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PROPERTIES OF LEXSEGMENT IDEALS

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Abstract

We show that any lexsegment ideal with linear resolution has linear quotients with respect to a suitable ordering of its minimal monomial generators. For completely lexsegment ideals with linear resolution we show that the decomposition function is regular. For arbitrary lexsegment ideals we compute the depth and the dimension. As application we characterize the Cohen–Macaulay lexsegment ideals.

Introduction

Let $S = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k. We order lexicographically the monomials of S such that $x_1 > x_2 > \cdots > x_n$. Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree d. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{\text{lex}} v$, the set

$$\mathcal{L}(u, v) = \{w \in \mathcal{M}_d \mid u \geq_{\text{lex}} w \geq_{\text{lex}} v\}$$

is called a lexsegment. A lexsegment ideal in S is a monomial ideal of S which is generated by a lexsegment. Lexsegment ideals have been introduced by Hulett and Martin [10]. Arbitrary lexsegment ideals have been studied by A. Aramova, E. De Negri, and J. Herzog in [1] and [4]. They characterized the lexsegment ideals which have a linear resolution.

Let $I \subset S$ be a monomial ideal and $G(I)$ its minimal monomial set of generators. I has linear quotients if there exists an ordering $u_1, \dots, u_m$ of the elements of $G(I)$ such that for all $2 \leq j \leq m$, the colon ideals $(u_1, \dots, u_{j-1}) : u_j$ are generated by a subset of $\{x_1, \dots, x_n\}$.

Ideals with linear quotients have a linear resolution, but, in general, the converse is not true. Therefore it is natural to ask whether lexsegment ideals with linear resolution have linear quotients. We positively answer this question. In Section 1 we show that any completely lexsegment ideal with linear resolution has linear quotients with respect to the following order of the generators. Given two monomials of degree d in S, $w = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $w' = x_1^{\beta_1} \cdots x_n^{\beta_n}$, we set $w \prec w'$ if $\alpha_1 < \beta_1$ or $\alpha_1 = \beta_1$ and $w >_{\text{lex}} w'$.

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Let $u, v \in \mathcal{M}_d$ which define the completely lexsegment ideal $I = (\mathcal{L}(u, v))$ with linear resolution. If $\mathcal{L}(u, v) = \{w_1, \ldots, w_r\}$, where $w_1 \prec w_2 \prec \cdots \prec w_r$, we show that I has linear quotients with respect to this ordering of the generators. The non-completely lexsegment ideal will be separately studied in Section 2.

For the monomial ideals with linear quotients one may consider the associated decomposition function defined in [9]. When this function has an additional property, namely it is regular, then one may apply the iterated mapping cone procedure developed in [9] (see also [5]) to get the explicit resolution of the ideal.

For the completely lexsegment ideals with linear resolution it will turn out that their decomposition function with respect to the ordering $\prec$ is regular. Therefore, we get the explicit resolutions for this class of ideals.

In the last section of our paper we study the depth and the dimension of lexsegment ideals. Our results show that one may compute these invariants just looking at the ends of the lexsegment. As an application, we characterize the Cohen–Macaulay lexsegment ideals.

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1. Completely lexsegment ideals with linear resolutions

In the theory of Hilbert functions or in extremal combinatorics usually one considers initial lexsegment ideals, that is ideals generated by an initial lexsegment $\mathcal{L}^i(v) = \{w \in \mathcal{M}_d \mid w \geq_{\text{lex}} v\}$. Initial lexsegment ideals are stable in the sense of Eliahou and Kervaire ([6], [2]) and they have linear quotients with respect to lexicographical order [11, Proposition 2.1].

One may also define the final lexsegment $\mathcal{L}^f(u) = \{w \in \mathcal{M}_d \mid u \geq_{\text{lex}} w\}$. Final lexsegment ideals are generated by final lexsegments. They are also stable in the sense of Eliahou and Kervaire with respect to $x_n > x_{n-1} > \cdots > x_1$. Therefore they have linear quotients.

Throughout this paper we use the following notations. If $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a monomial of S, we denote by $\nu_i(m)$ the exponent of the variable x_i in m, that is $\nu_i(m) = \alpha_i$, $i = 1, \ldots, n$. Also, we will denote $\max(m) = \max\{i \mid x_i \mid m\}$.

Hulett and Martin call a lexsegment L *completely lexsegment* if all the iterated shadows of L are again lexsegments. We recall that the shadow of a set T of monomials is the set $\text{Shad}(T) = \{vx_i \mid v \in T,\ 1 \leq i \leq n\}$. The i-th shadow is recursively defined as $\text{Shad}^i(T) = \text{Shad}(\text{Shad}^{i-1}(T))$. The initial lexsegments have the property that their shadow is again an initial lexsegment, a fact which is not true for arbitrary lexsegments. An ideal spanned by a completely lexsegment is called a *completely lexsegment ideal*. All the completely lexsegment ideals with linear resolution are determined in [1]:

Theorem 1.1 ([1]). *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d with $u \geq_{\text{lex}} v$, and let $I = (\mathcal{L}(u, v))$ be a completely lexsegment ideal. Then I has a linear resolution if and only if one of the following conditions holds*:
(a) $u = x_1^a x_2^{d-a}$, $v = x_1^a x_n^{d-a}$ *for some* a, $0 < a \leq d$;
(b) $b_1 < a_1 - 1$;
(c) $b_1 = a_1 - 1$ *and for the largest* $w <_{\text{lex}} v$, w *monomial of degree* d, *one has* $x_1 w / x_{\max(w)} \leq_{\text{lex}} u$.

Theorem 1.2. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, with $a_1 > 0$, and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d with $u \geq_{\text{lex}} v$, and let $I = (\mathcal{L}(u, v))$ be a completely lexsegment ideal. Then I has a linear resolution if and only if I has linear quotients.*

Proof. We have to prove that if I has a linear resolution then I has linear quotients, since the other implication is known [8]. By Theorem 1.1, since I has a linear resolution, one of the conditions (a), (b), (c) holds.

We define on the set of the monomials of degree d from S the following total order: for

$$w = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad w' = x_1^{\beta_1} \cdots x_n^{\beta_n},$$

we set

$$w \prec w' \quad \text{if} \quad \alpha_1 < \beta_1 \quad \text{or} \quad \alpha_1 = \beta_1 \quad \text{and} \quad w >_{\text{lex}} w'.$$

Let

$$\mathcal{L}(u, v) = \{w_1, \ldots, w_r\}, \quad \text{where} \quad w_1 \prec w_2 \prec \cdots \prec w_r.$$

We will prove that $I = (\mathcal{L}(u, v))$ has linear quotients with respect to this ordering of the generators.

Assume that u, v satisfy the condition (a) and $a < d$ (the case $a = d$ is trivial). Then I is isomorphic as S-module to the ideal generated by the final lexsegment $\mathcal{L}^f(x_2^{d-a}) \subset S$ and the ordering $\prec$ of its minimal generators coincides with the lexicographical ordering $>_{\text{lex}}$. The ideal $(\mathcal{L}^f(x_2^{d-a})) \cap k[x_2, \ldots, x_n]$ is the initial lexsegment ideal in $k[x_2, \ldots, x_n]$ defined by x_n^{d-a}, which has linear quotients with respect to $>_{\text{lex}}$. Hence I has linear quotients with respect to $\prec$ since it is the extension in the ring $k[x_1, \ldots, x_n]$ of a monomial ideal with linear quotients in $k[x_2, \ldots, x_n]$.

Next we assume that u, v satisfy the condition (b) or (c).

By definition, I has linear quotients with respect to the monomial generators $w_1, \ldots, w_r$ if the colon ideals $(w_1, \ldots, w_{i-1}) : w_i$ are generated by variables for all $i \geq 2$, that is for all $j < i$ there exists an integer $1 \leq k < i$ and an integer $l \in [n]$ such that $w_k / \gcd(w_k, w_i) = x_l$ and x_l divides $w_j / \gcd(w_j, w_i)$.

In other words, for any $w_j \prec w_i$, $w_j, w_i \in \mathcal{L}(u, v)$, we have to find a monomial $w' \in \mathcal{L}(u, v)$ such that

$$(*) \quad w' \prec w_i, \quad \frac{w'}{\gcd(w', w_i)} = x_l, \quad \text{for some} \quad l \in [n], \quad \text{and} \quad x_l \quad \text{divides} \quad \frac{w_j}{\gcd(w_j, w_i)}.$$

Let us fix $w_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $w_j = x_1^{\beta_1} \cdots x_n^{\beta_n}$, $w_i, w_j \in \mathcal{L}(u, v)$, such that $w_j \prec w_i$. By the definition of the ordering $\prec$, we must have

$$\beta_1 < \alpha_1 \quad \text{or} \quad \beta_1 = \alpha_1 \quad \text{and} \quad w_j >_{\text{lex}} w_i.$$

CASE 1: Let $\beta_1 < \alpha_1$. One may find an integer l, $2 \le l \le n$, such that $\alpha_s \ge \beta_s$ for all $s < l$ and $\alpha_l < \beta_l$ since, otherwise, $\deg(w_i) > \deg(w_j) = d$ which is impossible. We obviously have $\max(w_j) \ge l$. If $l \ge \max(w_i)$, one may take $\bar{w} = x_l w_i / x_1$ which satisfies the condition $(*)$ since the inequalities $\bar{w} \prec w_i$, $\bar{w} \le_{\text{lex}} w_i \le_{\text{lex}} u$ hold, and we will show that $\bar{w} \ge_{\text{lex}} w_j$. This will imply that $\bar{w} \ge_{\text{lex}} v$, hence $\bar{w} \in \mathcal{L}(u, v)$.

The inequality $\bar{w} \ge_{\text{lex}} w_j$ is obviously fulfilled if $\alpha_1 - 1 > \beta_1$ or if $\alpha_1 - 1 = \beta_1$ and at least one of the inequalities $\alpha_s \ge \beta_s$ for $2 \le s < l$, is strict. If $\alpha_1 - 1 = \beta_1$ and $\alpha_s = \beta_s$ for all $s < l$, comparing the degrees of w_i and w_j it results $d = \alpha_1 + \cdots + \alpha_l = \beta_1 + 1 + \beta_2 + \cdots + \beta_{l-1} + \alpha_l < (\beta_1 + 1) + \beta_2 + \cdots + \beta_l$. It follows that $d \ge \beta_1 + \beta_2 + \cdots + \beta_l > d - 1$, that is $\beta_1 + \beta_2 + \cdots + \beta_l = d$. This implies that $l = \max(w_j)$ and $\beta_l = \alpha_l + 1$, that is $\bar{w} = x_l w_i / x_1 = x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_l^{\alpha_l + 1} = x_1^{\beta_1} \cdots x_l^{\beta_l} = w_j$.

From now on, in Case 1, we may assume that $l < \max(w_i)$. We will show that at least one of the following monomials:

$$w' = \frac{x_l w_i}{x_{\max(w_i)}}, \quad w'' = \frac{x_l w_i}{x_1}$$

belongs to $\mathcal{L}(u, v)$. It is clear that both monomials are strictly less than w_i with respect to the ordering $\prec$. Therefore one of the monomials w', w'' will satisfy the condition $(*)$.

The following inequalities are fulfilled:

$$w' >_{\text{lex}} w_i \ge_{\text{lex}} v,$$

and

$$w'' <_{\text{lex}} w_i \le_{\text{lex}} u.$$

Let us assume, by contradiction, that $w' >_{\text{lex}} u$ and $w'' <_{\text{lex}} v$. Comparing the exponents of the variable x_1, we obtain $a_1 - 1 \le \alpha_1 - 1 \le b_1$. Since the ideal generated by $\mathcal{L}(u, v)$ has a linear resolution, we must have $b_1 = a_1 - 1$. Let z be the largest monomial of degree d such that $z <_{\text{lex}} v$. Then, by our assumption on w'', we also have the inequality $w'' \leq_{\text{lex}} z$.

Now we need the following

Lemma 1.3. *Let $m = x_1^{a_1} \cdots x_n^{a_n}$, $m' = x_1^{b_1} \cdots x_n^{b_n}$ be two monomials of degree d. If $m \leq_{\text{lex}} m'$ then $m/x_{\max(m)} \leq_{\text{lex}} m'/x_{\max(m')}$.*

Proof. The proof is immediate. □

Going back to the proof of our theorem, we apply the above lemma for the monomials w'' and z and we obtain $w''/x_{\max(w'')} \leq_{\text{lex}} z/x_{\max(z)}$, which implies that $x_1 w''/x_{\max(w'')} \leq_{\text{lex}} x_1 z/x_{\max(z)}$. By using condition (c) in Theorem 1.1 it follows that $x_1 w''/x_{\max(w'')} \leq_{\text{lex}} u$. On the other hand, $x_1 w''/x_{\max(w'')} = x_1 x_l w_i/(x_1 x_{\max(w_i)}) = x_l w_i/x_{\max(w_i)} = w'$. Therefore, it results $w' \leq_{\text{lex}} u$, which contradicts our assumption on w'.

Consequently, we have $w' \leq_{\text{lex}} u$ or $w'' \geq_{\text{lex}} v$, which proves that at least one of the monomials w', w'' belongs to $\mathcal{L}(u, v)$.

CASE 2: Let $\beta_1 = \alpha_1$ and $w_j >_{\text{lex}} w_i$. Then there exists l, $2 \leq l \leq n$, such that $\alpha_s = \beta_s$, for all $s < l$ and $\alpha_l < \beta_l$. If $\max(w_i) \leq l$, then, looking at the degrees of w_i and w_j, we get $d = \alpha_1 + \alpha_2 + \cdots + \alpha_l < \beta_1 + \beta_2 + \cdots + \beta_l$, contradiction. Therefore, $l < \max(w_i)$. We proceed in a similar way as in the previous case. Namely, exactly as in Case 1, it results that at least one of the following two monomials $w' = x_l w_i/x_{\max(w_i)}$, $w'' = x_l w_i/x_1$ belongs to $\mathcal{L}(u, v)$. It is clear that both monomials are strictly less than w_i with respect to the order $\prec$. □

EXAMPLE 1.4. Let $S = k[x_1, x_2, x_3]$. We consider the monomials: $u = x_1 x_2 x_3$ and $v = x_2 x_3^2$, $u >_{\text{lex}} v$, and let I be the monomial ideal generated by $\mathcal{L}(u, v)$. The minimal system of generators of the ideal I is

$$G(I) = \mathcal{L}(u, v) = \{x_1 x_2 x_3,\ x_1 x_3^2,\ x_2^3,\ x_2^2 x_3,\ x_2 x_3^2\}.$$

Since I verifies the condition (c) in Theorem 1.1, it follows that I is a completely lexsegment ideal with linear resolution. We denote the monomials from $G(I)$ as follows: $u_1 = x_1 x_2 x_3$, $u_2 = x_1 x_3^2$, $u_3 = x_2^3$, $u_4 = x_2^2 x_3$, $u_5 = x_2 x_3^2$, so $u_1 >_{\text{lex}} u_2 >_{\text{lex}} \cdots >_{\text{lex}} u_5$. The colon ideal $(u_1, u_2) : u_3 = (x_1 x_3)$ is not generated by a subset of $\{x_1, x_2, x_3\}$. This shows that I is not with linear quotients with respect to lexicographical order.

We consider now the order $\prec$ and check by direct computation that I has linear quotients. We label the monomials from $G(I)$ as follows: $u_1 = x_2^3$, $u_2 = x_2^2 x_3$, $u_3 = x_2 x_3^2$, $u_4 = x_1 x_2 x_3$, $u_5 = x_1 x_3^2$, so $u_1 \prec u_2 \prec \cdots \prec u_5$. Then $(u_1) : u_2 = (x_2)$, $(u_1, u_2) : u_3 = (x_2)$, $(u_1, u_2, u_3) : u_4 = (x_2, x_3)$, $(u_1, u_2, u_3, u_4) : u_5 = (x_2)$.

We further study the decomposition function of a completely lexsegment ideal with linear resolution. The decomposition function of a monomial ideal was introduced by J. Herzog and Y. Takayama in [9].

We recall the following notation. If $I \subset S$ is a monomial ideal with linear quotients with respect to the ordering $u_1, \ldots, u_m$ of its minimal generators, then we denote

$$\mathrm{set}(u_j) = \{k \in [n] \mid x_k \in (u_1, \ldots, u_{j-1}) : u_j\}$$

for $j = 1, \ldots, m$.

DEFINITION 1.5 ([9]). Let $I \subset S$ be a monomial ideal with linear quotients with respect to the sequence of minimal monomial generators $u_1, \ldots, u_m$ and set $I_j = (u_1, \ldots, u_j)$, for $j = 1, \ldots, m$. Let $M(I)$ be the set of all monomials in I. The map $g\colon M(I) \to G(I)$ defined as: $g(u) = u_j$, where j is the smallest number such that $u \in I_j$, is called *the decomposition function* of I.

We say that the decomposition function $g\colon M(I) \to G(I)$ is *regular* if $\mathrm{set}(g(x_s u)) \subseteq \mathrm{set}(u)$ for all $s \in \mathrm{set}(u)$ and $u \in G(I)$.

We show in the sequel that completely lexsegment ideals which have linear quotients with respect to $\prec$ have also regular decomposition functions.

In order to do this, we need some preparatory notations and results.

For an arbitrary lexsegment $\mathcal{L}(u, v)$ with the elements ordered by $\prec$, we denote by $I_{\prec w}$, the ideal generated by all the monomials $z \in \mathcal{L}(u, v)$ with $z \prec w$. $I_{\preceq} w$ will be the ideal generated by all the monomials $z \in \mathcal{L}(u, v)$ with $z \preceq w$.

Lemma 1.6. *Let $I = (\mathcal{L}(u, v))$ be a lexsegment ideal which has linear quotients with respect to the order $\prec$ of the generators. Then, for any $w \in \mathcal{L}(u, v)$, $1 \notin \mathrm{set}(w)$.*

Proof. Let us assume that $1 \in \mathrm{set}(w)$, that is $x_1 w \in I_{\prec w}$. It follows that there exists $w' \in \mathcal{L}(u, v)$, $w' \prec w$, and a variable x_j such that $x_1 w = x_j w'$. Obviously, we have $j \geq 2$. But this equality shows that $\nu_1(w') > \nu_1(w)$, which is impossible since $w' \prec w$. □

Lemma 1.7. *Let $I = (\mathcal{L}(u, v))$ be a completely lexsegment ideal which has linear quotients with respect to the ordering $\prec$ of the generators and let $g\colon M(I) \to G(I)$ the decomposition function of I with respect to the ordering $\prec$. If $w \in \mathcal{L}(u, v)$ and $s \in \mathrm{set}(w)$, then*

$$g(x_s w) = \begin{cases} \dfrac{x_s w}{x_1}, & \text{if} \quad x_s w \geq_{\mathrm{lex}} x_1 v, \\ \dfrac{x_s w}{x_{\max(w)}}, & \text{if} \quad x_s w <_{\mathrm{lex}} x_1 v. \end{cases}$$

Proof. Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$, $a_1 > 0$, and $w = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
In the first place we consider

$$x_s w \geq_{\text{lex}} x_1 v.$$

Since, by Lemma 1.6, we have $s \geq 2$, the above inequality shows that $\nu_1(w) \geq 1$. We have to show that $g(x_s w) = x_s w/x_1$, that is $x_s w/x_1 = \min_{\prec}\{w' \in \mathcal{L}(u, v) \mid x_s w \in I_{\preceq w'}\}$. It is clear that $v \leq_{\text{lex}} x_s w/x_1 <_{\text{lex}} w \leq_{\text{lex}} u$, hence $x_s w/x_1 \in \mathcal{L}(u, v)$. Let $w' \in \mathcal{L}(u, v)$ such that $x_s w \in I_{\preceq w'}$. We have to show that $x_s w/x_1 \preceq w'$. Let $w'' \in \mathcal{L}(u, v)$, $w'' \preceq w'$ such that $x_s w = w'' x_j$, for some variable x_j. Then $w'' = x_s w/x_j \succeq x_s w/x_1$ by the definition of our ordering $\prec$. This implies that $w' \succeq x_s w/x_1$.

Now we have to consider the second inequality,

$$x_s w <_{\text{lex}} x_1 v. \tag{1.1}$$

Since $s \in \text{set}(w)$, we have $x_s w \in I_{\prec w}$, that is there exists $w' \in \mathcal{L}(u, v)$, $w' \prec w$, and a variable x_j, $j \neq s$, such that

$$x_s w = x_j w'. \tag{1.2}$$

If $j = 1$, then $x_s w = x_1 w' \geq_{\text{lex}} x_1 v$, contradiction. Hence $j \geq 2$. We also note that $x_j \mid w$ since $j \neq s$, thus $j \leq \max(w)$. The following inequalities hold:

$$\frac{x_s w}{x_{\max(w)}} \geq_{\text{lex}} \frac{x_s w}{x_j} = w' \geq_{\text{lex}} v. \tag{1.3}$$

If $\nu_1(w) < a_1$, we obviously get $x_s w/x_{\max(w)} \leq_{\text{lex}} u$. Let $\nu_1(w) = a_1$. From the inequality (1.1) we obtain $a_1 \leq b_1 + 1$.

If $a_1 = b_1$ then $u = x_1^{a_1} x_2^{d-a_1}$ and $v = x_1^{a_1} x_n^{d-a_1}$ by Theorem 1.1. Since $w \leq_{\text{lex}} u$, by using Lemma 1.3, we have $x_s w/x_{\max(w)} \leq_{\text{lex}} x_s u/x_{\max(u)} = x_s u/x_2 \leq_{\text{lex}} u$, the last inequality being true by Lemma 1.6. Therefore, $x_s w/x_{\max(w)} \in \mathcal{L}(u, v)$.

If $a_1 = b_1 + 1$ then the condition (c) in Theorem 1.1 holds. Let z be the largest monomial with respect to the lexicographical order such that $z <_{\text{lex}} v$. Since $x_s w/x_1 <_{\text{lex}} v$ by hypothesis, we also have $x_s w/x_1 \leq_{\text{lex}} z$. By Lemma 1.3 we obtain $x_s w/(x_1 x_{\max(x_s w/x_1)}) \leq_{\text{lex}} z/x_{\max(z)}$. Next we apply the condition (c) from Theorem 1.1 and get the following inequalities:

$$x_1 \frac{x_s w}{x_1 x_{\max(x_s w/x_1)}} \leq_{\text{lex}} x_1 \frac{z}{x_{\max(z)}} \leq_{\text{lex}} u. \tag{1.4}$$

From the equality (1.1) we have $w' = x_s w/x_j$. As $j \neq 1$, $\nu_1(w') = \nu_1(w)$, and the inequality $w' \prec w$ gives $w' >_{\text{lex}} w$, that is $x_s w/x_j >_{\text{lex}} w$, which implies that $x_s >_{\text{lex}} x_j$. This shows that $s < j \leq \max(w)$. Now looking at the inequalities (1.4), we have

$$\frac{x_s w}{x_{\max(w)}} \leq_{\text{lex}} u. \tag{1.5}$$

From (1.5) and (1.3) we obtain $x_s w/x_{\max(w)} \in \mathcal{L}(u, v)$.

It remains to show that $x_s w/x_{\max(w)} = \min_{\prec}\{w' \in \mathcal{L}(u, v) \mid x_s w \in I_{\preceq w'}\}$. Let $\tilde{w} = \min_{\prec}\{w' \in \mathcal{L}(u, v) \mid x_s w \in I_{\preceq w'}\}$. We obviously have $\tilde{w} \preceq x_s w/x_{\max(w)} \prec w$. By the choice of $\tilde{w}$ we have

$$x_s w = x_t \tilde{w}$$

for some variable x_t.

If $t = s$ we get $w = \tilde{w}$ which is impossible since $\tilde{w} \prec w$. Therefore, $t \neq s$. Then $x_t \mid w$, so $t \leq \max(w)$. It follows that $\tilde{w} = x_s w/x_t \leq_{\text{lex}} x_s w/x_{\max(w)}$. If $t = 1$ we have $x_1\tilde{w} = x_s w <_{\text{lex}} x_1 v$, which implies that $\tilde{w} <_{\text{lex}} v$, contradiction. Therefore $t \neq 1$ and, moreover, $\tilde{w} \succeq x_s w/x_{\max(w)}$, the inequality being true by the definition of the ordering $\prec$. This yields $\tilde{w} = x_s w/x_{\max(w)}$. Therefore we have proved that $x_s w/x_{\max(w)} = g(x_s w)$. □

After this preparation, we prove the following

Theorem 1.8. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$, $u, v \in \mathcal{M}_d$, with $u \geq_{\text{lex}} v$, and $I = (\mathcal{L}(u, v))$ be a completely lexsegment ideal which has a linear resolution. Then the decomposition function g: $M(I) \to G(I)$ associated to the ordering $\prec$ of the generators from $G(I)$ is regular.*

Proof. Let $w \in \mathcal{L}(u, v)$ and $s \in \text{set}(w)$. We have to show that $\text{set}(g(x_s w)) \subset \text{set}(w)$.

Let $t \in \text{set}(g(x_s w))$. In order to prove that $t \in \text{set}(w)$, that is $x_t w \in I_{\prec w}$, we will consider the following two cases:

CASE 1: Let $x_s w \geq_{\text{lex}} x_1 v$. By Lemma 1.7, $g(x_s w) = x_s w/x_1$. Since $t \in \text{set}(g(x_s w))$, we have

$$\frac{x_t x_s w}{x_1} \in I_{\prec x_s w/x_1},$$

so there exists $w' \prec x_s w/x_1$, $w' \in \mathcal{L}(u, v)$, and a variable x_j, such that $x_t x_s w/x_1 = x_j w'$, that is

$$x_t x_s w = x_1 x_j w'. \tag{1.6}$$

By Lemma 1.6, $s, t \neq 1$ and, since $w' \prec x_s w/x_1$, we have $j \neq t$. Note also that $w' \prec w$ since $\nu_1(w') < \nu_1(w)$. If $j = s$ then $x_t w = x_1 w' \in I_{\prec w}$ and $t \in \text{set}(w)$.

Now let $j \neq s$. If $j = 1$, we have $x_t x_s w = x_1^2 w'$, which implies that $\nu_1(w') = \nu_1(w) - 2$. The following inequalities hold: $v <_{\text{lex}} x_1 w'/x_s <_{\text{lex}} w \leq_{\text{lex}} u$, the first one being true since $v \leq_{\text{lex}} w'$, so $\nu_1(v) \leq \nu_1(w')$. These inequalities show that $x_1 w'/x_s \in \mathcal{L}(u, v)$. But we also have $x_1 w'/x_s \prec w$, hence $x_1 w'/x_s \in I_{\prec w}$.

To finish this case we only need to treat the case $j \neq 1$, $j \neq s$. We are going to show that at least one of the monomials $x_1 w'/x_s$ or $x_j w'/x_s$ belongs to $I_{\prec w}$. In any case this will lead to the conclusion that $x_t w \in I_{\prec w}$ by using (1.6).

From the equality (1.6), we have $x_j \mid w$, hence $j \leq \max(w)$, and $\nu_1(w') = \nu_1(w) - 1$. Since $w' \prec x_s w/x_1$ and $\nu_1(w') = \nu_1(w) - 1 = \nu_1(x_s w/x_1)$, we get

$$w' >_{\text{lex}} \frac{x_s w}{x_1}, \tag{1.7}$$

which gives

$$\frac{x_1 w'}{x_s} >_{\text{lex}} v.$$

If the inequality

$$\frac{x_1 w'}{x_s} \leq_{\text{lex}} u \tag{1.8}$$

holds, then we get $x_1 w'/x_s \in \mathcal{L}(u, v)$. We also note that $\nu_1(x_1 w'/x_s) = \nu_1(w)$ and $x_1 w'/x_s >_{\text{lex}} w$ (by (1.7)). Therefore $x_1 w'/x_s \prec w$ and we may write $x_t w = x_j(x_1 w'/x_s) \in I_{\prec w}$. This implies that $t \in \text{set}(w)$.

Now we look at the monomial $x_j w'/x_s$ for which we have $\nu_1(x_j w'/x_s) = \nu_1(w') < \nu_1(w)$, so $x_j w'/x_s <_{\text{lex}} w \leq_{\text{lex}} u$. If the inequality

$$\frac{x_j w'}{x_s} \geq_{\text{lex}} v \tag{1.9}$$

holds, we obtain $x_j w'/x_s \in \mathcal{L}(u, v)$. Obviously we have $x_j w'/x_s \prec w$. By using (1.6), we may write $x_t w = x_1(x_j w'/x_s) \in I_{\prec w}$, which shows that $t \in \text{set}(w)$.

To finish the proof in Case 1 we need to consider the situation when both inequalities (1.8) and (1.9) fail. Hence, let

$$\frac{x_1 w'}{x_s} >_{\text{lex}} u$$

and

$$\frac{x_j w'}{x_s} <_{\text{lex}} v.$$

We will show that this inequalities cannot hold simultaneously. Comparing the exponents of x_1 in the monomials involved in the above inequalities, we obtain $\nu_1(w') = b_1 \geq a_1 - 1$. Since, by hypothesis, $x_s w >_{\text{lex}} x_1 v$, we have $\nu_1(w) > b_1$. On the other hand, $w \leq_{\text{lex}} u$ implies that $\nu_1(w) \leq a_1$. So $b_1 = a_1 - 1$ and $\mathcal{L}(u, v)$ satisfies the condition (c) in Theorem 1.1. Let, as usually, z be the largest monomial with respect to the lexicographical order such that $z <_{\text{lex}} v$.

Since $x_j w'/x_s <_{\text{lex}} v$, we have $x_j w'/x_s \leq_{\text{lex}} z$. By Lemma 1.3 and using the condition $x_1 z/x_{\max(z)} \leq_{\text{lex}} u$, we obtain: $x_1 x_j w'/(x_s x_{\max(x_j w'/x_s)}) \leq_{\text{lex}} u$. But our assumption was that $u <_{\text{lex}} x_1 w'/x_s$. Therefore, combining the last two inequalities, after cancellation, one obtains that $x_j <_{\text{lex}} x_{\max(x_j w'/x_s)} = x_{\max(x_t w/x_1)} = x_{\max(x_t w)}$. This leads to the inequality $j > \max(x_t w)$ and, since $j \leq \max(w)$, we get $\max(w) > \max(x_t w)$, which is impossible.

CASE 2: Let $x_s w <_{\text{lex}} x_1 v$. Then $g(x_s w) = x_s w/x_{\max(w)}$. In particular we have $x_s w/x_{\max(w)} \prec w$. Indeed, since $s \in \text{set}(w)$, we have $x_s w \in I_{\prec w}$, that is there exists $w' \in \mathcal{L}(u, v)$, $w' \prec w$, such that $x_s w \in I_{\preceq w'}$. By the definition of the decomposition function we have $g(x_s w) \preceq w'$ and next we get $g(x_s w) \prec w$. Since $\nu_1(x_s w/x_{\max(w)}) = \nu_1(w)$, the above inequality implies that $x_s w/x_{\max(w)} >_{\text{lex}} w$, that is $x_s >_{\text{lex}} x_{\max(w)}$ which means that $s < \max(w)$.

As $t \in \text{set}(g(x_s w))$, there exists $w' \prec x_s w/x_{\max(w)}$, $w' \in \mathcal{L}(u, v)$, and a variable x_j, such that

$$\frac{x_t x_s w}{x_{\max(w)}} = x_j w',$$

that is

$$x_t x_s w = x_j x_{\max(w)} w'. \tag{1.10}$$

As in the previous case, we would like to show that one of the monomials $x_{\max(w)} w'/x_s$ or $x_j w'/x_s$ belongs to $\mathcal{L}(u, v)$ and it is strictly less than w with respect to $\prec$. In this way we obtain $x_t w \in I_{\prec w}$ and $t \in \text{set}(w)$.

We begin our proof noticing that $s, t \neq 1$, by Lemma 1.6. The equality $j = t$ is impossible since $w' \neq x_s w/x_{\max(w)}$. If $j = s$, then $x_t w = w' x_{\max(w)} \in I_{\preceq w'}$. But $w' \prec x_s w/x_{\max(w)} \prec w$, hence $x_t w \in I_{\prec w}$.

Let $j \neq s, t$. From the equality (1.10) we have $x_j \mid w$, so $j \leq \max(w)$. We firstly consider $j = 1$. Then the equality (1.10) becomes

$$x_t x_s w = x_1 x_{\max(w)} w'. \tag{1.11}$$

Since $s < \max(w)$, we have $x_{\max(w)} w'/x_s <_{\text{lex}} w' \leq_{\text{lex}} u$. If the inequality $x_{\max(w)} w'/x_s \geq_{\text{lex}} v$ holds too, then $x_{\max(w)} w'/x_s \in \mathcal{L}(u, v)$ and, as $\nu_1(w') < \nu_1(w)$, it follows that $x_{\max(w)} w'/x_s \prec w$. From (1.11), we have $x_t w = x_1(x_{\max(w)} w'/x_s) \in I_{\prec w}$, hence $t \in \text{set}(w)$.

From the inequality $x_s w <_{\text{lex}} x_1 v$, we get

$$x_s w <_{\text{lex}} x_1 w',$$

so

$$\frac{x_1 w'}{x_s} >_{\text{lex}} w.$$

Let us assume that $x_1w'/x_s \leq_{\text{lex}} u$. Since $\nu_1(x_1w'/x_s) = \nu_1(w)$, by using the definition of the ordering $\prec$ we get $x_1w'/x_s \in I_{\prec w}$. Then we may write $x_tw = x_{\max(w)}(x_1w'/x_s) \in I_{\prec w}$.

It remains to consider that $x_{\max(w)}w'/x_s <_{\text{lex}} v$ and $x_1w'/x_s >_{\text{lex}} u$. Proceeding as in Case 1 we show that we reach a contradiction and this ends the proof for $j = 1$. We only need to notice that we have to consider $b_1 \leq a_1 - 1$. Indeed, we can not have $b_1 = a_1$ since one may find in $\mathcal{L}(u, v)$ at least two monomials, namely w and w', with $\nu_1(w') < \nu_1(w)$.

Finally, let $j \neq 1$. Recall that in the equality (1.10) we have $j \neq 1, t, s$ and $s < \max(w)$. From (1.10) we obtain $\nu_1(w) = \nu_1(w')$. Since $w' \prec x_sw/x_{\max(w)}$, we have $w' >_{\text{lex}} x_sw/x_{\max(w)}$, that is

$$w'x_{\max(w)} >_{\text{lex}} x_sw. \tag{1.12}$$

Replacing $w'x_{\max(w)}$ by x_tx_sw/x_j in (1.12), we get $x_t >_{\text{lex}} x_j$, which means $t < j$. It follows that: $x_{\max(w)}w'/x_s = x_tw/x_j >_{\text{lex}} w \geq_{\text{lex}} v$. Since $s < \max(w)$, as in the proof for $j = 1$, we have $x_{\max(w)}w'/x_s \leq_{\text{lex}} u$. Therefore $x_{\max(w)}w'/x_s \in \mathcal{L}(u, v)$. In addition, from (1.12), $x_{\max(w)}w'/x_s >_{\text{lex}} w$ and $\nu_1(x_{\max(w)}w'/x_s) = \nu_1(w)$, so $x_{\max(w)}w'/x_s \prec w$. In other words, we have got that $x_tw = x_j(x_{\max(w)}w'/x_s) \in I_{\prec w}$ and $t \in \text{set}(w)$. □

The general problem of determining the resolution of arbitrary lexsegment ideals is not completely solved. The resolutions of the lexsegment ideals with linear quotients are described in [9] using iterated mapping cones. We recall this construction from [9]. Suppose that the monomial ideal I has linear quotients with respect to the ordering $u_1, \ldots, u_m$ of its minimal generators. Set $I_j = (u_1, \ldots, u_j)$ and $L_j = (u_1, \ldots, u_j) : u_{j+1}$. Since $I_{j+1}/I_j \simeq S/L_j$, we get the exact sequences

$$0 \to S/L_j \to S/I_j \to S/I_{j+1} \to 0,$$

where the morphism $S/L_j \to S/I_j$ is the multiplication by u_{j+1}. Let $F^{(j)}$ be a graded free resolution of S/I_j, $K^{(j)}$ the Koszul complex associated to the regular sequence $x_{k_1}, \ldots, x_{k_l}$ with $k_i \in \text{set}(u_{j+1})$, and $\psi^{(j)}\colon K^{(j)} \to F^{(j)}$ a graded complex morphism lifting the map $S/L_j \to S/I_j$. Then the mapping cone $C(\psi^{(j)})$ of $\psi^{(j)}$ yields a free resolution of S/I_{j+1}. By iterated mapping cones we obtain step by step a graded free resolution of S/I.

Lemma 1.9 ([9]). *Suppose* $\deg u_1 \leq \deg u_2 \leq \cdots \leq \deg u_m$. *Then the iterated mapping cone* $\mathbb{F}$, *derived from the sequence* $u_1, \ldots, u_m$, *is a minimal graded free resolution of* S/I, *and for all* $i > 0$ *the symbols*

$$f(\sigma; u) \quad \textit{with} \quad u \in G(I), \quad \sigma \subset \text{set}(u), \quad |\sigma| = i - 1$$

form a homogeneous basis of the S*-module* F_i. *Moreover* $\deg(f(\sigma; u)) = |\sigma| + \deg(u)$.

Theorem 1.10 ([9]). *Let I be a monomial ideal of S with linear quotients, and $\mathbb{F}_\bullet$ the graded minimal free resolution of S/I. Suppose that the decomposition function $g\colon M(I) \to G(I)$ is regular. Then the chain map ∂ of $\mathbb{F}_\bullet$ is given by*

$$\partial(f(\sigma; u)) = -\sum_{s\in\sigma}(-1)^{\alpha(\sigma;s)} x_s f(\sigma \setminus s; u) + \sum_{s\in\sigma}(-1)^{\alpha(\sigma;s)} \frac{x_s u}{g(x_s u)} f(\sigma \setminus s; g(x_s u)),$$

if $\sigma \neq \emptyset$, and

$$\partial(f(\emptyset; u)) = u$$

otherwise. Here $\alpha(\sigma; s) = |\{t \in \sigma \mid t < s\}|$.

In our specific context we get the following

Corollary 1.11. *Let $I = (\mathcal{L}(u, v)) \subset S$ be a completely lexsegment ideal with linear quotients with respect to $\prec$ and $\mathbb{F}_\bullet$ the graded minimal free resolution of S/I. Then the chain map of $\mathbb{F}_\bullet$ is given by*

$$\partial(f(\sigma; w)) = -\sum_{s\in\sigma}(-1)^{\alpha(\sigma;s)} x_s f(\sigma \setminus s; w) + \sum_{\substack{s\in\sigma:\\ x_s w \geq_{\mathrm{lex}} x_1 v}} (-1)^{\alpha(\sigma;s)} x_1 f\left(\sigma \setminus s; \frac{x_s w}{x_1}\right)$$
$$+ \sum_{\substack{s\in\sigma:\\ x_s w <_{\mathrm{lex}} x_1 v}} (-1)^{\alpha(\sigma;s)} x_{\max(w)} f\left(\sigma \setminus s; \frac{x_s w}{x_{\max(w)}}\right),$$

if $\sigma \neq \emptyset$, and

$$\partial(f(\emptyset; w)) = w$$

otherwise. For convenience we set $f(\sigma; w) = 0$ if $\sigma \not\subseteq \operatorname{set} w$.

EXAMPLE 1.12. Let $u = x_1^2 x_2$ and $v = x_2^3$ be monomials in the polynomial ring $S = k[x_1, x_2, x_3]$. Then

$$\mathcal{L}(u, v) = \{x_2^3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_1^2x_2\}.$$

The ideal $I = (\mathcal{L}(u, v))$ is a completely lexsegment ideal with linear quotients with respect to this ordering of the generators. We denote $u_1 = x_2^3$, $u_2 = x_1x_2^2$, $u_3 = x_1x_2x_3$, $u_4 = x_1x_3^2$, $u_5 = x_1^2x_2$. We have $\operatorname{set}(u_1) = \emptyset$, $\operatorname{set}(u_2) = \{2\}$, $\operatorname{set}(u_3) = \{2\}$, $\operatorname{set}(u_4) = \{2\}$, $\operatorname{set}(u_5) = \{2, 3\}$. Let $\mathbb{F}_\bullet$ be the minimal graded free resolution of S/I.

Since $\max\{|\operatorname{set}(w)| \mid w \in \mathcal{L}(u, v)\} = 2$, we have $F_i = 0$, for all $i \geq 4$.

A basis for the S-module F_1 is $\{f(\emptyset; u_1), f(\emptyset; u_2), f(\emptyset; u_3), f(\emptyset; u_4), f(\emptyset; u_5)\}$.

A basis for the S-module F_2 is

$$\{f(\{2\}; u_2), f(\{2\}; u_3), f(\{2\}; u_4), f(\{2\}; u_5), f(\{3\}; u_5)\}.$$

A basis for the S-module F_3 is $\{f(\{2,3\}; u_5)\}$.

We have the minimal graded free resolution $\mathbb{F}_\bullet$:

$$0 \to S(-5) \xrightarrow{\partial_2} S(-4)^5 \xrightarrow{\partial_1} S(-3)^5 \xrightarrow{\partial_0} S \to S/I \to 0$$

where the maps are

$$\partial_0(f(\emptyset; u_i)) = u_i, \quad \text{for} \quad 1 \le i \le 5,$$

so

$$\partial_0 = \begin{pmatrix} x_2^3 & x_1x_2^2 & x_1x_2x_3 & x_1x_3^2 & x_1^2x_2 \end{pmatrix}.$$

$$\begin{aligned}
\partial_1(f(\{2\}; u_2)) &= -x_2\, f(\emptyset; u_2) + x_1\, f(\emptyset; u_1),\\
\partial_1(f(\{2\}; u_3)) &= -x_2\, f(\emptyset; u_3) + x_3\, f(\emptyset; u_2),\\
\partial_1(f(\{2\}; u_4)) &= -x_2\, f(\emptyset; u_4) + x_3\, f(\emptyset; u_3),\\
\partial_1(f(\{2\}; u_5)) &= -x_2\, f(\emptyset; u_5) + x_1\, f(\emptyset; u_2),\\
\partial_1(f(\{3\}; u_5)) &= x_3\, f(\emptyset; u_5) - x_1\, f(\emptyset; u_3),
\end{aligned}$$

so

$$\partial_1 = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ -x_2 & x_3 & 0 & x_1 & 0 \\ 0 & -x_2 & x_3 & 0 & -x_1 \\ 0 & 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & x_3 \end{pmatrix}.$$

$$\begin{aligned}
\partial_2(f(\{2,3\}; u_5)) &= -x_2\, f(\{3\}; u_5) + x_3\, f(\{2\}; u_5) + x_1\, f(\{3\}; u_2) - x_1\, f(\{2\}; u_3)\\
&= -x_2\, f(\{3\}; u_5) + x_3\, f(\{2\}; u_5) - x_1\, f(\{2\}; u_3),
\end{aligned}$$

since $\{3\} \not\subseteq \operatorname{set}(u_2)$, so

$$\partial_2 = \begin{pmatrix} 0 \\ -x_1 \\ 0 \\ x_3 \\ -x_2 \end{pmatrix}.$$

2. Non-completely lexsegment ideals with linear resolutions

Theorem 2.1. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_2^{b_2} \cdots x_n^{b_n}$ be monomials of degree d in S, $a_1 > 0$. Suppose that the ideal $I = (\mathcal{L}(u, v))$ is not completely lexsegment ideal. Then I has a linear resolution if and only if I has linear quotients.*

Proof. We only have to prove that if I has a linear resolution then I has linear quotients for a suitable ordering of its minimal monomial generators. By [1, Theorem 2.4], since I has a linear resolution, u and v have the form:

$$u = x_1 x_{l+1}^{a_{l+1}} \cdots x_n^{a_n}, \quad v = x_l x_n^{d-1}, \quad \text{for some} \quad l \geq 2.$$

Then the ideal $I = (\mathcal{L}(u, v))$ can be written as a sum of ideals $I = J + K$, where J is the ideal generated by all the monomials of $\mathcal{L}(u, v)$ which are not divisible by x_1 and K is generated by all the monomials of $\mathcal{L}(u, v)$ which are divisible by x_1. More precise, we have

$$J = (\{w \mid x_2^d \geq_{\text{lex}} w \geq_{\text{lex}} v\})$$

and

$$K = (\{w \mid u \geq_{\text{lex}} w \geq_{\text{lex}} x_1 x_n^{d-1}\}).$$

One may see that J is generated by the initial lexsegment $\mathcal{L}^i(v) \subset k[x_2, \ldots, x_n]$, and hence it has linear quotients with respect to lexicographical order $>_{\text{lex}}$. Let $G(J) = \{g_1 \prec \cdots \prec g_m\}$, where $g_i \prec g_j$ if and only if $g_i >_{\text{lex}} g_j$. The ideal K is isomorphic to the ideal generated by the final lexsegment $\mathcal{L}^f(u/x_1)$ of degree $d-1$. Since final lexsegments are stable with respect to the order $x_n > \cdots > x_1$ of the variables, it follows that the ideal K has linear quotients with respect to $>_{\overline{\text{lex}}}$, where by $\overline{\text{lex}}$ we mean the lexicographical order corresponding to $x_n > \cdots > x_1$. Let $G(K) = \{h_1 \prec \cdots \prec h_p\}$, where $h_i \prec h_j$if and only if $h_i >_{\overline{\text{lex}}} h_j$. We consider the following ordering of the monomials of $G(I)$:

$$G(I) = \{g_1 \prec \cdots \prec g_m \prec h_1 \prec \cdots \prec h_p\}.$$

We claim that, for this ordering of its minimal monomial generators, I has linear quotients. In order to check this, we firstly notice that $I_{\prec g} : g = J_{\prec g} : g$ for every $g \in G(J)$. Since J has linear quotients with respect to $\prec$ it follows that $J_{\prec g} : g$ is generated by variables. Now it is enough to show that, for any generator h of K, the colon ideal $I_{\prec h} : h$ is generated by variables. We note that

$$I_{\prec h} : h = J : h + K_{\prec h} : h.$$

Since K has linear quotients, we already know that $K_{\prec h} : h$ is generated by variables. Therefore we only need to prove that $J : h$ is generated by variables. We will show that $J : h = (x_2, \ldots, x_l)$ and this will end our proof. Let $m \in J : h$ be a monomial. It follows that $mh \in J$. Since h is a generator of K, h is of the form $h = x_1 x_{l+1}^{\alpha_{l+1}} \ldots x_n^{\alpha_n}$, that is $h \notin (x_2, \ldots, x_l)$. But this implies that m must be in the ideal $(x_2, \ldots, x_l)$. For the reverse inclusion, let $2 \leq t \leq l$. Then $x_t h = x_1 \gamma$ for some monomial γ, of degree d. Replacing h in the equality we get $\gamma = x_t x_{l+1}^{\alpha_{l+1}} \cdots x_n^{\alpha_n}$ which shows that γ is a generator of J. Hence $x_t h \in J$. $\square$

EXAMPLE 2.2. Let $I = (\mathcal{L}(u, v)) \subset k[x_1, \ldots, x_6]$ with $u = x_1x_3^2x_5$ and $v = x_2x_6^3$. I is not a completely lexsegment ideal as it follows applying [4, Theorem 2.3], but I has a linear resolution by [1, Theorem 2.4]. I has linear quotients if we order its minimal monomial generators as indicated in the proof of the above theorem. On the other hand, if we order the generators of I using the order relation defined in the proof of Theorem 1.2 we can easy see that I does not have linear quotients. Indeed, following the definition of the order relation from Theorem 1.2 we should take

$$G(I) = \{x_2^4 \prec x_2^3x_3 \prec \cdots \prec x_2x_6^3 \prec x_1x_3^2x_5 \prec x_1x_3^2x_6 \prec x_1x_3x_4^2 \prec \cdots \prec x_1x_6^3\}.$$

For $h = x_1x_3x_4^2$ one may easy check that $I_{\prec h} : h$ is not generated by variables.

EXAMPLE 2.3. Let $u = x_1x_3x_4,\ v = x_2x_4^2$ be monomials in $k[x_1, \ldots, x_4]$. The ideal $I = (\mathcal{L}(u, v)) \subset k[x_1, \ldots, x_4]$ is a non-completely lexsegment ideal, since it does not verify the condition [4, Theorem 2.3 (b)]. By [1, Theorem 2.4], I has a linear resolution and by the proof of Theorem 2.1, I has linear quotients with respect to the following ordering of its minimal monomial generators:

$$x_2^3,\ x_2^2x_3,\ x_2^2x_4,\ x_2x_3^2,\ x_2x_3x_4,\ x_2x_4^2,\ x_1x_4^2,\ x_1x_3x_4.$$

We note that $\operatorname{set}(x_1x_4^2) = \{2\}$ and $\operatorname{set}(g(x_1x_2x_4^2)) = \operatorname{set}(x_2x_4^2) = \{2, 3\} \not\subseteq \operatorname{set}(x_1x_4^2)$, so the decomposition function is not regular for this ordering of the generators.

3. Cohen–Macaulay lexsegment ideals

In this section we study the dimension and the depth of arbitrary lexsegment ideals. These results are applied to describe the lexsegments ideals which are Cohen–Macaulay. We begin with the study of the dimension. As in the previous sections, let $d \geq 2$ be an integer. We denote $\mathfrak{m} = (x_1, \ldots, x_n)$. It is clear that if $I = (\mathcal{L}(u, v)) \subset S$ is a lexsegment ideal of degree d then $\dim(S/I) = 0$ if and only if $I = \mathfrak{m}^d$.

Proposition 3.1. *Let $u = x_1^{a_1} \cdots x_n^{a_n},\ v = x_q^{b_q} \cdots x_n^{b_n},\ 1 \leq q \leq n,\ a_1, b_q > 0$, be two monomials of degree d such that $u \geq_{\text{lex}} v$ and let I be the lexsegment ideal generated by $\mathcal{L}(u, v)$. We assume that $I \neq \mathfrak{m}^d$. Then*

$$\dim(S/I) = \begin{cases} n - q, & \text{if} \quad 1 \leq q < n, \\ 1, & \text{if} \quad q = n. \end{cases}$$

Proof. For $q = 1$, we have $I \subset (x_1)$. Obviously (x_1) is a minimal prime of I and $\dim(S/I) = n - 1$.

Let $q = n$, that is $v = x_n^d$ and $\mathcal{L}(u, v) = \mathcal{L}^f(u)$. We may write the ideal I as a sum of two ideals, $I = J + K$, where $J = (x_1\mathcal{L}(u/x_1, x_n^{d-1}))$ and $K = (\mathcal{L}(x_2^d, x_n^d))$. Let $p \supset I$ be a monomial prime ideal. If $x_1 \in p$, then $J \subseteq p$. Since p also contains K, we have $p \supset (x_2, \ldots, x_n)$. Hence $p = (x_1, x_2, \ldots, x_n)$. If $x_1 \notin p$, we obtain $(x_2, \ldots, x_n) \subset p$. Hence, the only minimal prime ideal of I is $(x_2, \ldots, x_n)$. Therefore, $\dim(S/I) = 1$.

Now we consider $1 < q < n$ and write I as before, $I = J + K$, where $J = (x_1\mathcal{L}(u/x_1, x_n^{d-1}))$ and $K = (\mathcal{L}(x_2^d, v))$.

Firstly we consider $u = x_1^d$. Let $p \supset I$ be a monomial prime ideal. Then $p \ni x_1$ and, since $p \supset K$, we also have $p \supset (x_2, \ldots, x_q)$. Hence $(x_1, \ldots, x_q) \subset p$. Since $I \subset (x_1, \ldots, x_q)$, it follows that $(x_1, \ldots, x_q)$ is the only minimal prime ideal of I. Therefore $\dim(S/I) = n - q$.

Secondly, let $a_1 > 1$ and $u \neq x_1^d$. The lexsegment $\mathcal{L}(u/x_1, x_n^{d-1})$ contains the lexsegment $\mathcal{L}(x_2^{d-1}, x_n^{d-1})$. Let p be a monomial prime ideal which contains I and such that $x_1 \notin p$. Then $p \supset \mathcal{L}(x_2^{d-1}, x_n^{d-1})$ which implies that $(x_2, \ldots, x_n) \subset p$. Obviously we also have $I \subset (x_2, \ldots, x_n)$, hence $(x_2, \ldots, x_n)$ is a minimal prime ideal of I.

Let $p \supset I$ be a monomial prime ideal which contains x_1. Since $p \supset K$, we also have $(x_2, \ldots, x_q) \subset p$. This shows that $(x_1, \ldots, x_q)$ is a minimal prime ideal of I. In conclusion, for $a_1 > 1$, the minimal prime ideals of I are $(x_1, \ldots, x_q)$ and $(x_2, \ldots, x_n)$. Since $q \leq n - 1$, we get $\operatorname{ht}(I) = q$ and $\dim(S/I) = n - q$.

Finally, let $a_1 = 1$, that is $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$, for some $a_l > 0$, $l \geq 2$. As in the previous case, we obtain $(x_1, \ldots, x_q)$ a minimal prime ideal of I. Now we look for those minimal prime ideals of I which do not contain x_1.

If $a_l = d - 1$, the ideal $J = (x_1\mathcal{L}(u/x_1, x_n^{d-1}))$ becomes $J = (x_1\mathcal{L}(x_l^{d-1}, x_n^{d-1}))$. If $p \supset I$ is a monomial prime ideal such that $x_1 \notin p$, we get $(x_l, \ldots, x_n) \subset p$, and, since p contains K, we obtain $(x_2, \ldots, x_q) \subset p$. This shows that if $q < l$ then $(x_2, \ldots, x_q, x_l, \ldots, x_n)$ is a minimal prime ideal of I of height $q + n - l \geq q$, and if $q \geq l$, then $(x_2, \ldots, x_n)$ is a minimal prime ideal of height $n - 1 \geq q$. In both cases we may draw the conclusion that $\operatorname{ht}(I) = q$ and, consequently, $\dim(S/I) = n - q$.

The last case we have to consider is $a_l < d - 1$. Then $l < n$ and, with similar arguments as above, we obtain $\dim(S/I) = n - q$. □

In order to study the depth of arbitrary lexsegment ideals, we note that one can restrict to those lexsegments defined by monomials of the form $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$ of degree d with $a_1 > 0$ and $b_1 = 0$.

Indeed, if $a_1 = b_1$, then $I = (\mathcal{L}(u, v))$ is isomorphic, as an S-module, to the ideal generated by the lexsegment $\mathcal{L}(u/x_1^{a_1}, v/x_1^{b_1})$ of degree $d - a_1$. This lexsegment may be studied in the polynomial ring in a smaller number of variables.

If $a_1 > b_1$, then $I = (\mathcal{L}(u, v))$ is isomorphic, as an S-module, to the ideal generated by the lexsegment $\mathcal{L}(u', v')$, where $u' = u/x_1^{b_1}$ has $\nu_1(u') = a_1 - b_1 > 0$ and $v' = v/x_1^{b_1}$ has $\nu_1(v') = 0$.

Taking into account these remarks, from now on, we consider lexsegment ideals of ends $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_q^{b_q} \cdots x_n^{b_n}$, for some $q \geq 2$, $a_1, b_q > 0$.

The first step in the depth's study is the next

Proposition 3.2. *Let* $I = (\mathcal{L}(u, v))$, *where* $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_q^{b_q} \cdots x_n^{b_n}$, $q \geq 2$, $a_1, b_q > 0$. *Then* $\operatorname{depth}(S/I) = 0$ *if and only if* $x_n u/x_1 \geq_{\text{lex}} v$.

Proof. Let $x_n u/x_1 \geq_{\text{lex}} v$. We claim that $(I : (u/x_1)) = (x_1, \ldots, x_n)$. Indeed, for $1 \leq j \leq n$, the inequalities $u \geq_{\text{lex}} x_j u/x_1 \geq_{\text{lex}} x_n u/x_1 \geq_{\text{lex}} v$ hold. They show that $x_j u/x_1 \in I$ for $1 \leq j \leq n$. Therefore $(x_1, \ldots, x_n) \subseteq (I : (u/x_1))$. The other inclusion is obvious. We conclude that $(x_1, \ldots, x_n) \in \operatorname{Ass}(S/I)$, hence $\operatorname{depth}(S/I) = 0$.

For the converse, let us assume, by contradiction, that $x_n u/x_1 <_{\text{lex}} v$. We will show that $x_1 - x_n$ is regular on S/I. This will imply that $\operatorname{depth}(S/I) > 0$, which contradicts our hypothesis. We firstly notice that, from the above inequality, we have $a_1 - 1 = 0$, that is $a_1 = 1$. Therefore, u is of the form $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$, $l \geq 2$, $a_l > 0$. Moreover, we have $l \geq q$.

Let us suppose that $x_1 - x_n$ is not regular on S/I, that is there exists at least a polynomial $f \notin I$ such that $f(x_1 - x_n) \in I$. One may assume that all monomials of $\operatorname{supp}(f)$ do not belong to I. Let us choose such a polynomial $f = c_1 w_1 + \cdots + c_t w_t$, $c_i \in k$, $1 \leq i \leq t$, with $w_1 >_{\text{lex}} w_2 >_{\text{lex}} \cdots >_{\text{lex}} w_t$, $w_i \notin I$, $1 \leq i \leq t$.

Then $\operatorname{in}_{\text{lex}}((x_1 - x_n)f) = x_1 w_1 \in I$. It follows that there exists $\alpha \in G(I)$ such that

$$x_1 w_1 = \alpha \cdot \alpha' \tag{3.1}$$

for some monomial α'. We have $x_1 \nmid \alpha'$ since, otherwise, $w_1 \in I$, which is false. Hence α is a minimal generator of I which is divisible by x_1, that is α is of the form $\alpha = x_1 \gamma$, for some monomial γ such that $x_n^{d-1} \leq_{\text{lex}} \gamma \leq_{\text{lex}} u/x_1$. Looking at (3.1), we get $w_1 = \gamma \alpha'$. This equality shows that $x_1 \nmid w_1$. We claim that the monomial $x_n w_1$ does not cancel in the expansion of $f(x_1 - x_n)$. Indeed, it is clear that $x_n w_1$ cannot cancel by some monomial $x_n w_i$, $i \geq 2$. But it also cannot cancel by some monomial of the form $x_1 w_i$ since $x_n w_1$ is not divisible by x_1. Now we may draw the conclusion that there exists a monomial $w \notin I$ such that $w(x_1 - x_n) \in I$, that is $w x_1$, $w x_n \in I$.

Let $w \notin I$ be a monomial such that $w x_1, w x_n \in I$, let $\alpha, \beta \in \mathcal{L}(u, v)$ and α', β' monomials such that

$$x_1 w = \alpha \cdot \alpha' \tag{3.2}$$

and

$$x_n w = \beta \cdot \beta'. \tag{3.3}$$

As before, we get $x_1 \nmid w$, hence β must be a minimal generator of I such that $x_2^d \geq_{\text{lex}} \beta \geq_{\text{lex}} v$. By using (3.3), we can see that x_n does not divide β', hence $x_n \mid \beta$. It follows that w is divisible by β/x_n. w is also divisible by α/x_1. Therefore, $\delta = \operatorname{lcm}(\alpha/x_1, \beta/x_n) \mid w$. If $\deg \delta \geq d$ there exists a variable x_j, with $j \geq 2$, such that $(x_j \beta/x_n) \mid \delta$, thus $(x_j \beta/x_n) \mid w$.

It is obvious that $x_2^d \geq_{\text{lex}} x_j\beta/x_n \geq_{\text{lex}} \beta \geq_{\text{lex}} v$, hence $x_j\beta/x_n$ is a minimal generator of I which divides w, contradiction. This implies that δ has the degree $d-1$. This yields $\alpha/x_1 = \beta/x_n$. Then $\beta = x_n\alpha/x_1 \leq_{\text{lex}} x_n u/x_1 <_{\text{lex}} v$, contradiction. □

Next we are going to characterize the lexsegment ideals I such that depth $S/I > 0$, that is $x_n u/x_1 <_{\text{lex}} v$, which implies that u has the form $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$, for some $l \geq 2$, $a_l > 0$ and $l > q$, or $l = q$ and $a_q \leq b_q$. We denote $u' = u/x_1 = x_l^{a_l} \cdots x_n^{a_n}$. Then we have $x_n u' <_{\text{lex}} v$. From the proof of Proposition 3.2 we know that $x_1 - x_n$ is regular on S/I. Therefore

$$\operatorname{depth}(S/I) = \operatorname{depth}(S'/I') + 1,$$

where $S' = k[x_2, \ldots, x_n]$ and I' is the ideal of S' whose minimal monomial generating set is $G(I') = x_n\mathcal{L}(u', x_n^{d-1}) \cup \mathcal{L}^i(v)$.

Lemma 3.3. *In the above notations and hypotheses on the lexsegment ideal I, the following statements hold*:
(a) *If* $v = x_2^d$ *and* $l \geq 4$, *then* $\operatorname{depth}(S'/I') = l - 3$.
(b) *If* $v = x_2^{d-1}x_j$ *for some* $3 \leq j \leq n-2$ *and* $l \geq j+2$ *then* $\operatorname{depth}(S'/I') = l - j - 1$.
(c) $\operatorname{depth}(S'/I') = 0$ *in all the other cases.*

Proof. (a) Let $v = x_2^d$ and $l \geq 4$. The ideal $I' \subset S'$ is minimally generated by all the monomials $x_n\gamma$, where $x_n^{d-1} \leq_{\text{lex}} \gamma \leq_{\text{lex}} u'$, $\deg(\gamma) = d-1$, and by the monomial x_2^d. Then it is clear that $\{x_3, \ldots, x_{l-1}\}$ is a regular sequence on S'/I', hence

$$\operatorname{depth} S'/I' = \operatorname{depth} \frac{S'/I'}{(x_3, \ldots, x_{l-1})S'/I'} + l - 3.$$

We have

$$\frac{S'/I'}{(x_3, \ldots, x_{l-1})S'/I'} \cong \frac{k[x_2, x_l, \ldots, x_n]}{I' \cap k[x_2, x_l, \ldots, x_n]}.$$

In this way we may reduce the computation of $\operatorname{depth}(S'/I')$ to the case (c).

(b) Let $v = x_2^{d-1}x_j$, for some $3 \leq j \leq n-2$ and $l \geq j+2$. Hence I' is minimally generated by the following set of monomials

$$\{x_n\gamma \mid \gamma \text{ monomial of degree } d-1 \text{ such that } x_n^{d-1} \leq_{\text{lex}} \gamma \leq_{\text{lex}} u'\}$$
$$\cup \{x_2^d, x_2^{d-1}x_3, \ldots, x_2^{d-1}x_j\}.$$

Then $\{x_{j+1}, \ldots, x_{l-1}\}$ is a regular sequence on S'/I' and

$$\operatorname{depth} S'/I' = \operatorname{depth} \frac{S'/I'}{(x_{j+1}, \ldots, x_{l-1})S'/I'} + (l - j - 1).$$

Since

$$\frac{S'/I'}{(x_{j+1}, \ldots, x_{l-1})S'/I'} \cong \frac{k[x_2, \ldots, x_j, x_l, \ldots, x_n]}{I' \cap k[x_2, \ldots, x_j, x_l, \ldots, x_n]},$$

we may reduce the computation of $\operatorname{depth}(S'/I')$ to the case (c).

(c) In each of the cases that it remains to treat, we will show that $(x_2, \ldots, x_n) \in \operatorname{Ass}(S'/I')$, that is there exists a monomial $w \notin I'$ such that $I' : w = (x_2, \ldots, x_n)$. This implies that $\operatorname{depth}(S'/I') = 0$.

SUBCASE C_1: $v = x_2^d$, $l = 2$. Then $w = x_n^{d-1} \notin I'$ and $x_n^{d-1} \leq_{\text{lex}} x_j w/x_n = x_j x_n^{d-2} \leq_{\text{lex}} x_2 x_n^{d-2} \leq_{\text{lex}} x_l^{a_l} \cdots x_n^{a_n} = u'$, for all $2 \leq j \leq n$. Hence $\gamma = x_j w/x_n$ has the property that $x_n \gamma \in G(I')$. Therefore, $x_j \in I' : w$ for all $2 \leq j \leq n$. It follows that $I' : w = (x_2, \ldots, x_n)$.

SUBCASE C_2: $v = x_2^d$, $l = 3$. Then $w = x_2^{d-1} x_n^{d-1} \notin I'$. Indeed, $x_2^d \nmid w$ and if we assume that there exists $x_n^{d-1} \leq_{\text{lex}} \gamma \leq_{\text{lex}} u'$, $\deg \gamma = d - 1$, such that $x_n \gamma \mid w$, we obtain $x_n \gamma \mid x_n^{d-1}$ which is impossible.

We show that $x_j w \in I'$ for all $2 \leq j \leq n$. Indeed, $x_2 w = x_2^d x_n^{d-1} \in I'$. Let $3 \leq j \leq n$. Then $x_n^{d-1} \leq_{\text{lex}} x_j x_n^{d-2} \leq_{\text{lex}} x_3 x_n^{d-2} \leq_{\text{lex}} u'$. It follows that $\gamma = x_j x_n^{d-2}$ has the property that $x_n \gamma = x_j x_n^{d-1} \in G(I')$. Since $x_n \gamma \mid x_j w$, we have $x_j w \in I'$. This arguments shows that $I' : w = (x_2, \ldots, x_n)$.

SUBCASE C_3: $v = x_2^{d-1} x_j$ for some $3 \leq j \leq n - 1$ and $2 \leq l \leq j + 1$. Let us consider again the monomial $w = x_2^{d-1} x_n^{d-1}$. It is clear that $x_t w \in I$ for all $2 \leq t \leq j$. Let $t \geq j + 1$. Then $x_t w$ is divisible by $x_t x_n^{d-1}$. Since $x_t x_n^{d-2}$ satisfies the inequalities $x_n^{d-1} \leq_{\text{lex}} x_t x_n^{d-2} \leq_{\text{lex}} u'$, we have $x_t x_n^{d-1} \in G(I')$. It follows that $x_t w \in I'$ for $t \geq j+1$. Assume that $w \in I'$. Since $x_2^{d-1} x_t \nmid w$ for $2 \leq t \leq j$, we should have $x_n \gamma \mid w$ for some γ of degree $d - 1$ such that $x_n^{d-1} \leq_{\text{lex}} \gamma \leq_{\text{lex}} u'$. Since $\gamma \mid x_2^{d-1} x_n^{d-2}$ and $\gamma \leq_{\text{lex}} u'$, we get $l = 2$ and $a_2 = \nu_2(u') \geq \nu_2(\gamma)$. Let $\gamma = x_2^a x_n^{d-1-a}$, for some $a \geq 1$. In this case we change the monomial w. Namely, we consider the monomial $w' = x_2 x_n^{d-2}$ which does not belong to $G(I')$ since it has degree $d - 1$.

If $a_2 \geq 2$, for any j such that $2 \leq j \leq n$, we have $x_n^{d-1} <_{\text{lex}} x_j w'/x_n = x_2 x_j x_n^{d-3} <_{\text{lex}} x_l^{a_l} \cdots x_n^{a_n} = u'$. This shows that $x_j w' \in I'$ for $2 \leq j \leq n$ and hence, $I' : w = (x_2, \ldots, x_n)$.

If $a_2 = 1$, we take $w'' = x_n^{d-1} \notin I'$. For all j such that $2 \leq j \leq n$, we have $x_n^{d-1} \leq_{\text{lex}} x_j w''/x_n = x_j x_n^{d-2} \leq_{\text{lex}} x_2 x_n^{d-2} \leq_{\text{lex}} u'$. Therefore $x_j w'' \in I'$ for $2 \leq j \leq n$, hence $I' : w'' = (x_2, \ldots, x_n)$. In conclusion we have proved that in every case one may find a monomial $w \notin I'$ such that $I' : w = (x_2, \ldots, x_n)$.

SUBCASE C_4: Finally, let $v \leq_{\text{lex}} x_2^{d-1} x_n$. In this case, the ideal $I' : x_2^{d-1}$ obviously contains $(x_2, \ldots, x_n)$. Since the other inclusion is trivial, we get $I' : x_2^{d-1} = (x_2, \ldots, x_n)$. It is clear that $x_2^{d-1} \notin I'$. □

By using Lemma 3.3 we get:

Proposition 3.4. *Let* $I = (\mathcal{L}(u, v))$ *be a lexsegment ideal defined by the monomials* $u = x_1 x_l^{a_l} \cdots x_n^{a_n}$, $v = x_q^{b_q} \cdots x_n^{b_n}$ *where* $a_l, b_q > 0$, $l, q \geq 2$ *and* $x_n u/x_1 <_{\text{lex}} v$. *Then the following statements hold*:

(a) *If* $v = x_2^d$ *and* $l \geq 4$ *then* $\operatorname{depth}(S/I) = l - 2$;

(b) *If* $v = x_2^{d-1} x_j$ *for some* $3 \leq j \leq n-2$ *and* $l \geq j + 2$ *then* $\operatorname{depth}(S/I) = l - j$;

(c) $\operatorname{depth}(S/I) = 1$ *in all the other cases*.

Proof. Since $x_1 - x_n$ is regular on S/I if $x_n u/x_1 <_{\text{lex}} v$, we have $\operatorname{depth}(S/I) = \operatorname{depth}(S'/I') + 1$. The conclusion follows applying Lemma 3.3. □

As a consequence of the results of this section we may characterize the Cohen–Macaulay lexsegment ideals.

In the first place, we note that the only Cohen–Macaulay lexsegment ideal such that $\dim(S/I) = 0$ is $I = \mathfrak{m}^d$. Therefore it remains to consider Cohen–Macaulay ideals I with $\dim(S/I) \geq 1$.

Theorem 3.5. *Let* $n \geq 3$ *be an integer, let* $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$, *with* $a_1 > b_1 \geq 0$, *monomials of degree* d, *and* $I = (\mathcal{L}(u, v)) \subset S$ *the lexsegment ideal defined by* u *and* v. *We assume that* $\dim(S/I) \geq 1$. *Then* I *is Cohen–Macaulay if and only if one of the following conditions is fulfilled*:

(a) $u = x_1 x_n^{d-1}$ *and* $v = x_2^d$;

(b) $v = x_{n-1}^a x_n^{d-a}$ *for some* $a > 0$ *and* $x_n u/x_1 <_{\text{lex}} v$.

Proof. Let u, v be as in (a). Then $\dim(S/I) = n - 2$, by Proposition 3.1 and $\operatorname{depth}(S/I) = n - 2$ by using (a) in Proposition 3.4 for $n \geq 4$ and (c) for $n = 3$.

Let u, v as in (b). Then $\dim(S/I) = 1$ by Proposition 3.1. By using Proposition 3.4 (c), we obtain $\operatorname{depth}(S/I) = 1$, hence S/I is Cohen–Macaulay.

For the converse, in the first place, let us take I to be Cohen–Macaulay of $\dim(S/I) = 1$. By Proposition 3.1 we have $q = n$ or $q = n - 1$. If $q = n$, then $v = x_n^d$ and $x_n u/x_1 \geq_{\text{lex}} v$. By Proposition 3.2, $\operatorname{depth}(S/I) = 0$, so I is not Cohen–Macaulay.

Let $q = n - 1$, that is $v = x_{n-1}^a x_n^{d-a}$ for some $a > 0$. By Proposition 3.2, since $\operatorname{depth}(S/I) > 0$, we must have $x_n u/x_1 <_{\text{lex}} v$, thus we get (b).

Finally, let $\dim(S/I) \geq 2$, that is $q \leq n - 2$. By using Proposition 3.4, we obtain $q = 2$. Therefore $\dim(S/I) = \operatorname{depth}(S/I) = n - 2$. Using again Proposition 3.4 (a), (b), it follows that $u = x_1 x_n^{d-1}$ and $v = x_2^d$. □

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