \quad \text{in} \quad \Omega \quad \text{and} \quad \int_{\partial\Omega} \xi \cdot \nu \, d\sigma = 0. \tag{1.8}$$

Then, there exists a unique solution $H(x) \in H^1(\Omega)$ to

$$\begin{cases} \operatorname{curl} H = \xi, & \text{in} \quad \Omega, \\ \operatorname{div} H = 0, & \text{in} \quad \Omega, \\ H \cdot \nu = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{1.9}$$

with the estimates ([11, 17]):

$$\|H\|_{H^1(\Omega)} \le C_1 \|\xi\|, \quad \sqrt{\lambda_1}\|H\| \le \|\xi\|, \tag{1.10}$$

for some positive constant λ_1, $C_1 > 0$ independent of ξ and the first Dirichlet eigenvalue λ_1 of $-\Delta$ in Ω depending only on $|\Omega|$. Here $\|\cdot\|$ stands for the $L^2(\Omega)$ norm of scalar valued functions and vector valued functions throughout this paper, and we sometimes suppress Ω for function spaces $H^1(\Omega)$ etc.

For this H, we define the nonlinear operator

$$B\colon L^2(\Omega) \to L^2(\Omega)$$

by

$$\begin{aligned}(B\xi)(x) := {} & A_t(x, T) + \nabla\phi(x, T) + \operatorname{curl}^2 \chi \\ & + \operatorname{Re}\left[\left(\frac{i}{k}\nabla\psi(x, T) + \psi(x, T)A(x, T)\right)\bar{\psi}(x, T)\right],\end{aligned} \tag{1.11}$$

where ψ, A, ϕ are those which can be found as the unique solution of the system (1.1)–(1.5). We will proceed to study the operator equation of the second kind in the space $L^2(\Omega)$:

$$\xi = B\xi. \tag{1.12}$$

The relation between the inverse problem (1.1)–(1.7) and solvability of the nonlinear equation (1.12) is revealed in the following assertion.

Theorem 1.2. *If equation* (1.12) *has a solution, then there exists a solution of the inverse problem* (1.1)–(1.7).

Proof. The proof is the same as that in [13, pp. 244–245], hence, we omit it here. □

We will use the Tikhonov fixed point theorem to prove that (1.12) has a solution. For the reader's convenience, we recall the Tikhonov theorem.

Theorem 1.3 (Tikhonov theorem). *Let D be a nonempty closed convex subset of a separable reflexive Banach space* X *and let B: $D \to D$ be a weakly continuous mapping* (*i.e. if $x_n \in D$, $x_n \rightharpoonup x$ weakly in* X*, then $Bx_n \rightharpoonup Bx$ weakly in* X *as well*)*. Then B has at least one fixed point in D.*

Now we are in a position to state our main theorem:

Theorem 1.4. *Let* (H1)–(H3) *be satisfied and the constant λ_1 is large enough, then there exists a unique solution $\{\psi, A, \phi, H\}$ to the inverse problem* (1.1)–(1.7)*. Moreover, let $(\psi_i, A_i, \phi_i, H_i)$ $(i = 1, 2)$ be the unique solution to the inverse problem* (1.1)–(1.7) *corresponding to the input data $(\psi_{0i}, A_{0i}, \chi_i)$, then*

$$\begin{aligned} &\|\psi_1 - \psi_2\|_{H^1} + \|A_1 - A_2\|_{H^1} + \|\phi_1 - \phi_2\|_{H^1} + \|H_1 - H_2\|_{H^1} \\ &\quad \le O(1)(\|\psi_{01} - \psi_{02}\|_{H^1} + \|A_{01} - A_{02}\|_{H^2} + \|\chi_1 - \chi_2\|_{H^2}) \quad \textit{in} \quad (0, T). \end{aligned} \tag{1.13}$$

REMARK 2. (i) Similar results can be proved when initial data $\psi_0 \in H^1(\Omega)$ and $A_0 \in H^1(\Omega)$, $\operatorname{div} A_0 = 0$ in Ω, $A_0 \cdot \nu = 0$ on $\partial\Omega$ are given only.
(ii) The assumption that λ_1 is large enough was needed in [1] for inverse parabolic problem with final overdetermination.
(iii) The largeness required for λ_1 can be calculated clearly in the following proofs.

In the next section, we give some preliminaries to the proof of Theorem 1.4, which is given in Section 3.

2. Preliminaries

In this section, we provide some estimates for the solution (ψ, A, ϕ) to (1.1)–(1.5). To begin with, we state the maximum principle for $|\psi|$ and the Gibbs free energy:

$$G(\psi, A) := \frac{1}{2}\int_\Omega \left|\frac{i}{k}\nabla\psi + \psi A\right|^2 dx + \frac{1}{2}\int_\Omega (\mathrm{curl}\, A - H)^2 dx + \frac{1}{4}\int_\Omega (|\psi|^2 - 1)^2 dx. \tag{2.1}$$

Lemma 2.1.

$$|\psi| \le 1 \quad in \quad \Omega \times (0, T), \tag{2.2}$$

$$G(\psi, A) + \int_0^t \int_\Omega \eta|\psi_t + ik\phi\psi|^2 dx\, ds + \int_0^t\int_\Omega [A_t^2 + (\nabla\phi)^2]\, dx\, ds = G(\psi_0, A_0) \quad in \quad (0, T). \tag{2.3}$$

Proof. Since the proof can be found in [3], we omit the details here. □

Corollary 2.2.

$$\|\mathrm{curl}\, A\| \le \|\mathrm{curl}\, A_0\| + \left\|\frac{i}{k}\nabla\psi_0 + \psi_0 A_0\right\| + \||\psi_0|^2 - 1\| + \frac{2}{\sqrt{\lambda_1}}\|\xi\|, \tag{2.4}$$

$$\|\nabla\psi\| \le \left(1 + \frac{1}{\sqrt{\lambda_1}}\right)k\left[\|\mathrm{curl}\, A_0\| + \left\|\frac{i}{k}\nabla\psi_0 + \psi_0 A_0\right\| + \||\psi_0|^2 - 1\|\right] + \left(1 + \frac{2}{\sqrt{\lambda_1}}\right)\frac{k}{\sqrt{\lambda_1}}\|\xi\| \tag{2.5}$$

in $(0, T)$ *and*

$$\int_0^T\int_\Omega \eta|\psi_t|^2 dx\, dt \le 2\left(1 + \frac{\eta k^2}{\lambda_1}\right)G(\psi_0, A_0). \tag{2.6}$$

Proof. (2.4), (2.5) and (2.6) easily follow from (2.1), (2.2), (2.3) and the following Poincaré type inequalities:

$$\sqrt{\lambda_1}\|A\| \le \|\mathrm{curl}\, A\|, \quad \sqrt{\lambda_1}\|\phi\| \le \|\nabla\phi\| \quad \text{in} \quad (0, T).$$ □

Lemma 2.3. *ϕ satisfies*

$$-\Delta\phi = \operatorname{div}\operatorname{Re}\left[\left(\frac{i}{k}\nabla\psi + \psi A\right)\bar{\psi}\right], \quad \int_\Omega \phi\, dx = 0, \tag{2.7}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.8}$$

and

$$\|\nabla\phi(\cdot, t)\| \le \left\|\frac{i}{k}\nabla\psi(\cdot, t) + \psi(\cdot, t)A(\cdot, t)\right\|, \tag{2.9}$$

$$\|\nabla\phi_t\| \le \left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^6}\|\psi_t\|_{L^3} + \frac{1}{k}\|\nabla\psi_t\| + \|A\|_{L^6}\|\psi_t\|_{L^3} + \|A_t\| \tag{2.10}$$

in $(0, T)$.

Proof. By applying div to both sides of (1.2), we have (2.7). The proof of (2.8) is given in [6]. Multiplying (2.7) by ϕ and integrating by parts imply (2.9). Finally, we have (2.10) by differentiating (2.7) with respect to time t and then multiplying by ϕ_t and integrating by parts. □

In the following proofs, we will frequently use the following inequalities:

$$\|\nabla\psi\|_{L^3} \le C_0\|\nabla\psi\|^{1/2}\|\Delta\psi\|^{1/2}, \quad \|\psi_t\|_{L^3} \le C_0\|\psi_t\|^{1/2}\|\nabla\psi_t\|^{1/2} + C_0\|\psi_t\| \quad \text{in} \quad (0, T)$$

which follow from Gagliardo–Nirenberg inequality and the

$$\|\nabla\psi\|_{L^6} \le C_0\|\Delta\psi\|, \quad \|A\|_{L^6} \le C_0\|\operatorname{curl} A\| \quad \text{in} \quad (0, T)$$

which follow from [17] and Poincaré inequality, C_0 denotes an absolute positive constant throughout this paper.

Lemma 2.4.

$$\|\Delta\psi\| \le 2\eta k^2\|\psi_t + ik\phi\psi\| + 2k^2|\Omega|^{1/2} + 2k^2\|A\|_{L^4}^2 + 4C_0^2k^2\|A\|_{L^6}^2\|\nabla\psi\|. \tag{2.11}$$

Proof. We rewrite (1.1) in the form:

$$\eta\psi_t + i\eta k\phi\psi - \frac{1}{k^2}\Delta\psi + \frac{2}{k}iA\nabla\psi + A^2\psi + (|\psi|^2 - 1)\psi = 0. \tag{2.12}$$

Multiplying (2.12) by $-\Delta\bar{\psi}$ and integrating by parts, then taking the real part, we have

$$\frac{1}{k^2}\|\Delta\psi\|^2 \le \eta\|\psi_t + ik\phi\psi\|\,\|\Delta\psi\| + 2\|\Delta\psi\|\,|\Omega|^{1/2} + \|A\|_{L^4}^2\|\Delta\psi\| + \frac{2}{k}\|A\|_{L^6}\|\nabla\psi\|_{L^3}\|\Delta\psi\|$$

and hence

$$\|\Delta\psi\| \le \eta k^2\|\psi_t + ik\phi\psi\| + k^2|\Omega|^{1/2} + k^2\|A\|_{L^4}^2 + 2k\|A\|_{L^6}\cdot C_0\|\nabla\psi\|^{1/2}\|\Delta\psi\|^{1/2}.$$

Then, we have (2.11) if we estimate the last term as follows:

$$2k\|A\|_{L^6}C_0\|\nabla\psi\|^{1/2}\|\Delta\psi\|^{1/2} \le \frac{1}{2}\|\Delta\psi\| + 2C_0^2k^2\|A\|_{L^6}^2\|\nabla\psi\|. \qquad \square$$

Lemma 2.5. *If λ_1 is large enough, then*

$$\begin{aligned}\|A_t(\cdot\,,T)\| \le \|\xi\|e^{-d_*T/2} + \|\mathrm{curl}^2 A_0\| + 3\left\|\frac{i}{k}\nabla\psi_0 + \psi_0A_0\right\| \\ + \frac{1}{\sqrt{\eta}}\left\|\left(\frac{i}{k}\nabla + A_0\right)^2\psi_0 + (|\psi_0|^2-1)\psi_0\right\| \\ + \frac{8}{\sqrt{\eta}}(2+2d_1+d_6+\eta)^{1/2}G(\psi_0, A_0) \quad \text{in} \quad (0,T),\end{aligned} \tag{2.13}$$

where d_1, d_6 and d_* are some positive constants depending on the initial data.

Proof. Applying $\partial/\partial t$ to (1.2), we see that

$$A_{tt} + \nabla\phi_t + \mathrm{curl}^2 A_t + \mathrm{Re}\left[\left(\frac{i}{k}\nabla\psi + \psi A\right)\bar{\psi}_t + \left(\frac{i}{k}\nabla\psi_t + \psi_tA + \psi A_t\right)\bar{\psi}\right] = 0 \quad \text{in} \quad (0,T).$$

Multiplying the above equation by A_t and integrating by parts, and using

$$\int_\Omega \frac{i}{k}\nabla\psi_t\cdot\bar{\psi}A_t\,dx = -\frac{i}{k}\int_\Omega \psi_t\nabla\bar{\psi}\cdot A_t\,dx,$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega A_t^2\,dx + \int_\Omega(\mathrm{curl}\,A_t)^2\,dx + \int_\Omega|\psi|^2A_t^2\,dx \\ &+ \mathrm{Re}\int_\Omega\left[\left(\frac{i}{k}\nabla\psi + \psi A\right)\bar{\psi}_tA_t + \left(-\frac{i}{k}\nabla\bar{\psi} + \bar{\psi}A\right)\psi_tA_t\right]dx = 0 \quad \text{in} \quad (0,T)\end{aligned}$$

and hence

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega A_t^2\,dx + \int_\Omega (\text{curl } A_t)^2\,dx + \int_\Omega |\psi|^2 A_t^2\,dx \\
&= -2\,\text{Re}\int_\Omega \left(\frac{i}{k}\nabla\psi + \psi A\right)\bar{\psi}_t A_t\,dx \\
&\le 2\left\|\frac{i}{k}\nabla\psi + \psi A\right\| \|\psi_t\|_{L^3}\|A_t\|_{L^6} \\
&\le 2\left\|\frac{i}{k}\nabla\psi + \psi A\right\| \|\psi_t\|_{L^3}\cdot C_0\|\text{curl } A_t\| \\
&\le \frac{1}{6}\|\text{curl } A_t\|^2 + 6C_0^2\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^2 \|\psi_t\|_{L^3}^2 \qquad (2.14)\\
&\le \frac{1}{6}\|\text{curl } A_t\|^2 + 6C_0^3\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^2 (\|\psi_t\|\,\|\nabla\psi_t\| + \|\psi_t\|^2) \\
&\le \frac{1}{6}\|\text{curl } A_t\|^2 + \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 \\
&\quad + \left[(3C_0^3)^2 12k^2\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^4 + 6C_0^3\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^2\right]\|\psi_t\|^2 \\
&\le \frac{1}{6}\|\text{curl } A_t\|^2 + \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + d_1\|\psi_t\|^2 \quad \text{in} \quad (0, T)
\end{aligned}
$$

for some constant $d_1 > 0$ which can be bounded as follows:

$$
d_1 \le (3C_0^3)^2 12k^2 \cdot 4G^2(\psi_0, A_0) + 6C_0^3 \cdot 2G(\psi_0, A_0).
$$

Now, differentiating (2.12) with respect to time t, we have

$$
\begin{aligned}
&\eta\psi_{tt} + i\eta k\phi\psi_t + i\eta k\phi_t\psi - \frac{1}{k^2}\Delta\psi_t + 2\left(\frac{i}{k}\nabla\psi + \psi A\right)A_t \\
&+ \frac{2}{k} iA\nabla\psi_t + A^2\psi_t + 2|\psi|^2\psi_t + \psi^2\bar{\psi}_t = \psi_t.
\end{aligned}
$$

Multiplying the above equation by $\bar{\psi}_t$ and integrating by parts, then taking the real part, we obtain

$$
\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int_\Omega |\psi_t|^2\,dx + \frac{1}{k^2}\int_\Omega |\nabla\psi_t|^2\,dx + \int_\Omega |\psi_t|^2 A^2\,dx + \int_\Omega |\psi|^2|\psi_t|^2\,dx \\
&\le \eta k\|\phi_t\|\,\|\psi_t\| + 2\left\|\frac{i}{k}\nabla\psi + \psi A\right\| \|A_t\|_{L^6}\|\psi_t\|_{L^3} \\
&\quad + \frac{2}{k}\|\nabla\psi_t\|\,\|A\|_{L^6}\|\psi_t\|_{L^3} + \|\psi_t\|^2 \quad \text{in} \quad (0, T).
\end{aligned}
$$

Then, it follows from (2.10) that

$$
\begin{aligned}
&\frac{\eta}{2}\frac{d}{dt}\int_\Omega |\psi_t|^2 dx + \frac{1}{k^2}\int_\Omega |\nabla\psi_t|^2 dx + \int_\Omega |\psi_t|^2 A^2 dx + \int_\Omega |\psi|^2|\psi_t|^2 dx \\
&\le \frac{\eta k}{\sqrt{\lambda_1}}\left[\left\|\frac{i}{k}\nabla\psi + \psi A\right\|_{L^6}\|\psi_t\|_{L^3} + \frac{1}{k}\|\nabla\psi_t\| + \|A\|_{L^6}\|\psi_t\|_{L^3} + \|A_t\|\right]\|\psi_t\| \\
&\quad + 2\left\|\frac{i}{k}\nabla\psi + \psi A\right\|\|A_t\|_{L^6}\|\psi_t\|_{L^3} + \frac{2}{k}\|\nabla\psi_t\|\,\|A\|_{L^6}\|\psi_t\|_{L^3} + \|\psi_t\|^2 \\
&\le \frac{\eta}{\sqrt{\lambda_1}}\|\nabla\psi\|_{L^6}\|\psi_t\|_{L^3}\|\psi_t\| + 2\frac{\eta k}{\sqrt{\lambda_1}}\|A\|_{L^6}\|\psi_t\|_{L^3}\|\psi_t\| \qquad (2.15)\\
&\quad + \frac{\eta}{\sqrt{\lambda_1}}\|\nabla\psi_t\|\,\|\psi_t\| + \frac{\eta k}{\sqrt{\lambda_1}}\|A_t\|\,\|\psi_t\| \\
&\quad + 2\left\|\frac{i}{k}\nabla\psi + \psi A\right\|\|A_t\|_{L^6}\|\psi_t\|_{L^3} + \frac{2}{k}\|\nabla\psi_t\|\,\|A\|_{L^6}\|\psi_t\|_{L^3} + \|\psi_t\|^2 \\
&=: \sum_{i=1}^{7} I_i \quad \text{in} \quad (0, T).
\end{aligned}
$$

Here, each term I_i can be estimated as follows

$$
\begin{aligned}
I_1 &\le \frac{\eta}{\sqrt{\lambda_1}}C_0\|\Delta\psi\|\cdot C_0(\|\psi_t\|^{1/2}\|\nabla\psi_t\|^{1/2} + \|\psi_t\|)\|\psi_t\| \\
&\le \frac{\eta}{\sqrt{\lambda_1}}C_0^2\|\Delta\psi\|\,\|\nabla\psi_t\|^{1/2}\|\psi_t\|^{3/2} + \frac{\eta}{\sqrt{\lambda_1}}C_0^2\|\Delta\psi\|\,\|\psi_t\|^2 \\
&\le \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + \left[\frac{3}{4}(3k^2)^{1/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3}C_0^{8/3}\|\Delta\psi\|^{4/3} + \frac{\eta}{\sqrt{\lambda_1}}C_0^2\|\Delta\psi\|\right]\|\psi_t\|^2 \\
&\le \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + \left[\frac{3}{4}(3k^2)^{1/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3}C_0^{8/3}\left(\frac{1}{3} + \frac{2}{3}\|\Delta\psi\|^2\right)\right. \\
&\qquad\qquad \left. + \frac{\eta}{\sqrt{\lambda_1}}C_0^2\left(\frac{1}{2} + \frac{1}{2}\|\Delta\psi\|^2\right)\right]\|\psi_t\|^2 \\
&= \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + \left[\frac{1}{2}(3k^2)^{1/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3}C_0^{8/3} + \frac{1}{2}\frac{\eta}{\sqrt{\lambda_1}}C_0^2\right]\|\Delta\psi\|^2\|\psi_t\|^2 \\
&\quad + \left[\frac{1}{4}(3k^2)^{1/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3}C_0^{8/3} + \frac{1}{2}\frac{\eta}{\sqrt{\lambda_1}}C_0^2\right]\|\psi_t\|^2 \quad \text{in} \quad (0, T)
\end{aligned}
$$

and we have

$$
\|\Delta\psi\|^2 \le 8\eta^2k^4\|\psi_t + ik\phi\psi\|^2 + 32k^4|\Omega| + 8k^4\|A\|_{L^4}^4 + 16C_0^4k^4\|A\|_{L^6}^4\|\nabla\psi\|^2 \quad \text{in} \quad (0, T).
$$

Hence

$$
\begin{aligned}
I_1 \leq\ & \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + \left[(3k^2)^{1/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3} C_0^{8/3} + \frac{\eta}{\sqrt{\lambda_1}}C_0^2\right]4\eta^2k^4\|\psi_t + ik\phi\psi\|^2\|\psi_t\|^2 \\
& + \left\{\left[(3k^2)^{16/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3} C_0^{8/3} + \frac{\eta}{\sqrt{\lambda_1}}C_0^2\right]\right. \\
& \qquad \times (8k^4|\Omega| + 4k^4\|A\|_{L^4}^4 + 8C_0^4k^4\|A\|_{L^6}^4\|\nabla\psi\|^2) \\
& \qquad \left. + \left[\frac{1}{4}(3k^2)^{1/3}\left(\frac{\eta}{\sqrt{\lambda_1}}\right)^{4/3} C_0^{8/3} + \frac{1}{2}\frac{\eta}{\sqrt{\lambda_1}}C_0^2\right]\right\}\|\psi_t\|^2 \\
=\ & \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + d_0\|\eta\psi_t + i\eta k\phi\psi\|^2\|\psi_t\|^2 + d_2\|\psi_t\|^2 \quad \text{in} \quad (0, T),
\end{aligned}
$$

where d_0 and d_2 are positive constants such that $d_0 = O(1/\sqrt{\lambda_1})$, $d_2 = O(1/\sqrt{\lambda_1})$ $(\lambda_1 \to \infty)$. As for I_2 and I_3, we have

$$
\begin{aligned}
I_2 &\leq 2\frac{\eta k}{\sqrt{\lambda_1}}\|A\|_{L^6} \cdot C_0\|\psi_t\|^{3/2}\|\nabla\psi_t\|^{1/2} \\
&\leq \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + \frac{3}{4}(3C_0k^2)^{1/3}\left(\frac{2\eta k}{\sqrt{\lambda_1}}\right)^{4/3}\|A\|_{L^6}^{4/3}\|\psi_t\|^2 \\
&\leq \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + d_3\|\psi_t\|^2 \quad \text{in} \quad (0, T),
\end{aligned}
$$

and

$$
I_3 \leq \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + 3k^2\frac{\eta^2}{\lambda_1}\|\psi_t\|^2 = \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + d_4\|\psi_t\|^2 \quad \text{in} \quad (0, T),
$$

where $d_3 = O((1/\sqrt{\lambda_1})^{4/3})$ and $d_4 = O(1/\lambda_1)$. Also, for I_4, I_5 and I_6, we have

$$
\begin{aligned}
I_4 &\leq \frac{\eta k}{\lambda_1}\|\text{curl } A_t\|\,\|\psi_t\| \leq \frac{1}{6}\|\text{curl } A_t\|^2 + \frac{6}{4}\frac{(\eta k)^2}{\lambda_1^2}\|\psi_t\|^2 \\
&\leq \frac{1}{6}\|\text{curl } A_t\|^2 + d_5\|\psi_t\|^2 \quad \text{in} \quad (0, T), \\
I_5 &\leq \frac{1}{6}\|\text{curl } A_t\|^2 + \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 \\
&\quad + \left[(3C_0^3)^2 12k^2\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^4 + 6C_0^4\left\|\frac{i}{k}\nabla\psi + \psi A\right\|^2\right]\|\psi_t\|^2 \\
&\leq \frac{1}{6}\|\text{curl } A_t\|^2 + \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + d_1\|\psi_t\|^2 \quad \text{in} \quad (0, T),
\end{aligned}
$$

$$
\begin{aligned}
I_6 &\le \frac{2}{k}\|\nabla\psi_t\|\cdot C_0\|\mathrm{curl}\,A\|\cdot C_0(\|\psi_t\|^{1/2}\|\nabla\psi_t\|^{1/2}+\|\psi_t\|)\\
&\le \frac{2}{k}C_0^2\|\mathrm{curl}\,A\|\,\|\psi_t\|\,\|\nabla\psi_t\| + \frac{2}{k}C_0^2\|\mathrm{curl}\,A\|\,\|\psi_t\|^{1/2}\|\nabla\psi_t\|^{3/2}\\
&\le \frac{1}{24}\frac{1}{k^2}\|\nabla\psi_t\|^2 + 24C_0^4\|\mathrm{curl}\,A\|^2\|\psi_t\|^2\\
&\quad + \frac{1}{24}\frac{1}{k^2}\|\nabla\psi_t\|^2 + 4(18)^3C_0^8k^2\|\mathrm{curl}\,A\|^4\|\psi_t\|^2\\
&= \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + [24C_0^4\|\mathrm{curl}\,A\|^2 + 4(18)^3C_0^8k^2\|\mathrm{curl}\,A\|^4]\|\psi_t\|^2\\
&= \frac{1}{12}\frac{1}{k^2}\|\nabla\psi_t\|^2 + d_6\|\psi_t\|^2 \quad \text{in} \quad (0,T)
\end{aligned}
$$

where d_1, d_5 and d_6 are positive constants such that $d_1 = O(1/\lambda_1)$, $d_5 = O(1/\lambda_1^2)$ $(\lambda_1 \to \infty)$ and d_6 can be bounded by (2.4).

Using these estimates for (2.15) and combining (2.14), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (A_t^2 + \eta|\psi_t|^2)\,dx + \frac{1}{2}\int_\Omega\left[(\mathrm{curl}\,A_t)^2 + \frac{1}{k^2}|\nabla\psi_t|^2\right]\\
&\le d_0\|\eta\psi_t + i\eta k\phi\psi\|^2\|\psi_t\|^2 + \left(1+\sum_{i=1}^{6} d_i\right)\|\psi_t\|^2 \quad \text{in} \quad (0,T).
\end{aligned} \tag{2.16}
$$

Then, adding $(1/2)\eta\int_\Omega|\psi_t|^2dx$ on both sides of (2.16) and setting $d_* := \min(\lambda_1,\eta)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (A_t^2 + \eta|\psi_t|^2)\,dx + d_*\int_\Omega (A_t^2 + \eta|\psi_t|^2)\,dx\\
&\le 2d_0\|\eta\psi_t + i\eta k\phi\psi\|^2\|\psi_t\|^2 + 2\left(1+\eta+\sum_{i=1}^{6} d_i\right)\|\psi_t\|^2\\
&\le \frac{2d_0}{\eta}\|\eta\psi_t + i\eta k\phi\psi\|^2\cdot\int_\Omega (A_t^2 + \eta|\psi_t|^2)\,dx + 2\left(1+\eta+\sum_{i=1}^{6} d_i\right)\|\psi_t\|^2\\
&\qquad\qquad \text{in} \quad (0,T).
\end{aligned}
$$

Solving this inequality with respect to $\int_\Omega (A_t^2 + \eta|\psi_t|^2)\,dx$, we have

$$
\begin{aligned}
&\int_\Omega (A_t^2 + \eta|\psi_t|^2)\,dx\\
&\le \int (A_t^2(x,0) + \eta|\psi_t(x,0)|^2)\,dx\cdot e^{-d*t}\cdot\exp\left(\int_0^t \frac{2d_0}{\eta}\|\eta\psi_t + i\eta k\phi\psi\|^2\,dt\right)\\
&\quad + 2\left(1+\eta+\sum_{i=1}^{6} d_i\right)\int_0^T \|\psi_t\|^2dt\cdot\exp\left(\frac{2d_0}{\eta}\int_0^T \|\eta\psi_t + i\eta k\phi\psi\|^2dt\right)\\
&\qquad\qquad \text{in} \quad (0,T).
\end{aligned}
$$

This implies that if λ_1 is large enough,

$$
\begin{aligned}
&\|A_t(\cdot, T)\| \\
&\le (\|A_t(\cdot,0)\| + \sqrt{\eta}\|\psi_t(\cdot,0)\|)e^{-d_* T/2} \\
&\quad + 2\left(1+\eta+\sum_{i=1}^{6} d_i\right)^{1/2} \left(\int_0^T \|\psi_t\|^2 dt\right)^{1/2} \\
&\quad \times \exp\left(\frac{d_0}{\eta}\int_0^T \|\eta\psi_t + i\eta k\phi\psi\|^2 dt\right) \\
&\le \left(\|\mathrm{curl}^2\, A_0\| + 2\left\|\frac{i}{k}\nabla\psi_0 + \psi_0 A_0\right\| + \|\xi\| + \sqrt{\eta}k\cdot\frac{1}{\sqrt{\lambda_1}}\left\|\frac{i}{k}\nabla\psi_0 + \psi_0 A_0\right\|\right. \\
&\quad \left. + \frac{1}{\sqrt{\eta}}\left\|\left(\frac{i}{k}\nabla + A_0\right)^2\psi_0 + (|\psi_0|^2-1)\psi_0\right\|\right)e^{-d_* T/2} \\
&\quad + \frac{2}{\sqrt{\eta}}(2+2d_1+d_6+\eta)^{1/2}\cdot 2\left(1+\frac{\eta k^2}{\lambda_1}\right)G(\psi_0, A_0)\cdot\exp\left(\frac{d_0}{\eta}G(\psi_0, A_0)\right) \\
&\le \|\xi\|\cdot e^{-d_* T/2} + \|\mathrm{curl}^2\, A_0\| + 3\left\|\frac{i}{k}\nabla\psi_0 + \psi_0 A_0\right\| \\
&\quad + \frac{1}{\sqrt{\eta}}\left\|\left(\frac{i}{k}\nabla + A_0\right)^2\psi_0 + (|\psi_0|^2-1)\,\psi_0\right\| \\
&\quad + \frac{8}{\sqrt{\eta}}(2+2d_1+d_6+\eta)^{1/2}G(\psi_0, A_0).
\end{aligned}
$$

This completes the proof of the lemma. □

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 using the estimates in Section 2. Let

$$
D := \left\{h \in L^2(\Omega)\ \middle|\ \mathrm{div}\, h = 0 \text{ in } \Omega,\ \int_{\partial\Omega} h\cdot\nu\, d\sigma = 0,\ \|h\| \le R\right\}, \tag{3.1}
$$

where $R := R(\psi_0, A_0, \chi, \eta, k, T)$ is a positive constant which will be specified later in the proof of the next lemma.

Lemma 3.1. *If λ_1 is large enough, then there exists a positive constant R such that B maps D into itself.*

Proof. First of all, it is easy to show that

$$
\mathrm{div}(B\xi) = 0 \quad \text{in} \quad \Omega, \quad \int_{\partial\Omega} \xi\cdot\nu\, d\sigma = 0.
$$

Then together with this and (1.1), (2.3), (2.9), (2.13), we see that

$$\begin{aligned}\|B\xi\| &\le \|A_t(\,\cdot\,,T)\| + \|\nabla\phi(\,\cdot\,,T)\| + \left\|\left(\frac{i}{k}\nabla\psi + \psi A\right)(\,\cdot\,,T)\right\| + \|\mathrm{curl}^2\,\chi\| \\ &\le \|A_t(\,\cdot\,,T)\| + 2\left\|\left(\frac{i}{k}\nabla\psi + \psi A\right)(\,\cdot\,,T)\right\| + \|\mathrm{curl}^2\,\chi\| \\ &\le \|\xi\|e^{-d_*T/2} + O(1) \\ &\le \mathrm{Re}^{-d_*T/2} + O(1) \\ &\le R\end{aligned}$$

if we take

$$R := \frac{O(1)}{1 - e^{-d_*T/2}},$$

where $O(1)$ is some positive constant independent of ξ and bounded as $\lambda_1 \to \infty$. We will use the notation $O(1)$ to denote such constant in the rest of the paper. □

Lemma 3.2. *B is weakly continuous from D to D.*

Proof. Let $\xi_n \in D$ and $\xi_n \rightharpoonup \xi$ weakly in D, then $\xi \in D$. Also, let (ψ_n, A_n, ϕ_n) be the corresponding unique solution to (1.1)–(1.5). Then it follows from [10] that ψ_n, A_n, $\phi_n \in L^\infty(0,T;H^2(\Omega))$ and ψ_{nt}, $A_{nt} \in L^\infty(0,T;L^2(\Omega))$ are uniformly bounded in n.

Hence, by the standard weak convergence argument, it is easy to prove that $B\xi_n \rightharpoonup B\xi$ weakly in D. □

Lemma 3.3. *If λ_1 is large enough, then*

$$\|\psi, A, \phi\|_{L^\infty(0,T;H^2(\Omega))} \le O(1), \tag{3.2}$$

$$\|\psi_t, A_t\|_{L^\infty(0,T;L^2(\Omega))} \le O(1), \tag{3.3}$$

$$\|\psi_t, A_t, \phi_t\|_{L^2(0,T;H^1(\Omega))} \le O(1). \tag{3.4}$$

Proof. From Lemma 3.1 we know that if λ_1 is large enough, then

$$\|\xi\| \le R$$

and (3.2), (3.3), (3.4) follow from the same proofs as in [10], and so we omit the details here. □

Let $(\psi_i, A_i, \phi_i, H_i, \xi_i)$ ($\xi_i = \mathrm{curl}\,H_i$, $i = 1,2$) be the solutions to the inverse problem (1.1)–(1.7) corresponding to the input data $(\psi_{0i}, A_{0i}, \chi_i)$ ($i = 1, 2$). Also, let

$$\psi := \psi_1 - \psi_2,\quad A := A_1 - A_2,\quad \phi := \phi_1 - \phi_2,\quad H := H_1 - H_2,\quad \xi := \xi_1 - \xi_2,$$
$$\psi_0 := \psi_{01} - \psi_{02},\quad A_0 := A_{01} - A_{02},\quad \chi := \chi_1 - \chi_2.$$

Then we can estimate as follows.

Lemma 3.4.

$$\|\nabla\phi\| \le O(1)(\|A\| + \|\psi\|_{H^1}) \quad in \quad (0, T). \tag{3.5}$$

Proof. It follows from (2.7) that

$$-\Delta\phi = \operatorname{div} \operatorname{Re}\left[\left(\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right)\bar{\psi} + \left(\frac{i}{k}\nabla\psi + \psi A_1 + \psi_2 A\right)\bar{\psi}_2\right], \tag{3.6}$$

$$\int_\Omega \phi\, dx = 0, \quad \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0. \tag{3.7}$$

Then, multiplying (3.6) by ϕ and integrating by parts imply

$$\begin{aligned} \|\nabla\phi\| &\le \left\|\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right\|_{L^6} \|\psi\|_{L^3} + \frac{1}{k}\|\nabla\psi\| + \|\psi\|\,\|A_1\|_{L^\infty} + \|A\| \\ &\le O(1)(\|A\| + \|\psi\|_{H^1}) \quad \text{in} \quad (0, T). \end{aligned} \tag{3.8}$$

This completes the proof. □

Lemma 3.5.

(3.9)

$$\|\psi\|_{H^1} + \|A\| + \|\operatorname{curl} A\| \le O(1)(\|\psi_0\|_{H^1} + \|A_0\| + \|\operatorname{curl} A_0\|) + O(1)\frac{1}{\sqrt{\lambda_1}}\|\xi\|,$$

$$\|\psi_t\|_{L^2(\Omega\times(0,T))} \le O(1)(\|\psi_0\|_{H^1} + \|A_0\| + \|\operatorname{curl} A_0\|) + O(1)\frac{1}{\sqrt{\lambda_1}}\|\xi\| \tag{3.10}$$

in $(0, T)$.

Proof. Substracting the equation (2.12) for each ψ_i $(i = 1, 2)$, we have

$$\begin{aligned} &\eta\psi_t + i\eta k\phi_1\psi + i\eta k\phi\psi_2 - \frac{1}{k^2}\Delta\psi + \frac{2i}{k}A_1\nabla\psi + \frac{2i}{k}A\nabla\psi_2 \\ &\quad + A_1^2\psi + (A_1 + A_2)A\psi_2 + |\psi_1|^2\psi + \bar{\psi}_1\psi_2\bar{\psi} + \psi_2^2\bar{\psi} - \psi = 0. \end{aligned} \tag{3.11}$$

Multiplying (3.11) by $\bar{\psi}$ and integrating by parts, then taking the real part, we have

$$\begin{aligned} &\frac{\eta}{2}\frac{d}{dt}\int_\Omega |\psi|^2 dx + \frac{1}{k^2}\int_\Omega |\nabla\psi|^2 dx \\ &\le \eta k\|\phi\|\,\|\psi\| + \frac{2}{k}\|A_1\|_{L^\infty}\|\nabla\psi\|\,\|\psi\| \\ &\quad + \frac{2}{k}\|A\|\,\|\nabla\psi_2\|_{L^6}\|\psi\|_{L^3} + \|A_1 + A_2\|_{L^\infty}\|A\|\,\|\psi\| + 3\|\psi\|^2 \quad \text{in} \quad (0, T). \end{aligned}$$

The first term of the right hand side can be bounded by (3.5) and hence

$$\frac{d}{dt}\int_\Omega |\psi|^2 dx + \frac{1}{\eta k^2}\int_\Omega |\nabla\psi|^2 dx \le O(1)(\|A\|^2 + \|\psi\|_{H^1}^2) \quad \text{in} \quad (0, T). \tag{3.12}$$

On the other hand, multiplying (3.11) by $\bar{\psi}_t$ and integrating by parts, then taking the real part, we see that

$$\begin{aligned}
&\frac{1}{2k^2}\frac{d}{dt}\int_\Omega |\nabla\psi|^2 dx + \eta\int_\Omega |\psi_t|^2 dx \\
&\le \eta k\|\phi_1\|_{L^\infty}\|\psi\|\,\|\psi_t\| + \eta k\|\phi\|\,\|\psi_t\| \\
&\quad + \frac{2}{k}\|A_1\|_{L^\infty}\|\nabla\psi\|\,\|\psi_t\| + \frac{2}{k}\|A\|_{L^3}\|\nabla\psi_2\|_{L^6}\|\psi_t\| \\
&\quad + \|A_1\|_{L^\infty}^2\|\psi\|\,\|\psi_t\| + \|A_1 + A_2\|_{L^\infty}\|A\|\,\|\psi_t\| + 4\|\psi\|\,\|\psi_t\| \\
&\le O(1)(\|\psi\| + \|\phi\| + \|\nabla\psi\| + \|A\|_{L^3} + \|A\|)\|\psi_t\| \\
&\le O(1)(\|\psi\|_{H^1} + \|A\|_{L^3})\|\psi_t\| \\
&\le \frac{\eta}{2}\|\psi_t\|^2 + O(1)(\|\psi\|_{H^1}^2 + \|A\|_{L^3}^2) \quad \text{in} \quad (0, T).
\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (|\psi|^2 + |\nabla\psi|^2)\, dx + O(1)\int_\Omega |\psi_t|^2 dx \\
&\le O(1)(\|A\|_{L^3}^2 + \|\psi\|_{H^1}^2) \quad \text{in} \quad (0, T).
\end{aligned} \tag{3.14}$$

Now, substracting equation (1.2) for each A_i $(i = 1, 2)$, we have

$$\begin{aligned}
&A_t + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\left(\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right)\bar{\psi} + \operatorname{Re}\left(\frac{i}{k}\nabla\psi + \psi_1 A + \psi A_2\right)\bar{\psi}_2 \\
&= \operatorname{curl} H.
\end{aligned} \tag{3.15}$$

Testing (3.15) by A we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega A^2 dx + \int_\Omega (\operatorname{curl} A)^2 dx \\
&\le \int_\Omega H \operatorname{curl} A\, dx + \left\|\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right\|_{L^6}\|\psi\|_{L^3}\|A\| \\
&\quad + \frac{1}{k}\|\nabla\psi\|\,\|A\| + \|A\|^2 + \|\psi\|\,\|A\|\,\|A_2\|_{L^\infty} \\
&\le \frac{1}{2}\|\operatorname{curl} A\|^2 + O(1)(\|A\|^2 + \|\psi\|_{H^1}^2) + O(1)\frac{1}{\lambda_1}\|\xi\|^2
\end{aligned}$$

in $(0, T)$. Hence, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega A^2 dx + \int_\Omega (\operatorname{curl} A)^2 dx \\ \le O(1)(\|A\|^2 + \|\psi\|_{H^1}^2) + O(1)\frac{1}{\lambda_1}\|\xi\|^2 \quad \text{in} \quad (0, T).\end{aligned} \tag{3.16}$$

Multiplying (3.15) by $\operatorname{curl}(\operatorname{curl} A - H)$ and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (\operatorname{curl} A - H)^2 dx + \int_\Omega |\operatorname{curl}(\operatorname{curl} A - H)|^2 dx \\ &\le \left(\left\|\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right\|_{L^6}\|\psi\|_{L^3} + \frac{1}{k}\|\nabla\psi\| + \|A\| + \|\psi\|\,\|A_2\|_{L^\infty}\right)\|\operatorname{curl}(\operatorname{curl} A - H)\| \\ &\le \frac{1}{2}\|\operatorname{curl}(\operatorname{curl} A - H)\|^2 + O(1)(\|A\|^2 + \|\psi\|_{H^1}^2) \quad \text{in} \quad (0, T)\end{aligned}$$

and hence

$$\begin{aligned}\frac{d}{dt}\int_\Omega (\operatorname{curl} A - H)^2 dx + \int_\Omega |\operatorname{curl}(\operatorname{curl} A - H)|^2 dx \\ \le O(1)(\|A\|^2 + \|\psi\|_{H^1}^2) \quad \text{in} \quad (0, T).\end{aligned} \tag{3.17}$$

Combining (3.14), (3.16) and (3.17), we have

$$\begin{aligned}\frac{d}{dt}(\|\psi\|_{H^1}^2 + \|A\|^2 + \|\operatorname{curl} A - H\|^2) + O(1)\|\psi_t\|^2 \\ \le O(1)\frac{1}{\lambda_1}\|\xi\|^2 + O(1)(\|\psi\|_{H^1}^2 + \|A\|^2 + \|\operatorname{curl} A - H\|^2) \quad \text{in} \quad (0, T).\end{aligned} \tag{3.18}$$

Then, by the Gronwall's inequality, we have

$$\begin{aligned}&\|\psi\|_{H^1}^2 + \|A\|^2 + \|\operatorname{curl} A - H\|^2 \\ &\le O(1)(\|\psi_0\|_{H^1}^2 + \|A_0\|^2 + \|\operatorname{curl} A_0 - H\|^2) + O(1)\frac{1}{\lambda_1}\|\xi\|^2 \quad \text{in} \quad (0, T)\end{aligned}$$

and hence

$$\begin{aligned}&\|\psi\|_{H^1}^2 + \|A\|^2 + \|\operatorname{curl} A\|^2 \\ &\le O(1)(\|\psi_0\|_{H^1}^2 + \|A_0\|^2 + \|\operatorname{curl} A_0\|^2) + O(1)\frac{1}{\lambda_1}\|\xi\|^2 \quad \text{in} \quad (0, T).\end{aligned}$$

This proves (3.9).

As for (3.10), it follows by integrating (3.18) over $(0, T)$ and using (3.9). □

Lemma 3.6. *If λ_1 is large enough, then*

$$\|\xi\| \le O(1)(\|A_0\| + \|\text{curl } A_0\| + \|\text{curl}^2 A_0\| + \|\chi\| + \|\text{curl}^2 \chi\| + \|\psi_0\|_{H^1}) \quad \textit{in} \quad (0, T). \tag{3.19}$$

Proof. Applying $\partial/\partial t$ to (3.15), we have

$$\begin{aligned} &A_{tt} + \nabla\phi_t + \text{curl}^2 A_t + \text{Re}\left[\left(\frac{i}{k}\nabla\psi_{1t} + \psi_{1t}A_1 + \psi_1 A_{1t}\right)\bar{\psi}\right] \\ &+ \text{Re}\left[\left(\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right)\bar{\psi}_t\right] + \text{Re}\left[\left(\frac{i}{k}\nabla\psi_t + \psi_1 A_t + \psi_{1t}A + \psi_t A_2 + \psi A_{2t}\right)\bar{\psi}_2\right] \\ &+ \text{Re}\left[\left(\frac{i}{k}\nabla\psi + \psi_1 A + \psi A_2\right)\bar{\psi}_{2t}\right] = 0 \quad \text{in} \quad (0, T). \end{aligned} \tag{3.20}$$

Testing (3.20) by A_t and using

$$\int_\Omega \frac{i}{k}\nabla\psi_t \cdot \bar{\psi}_2 A_t \, dx = -\int_\Omega \frac{i}{k}\psi_t A_t \nabla\bar{\psi}_2 \, dx, \tag{3.21}$$

we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega A_t^2 dx + \int_\Omega (\text{curl } A_t)^2 dx \\ &\le \frac{1}{k}\|\nabla\psi_{1t}\|\,\|\psi\|_{L^3}\|A_t\|_{L^6} + \|\psi_{1t}\|\,\|A_1\|_{L^\infty}\|\psi\|_{L^3}\|A_t\|_{L^6} \\ &\quad + \|A_{1t}\|_{L^6}\|\psi\|_{L^3}\|A_t\| + \left\|\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right\|_{L^6}\|\psi_t\|\,\|A_t\|_{L^3} \\ &\quad + \frac{1}{k}\|\nabla\psi_2\|_{L^6}\|\psi_t\|\,\|A_t\|_{L^3} + \|A_t\|^2 + \|\psi_{1t}\|_{L^6}\|A\|_{L^3}\|A_t\| \\ &\quad + \|\psi_t\|\,\|A_2\|_{L^\infty}\|A_t\| + \|\psi\|_{L^6}\|A_{2t}\|_{L^3}\|A_t\| \\ &\quad + \left(\frac{1}{k}\|\nabla\psi\| + \|A\| + \|\psi\|\,\|A_2\|_{L^\infty}\right)\|\psi_{2t}\|_{L^6}\|A_t\|_{L^3} \\ &\le O(1)\|\psi_{1t}\|_{H^1}\|\psi\|_{H^1}\|\text{curl } A_t\| + O(1)\|A_{1t}\|_{H^1}\|\psi\|_{H^1}\|A_t\| \\ &\quad + O(1)\|\psi_t\|\,\|A_t\|_{L^3} + O(1)\|A_t\|^2 + O(1)\|\psi_{1t}\|_{H^1}\|A\|_{L^3}\|A_t\| \\ &\quad + O(1)\|\psi_t\|\,\|A_t\| + O(1)\|A_{2t}\|_{H^1}\|\psi\|_{H^1}\|A_t\| \\ &\quad + O(1)\|\psi_{2t}\|_{H^1}(\|\psi\|_{H^1} + \|A\|)\|A_t\|_{L^3} \quad \text{in} \quad (0, T). \end{aligned}$$

Then, using $\sqrt{\lambda_1}\|A_t\| \le \|\text{curl } A_t\|$, $\|A_t\|_{L^3} \le O(1)\|\text{curl } A_t\|$, and reminding ε-Cauchy inequality: $2ab \le \varepsilon a^2 + (1/\varepsilon)b^2$ and if λ_1 is large enough, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega A_t^2\,dx + \int_\Omega (\operatorname{curl} A_t)^2\,dx \le O(1)(\|\psi_{1t}\|_{H^1}^2 + \|\psi_{2t}\|_{H^1}^2 + \|A_{1t}\|_{H^1}^2 + \|A_{2t}\|_{H^1}^2),\\ (\|A\|_{H^1}^2 + \|\psi\|_{H^1}^2) + O(1)\|\psi_t\|^2 \quad \text{in} \quad (0,T)\end{aligned}$$

and hence

$$\begin{aligned}\frac{d}{dt}\int_\Omega A_t^2\,dx + \lambda_1\int_\Omega A_t^2\,dx \le O(1)(\|\psi_{1t}\|_{H^1}^2 + \|\psi_{2t}\|_{H^1}^2 + \|A_{1t}\|_{H^1}^2 + \|A_{2t}\|_{H^1}^2),\\ (\|A\|^2 + \|\operatorname{curl} A\|^2 + \|\psi\|_{H^1}^2) + O(1)\|\psi_t\|^2 \quad \text{in} \quad (0,T).\end{aligned}$$

Solving this inequality implies

$$\begin{aligned}\|A_t(\cdot\,,T)\|^2 \le \|A_t(\cdot\,,0)\|^2 e^{-\lambda_1 T} + O(1)\sup_{0\le t\le T}(\|A\|^2 + \|\operatorname{curl} A\|^2 + \|\psi\|_{H^1}^2)\\ \times\int_0^T (\|\psi_{1t}\|_{H^1}^2 + \|\psi_{2t}\|_{H^1}^2 + \|A_{1t}\|_{H^1}^2 + \|A_{2t}\|_{H^1}^2)\,dt\\ + O(1)\int_0^T \|\psi_t\|^2\,dt \quad \text{in} \quad (0,T),\end{aligned}$$

and hence by Lemma 3.3 and Lemma 3.5, we have

$$\begin{aligned}\|A_t(\cdot\,,T)\| &\le \|A_t(\cdot\,,0)\|e^{-\lambda_1 T/2}\\ &\quad + O(1)\sup_{0\le t\le T}(\|A\| + \|\operatorname{curl} A\| + \|\psi\|_{H^1}) + O(1)\|\psi_t\|_{L^2(\Omega\times(0,T))}\\ &\le \|A_t(\cdot\,,0)\|e^{-\lambda_1 T/2}\\ &\quad + O(1)(\|\psi_0\|_{H^1} + \|A_0\| + \|\operatorname{curl} A_0\|) + O(1)\frac{1}{\sqrt{\lambda_1}}\|\xi\|.\end{aligned} \tag{3.22}$$

Now using (3.15) at time $t=0$, we have

$$\begin{aligned}\|A_t(\cdot\,,0)\| \le \|\operatorname{curl}^2 A_0\| + \|\nabla\phi(\cdot\,,0)\| + \left\|\frac{i}{k}\nabla\psi_{01} + \psi_{01}A_{01}\right\|_{L^6}\|\psi_0\|_{L^3}\\ + \frac{1}{k}\|\nabla\psi_0\| + \|A_0\| + \|A_{02}\|_{L^\infty}\|\psi_0\| + \|\xi\|.\end{aligned} \tag{3.23}$$

Also, using (3.5) at the time $t=0$ and $t=T$, we have

$$\|\nabla\phi(\cdot\,,0)\| \le O(1)(\|A_0\| + \|\psi_0\|_{H^1}), \tag{3.24}$$

$$\|\nabla\phi(\cdot\,,T)\| \le O(1)(\|\psi(\cdot\,,T)\|_{H^1} + \|\chi\|). \tag{3.25}$$

Moreover, using (3.15) at the final time $t = T$, we have

$$
\begin{aligned}
\|\xi\| \le\ & \|A_t(\,\cdot\,, T)\| + \|\nabla\phi(\,\cdot\,, T)\| + \|\mathrm{curl}^2\,\chi\| \\
& + \left\|\frac{i}{k}\nabla\psi_1 + \psi_1 A_1\right\|_{L^6}\|\psi\|_{L^3} + \frac{1}{k}\|\nabla\psi\| + \|\chi\| + \|\chi_2\|_{L^\infty}\|\psi\|.
\end{aligned} \tag{3.26}
$$

By combining (3.22)–(3.26) and using Lemma 3.5, we have

$$\|\xi\| \le O(1)(\|A_0\| + \|\mathrm{curl}\ A_0\| + \|\mathrm{curl}^2\ A_0\| + \|\chi\| + \|\mathrm{curl}^2\ \chi\| + \|\psi_0\|_{H^1})$$

for large enough λ_1.

This proves (3.19). □

Based on what we have obtained, we can quickly give the proof of Theorem 1.4 as follows.

Proof of Theorem 1.4. By Tikhonov's fixed point theorem, the existence part of Theorem 1.4 follows from Lemmas 3.1, 3.2 and Theorem 1.2. The uniqueness and stability parts of Theorem 1.4 is a corollary of (1.13) which follows from (3.5), (3.9) and (3.19). □

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