Yousofzadeh, M.
Osaka J. Math.
46 (2009), 611–643

FIXED POINT SUBALGEBRAS OF ROOT GRADED LIE ALGEBRAS

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(Received April 9, 2008)

Abstract

We study the subalgebra of fixed points of a root graded Lie algebra under a certain class of finite order automorphisms. As the centerless core of extended affine Lie algebras or equivalently irreducible centerless Lie tori are examples of root graded Lie algebras, our work is an extension of some recent result about the subalgebra of fixed points of a Lie torus under a certain finite order automorphism.

0. Introduction

In 1955, A. Borel and G.D. Mostow [7] proved that the fixed point subalgebra (or f.p.s. for the sake of brevity) of a finite dimensional simple Lie algebra over a field $\mathbb{F}$ of characteristic zero, under a finite order automorphism, is a reductive Lie algebra. As a finite dimensional simple Lie algebra is an extended affine Lie algebra of nullity zero, a natural question which arises is that, what is the f.p.s. of an extended affine Lie algebra under a certain finite order automorphism? In 2005, S. Azam, S. Berman and M. Yousofzadeh [5] considered and showed that such a subalgebra has a reductive-like structure, namely it is decomposed into a sum of extended affine Lie algebras (up to existence of some isolated root spaces), a subspace of the center and a subspace contained in the centralizer of the core. Since the centerless core of an extended affine Lie algebra is a centerless irreducible Lie torus, a second question arises: What we can say about the fixed points of a Lie torus under automorphisms of similar nature. In 2006, S. Azam and V. Khalili [4] studied the f.p.s. of a centerless irreducible Lie torus $\mathcal{L}$ under a certain class of finite order automorphisms. They showed that the centerless core of the f.p.s. of $\mathcal{L}$ under an automorphism in the stated class is a direct sum of centerless irreducible Lie tori. In this article, we consider a similar question for a much more general class of Lie algebras, namely, the class of (R, S, Λ)-graded Lie algebras. An (R, S, Λ)-graded Lie algebra, for a finite root system R with a subsystem S and abelian group Λ, is a $\mathcal{Q}(R)$-graded Lie algebra whose support contains in R and that is generated by the homogeneous spaces of degree not equal zero. We study the subalgebra of fixed points of an (R, S, Λ)-graded Lie algebra, with respect to a certain automorphism.

2000 Mathematics Subject Classification. Primary 17B40, 17B70; Secondary 17B67.

This research was supported the center of excellence for Mathematics at the University of Isfahan and was in part supported by a grant from IPM (86170015).

We arrange the paper as follows: Section 1 is devoted to preliminary definitions and results we need throughout the work. In the first subsection of Section 2, we are exclusively concerned with the f.p.s. of an (R, Λ)-graded Lie algebra $\mathcal{L}$ equipped with a non-degenerate symmetric invariant graded bilinear form under a finite order automorphism satisfying certain properties. In this situation, we get the same result as in [4] for this much large class, more precisely, we prove that the centerless core of the f.p.s. of an algebra in this class is a direct sum of irreducible Lie tori. In the second subsection of Section 2, we focuse on the general case when we study the f.p.s. of an (R, S, Λ)-graded Lie algebra $\mathcal{L}$ under a certain finite order automorphism σ for a finite root system R, a subsystem S of R and an abelian group Λ. We prove that the core of the f.p.s. of $\mathcal{L}$ under σ is a sum of a root graded Lie algebra $\mathfrak{L}$ with a grading subalgebra $\mathfrak{g}$ and a subspace $\mathcal{K}$ whose normalizer contains $\mathfrak{L}$. We also prove that the f.p.s. of $\mathcal{L}$ is decomposed into its core, a subspace of the centralizer of the core and a subspace of the centralizer of $\mathfrak{g}$. We conclude our work with Section 3 allocated to examples.

1. Root graded Lie algebras

Throughout this work Λ is an abelian additive group and $\mathbb{F}$ is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over $\mathbb{F}$. In the present paper, we denote the dual space of a vector space V by $V^\star$. If $x \in V$ and $f \in V^\star$, we denote by $\langle x, f\rangle$, the image of x under f. If a finite dimensional vector space V is equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ and R is a subset of V, we set $R^\times := \{\alpha \in R \mid (\alpha, \alpha) \neq 0\}$ and $R^0 := R \setminus R^\times$. Also for $\alpha \in V^\star$, we take t_α to be the unique element in V representing α through the form. The form $(\cdot, \cdot)$ induces a form on $V^\star$, denoted again by $(\cdot, \cdot)$, by letting $(\alpha, \beta) := (t_\alpha, t_\beta)$ for $\alpha, \beta \in V^\star$. For a set S, we take $\#(S)$ to be the cardinal of S and id_S to be the identity map on S. For a subset S of a vector space, we denote by $\langle S\rangle$, the $\mathbb{Z}$-span of S and by $\mathcal{Q}(S)$, the $\mathbb{Q}$-span of S. For a finite dimensional Lie algebra $\mathcal{G}$, we use κ for Killing form of $\mathcal{G}$. Also for an algebra $\mathcal{A}$ and a subset S of $\mathcal{A}$, we mean by $Z(\mathcal{A})$, the center of $\mathcal{A}$ and by $C_\alpha a(S)$, the centralizer of S in $\mathcal{A}$.

DEFINITION 1.1. Let V be a vector space. V is called Λ-*graded* if $V = \bigoplus_{\lambda\in\Lambda} V^\lambda$ where V^λ's are subspaces of V. The *support* of V (with respect to the Λ-grading) is by definition the set $\mathrm{supp}(V) := \{\lambda \in \Lambda \mid V^\lambda \neq \{0\}\}$. For $\lambda \in \Lambda$, V^λ is called the *homogeneous subspace of V of degree* λ and $x \in V^\lambda$ is called a *homogeneous element of degree* λ. Let G be another abelian group, we say two gradings $V = \bigoplus_{g\in G} V_g$ and $V = \bigoplus_{\lambda\in\Lambda} V^\lambda$ are *compatible* if for all $g \in G$, $V_g = \bigoplus_{\lambda\in\Lambda} V_g^\lambda$ where $V_g^\lambda := V_g \cap V^\lambda$ for $\lambda \in \Lambda$. An algebra $(\mathcal{A}, \cdot)$ is called a Λ-*graded algebra* if $\mathcal{A} = \bigoplus_{\lambda\in\Lambda} \mathcal{A}^\lambda$ is a Λ-graded vector space satisfying $\mathcal{A}^\lambda \cdot \mathcal{A}^\mu \subseteq \mathcal{A}^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$.

DEFINITION 1.2. Let V be a finite dimensional nontrivial vector space over the field $\mathbb{F}$. A subset R of V is called a *finite root system* in V if the following conditions hold:
(i) R is finite, $0 \in R$ and R spans V.
(ii) For each $0 \neq \alpha \in R$, there exists $\check{\alpha} \in V^\star$ such that $\langle \alpha, \check{\alpha} \rangle = 2$ and such that the reflection w_α of V defined by

$$v \mapsto v - \langle v, \check{\alpha} \rangle \alpha; \quad v \in V$$

stabilizes R.
(iii) $\langle \beta, \check{\alpha} \rangle \in \mathbb{Z}$ for all $\beta, \alpha \in R \setminus \{0\}$.

Each element of R is called a *root* and the dimension of V is called the *rank* of R. A root α is said to be *divisible* or *indivisible* according to whether $\alpha/2$ is a root or not. We set $R_{\text{ind}} := \{0\} \cup \{\alpha \in R \mid \alpha/2 \notin R\}$. The root system R is called *indivisible* (*divisible*) if $R = R_{\text{ind}}$ ($R \neq R_{\text{ind}}$). For a subset S of R, we set $S^\times := S \setminus \{0\}$. The root system R is called *irreducible* if $R^\times$ cannot be written as a disjoint union of nonempty subsets A and B of $R^\times$ such that $\langle \beta, \check{\alpha} \rangle = \langle \alpha, \check{\beta} \rangle = 0$ for $\alpha \in A$ and $\beta \in B$. A subset $S \subseteq R$ is called a *subsystem* of R if $0 \in S$ and $w_\alpha(\beta) \in S$ for all $\alpha, \beta \in S^\times$.

DEFINITION 1.3 ([10, Section 2.9]). Let R be a finite root system and $\mathcal{L} = \bigoplus_{\alpha \in \mathcal{Q}(R)} \mathcal{L}_\alpha$ be a $\mathcal{Q}(R)$-graded Lie algebra with $\text{supp}(\mathcal{L}) \subseteq R$. In this situation, a nonzero element $e \in \mathcal{L}_\alpha$, $\alpha \in R^\times$, is called *invertible* (an invertible element of the $\mathcal{Q}(R)$-graded Lie algebra), if there exists $f \in \mathcal{L}_{-\alpha}$ such that $h := [f, e] \in \mathcal{L}_0$ operates diagonally on $\mathcal{L}$ that means $\operatorname{ad} h_{|\mathcal{L}_\beta} = \langle \beta, \check{\alpha} \rangle \operatorname{id}_{\mathcal{L}_\beta}$ for all $\beta \in \mathcal{Q}(R)$. It is proved that f with this property is unique and so we refer to f as the *inverse* of e and denote it by e^{-1}.

REMARK 1.4. Let R be a finite root system and $\mathcal{L} = \bigoplus_{\alpha \in \mathcal{Q}(R)} \mathcal{L}_\alpha$ be a $\mathcal{Q}(R)$-graded Lie algebra. If for $\alpha, \beta \in R^\times$, $e_\alpha \in \mathcal{L}_\alpha$, $e_\beta \in \mathcal{L}_\beta$ are invertible elements and $h_\alpha := [e_\alpha^{-1}, e_\alpha]$, $h_\beta := [e_\beta^{-1}, e_\beta]$, then since $h_\beta \in \mathcal{L}_0$, we have

$$[h_\alpha, h_\beta] = \langle 0, \check{\alpha} \rangle h_\beta = 0.$$

DEFINITION 1.5. Let $\mathcal{H}$ be an abelian Lie algebra. We say an $\mathcal{H}$-module $\mathcal{M}$ has a *weight space decomposition with respect to* $\mathcal{H}$, if

$$\mathcal{M} = \bigoplus_{\alpha \in \mathcal{H}^\star} \mathcal{M}_\alpha \quad \text{where} \quad \mathcal{M}_\alpha := \{x \in \mathcal{M} \mid h \cdot x = \alpha(h)x, \ \forall h \in \mathcal{H}\}; \quad \alpha \in \mathcal{H}^\star.$$

DEFINITION 1.6. Let R be a finite root system and $\mathcal{L} = \bigoplus_{\alpha \in \mathcal{Q}(R)} \mathcal{L}_\alpha$ be a $\mathcal{Q}(R)$-graded Lie algebra such that $\text{supp}(\mathcal{L}) \subseteq R$. The *core* of $\mathcal{L}$ is defined to be the subalgebra $\mathcal{L}_c$ of $\mathcal{L}$ generated by $\mathcal{L}_\alpha$, $\alpha \in R^\times$. The core modulo its center is called the *centerless core* of $\mathcal{L}$. Now let $\mathcal{L}$ be a Lie algebra equipped with compatible $\mathcal{Q}(R)$- and Λ-gradings $\mathcal{L} = \bigoplus_{\alpha\in\mathcal{Q}(R)} \mathcal{L}_\alpha$, $\mathcal{L} = \bigoplus_{\lambda\in\Lambda} \mathcal{L}^\lambda$ such that $\{\alpha \in \mathcal{Q}(R) \mid \mathcal{L}_\alpha \neq \{0\}\} \subseteq R$. We call an element $\lambda \in \Lambda$ with $\mathcal{L}_0^\lambda \neq \{0\}$, an *isotropic root* of $\mathcal{L}$. An isotropic root λ of $\mathcal{L}$ is called *isolated* if there are not $\alpha \in R^\times$ and $\delta \in \Lambda$ such that $\mathcal{L}_\alpha^{\lambda+\delta} \neq \{0\}$, $\mathcal{L}_\alpha^\delta \neq \{0\}$. An isotropic root λ is called *non-isolated* if it is not isolated. We denote the set of all isolated roots of $\mathcal{L}$ by $\Lambda(\mathcal{L})_{\text{iso}}$. We know that $\mathcal{L}_c$ inherits the compatible $\mathcal{Q}(R)$- and Λ-gradings. Take Λ_c to be the support of $\mathcal{L}_c$ with respect to the Λ-grading. We call the subspace $\mathcal{I} := \sum_{\lambda\in\Lambda(\mathcal{L})_{\text{iso}}\setminus\Lambda_c} \mathcal{L}_0^\lambda$ of $\mathcal{L}$, the *isolated subspace* of $\mathcal{L}$ with respect to the compatible gradings (we define $\mathcal{I} := \{0\}$ if $\Lambda(\mathcal{L})_{\text{iso}} \setminus \Lambda_c = \emptyset$).

DEFINITION 1.7 ([10, Section 2.9]). Let R be a finite root system and S be a subsystem of R. An (R, S, Λ)-graded Lie algebra is a Lie algebra $\mathcal{L}$ equipped with compatible $\mathcal{Q}(R)$- and Λ-gradings $\mathcal{L} = \bigoplus_{\alpha\in\mathcal{Q}(R)} \mathcal{L}_\alpha$, $\mathcal{L} = \bigoplus_{\lambda\in\Lambda} \mathcal{L}^\lambda$ such that the following conditions hold:

- $\{\alpha \in \mathcal{Q}(R) \mid \mathcal{L}_\alpha \neq \{0\}\} \subseteq R$.
- For every $\alpha \in S^\times$, the homogeneous space $\mathcal{L}_\alpha^0$ contains an invertible element of the $\mathcal{Q}(R)$-graded Lie algebra $\mathcal{L}$.
- $\mathcal{L}_0 = \sum_{\alpha\in R^\times}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.

For an (R, S, Λ)-graded Lie algebra $\mathcal{L}$, a family $\{e_\alpha \mid \alpha \in S^\times\}$ of invertible elements of $e_\alpha \in \mathcal{L}_\alpha^0$, $\alpha \in S^\times$, is called a *splitting family*. An (R, S, Λ)-graded Lie algebra is called (R, Λ)-*graded* if $S = R_{\text{ind}}$ and is called an (R, S)-*graded* Lie algebra if $\Lambda = \{0\}$. An (R, S)-graded Lie algebra is called an R-*graded* Lie algebra if $S = R_{\text{ind}}$. An (R, Λ)-graded Lie algebra $\mathcal{L}$ is called a *Lie torus* of type (R, Λ) if for each $\alpha \in R^\times$ and $\lambda \in \Lambda$, $\dim \mathcal{L}_\alpha^\lambda \leq 1$ and $\mathcal{L}_\alpha^\lambda$ contains an invertible element of the $\mathcal{Q}(R)$-graded Lie algebra $\mathcal{L}$ if $\mathcal{L}_\alpha^\lambda \neq \{0\}$. The Lie torus $\mathcal{L}$ is called *irreducible* if R is an irreducible finite root system.

Now let S be a subsystem of a finite root system R and $\mathcal{L} = \bigoplus_{\alpha\in\mathcal{Q}(R),\lambda\in\Lambda} \mathcal{L}_\alpha^\lambda$ be an (R, S, Λ)-graded Lie algebra. If $e \in \mathcal{L}_\alpha^\lambda$, $\alpha \in R^\times$, $\lambda \in \Lambda$ is an invertible element of the $\mathcal{Q}(R)$-graded Lie algebra $\mathcal{L}$ with the inverse f, then the uniqueness of the inverse implies that $f \in \mathcal{L}_{-\alpha}^{-\lambda}$.

Lemma 1.8. *Let R be a finite root system and $\mathcal{L} = \bigoplus_{\alpha\in R,\lambda\in\Lambda} \mathcal{L}_\alpha^\lambda$ be a Lie torus of type (R, Λ), then*

i) *$Z(\mathcal{L}) \subseteq \mathcal{L}_0$ and $\mathcal{L}/Z(\mathcal{L})$ is a centerless Lie torus of type (R, Λ).*

ii) *$\mathcal{L}$ is a sum of irreducible Lie tori. Moreover, if $\mathcal{L}$ is centerless, $\mathcal{L}$ is a direct sum of centerless irreducible Lie tori.*

Proof. i) Let $\alpha \in R^\times$, $\lambda \in \Lambda$ and $\mathcal{L}_\alpha^\lambda \neq \{0\}$, then since $\mathcal{L}_\alpha^\lambda$ is a one dimensional subspace of $\mathcal{L}$ containing an invertible element, one finds that $\mathcal{L}_\alpha^\lambda \cap Z(\mathcal{L}) = \{0\}$. This implies that $Z(\mathcal{L}) \subseteq \mathcal{L}_0$. Next it is easy to see that $Z(\mathcal{L}) = \bigoplus_{\alpha\in R,\lambda\in\Lambda}(\mathcal{L}_\alpha^\lambda \cap Z(\mathcal{L}))$ and then the second statement follows.

ii) Let $R = \biguplus_{i=1}^{p} R_i$ be the decomposition of R into irreducible finite root systems. For $1 \leq i \leq p$, define $\mathcal{L}_i := \sum_{\alpha \in R_i^\times} \mathcal{L}_\alpha \oplus \sum_{\alpha \in R_i^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$. Now let $1 \leq i \neq j \leq p$, then since for $\alpha \in R_i^\times$, $\beta \in R_j^\times$, $\alpha + \beta \notin R$, it follows that $\mathcal{L}_i$ is a Lie subalgebra of $\mathcal{L}$ and that

$$[\mathcal{L}_i, \mathcal{L}_j] = \{0\}; \quad 1 \leq i \neq j \leq p. \tag{1.9}$$

Now it is clear that the subalgebra $\mathcal{L}_i$, $1 \leq i \leq p$, inherits the compatible gradings on $\mathcal{L}$ and that $\mathcal{L}_i$, $1 \leq i \leq p$, is an irreducible Lie torus. This completes the proof of the first statement as $\mathcal{L} = \sum_{i=1}^{p} \mathcal{L}_i$. Next let the Lie algebra $\mathcal{L}$ be centerless and $\sum_{i=1}^{p} x_i = 0$ with $x_i \in \mathcal{L}_i$ for $1 \leq i \leq p$, then (1.9) implies that

$$[x_i, \mathcal{L}_j] \subseteq [\mathcal{L}_i, \mathcal{L}_j] = \{0\} \quad \text{and} \quad [x_i, \mathcal{L}_i] \subseteq \left[\sum_{i \neq j=1}^{p} \mathcal{L}_j, \mathcal{L}_i \right] = \{0\}; \quad 1 \leq i \leq p,$$

which means that for $1 \leq i \leq p$, $x_i \in Z(\mathcal{L}) = \{0\}$. Therefore $\mathcal{L}$ is direct sum of irreducible Lie tori $\mathcal{L}_i$'s $(1 \leq i \leq p)$ which are centerless using (1.9). □

To study a class of root graded Lie algebras containing a so-called grading subalgebra, from now on we assume that $\mathcal{L}$ is an (R, S, Λ)-graded Lie algebra where R is a finite root system in a vector space and S is a subsystem of R satisfying

(⋆) $\mathcal{Q}(S) = \mathcal{Q}(R)$ and there exists a base $\{\alpha_i \mid 1 \leq i \leq l\}$ of S such that for $1 \leq i \neq j \leq l$, $\{\alpha_i + n\alpha_j \mid n \in \mathbb{Z}\} \cap S = \{\alpha_i + n\alpha_j \mid n \in \mathbb{Z}\} \cap R$.

We fix a base $\{\alpha_i \mid 1 \leq i \leq l\}$ of S_{ind} satisfying the condition stated in (⋆) and for each $1 \leq i \leq l$, take e_i to be a fixed invertible element of $\mathcal{L}$ contained in $\mathcal{L}^0_{\alpha_i}$. One can see that $\{-e_i, f_i := e_i^{-1}, h_i := [-e_i, f_i] \mid 1 \leq i \leq l\}$ satisfies Serre's relations, so the subalgebra $\mathcal{G}$ of $\mathcal{L}$ generated by $\{e_i, f_i, h_i\}$ is a finite dimensional split semisimple Lie algebra with splitting Cartan subalgebra $\mathcal{H} := \bigoplus_{i=1}^{l} \mathbb{F}h_i$ and the root system S_{ind}. We refer to $\mathcal{G}$ as a *grading subalgebra* of $\mathcal{L}$. One knows from the finite dimensional theory that S_{ind} can be identified as a finite root system in $\mathcal{H}^\star$ and

$$\langle \beta, \check{\alpha} \rangle = 2\kappa(t_\beta, t_\alpha)/\kappa(t_\alpha, t_\alpha), \quad \alpha, \beta \in S^\times. \tag{1.10}$$

For $1 \leq i \leq l$, set $\dot{h}_i := 2t_{\alpha_i}/\kappa(t_{\alpha_i}, t_{\alpha_i}) \in \mathcal{H}$ and let $1 \leq j \leq l$. Since $\mathcal{G}_{\alpha_j} = \{x \in \mathcal{G} \mid [h, x] = \alpha_j(h)x, \forall h \in \mathcal{H}\} = \mathbb{F}e_j \subseteq \mathcal{L}_{\alpha_j}$, (1.10) and the invertibility of e_i imply that

$$\begin{aligned}[\dot{h}_i, e_j] = \alpha_j(\dot{h}_i)e_j = 2(\kappa(t_{\alpha_j}, t_{\alpha_i})/\kappa(t_{\alpha_i}, t_{\alpha_i}))e_j = \langle \alpha_j, \check{\alpha}_i \rangle e_j &= [h_i, e_j] \\ &= \alpha_j(h_i)e_j.\end{aligned}$$

Therefore we have

$$\alpha_j(\dot{h}_i) = \alpha_j(h_i); \quad 1 \leq i, j \leq l.$$

It follows from this together with the facts that $\mathcal{H}$ is finite dimensional and $\mathcal{H}^\star$ is spanned by $\{\alpha_j \mid 1 \leq j \leq l\}$ that

$$h_i = \dot{h}_i = 2t_{\alpha_i}/\kappa(t_{\alpha_i}, t_{\alpha_i}); \quad 1 \leq i \leq l. \tag{1.11}$$

We recall that we identified S_{ind} as a finite root system in $\mathcal{H}^\star$. Therefore $\mathcal{Q}(R) = \mathcal{Q}(S)$ is identified as a subset of $\mathcal{H}^\star$. Now we have the following lemma:

Lemma 1.12. (i) *Let $\beta \in R$, $\alpha \in S^\times$, then $\langle \beta, \check{\alpha}\rangle = 2\kappa(t_\beta, t_\alpha)/\kappa(t_\alpha, t_\alpha)$.*
(ii) *There is a splitting family $\{e_\alpha \mid \alpha \in S^\times\}$ of $\mathcal{L}$ such that*

$$\{e_\alpha, e_\alpha^{-1}, [e_\alpha^{-1}, e_\alpha] \mid \alpha \in S_{\text{ind}}^\times\} \subseteq \mathcal{G}.$$

Proof. (i) Let $\beta \in R$, $\alpha \in S^\times$. Since $\mathcal{Q}(R) = \mathcal{Q}(S)$, there exist rational numbers $r_1, \ldots, r_l$ such that $\beta = \sum_{i=1}^l r_i\alpha_i$ (identified as an element of $\mathcal{H}^\star$). Now since $\alpha_1, \ldots, \alpha_l \in S$, (1.10) implies that

$$\begin{aligned}\langle \beta, \check{\alpha}\rangle = \left\langle \sum_{i=1}^l r_i\alpha_i, \check{\alpha}\right\rangle = \sum_{i=1}^l r_i\langle \alpha_i, \check{\alpha}\rangle &= 2\kappa\left(\sum_{i=1}^l r_i t_{\alpha_i}, t_\alpha\right)\Big/\kappa(t_\alpha, t_\alpha)\\ &= 2\kappa(t_\beta, t_\alpha)/\kappa(t_\alpha, t_\alpha).\end{aligned}$$

(ii) Let $\alpha = \sum_{i=1}^l s_i\alpha_i \in S^\times$ and $\beta \in R$. Then by (i), we have

$$\begin{aligned}\langle \beta, \check{\alpha}\rangle = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} &= \frac{2\kappa\left(t_\beta, \sum_{i=1}^l s_i t_{\alpha_i}\right)}{\kappa(t_\alpha, t_\alpha)}\\ &= \sum_{i=1}^l \frac{2s_i\kappa(t_\beta, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)}\\ &= \sum_{i=1}^l \frac{2s_i\kappa(t_{\alpha_i}, t_{\alpha_i})\kappa(t_\beta, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)\kappa(t_{\alpha_i}, t_{\alpha_i})}\\ &= \sum_{i=1}^l \frac{s_i\kappa(t_{\alpha_i}, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)}\langle \beta, \check{\alpha}_i\rangle.\end{aligned} \tag{1.13}$$

Also using (1.11), we have

$$h'_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} = \sum_{i=1}^l \frac{2s_i t_{\alpha_i}}{\kappa(t_\alpha, t_\alpha)} = \sum_{i=1}^l \frac{s_i\kappa(t_{\alpha_i}, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)}h_i. \tag{1.14}$$

Now let $x \in \mathcal{L}_\beta$, then (1.14) together with (1.13) implies that

$$
\begin{aligned}
[h'_\alpha, x] &= \left[\sum_{i=1}^{l} \frac{s_i \kappa(t_{\alpha_i}, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)} h_i, x\right] = \sum_{i=1}^{l} \frac{s_i \kappa(t_{\alpha_i}, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)} [h_i, x] \\
&= \sum_{i=1}^{l} \frac{s_i \kappa(t_{\alpha_i}, t_{\alpha_i})}{\kappa(t_\alpha, t_\alpha)} \langle \beta, \check{\alpha}_i \rangle x \\
&= \langle \beta, \check{\alpha} \rangle x.
\end{aligned} \tag{1.15}
$$

We also know that $\alpha \in S^\times_{\text{ind}}$, then $[\mathcal{G}_{-\alpha}, \mathcal{G}_{+\alpha}] = \mathbb{F}t_\alpha$ and so there exist $e'_{\pm\alpha} \in \mathcal{G}_{\pm\alpha} \subseteq \mathcal{L}^0_{\pm\alpha}$ such that $[e'_{-\alpha}, e'_{+\alpha}] = h'_\alpha$ which together with (1.15) implies that $e'_{+\alpha}$ is an invertible element of $\mathcal{L}$. This completes the proof. □

From now on we work with a splitting family $\{e_\alpha \mid \alpha \in S^\times\}$ of $\mathcal{L}$ satisfying the condition that $\{e_\alpha \mid \alpha \in S^\times_{\text{ind}}\} \subseteq \mathcal{G}$ with $h_\alpha := [e_\alpha^{-1}, e_\alpha] = 2t_\alpha/\kappa(t_\alpha, t_\alpha)$, $\alpha \in S^\times_{\text{ind}}$. Since we have identified $R \subseteq \mathcal{Q}(S)$ as a subset of $\mathcal{H}^\star$, Lemma 1.12 (i) implies that

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = 2\kappa(t_\beta, t_\alpha)/\kappa(t_\alpha, t_\alpha) = \langle \beta, \check{\alpha} \rangle; \quad \alpha \in S^\times_{\text{ind}},\ \beta \in R.$$

Using this and the same argument as in [10, Proposition 2.11], for $\beta \in R$, we have

$$
\begin{aligned}
\mathcal{L}_\beta &= \{x \in \mathcal{L} \mid [h_\alpha, x] = \langle \beta, \check{\alpha} \rangle x, \text{ for all } \alpha \in S_{\text{ind}}\} \\
&= \{x \in \mathcal{L} \mid [h, x] = \beta(h)x, \text{ for all } h \in \mathcal{H}\}.
\end{aligned} \tag{1.16}
$$

2. The subalgebra of fixed points

This section deals with the study of the f.p.s. of a root graded Lie algebra under a certain finite order automorphism, a topic inspired by the work of S. Azam and V. Khalili [4]. They study the f.p.s. of a centerless irreducible Lie torus, an element of the class of root graded Lie algebras equipped with a symmetric non-degenerate invariant graded bilinear form, under an automorphism satisfying some properties. In the first subsection, we consider a triple $(\mathcal{L}, (\cdot, \cdot), \mathcal{H})$, where $\mathcal{L}$ is an (R, Λ)-graded Lie algebra equipped with a symmetric non-degenerate invariant graded bilinear form $(\cdot, \cdot)$ and $\mathcal{H}$ is a Cartan subalgebra of a grading subalgebra of $\mathcal{L}$. We let σ be an automorphism of $\mathcal{L}$ and take $R(\sigma)$ to be the root system of the f.p.s. of $\mathcal{L}$, $\mathcal{L}(\sigma)$, with respect to $\mathcal{H}(\sigma) := \mathcal{H} \cap \mathcal{L}(\sigma)$. We set some conditions on σ, extending the conditions in [4], among them invariancy of the form under σ and that the elements of $R(\sigma)$ are non-isotropic. These two conditions guarantee the existence of a subalgebra of $\mathcal{L}(\sigma)$ we will call a σ-splitting subalgebra. The existence of such a subalgebra is needed in the study of the general case as well, when we work with an (R, S, Λ)-graded Lie algebra, not necessarily equipped with a symmetric non-degenerate invariant graded bilinear form. In this situation we replace the two conditions stated above with two new appropriate conditions ((GC3) and (GC4) in Subsection 2.2). Throughout this section we suppose that n is a positive integer and the field $\mathbb{F}$ contains a primitive n-th root ζ of unity.

2.1. The subalgebra of fixed points of an (R, Λ)-graded Lie algebra. Throughout this subsection, R is a finite root system and $\mathcal{L} = \bigoplus_{\alpha\in R,\lambda\in\Lambda} \mathcal{L}_\alpha^\lambda$ is an (R, Λ)-graded Lie algebra equipped with a non-degenerate symmetric invariant graded bilinear form $(\cdot\,,\,\cdot)$, that is, $(\cdot\,,\,\cdot)$ is a non-degenerate symmetric invariant bilinear form satisfying

$$(\mathcal{L}^\lambda, \mathcal{L}^\mu) = \{0\} \quad \text{unless} \quad \lambda + \mu = 0.$$

Fix a grading subalgebra $\mathcal{G}$ of $\mathcal{L}$ with a splitting Cartan subalgebra $\mathcal{H}$. Suppose that *the restriction of the form* $(\cdot\,,\,\cdot)$ *to* $\mathcal{G}$ *is nonzero*. As in the previous section, we may assume that $R \subseteq \mathcal{H}^\star$ and that $\mathcal{L} = \bigoplus_{\alpha\in R} \mathcal{L}_\alpha$ with $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x,\ \forall h \in \mathcal{H}\}$ for $\alpha \in R$. Now it follows from the invariancy and the non-degeneracy of the form that

$$\text{for} \quad \alpha, \beta \in \operatorname{supp}(\mathcal{L}), \quad (\mathcal{L}_\alpha, \mathcal{L}_\beta) = \{0\} \quad \text{unless} \quad \alpha + \beta = 0.$$

We know that $\mathcal{G}$ is a finite dimensional split semisimple Lie algebra. Suppose that $\mathcal{G} = \bigoplus_{i=1}^l \mathcal{G}^i$ is the decomposition of $\mathcal{G}$ into simple ideals, then for $1 \leq i \leq l$, $\mathcal{G}^i$ is a finite dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathcal{H}^i := \mathcal{H}\cap\mathcal{G}^i$ and $\mathcal{H} = \bigoplus_{i=1}^l \mathcal{H}^i$. Let $1 \leq i \leq l$ and take R^i to be the root system of $\mathcal{G}^i$. Now the invariancy of the form implies that

$$(\mathcal{G}^i, \mathcal{G}^j) = \{0\}; \quad 1 \leq i \neq j \leq l. \tag{2.1}$$

Next for $1 \leq i \leq l$, put $(\cdot\,,\,\cdot)_i := (\cdot\,,\,\cdot)_{|_{\mathcal{G}^i\times\mathcal{G}^i}}$, then (2.1) implies that

$$(\cdot\,,\,\cdot)_{|_{\mathcal{G}\times\mathcal{G}}} = \bigoplus_{i=1}^l (\cdot\,,\,\cdot)_i \quad \text{and} \quad (\cdot\,,\,\cdot)_{|_{\mathcal{H}\times\mathcal{H}}} = \bigoplus_{i=1}^l (\cdot\,,\,\cdot)_{i|_{\mathcal{H}^i\times\mathcal{H}^i}}.$$

Lemma 2.2. *For $1 \leq i \leq l$, $(\cdot\,,\,\cdot)_i$ is a scalar multiple of Killing form κ_i of $\mathcal{G}^i$. Also if $(\cdot\,,\,\cdot)_i$ is nonzero, $(\cdot\,,\,\cdot)_{|_{\mathcal{G}^i\times\mathcal{G}^i}}$ and $(\cdot\,,\,\cdot)_{|_{\mathcal{H}^i\times\mathcal{H}^i}}$ are non-degenerate and $(\alpha, \alpha) \neq 0$ for $\alpha \in \mathcal{Q}(R^i)\setminus\{0\}$. In particular if $\mathcal{G}$ is simple, $(\cdot\,,\,\cdot)_{|_{\mathcal{G}\times\mathcal{G}}}$ and $(\cdot\,,\,\cdot)_{|_{\mathcal{H}\times\mathcal{H}}}$ are non-degenerate and $(\alpha, \alpha) \neq 0$ for $\alpha \in \mathcal{Q}(R)\setminus\{0\}$.*

Proof. The first statement is immediate as for $1 \leq i \leq l$, $\mathcal{G}^i$ is a centroid-simple Lie algebra. Now let $1 \leq i \leq l$ and $(\cdot\,,\,\cdot)_i$ be nonzero, then $(\cdot\,,\,\cdot)_i$ is a nonzero scalar multiple of Killing form κ_i. We know that κ_i is non-degenerate on $\mathcal{G}^i$ and $\mathcal{H}^i$ and that it is positive definite on $\mathcal{Q}(R^i)$, so we are done. □

From now on we assume that the restriction of the form on $\mathcal{H}$ is non-degenerate. Suppose $\sigma : \mathcal{L} \to \mathcal{L}$ is an automorphism of $\mathcal{L}$ satisfying
(C1) $\sigma^n = \mathrm{id}_{\mathcal{L}}$ (σ is of order n),
(C2) $\sigma(\mathcal{H}) \subset \mathcal{H}$,
(C3) $(\sigma(x), \sigma(y)) = (x, y)$ for $x, y \in \mathcal{L}$,
(C4) $\sigma(\mathcal{L}^\lambda) \subseteq \mathcal{L}^\lambda$ for $\lambda \in \Lambda$.

Since σ is of finite order n and $\sigma(\mathcal{H}) \subseteq \mathcal{H}$, we have

$$\mathcal{L} = \bigoplus_{i=0}^{n-1} \mathcal{L}_{\bar{i}} \quad \text{and} \quad \mathcal{H} = \bigoplus_{i=0}^{n-1} \mathcal{H}_{\bar{i}} \quad \text{where for} \quad 0 \le i \le n-1,$$

$$\mathcal{L}_{\bar{i}} := \{x \in \mathcal{L} \mid \sigma(x) = \zeta^i x\} \quad \text{and} \quad \mathcal{H}_{\bar{i}} := \mathcal{H} \cap \mathcal{L}_{\bar{i}}.$$

Set

$$\mathcal{L}(\sigma) := \mathcal{L}_{\bar{0}}, \quad \mathcal{H}(\sigma) := \mathcal{H}_{\bar{0}}. \tag{2.3}$$

Now using (C3), one concludes that

$$(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}) = (\mathcal{H}_{\bar{i}}, \mathcal{H}_{\bar{j}}) = \{0\} \quad \text{unless} \quad \overline{i+j} = \bar{0}. \tag{2.4}$$

Since the form on $\mathcal{H}$ is non-degenerate, for each $h \in \mathcal{H}$, there is a unique element of $\mathcal{H}^\star$, say $h^\star$, such that $t_{h^\star} = h$. Now as σ is an automorphism of $\mathcal{H}$, it induces an automorphism of $\mathcal{H}^\star$, denoted again by σ, as follows:

$$\sigma : \mathcal{H}^\star \to \mathcal{H}^\star, \quad \alpha \mapsto \sigma(t_\alpha)^\star; \quad \alpha \in \mathcal{H}^\star,$$

i.e., using (C3), we have $\sigma(\alpha)(h) = (\sigma(t_\alpha), h) = (t_\alpha, \sigma^{-1}(h)) = \alpha(\sigma^{-1}(h))$ for $\alpha \in \mathcal{H}^\star$ and $h \in \mathcal{H}$. Thus we have

$$\sigma^i(\alpha)(h) = \alpha(h); \quad \alpha \in \mathcal{H}^\star, \ h \in \mathcal{H}(\sigma), \ 0 \le i \le n. \tag{2.5}$$

Next since σ is an automorphism of $\mathcal{H}^\star$ of finite order n, we have

$$\mathcal{H}^\star = \bigoplus_{i=0}^{n-1} (\mathcal{H}^\star)_{\bar{i}} \quad \text{where} \quad (\mathcal{H}^\star)_{\bar{i}} := \{\alpha \in \mathcal{H}^\star \mid \sigma(\alpha) = \zeta^i \alpha\}; \quad 0 \le i \le n-1.$$

Take $\mathcal{H}^\star(\sigma) := (\mathcal{H}^\star)_{\bar{0}}$ and $\mathcal{H}^\star(c) := \sum_{i=1}^{n-1} (\mathcal{H}^\star)_{\bar{i}}$, then $\mathcal{H}^\star = \mathcal{H}^\star(\sigma) \oplus \mathcal{H}^\star(c)$. Let $\pi : \mathcal{H}^\star \to \mathcal{H}^\star(\sigma)$ be the natural projection map. Since for $\alpha \in \mathcal{H}^\star = \bigoplus_{i=0}^{n-1} (\mathcal{H}^\star)_{\bar{i}}$, $\alpha - \sigma(\alpha) \in \mathcal{H}^\star(c)$, $\pi(\alpha) = \pi(\sigma^i(\alpha))$ for all $0 \le i \le n-1$ and so $n\pi(\alpha) = \sum_{i=0}^{n-1} \pi(\sigma^i(\alpha)) = \pi\left(\sum_{i=0}^{n-1} \sigma^i(\alpha)\right) = \sum_{i=0}^{n-1} \sigma^i(\alpha)$. Therefore

$$\pi(\alpha) = \frac{1}{n} \sum_{i=0}^{n-1} \sigma^i(\alpha); \quad \alpha \in \mathcal{H}^\star. \tag{2.6}$$

Identifying $\mathcal{H}(\sigma)^\star$ with $\mathcal{H}^\star(\sigma)$ by letting any element of $\mathcal{H}(\sigma)^\star$ acts as zero on $\sum_{i=1}^{n-1}\mathcal{H}_{\bar{i}}$ and using (2.6) together with (2.5), we may assume that

$$\pi : \mathcal{H}^\star \to \mathcal{H}(\sigma)^\star; \quad \alpha \mapsto \alpha_{|\mathcal{H}(\sigma)}, \quad \alpha \in \mathcal{H}^\star. \tag{2.7}$$

Lemma 2.8. *If R is an irreducible finite root system, $(\pi(\alpha), \pi(\alpha)) \neq 0$ for $\alpha \in R$ with $\pi(\alpha) \neq 0$.*

Proof. One can see that for $\alpha \in R$, $\sigma(\mathcal{L}_\alpha) \subseteq \mathcal{L}_{\sigma(\alpha)}$ which implies that the support of $\mathcal{L}$ with respect to the $\mathcal{Q}(R)$-grading on $\mathcal{L}$ is preserved by the automorphism σ, in particular, $\sigma(R_{\text{ind}}) \subseteq R$. Therefore $\sigma^i(\alpha) \subseteq \mathcal{Q}(R)$ for $0 \leq i \leq n-1$ and $\alpha \in R$. Using this together with (2.6), for $\alpha \in R$, we have $\pi(\alpha) = (1/n)\sum_{i=0}^{n-1}\sigma^i(\alpha) \in \mathcal{Q}(R)$. Now we are done contemplating Lemma 2.2. □

We know from (1.16) that $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha\in R}\mathcal{L}_\alpha$ with respect to $\mathcal{H}$. Now since $\mathcal{H}(\sigma) \subseteq \mathcal{H}$, one gets that $\mathcal{L}$ is an $\mathcal{H}(\sigma)$-module having the weight space decomposition $\mathcal{L} = \sum_{\alpha\in R}\mathcal{L}_{\pi(\alpha)}$ with respect to $\mathcal{H}(\sigma)$ where

$$\mathcal{L}_{\pi(\alpha)} = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x,\ \forall h \in \mathcal{H}(\sigma)\} = \bigoplus_{\substack{\beta\in R,\\ \pi(\alpha)=\pi(\beta)}} \mathcal{L}_\beta; \quad \alpha \in R. \tag{2.9}$$

Now suppose that $\alpha \in R$, $x \in \mathcal{L}_{\pi(\alpha)}$ and $h \in \mathcal{H}(\sigma)$, then since σ is an automorphism, we have

$$\alpha(h)\sigma(x) = \sigma(\alpha(h)x) = \sigma([h, x]) = [\sigma(h), \sigma(x)] = [h, \sigma(x)]$$

which implies that $\sigma(\mathcal{L}_{\pi(\alpha)}) \subseteq \mathcal{L}_{\pi(\alpha)}$. Therefore we have

$$\mathcal{L}_{\pi(\alpha)} = \bigoplus_{i=0}^{n-1}\mathcal{L}_{\pi(\alpha),\bar{i}} \quad \text{where} \quad \mathcal{L}_{\pi(\alpha),\bar{i}} := \mathcal{L}_{\pi(\alpha)} \cap \mathcal{L}_{\bar{i}}; \quad 0 \leq i \leq n-1.$$

It then follows that

$$\mathcal{L}(\sigma) = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathcal{L}(\sigma)_{\pi(\alpha)} \quad \text{with} \quad \mathcal{L}(\sigma)_{\pi(\alpha)} := \mathcal{L}_{\pi(\alpha),\bar{0}}; \quad \alpha \in R. \tag{2.10}$$

Set

$$R(\sigma) := \{\pi(\alpha) \in \pi(R) \mid \mathcal{L}(\sigma)_{\pi(\alpha)} \neq \{0\}\}. \tag{2.11}$$

Since by (2.4), the form restricted to $\mathcal{L}(\sigma)$ is non-degenerate, for $\pi(\alpha), \pi(\beta) \in R(\sigma)$ we have

$$(\mathcal{L}(\sigma)_{\pi(\alpha)}, \mathcal{L}(\sigma)_{\pi(\beta)}) = \{0\} \quad \text{unless} \quad \pi(\alpha + \beta) = 0. \tag{2.12}$$

Next note that (C4) implies that $\mathcal{L}(\sigma)$ inherits the Λ-grading on $\mathcal{L}$, i.e.,

$$\mathcal{L}(\sigma)=\bigoplus_{\lambda\in\Lambda}\mathcal{L}(\sigma)^{\lambda}\quad\text{where}\quad \mathcal{L}(\sigma)^{\lambda}:=\mathcal{L}(\sigma)\cap\mathcal{L}^{\lambda};\quad \lambda\in\Lambda. \tag{2.13}$$

Lemma 2.14. *This grading is compatible with the $\mathcal{Q}(R(\sigma))$-grading on $\mathcal{L}(\sigma)$ stated in* (2.10), *i.e.,* $\mathcal{L}(\sigma)_{\pi(\alpha)}=\sum_{\lambda\in\Lambda}\mathcal{L}(\sigma)^{\lambda}_{\pi(\alpha)}=\sum_{\lambda\in\Lambda}(\mathcal{L}(\sigma)^{\lambda}\cap\mathcal{L}(\sigma)_{\pi(\alpha)})$ *for* $\alpha\in R$.

Proof. Let $\alpha\in R$ and $x\in\mathcal{L}(\sigma)_{\pi(\alpha)}$. Using (2.9), we have $x=\sum_{\beta\in R}x_\beta$ where for $\beta\in R$, $x_\beta\in\mathcal{L}_\beta$ and $x_\beta=0$ if $\pi(\beta)\neq\pi(\alpha)$. Since for $\beta\in R$, $\mathcal{L}_\beta=\bigoplus_{\lambda\in\Lambda}\mathcal{L}^{\lambda}_\beta=\bigoplus_{\lambda\in\Lambda}(\mathcal{L}_\beta\cap\mathcal{L}^{\lambda})$, one gets that $x_\beta=\sum_{\lambda\in\Lambda}x^{\lambda}_\beta$ where for $\lambda\in\Lambda$, $x^{\lambda}_\beta\in\mathcal{L}_\beta\cap\mathcal{L}^{\lambda}\subseteq\mathcal{L}_{\pi(\alpha)}\cap\mathcal{L}^{\lambda}$ if $\pi(\beta)=\pi(\alpha)$ and $x^{\lambda}_\beta=0$ otherwise. Next set

$$x^{\lambda}:=\sum_{\beta\in R}x^{\lambda}_\beta\in\mathcal{L}_{\pi(\alpha)}\cap\mathcal{L}^{\lambda};\quad \lambda\in\Lambda, \tag{2.15}$$

then $x=\sum_{\lambda\in\Lambda}x^{\lambda}$ and since $\sigma(x)=x$, we have $\sum_{\lambda\in\Lambda}x^{\lambda}=\sum_{\lambda\in\Lambda}\sigma(x^{\lambda})$. Contemplating (C4), we have $\sigma(x^{\lambda})=x^{\lambda}$, $\lambda\in\Lambda$, which together with (2.15) implies that $x^{\lambda}\in\mathcal{L}(\sigma)^{\lambda}_{\pi(\alpha)}$ for $\lambda\in\Lambda$. This completes the proof. □

The fifth condition on σ is as follows:
(C5) $C_{\mathcal{L}(\sigma)^0}(\mathcal{H}(\sigma))=\mathcal{H}(\sigma)$.

Let $\alpha\in R$, $\lambda\in\Lambda$, $x\in\mathcal{L}(\sigma)^{\lambda}_{\pi(\alpha)}$, $y\in\mathcal{L}(\sigma)^{-\lambda}_{\pi(-\alpha)}$ and $h\in\mathcal{H}(\sigma)$, then

$$([x,y],h)=(x,[y,h])=\alpha(h)(x,y)=\pi(\alpha)(h)(x,y)=(t_{\pi(\alpha)}(x,y),h).$$

But $[x,y]\in\mathcal{L}(\sigma)^{0}_{\pi(0)}=C_{\mathcal{L}(\sigma)^0}(\mathcal{H}(\sigma))=\mathcal{H}(\sigma)$ and by (2.4), the form is non-degenerate on $\mathcal{H}(\sigma)$, so

$$[x,y]=(x,y)t_{\tilde\alpha};\quad x\in\mathcal{L}(\sigma)^{\lambda}_{\tilde\alpha},\ y\in\mathcal{L}(\sigma)^{-\lambda}_{-\tilde\alpha},\ \tilde\alpha\in R(\sigma),\ \lambda\in\Lambda. \tag{2.16}$$

Proposition 2.17. *If $R(\sigma)^{\times}\neq\emptyset$, $R(\sigma)^{\times}\cup\{0\}$ is a finite root system in its $\mathbb{F}$-span.*

Proof. Consider the triple $(\mathcal{L}(\sigma),(\cdot,\cdot)_{|\mathcal{L}(\sigma)\times\mathcal{L}(\sigma)},\mathcal{H}(\sigma))$. We know that the form is symmetric, non-degenerate and invariant on $\mathcal{L}(\sigma)$. Also using the fact that $R(\sigma)^{\times}\neq\emptyset$ together with (2.10), the non-degeneracy of the form on $\mathcal{H}$ and (2.4), one gets that $\mathcal{H}(\sigma)$ is a nontrivial finite dimensional abelian subalgebra of $\mathcal{L}(\sigma)$ such that $\mathrm{ad}_{\mathcal{L}(\sigma)}(h)$ is diagonalizable for all $h\in\mathcal{H}(\sigma)$ and that $(\cdot,\cdot)_{|\mathcal{H}(\sigma)\times\mathcal{H}(\sigma)}$ is non-degenerate. Next, since the form is graded and non-degenerate on $\mathcal{L}(\sigma)$, (2.12) implies that for $\tilde\alpha,\tilde\beta\in R(\sigma)$, $\lambda,\mu\in\Lambda$,

$$(\mathcal{L}(\sigma)^{\lambda}_{\tilde\alpha},\mathcal{L}(\sigma)^{\mu}_{\tilde\beta})=\{0\}\quad\text{unless}\quad\lambda+\mu=0,\ \tilde\alpha+\tilde\beta=0\quad\text{and}\quad\mathcal{L}(\sigma)^{\lambda}_{\tilde\alpha}\neq\{0\}. \tag{2.18}$$

So by (2.16) there exist $e_{\pm\tilde{\alpha}} \in \mathcal{L}(\sigma)_{\pm\tilde{\alpha}}$ such that $[e_{\tilde{\alpha}}, e_{-\tilde{\alpha}}] = t_{\tilde{\alpha}}$. Also since $R(\sigma)$ is finite, $\mathrm{ad}_{\mathcal{L}(\sigma)}(x)$ is locally nilpotent on $\mathcal{L}(\sigma)$ for any $\tilde{\alpha} \in R(\sigma)^\times$ and $x \in \mathcal{L}_{\tilde{\alpha}}$. These all together with [5, Proposition 1.4] imply that $R(\sigma)^\times \cup \{0\}$ is a finite root system. □

Lemma 2.19. i) *For* $\tilde{\alpha} \in R(\sigma)^\times$ *and* $\lambda \in \Lambda$, $[\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}, \mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}] = \{0\}$.
ii) *For* $\tilde{\alpha} \in R(\sigma)^\times$ *and* $\lambda \in \Lambda$, $\dim(\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}) \leq 1$.

Proof. i) Let $\tilde{\alpha} \in R(\sigma)^\times$ and $\lambda \in \Lambda$. If $\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}} = \{0\}$, there is nothing to prove, so let $\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}} \neq \{0\}$ and $0 \neq z \in [\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}, \mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}] \subseteq \mathcal{L}(\sigma)^{2\lambda}_{2\tilde{\alpha}}$. Then (2.16) together with (2.18) implies that there is a subspace $S := \mathbb{F}x \oplus \mathbb{F}h \oplus \mathbb{F}y$ of $\mathcal{L}(\sigma)$ isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ with $x \in \mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}$, $y \in \mathcal{L}(\sigma)^{-\lambda}_{-\tilde{\alpha}}$ and $h = [x, y] \in \mathcal{H}(\sigma)$. Since $y \in \mathcal{L}(\sigma)^{-\lambda}_{-\tilde{\alpha}}$ and $[y, z] \in \mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}$, (2.16) implies that

$$[y, [y, z]] = (y, [y, z])t_{\tilde{\alpha}} = ([y, y], z)t_{\tilde{\alpha}} = 0.$$

This together with the fact that $3\tilde{\alpha}, -3\tilde{\alpha} \notin R(\sigma)$ (Proposition 2.17), implies that $\mathcal{M} := \mathbb{F}z \oplus \mathbb{F}[y, z]$ is a 2-dimensional S-module which is a contradiction by the $\mathfrak{sl}_2$-module theory.

ii) Let $\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}} \neq \{0\}$ and consider the $\mathfrak{sl}_2$-triple (x, h, y) introduced in the previous part. Let $e \in \mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}$ with $[e, y] = 0$, then by part i), we have

$$\tilde{\alpha}(h)e = [h, e] = [[x, y], e] = [[x, e], y] + [[e, y], x] = 0,$$

therefore by (2.16), the map $\mathrm{ad}\, y \colon \mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}} \to \mathbb{F}t_{\tilde{\alpha}}$ is a nonzero injective map and so $\dim(\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}) = \dim(\mathbb{F}t_{\tilde{\alpha}}) = 1$. □

Now we are ready to set our last assumption on σ:
(C6) $(\pi(\alpha), \pi(\alpha)) \neq 0$ for $\alpha \in R$ with $\pi(\alpha) \neq 0$.

Theorem 2.20. *Let $\mathcal{L}$ be an (R, Λ)-graded Lie algebra equipped with a non-degenerate symmetric invariant graded bilinear form $(\cdot, \cdot)$. Fix a grading subalgebra $\mathcal{G}$ of $\mathcal{L}$ with a splitting Cartan subalgebra $\mathcal{H}$ and suppose that the restriction of the form $(\cdot, \cdot)$ on $\mathcal{H}$ is non-degenerate. Let σ be an automorphism of $\mathcal{L}$ satisfying (C1)–(C6). If $R(\sigma)^\times \neq \emptyset$, $R(\sigma)$ is a finite root system and $\mathcal{L}(\sigma)_c$ is a Lie torus of type $(R(\sigma), \Lambda)$. Moreover $\mathcal{L}(\sigma)_c/Z(\mathcal{L}(\sigma)_c)$ is a direct sum of centerless irreducible Lie tori.*

Proof. Use Proposition 2.17 together with (C6) to conclude that $R(\sigma)$ is a finite root system. Next using Lemma 2.14, we get that $\mathcal{L}(\sigma)$ is equipped with compatible $\mathcal{Q}(R(\sigma))$- and Λ-gradings. Now note that for each $\tilde{\alpha} \in R(\sigma)^\times$ and $\lambda \in \Lambda$ with $\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}} \neq \{0\}$, (2.16) and (2.18) imply that $\mathcal{L}(\sigma)^\lambda_{\tilde{\alpha}}$ contains an invertible element. This together with Lemma 2.19 and that $\mathcal{L}(\sigma)_c = \sum_{\tilde{\alpha} \in R(\sigma)^\times} [\mathcal{L}(\sigma)_{\tilde{\alpha}}, \mathcal{L}(\sigma)_{-\tilde{\alpha}}] + \bigoplus_{\tilde{\alpha} \in R(\sigma)^\times} \mathcal{L}(\sigma)_{\tilde{\alpha}}$ implies that $\mathcal{L}(\sigma)_c$ is a Lie torus of type $(R(\sigma), \Lambda)$. Now using Lemma 1.8, we are done. □

Using Lemmas 2.2, 2.8, Theorem 2.20 and [11, Theorem 7.1], we have the following:

Corollary 2.21. *Let R be an irreducible finite root system and $\mathcal{L}$ be an (R,Λ)-graded Lie algebra equipped with a non-degenerate symmetric invariant graded bilinear form $(\cdot\,,\,\cdot)$ (e.g., $\mathcal{L}$ is an irreducible Lie torus of type (R,Λ)). Fix a grading subalgebra $\mathcal{G}$ of $\mathcal{L}$ with a splitting Cartan subalgebra $\mathcal{H}$ such that $(\cdot,\cdot)_{|\mathcal{G}\times\mathcal{G}}$ is nonzero. Let σ be an automorphism of $\mathcal{L}$ satisfying $(C1)$–$(C5)$. If $R(\sigma)^\times \neq \emptyset$, $R(\sigma)$ is a finite root system and $\mathcal{L}(\sigma)_c$ is a Lie torus of type $(R(\sigma),\Lambda)$. Moreover $\mathcal{L}(\sigma)_c/Z(\mathcal{L}(\sigma)_c)$ is a direct sum of centerless irreducible Lie tori.*

2.2. The general case. In this subsection, we are concerned with the study of the subalgebra of fixed points of an (R,S,Λ)-graded Lie algebra under an automorphism satisfying certain properties. We fix an (R,S,Λ)-graded Lie algebra $\mathcal{L}=\bigoplus_{\alpha\in\mathcal{Q}(R),\lambda\in\Lambda}\mathcal{L}_\alpha^\lambda$ for a finite root system R and a subsystem S of R satisfying $(\star)$. We also fix a base $\{\alpha_i \mid 1\leq i\leq l\}$ of S satisfying the property stated in $(\star)$ and a set of invertible elements $e_i\in\mathcal{L}^0_{\alpha_i}$, $1\leq i\leq l$. Then the subalgebra $\mathcal{G}$ of $\mathcal{L}^0$ generated by $\{e_i, h_i := [e_i^{-1}, e_i], e_i^{-1},\ 1\leq i\leq l\}$ is a finite dimensional split semisimple Lie algebra with splitting Cartan subalgebra $\mathcal{H}=\bigoplus_{i=1}^l \mathbb{F}h_i$ and the root system S_{ind} (see the previous section). Consider an automorphism σ of $\mathcal{L}$ satisfying the conditions (GC1)–(GC5) describing below: We start with

(GC1) $\sigma^n=\mathrm{id}_{\mathcal{L}}$ (σ is of order n).

Since σ is of finite order n, we have

$$\mathcal{L}=\bigoplus_{i=0}^{n-1}\mathcal{L}_{\bar{i}}\quad\text{where}\quad \mathcal{L}_{\bar{i}} := \{x\in\mathcal{L}\mid \sigma(x)=\zeta^i x\};\quad 0\leq i\leq n-1.$$

Set

$$\mathcal{L}(\sigma):=\mathcal{L}_{\bar{0}},\quad \mathcal{G}(\sigma):=\mathcal{G}\cap\mathcal{L}(\sigma),\quad \mathcal{H}(\sigma):=\mathcal{H}\cap\mathcal{L}(\sigma). \tag{2.22}$$

We know from (1.16) that $\mathcal{L}$ has a weight space decomposition $\mathcal{L}=\bigoplus_{\alpha\in R}\mathcal{L}_\alpha$ with respect to $\mathcal{H}$. Now since $\mathcal{H}(\sigma)\subseteq\mathcal{H}$, one gets that $\mathcal{L}$ has a weight space decomposition $\mathcal{L}=\sum_{\alpha\in R}\mathcal{L}_{\pi(\alpha)}$ with respect to $\mathcal{H}(\sigma)$ where

$$\pi:\mathcal{H}^\star\to\mathcal{H}(\sigma)^\star;\quad \alpha\mapsto\alpha_{|\mathcal{H}(\sigma)},\quad \alpha\in\mathcal{H}^\star\quad\text{and}$$

$$\mathcal{L}_{\pi(\alpha)}:=\{x\in\mathcal{L}\mid [h,x]=\alpha(h)x,\ \forall h\in\mathcal{H}(\sigma)\}=\bigoplus_{\substack{\beta\in R,\\ \pi(\alpha)=\pi(\beta)}}\mathcal{L}_\beta;\quad \alpha\in R. \tag{2.23}$$

Also since $\mathcal{L}(\sigma)$ and $\mathcal{G}(\sigma)$ are two $\mathcal{H}(\sigma)$-submodules of $\mathcal{L}$, we have

$$\begin{aligned}&\mathcal{L}(\sigma)=\bigoplus_{\pi(\alpha)\in\pi(R)}\mathcal{L}(\sigma)_{\pi(\alpha)}\quad\text{and}\quad\mathcal{G}(\sigma)=\bigoplus_{\pi(\alpha)\in\pi(R)}\mathcal{G}(\sigma)_{\pi(\alpha)}\quad\text{where}\\&\mathcal{L}(\sigma)_{\pi(\alpha)}:=\mathcal{L}(\sigma)\cap\mathcal{L}_{\pi(\alpha)}\quad\text{and}\quad\mathcal{G}(\sigma)_{\pi(\alpha)}:=\mathcal{G}(\sigma)\cap\mathcal{L}_{\pi(\alpha)};\quad\alpha\in R.\end{aligned}\tag{2.24}$$

Set

$$\begin{aligned}&R_{\mathcal{L}(\sigma)}:=\{\pi(\alpha)\in\pi(R)\mid\mathcal{L}(\sigma)_{\pi(\alpha)}\neq\{0\}\}\quad\text{and}\\&R_{\mathcal{G}(\sigma)}:=\{\pi(\alpha)\in\pi(R)\mid\mathcal{G}(\sigma)_{\pi(\alpha)}\neq\{0\}\}.\end{aligned}\tag{2.25}$$

Now consider the following assumptions on σ:
(GC2) $\mathcal{H}(\sigma)$ is self-centralizing in $\mathcal{G}(\sigma)$.
(GC3) $R^{\times}_{\mathcal{G}(\sigma)}\neq\emptyset$ and the restriction of Killing form $\kappa(\cdot\,,\,\cdot)$ of $\mathcal{G}$ to $\mathcal{G}(\sigma)$, denoted by $(\cdot\,,\,\cdot)$, is non-degenerate.

Since $(\cdot\,,\,\cdot)$ is invariant and non-degenerate on $\mathcal{G}(\sigma)$, for $\tilde{\alpha},\tilde{\beta}\in R_{\mathcal{G}(\sigma)}$, we have $(\mathcal{G}(\sigma)_{\tilde{\alpha}},\mathcal{G}(\sigma)_{\tilde{\beta}})=\{0\}$ unless $\tilde{\alpha}+\tilde{\beta}=0$. This implies that

$$\begin{aligned}&\text{the restriction of the form }(\cdot\,,\,\cdot)\text{ to }\mathcal{G}(\sigma)_{\tilde{\alpha}}+\mathcal{G}(\sigma)_{-\tilde{\alpha}},\ \tilde{\alpha}\in R_{\mathcal{G}(\sigma)},\\&\text{is non-degenerate,}\end{aligned}\tag{2.26}$$

in particular (GC2) implies that the restriction of the form $(\cdot\,,\,\cdot)$ to $\mathcal{H}(\sigma)=\mathcal{G}(\sigma)_0$ is non-degenerate. Transfer the form $(\cdot\,,\,\cdot)_{|\mathcal{H}(\sigma)\times\mathcal{H}(\sigma)}$ to a form on $\mathcal{H}(\sigma)^{\star}$, denoted again by $(\cdot\,,\,\cdot)$, by setting $(\gamma,\eta):=(t_\gamma,t_\eta)$ for $\gamma,\eta\in\mathcal{H}(\sigma)^{\star}$. The next assumptions on σ are as follow:
(GC4) There is a finite root system $R(\sigma)$, containing $R_{\mathcal{L}(\sigma)}$, in a subspace of $\mathcal{H}(\sigma)^{\star}$ such that

$$(\dot{\alpha},\dot{\alpha})\neq 0\quad\text{and}\quad\langle\dot{\beta},\check{\dot{\alpha}}\rangle=2(\dot{\beta},\dot{\alpha})/(\dot{\alpha},\dot{\alpha})\quad\text{for}\quad\dot{\alpha},\dot{\beta}\in R(\sigma)^{\times}=R(\sigma)\setminus\{0\}.$$

(GC5) $\sigma(\mathcal{L}^{\lambda})\subseteq\mathcal{L}^{\lambda}$ for $\lambda\in\Lambda$.

We note that (GC5) implies that $\mathcal{L}(\sigma)$ inherits the Λ-grading on $\mathcal{L}$, i.e.,

$$\mathcal{L}(\sigma)=\bigoplus_{\lambda\in\Lambda}\mathcal{L}(\sigma)^{\lambda}\quad\text{where}\quad\mathcal{L}(\sigma)^{\lambda}:=\mathcal{L}(\sigma)\cap\mathcal{L}^{\lambda};\quad\lambda\in\Lambda,$$

also using (GC4) together with (2.24), one gets that $\mathcal{L}(\sigma)$ is a $\mathcal{Q}(R(\sigma))$-graded Lie algebra. Now using the same argument as in Lemma 2.14, we have

Lemma 2.27. *These two gradings are compatible.*

We know that $Z(\mathcal{L}(\sigma))$ inherits these compatible gradings, in other words,

$$Z(\mathcal{L}(\sigma)) = \bigoplus_{\lambda\in\Lambda}\bigoplus_{\alpha\in\mathcal{Q}(R(\sigma))} Z(\mathcal{L}(\sigma))^{\lambda}_{\alpha} \tag{2.28}$$

where, for $\lambda \in \Lambda$ and $\alpha \in \mathcal{Q}(R(\sigma))$, $Z(\mathcal{L}(\sigma))^{\lambda}_{\alpha} := Z(\mathcal{L}(\sigma)) \cap \mathcal{L}(\sigma)^{\lambda}_{\alpha}$.

DEFINITION 2.29. We call a finite dimensional split semisimple subalgebra $\mathfrak{L}$ of $\mathcal{L}(\sigma)$, a *σ-splitting subalgebra* of $\mathcal{L}(\sigma)$ if $\mathfrak{L}$ satisfies the following conditions:
(1) There is a splitting Cartan subalgebra $\mathcal{C}$ of $\mathfrak{L}$ with $\mathcal{C} \subseteq \mathcal{H}(\sigma) + Z(\mathcal{L}(\sigma))^0_0$, called a *$\sigma$-splitting Cartan subalgebra of $\mathfrak{L}$.*
(2) For each root α of the root system $\Delta_{\mathfrak{L}}$ of $\mathfrak{L}$ with respect to $\mathcal{C}$, there exist $\beta_\alpha \in R(\sigma)$ and $\lambda_\alpha \in \Lambda$ such that
  (a) $\mathfrak{L}_\alpha \subseteq \mathcal{L}(\sigma)^{\lambda_\alpha}_{\beta_\alpha}$,
  (b) $\mathcal{S}_{\mathfrak{L}} := \{\beta_\alpha \in R(\sigma) \mid \alpha \in \Delta_{\mathfrak{L}}\}$ is a subsystem of $R(\sigma)$,
  (c) the map $\alpha \mapsto \beta_\alpha$ defines an isomorphism between $\Delta_{\mathfrak{L}}$ and $\mathcal{S}_{\mathfrak{L}}$.

REMARK 2.30. (i) We drew the attention of the reader to the point that if a finite dimensional split semisimple Lie subalgebra $\mathfrak{L}$ of $\mathcal{L}(\sigma)$ satisfies the conditions (1)–(2) (a) of a σ-splitting subalgebra, different weight spaces of $\mathfrak{L}$ with respect to $\mathcal{C}$ are contained in different weight spaces of $\mathcal{L}(\sigma)$ with respect to $\mathcal{H}(\sigma)$. Indeed, if α, β are two roots of $\Delta_{\mathfrak{L}}$ such that $\mathfrak{L}_\alpha, \mathfrak{L}_\beta \subseteq \mathcal{L}(\sigma)_\gamma$ for some $\gamma \in R(\sigma)$, then for $0 \neq x \in \mathfrak{L}_\alpha$, $0 \neq y \in \mathfrak{L}_\beta$, we have

$$\begin{aligned}&\alpha(h+z)x = [h+z, x] = \gamma(h)x, \quad \beta(h+z)y = [h+z, y] = \gamma(h)y\\ &\text{where}\quad h \in \mathcal{H}(\sigma),\ z \in Z(\mathcal{L}(\sigma)),\ h+z \in \mathcal{C},\end{aligned}$$

which implies $\alpha(\dot{h}) = \beta(\dot{h})$ for $\dot{h} \in \mathcal{C}$. Therefore $\alpha = \beta$.
(ii) For a σ-splitting subalgebra $\mathfrak{L}$ of $\mathcal{L}(\sigma)$, the conditions (2) (b) and (2) (c) of the definition imply that if B is a base of $\Delta_{\mathfrak{L}}$, then $\{\beta_\alpha \mid \alpha \in B\}$ is a base of $\mathcal{S}_{\mathfrak{L}}$.
(iii) Let $\mathfrak{L}$ be a σ-splitting subalgebra of $\mathcal{L}(\sigma)$ and $\alpha \in R(\sigma)^\times$ with $\mathfrak{L} \cap \mathcal{L}(\sigma)_\alpha \neq \{0\}$, then there is $\gamma \in \Delta^\times_{\mathfrak{L}}$ such that $\mathfrak{L} \cap \mathcal{L}(\sigma)_\alpha = \mathfrak{L}_\gamma$. So from the finite dimensional theory we have $[\mathfrak{L} \cap \mathcal{L}(\sigma)_\alpha, \mathfrak{L} \cap \mathcal{L}(\sigma)_\beta] = \mathfrak{L} \cap \mathcal{L}(\sigma)_{\alpha+\beta}$ where $\alpha, \beta, \alpha+\beta \in R(\sigma)^\times$ with $\mathfrak{L} \cap \mathcal{L}(\sigma)_\alpha \neq \{0\}$, $\mathfrak{L} \cap \mathcal{L}(\sigma)_\beta \neq \{0\}$, $\mathfrak{L} \cap \mathcal{L}(\sigma)_{\alpha+\beta} \neq \{0\}$.

Lemma 2.31. *Let $\mathfrak{L}_1, \mathfrak{L}_2$ be two σ-splitting subalgebras of $\mathcal{L}(\sigma)$ with σ-splitting Cartan subalgebras $\mathcal{C}_1, \mathcal{C}_2$ and the root systems $\Delta_{\mathfrak{L}_1}, \Delta_{\mathfrak{L}_2}$, respectively, such that $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, then $\mathcal{S}_{\mathfrak{L}_1}$ is a subsystem of $\mathcal{S}_{\mathfrak{L}_2}$, moreover, if $\mathcal{S}_{\mathfrak{L}_1} = \mathcal{S}_{\mathfrak{L}_2}$, then $\mathfrak{L}_1 = \mathfrak{L}_2$.*

Proof. Let $\beta \in \mathcal{S}^\times_{\mathfrak{L}_1}$, then there exists $\alpha \in \Delta^\times_{\mathfrak{L}_1}$ such that $(\mathfrak{L}_1)_\alpha \subseteq \mathcal{L}(\sigma)_\beta$. Consider a nonzero $x \in (\mathfrak{L}_1)_\alpha \subseteq \mathfrak{L}_2$, then $x = \sum_{\gamma\in\Delta_{\mathfrak{L}_2}} x_\gamma$ with $x_\gamma \in (\mathfrak{L}_2)_\gamma$, $\gamma \in \Delta_{\mathfrak{L}_2}$. But for each $\gamma \in \Delta_{\mathfrak{L}_2}$, there is $\beta_\gamma \in \mathcal{S}_{\mathfrak{L}_2}$ such that $(\mathfrak{L}_2)_\gamma \subseteq \mathcal{L}(\sigma)_{\beta_\gamma}$. Therefore $x = \sum_{\gamma\in\Delta_{\mathfrak{L}_2}} x_\gamma$

with $x_\gamma \in \mathcal{L}(\sigma)_{\beta_\gamma}$, $\gamma \in \Delta_{\mathfrak{L}_2}$. Now since $x \in \mathcal{L}(\sigma)_\beta$, there is $\gamma \in \Delta_{\mathfrak{L}_2}$ such that $x = x_\gamma$ and $\beta = \beta_\gamma \in \mathcal{S}_{\mathfrak{L}_2}$. This completes the proof of the first statement. For the second statement suppose that $\alpha \in \Delta^\times_{\mathfrak{L}_2}$, then since $\mathcal{S}_{\mathfrak{L}_1} = \mathcal{S}_{\mathfrak{L}_2}$, one finds $\beta \in \mathcal{S}_{\mathfrak{L}_1} = \mathcal{S}_{\mathfrak{L}_2}$ and $\gamma \in \Delta^\times_{\mathfrak{L}_1}$ such that $(\mathfrak{L}_2)_\alpha \subseteq \mathcal{L}(\sigma)_\beta$ and $(\mathfrak{L}_1)_\gamma \subseteq \mathcal{L}(\sigma)_\beta$. Now as $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, we have $(\mathfrak{L}_2)_\alpha + (\mathfrak{L}_1)_\gamma \subseteq \mathfrak{L}_2 \cap \mathcal{L}(\sigma)_\beta$. But by Remark 2.30 (iii), $\dim(\mathfrak{L}_2 \cap \mathcal{L}(\sigma)_\beta) = 1$, so $(\mathfrak{L}_2)_\alpha = (\mathfrak{L}_1)_\gamma$. It means that for each $\alpha \in \Delta^\times_{\mathcal{L}_2}$, there is $\gamma \in \Delta^\times_{\mathcal{L}_1}$ such that $(\mathfrak{L}_2)_\alpha = (\mathfrak{L}_1)_\gamma$, so as $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, we are done. □

Proposition 2.32. *The derived algebra $\mathcal{G}'(\sigma)$ of $\mathcal{G}(\sigma)$ is a σ-splitting subalgebra of $\mathcal{L}(\sigma)$.*

Proof. Using (GC2), (GC3) and (2.24) together with the fact that $R_{\mathcal{G}(\sigma)}$ (see (2.25)) is a finite set, we conclude that $(\mathcal{G}(\sigma), (\cdot\,,\,\cdot), \mathcal{H}(\sigma))$ satisfies the following:

- $(\cdot\,,\,\cdot)$ is a non-degenerate symmetric invariant bilinear form on $\mathcal{G}(\sigma)$.
- $\mathcal{H}(\sigma)$ is a nontrivial finite dimensional abelian subalgebra of $\mathcal{G}(\sigma)$ which is self-centralizing and ad-diagonalizable.
- If $\pi(\alpha) \in R^\times_{\mathcal{G}(\sigma)} = R_{\mathcal{G}(\sigma)} \setminus \{0\}$ and $x \in \mathcal{G}(\sigma)_{\pi(\alpha)}$, $\mathrm{ad}_{\mathcal{G}(\sigma)}\, x$ acts locally nilpotently on $\mathcal{G}(\sigma)$.

Now let $\pi(\alpha) \in R_{\mathcal{G}(\sigma)}$, $x \in \mathcal{G}(\sigma)_{\pi(\alpha)}$ and $y \in \mathcal{G}(\sigma)_{-\pi(\alpha)}$, then by (GC2), $[x, y] \in \mathcal{G}(\sigma)_0 = \mathcal{H}(\sigma)$. Also for $h \in \mathcal{H}(\sigma)$, we have

$$(h, [x, y]) = ([h, x], y) = \pi(\alpha)(h)(x, y) = (t_{\pi(\alpha)}, h)(x, y) = (h, t_{\pi(\alpha)}(x, y)).$$

Now since by (2.26), the form is non-degenerate on $\mathcal{H}(\sigma)$, we have $[x, y] = t_{\pi(\alpha)}(x, y)$ and so using (2.26) again, we have

$$[\mathcal{G}_{\pi(\alpha)}, \mathcal{G}_{-\pi(\alpha)}] = \mathbb{F}t_{\pi(\alpha)}, \quad \pi(\alpha) \in R_{\mathcal{G}(\sigma)}. \tag{2.33}$$

Now using [5, Propositions 1.4 and 1.5] together with (GC4) and (2.33), one concludes that $R_{\mathcal{G}(\sigma)}$ is an indivisible subsystem of $R(\sigma)$ and that for $\pi(\alpha) \in R^\times_{\mathcal{G}(\sigma)}$, $\dim(\mathcal{G}_{\pi(\alpha)}) = 1$. Now it follows from these and Serre's theorem that $\mathcal{G}'(\sigma)$ is a finite dimensional split semisimple subalgebra of $\mathcal{L}(\sigma)$ with splitting Cartan subalgebra $\mathcal{H}(\sigma) \cap \mathcal{G}'(\sigma)$ and the root system $R_{\mathcal{G}(\sigma)}$. Now for each $\dot\alpha \in R_{\mathcal{G}(\sigma)}$, define $\beta_{\dot\alpha} := \dot\alpha \in R(\sigma)$, then $\mathcal{G}'(\sigma)_{\dot\alpha} \subseteq \mathcal{G}(\sigma)_{\dot\alpha} \subseteq \mathcal{L}(\sigma)^0_{\beta_{\dot\alpha}}$. All together imply that $\mathcal{G}'(\sigma)$ is a σ-splitting subalgebra of $\mathcal{L}(\sigma)$. □

One knows that the dimension of a σ-splitting subalgebra of $\mathcal{L}(\sigma)$ is at most $\#(R(\sigma)^\times) + \mathrm{rank}(R(\sigma))$ and $\mathcal{G}'(\sigma)$ is a σ-splitting subalgebra of $\mathcal{L}(\sigma)$. Let us fix a maximal σ-splitting subalgebra $\mathfrak{g}$ of $\mathcal{L}(\sigma)$ with a σ-splitting Cartan subalgebra $\mathcal{C}$ and the root system $\Delta_{\mathfrak{g}}$. Let $\{\dot\alpha_1, \ldots, \dot\alpha_m\}$ be a base of $\Delta_{\mathfrak{g}}$, then for each $\alpha \in \Delta_{\mathfrak{g}}$, there exist

$\lambda_\alpha \in \Lambda$ and $\beta_\alpha \in R(\sigma)$ such that $\mathfrak{g}_\alpha \subseteq \mathcal{L}(\sigma)^{\lambda_\alpha}_{\beta_\alpha}$. Set

$$\alpha_i := \beta_{\dot{\alpha}_i} \quad \text{and} \quad \delta_i := \lambda_{\dot{\alpha}_i}; \quad 1 \leq i \leq m, \tag{2.34}$$

and define

$$\mathcal{R} := R(\sigma) \cap \operatorname{span}_{\mathbb{Z}}\{\alpha_i \mid 1 \leq i \leq m\} = R(\sigma) \cap \operatorname{span}_{\mathbb{Z}} \mathcal{S}_{\mathfrak{g}}. \tag{2.35}$$

One can easily prove the following lemma:

Lemma 2.36. *$\mathcal{R}$ is a subsystem of $R(\sigma)$.*

Proposition 2.37. *Recall that $\mathfrak{g}$ is a maximal σ-splitting subalgebra of $\mathcal{L}(\sigma)$ with a σ-splitting Cartan subalgebra $\mathcal{C}$ and the root system $\Delta_{\mathfrak{g}}$, also $\mathcal{S} := \mathcal{S}_{\mathfrak{g}} = \{\beta_\alpha \in R(\sigma) \mid \alpha \in \Delta_{\mathfrak{g}}\}$ (see* Definition 2.29*). Consider* (2.35) *and set*

$$\tilde{\mathcal{L}} := \textit{the subalgebra of } \mathcal{L}(\sigma) \textit{ generated by } \mathcal{L}(\sigma)_\alpha, \quad \alpha \in \mathcal{R}^\times. \tag{2.38}$$

Then $\tilde{\mathcal{L}}$ is an $(\mathcal{R}, \mathcal{S}, \Lambda)$-graded Lie algebra.

Proof. We first consider (2.34) and note that by Remark 2.30 (ii), $\{\alpha_i \mid 1 \leq i \leq m\}$ is a base of $\mathcal{S}$ and so $\mathcal{S}$ is a subsystem of $\mathcal{R}$. Now the proof is carried out in steps:

STEP 1. $\tilde{\mathcal{L}} = \sum_{\gamma \in \mathcal{R}^\times} [\mathcal{L}(\sigma)_\gamma, \mathcal{L}(\sigma)_{-\gamma}] + \sum_{\gamma \in \mathcal{R}^\times} \mathcal{L}(\sigma)_\gamma$: It is enough to show that $[\mathcal{L}(\sigma)_\alpha, \mathcal{L}(\sigma)_\beta] \subseteq \sum_{\gamma \in \mathcal{R}^\times} \mathcal{L}(\sigma)_\gamma$ for $\alpha, \beta \in \mathcal{R}^\times$ with $\alpha + \beta \neq 0$. Let $\alpha, \beta \in \mathcal{R}^\times$ be such that $\alpha + \beta \neq 0$. If $[\mathcal{L}(\sigma)_\alpha, \mathcal{L}(\sigma)_\beta] = \{0\}$, then there is nothing to prove so let $[\mathcal{L}(\sigma)_\alpha, \mathcal{L}(\sigma)_\beta] \neq \{0\}$, then since $\{0\} \neq [\mathcal{L}(\sigma)_\alpha, \mathcal{L}(\sigma)_\beta] \subseteq \mathcal{L}(\sigma)_{\alpha+\beta}$ and $\alpha + \beta \in \operatorname{span}_{\mathbb{Z}}\{\alpha, \beta\} \subseteq \operatorname{span}_{\mathbb{Z}}\{\alpha_i \mid 1 \leq i \leq m\}$, $\alpha + \beta \in R(\sigma) \cap \operatorname{span}_{\mathbb{Z}}\{\alpha_i \mid 1 \leq i \leq m\} = \mathcal{R}$. Therefore $[\mathcal{L}(\sigma)_\alpha, \mathcal{L}(\sigma)_\beta] \subseteq \sum_{\gamma \in \mathcal{R}^\times} \mathcal{L}(\sigma)_\gamma$.

STEP 2. $\tilde{\mathcal{L}}$ is a $\mathcal{Q}(\mathcal{R})$-graded Lie algebra with the support containing in $\mathcal{R}$: Define

$$\tilde{\mathcal{L}}_\alpha := \begin{cases} \mathcal{L}(\sigma)_\alpha, & \alpha \in \mathcal{R}^\times, \\ \displaystyle\sum_{\gamma \in \mathcal{R}^\times} [\mathcal{L}(\sigma)_\gamma, \mathcal{L}(\sigma)_{-\gamma}], & \alpha = 0, \\ 0, & \alpha \in \mathcal{Q}(\mathcal{R}) \setminus \mathcal{R}. \end{cases} \tag{2.39}$$

With the same argument as in Step 1, one gets that $\tilde{\mathcal{L}} = \bigoplus_{\alpha \in \mathcal{R}} \tilde{\mathcal{L}}_\alpha$ is a $\mathcal{Q}(\mathcal{R})$-graded Lie algebra with the desired property.

STEP 3. Consider Lemma 2.27 and let $\alpha = \sum_{i=1}^m r_i \alpha_i \in \mathcal{R}$ $(r_1, \ldots, r_m \in \mathbb{Z})$, $\lambda \in \Lambda$. For $x \in \mathcal{L}(\sigma)^\lambda_\alpha$ define $\deg_n x := \lambda - \sum_{i=1}^m r_i \delta_i$. This defines a Λ-grading on $\tilde{\mathcal{L}}$: We know that for $\alpha \in R(\sigma)^\times$, $\mathcal{L}(\sigma)_\alpha = \sum_{\lambda \in \Lambda} \mathcal{L}(\sigma)^\lambda_\alpha$, so we have $\tilde{\mathcal{L}} = \bigoplus_{\nu \in \Lambda} \tilde{\mathcal{L}}^\nu$ where for $\nu \in \Lambda$,

$$\tilde{\mathcal{L}}^\nu := \sum_{\substack{\alpha = \sum_{i=1}^m r_i \alpha_i \in \mathcal{R}^\times,\ \lambda \in \Lambda, \\ \nu = \lambda - \sum_{i=1}^m r_i \delta_i}} \mathcal{L}(\sigma)^\lambda_\alpha + \sum_{\substack{\gamma \in \mathcal{R}^\times, \\ \lambda, \mu \in \Lambda, \lambda + \mu = \nu}} [\mathcal{L}(\sigma)^\lambda_\gamma, \mathcal{L}(\sigma)^\mu_{-\gamma}].$$

Now it is easy to see that $[\tilde{\mathcal{L}}^{\nu}, \tilde{\mathcal{L}}^{\nu'}] \subseteq \tilde{\mathcal{L}}^{\nu+\nu'}$ for $\nu, \nu' \in \Lambda$.

STEP 4. The gradings introduced in Steps 2 and 3 are compatible, i.e., for $\alpha \in \mathcal{R}$, $\tilde{\mathcal{L}}_\alpha = \sum_{\nu\in\Lambda} \tilde{\mathcal{L}}_\alpha^\nu = \sum_{\nu\in\Lambda}(\tilde{\mathcal{L}}_\alpha \cap \tilde{\mathcal{L}}^\nu)$: Because of (2.39), it is enough to prove the statement for $\alpha \in \mathcal{R}^\times$. Let $\alpha = \sum_{i=1}^m r_i\alpha_i \in \mathcal{R}^\times$ and $x \in \tilde{\mathcal{L}}_\alpha = \mathcal{L}(\sigma)_\alpha$, then by Lemma 2.27, $x = \sum_{\lambda\in\Lambda} x_\alpha^\lambda$ with $x_\alpha^\lambda \in \mathcal{L}(\sigma)_\alpha^\lambda$. This completes the proof as for $\lambda \in \Lambda$, $x_\alpha^\lambda \in \tilde{\mathcal{L}}_\alpha$ and with respect to the Λ-grading on $\tilde{\mathcal{L}}$, x_α^λ is homogeneous of degree $\deg_n x_\alpha^\lambda = \lambda - \sum_{i=1}^m r_i\delta_i$. In other words, $x = \sum_{\lambda\in\Lambda} x_\alpha^\lambda$ with $x_\alpha^\lambda \in \tilde{\mathcal{L}}_\alpha \cap \tilde{\mathcal{L}}^{\lambda - \sum_{i=1}^m r_i\delta_i}$.

STEP 5. For each $\alpha \in \mathcal{S}^\times$, $\tilde{\mathcal{L}}_\alpha^0$ contains an invertible element: Suppose that $\dot{e}_i, \dot{f}_i, \dot{h}_i$, $1 \le i \le m$, are Chevalley generators of $\mathfrak{g}$ corresponding to the base $\{\dot{\alpha}_i \mid 1 \le i \le m\}$. We know that for $1 \le i \le m$, $\dot{e}_i \in \mathcal{L}(\sigma)_{\alpha_i}^{\delta_i}$ (see (2.34)), so $\deg_n(\dot{e}_i) = 0$. Also since the σ-splitting Cartan subalgebra of $\mathfrak{g}$ is a subset of $\mathcal{H}(\sigma) + Z(\mathcal{L}(\sigma))_0^0$, $\dot{f}_i \in \mathcal{L}(\sigma)_{-\alpha_i}^{-\delta_i}$ and so $\deg_n(\dot{f}_i) = \deg_n(\dot{h}_i) = 0$. Now since the generating set $\{\dot{e}_i, \dot{f}_i, \dot{h}_i \mid 1 \le i \le m\}$ of $\mathfrak{g}$ is a subset of $\tilde{\mathcal{L}}^0$, we have

$$\mathfrak{g} \subseteq \tilde{\mathcal{L}}^0. \tag{2.40}$$

Next note that for $\dot{\alpha} \in \Delta_\mathfrak{g}$, there exists $\beta_{\dot{\alpha}} \in R(\sigma)$ such that $\mathfrak{g}_{\dot{\alpha}} \subseteq \mathcal{L}(\sigma)_{\beta_{\dot{\alpha}}}$ and the map $\dot{\alpha} \mapsto \beta_{\dot{\alpha}}$ defines an isomorphism between $\Delta_\mathfrak{g}$ and $\mathcal{S} = \{\beta_{\dot{\alpha}} \in R(\sigma) \mid \dot{\alpha} \in \Delta_\mathfrak{g}\}$. The inverse of this isomorphism defines an isomorphism $\alpha \mapsto \tilde{\alpha}$ between $\mathcal{S}$ and $\Delta_\mathfrak{g}$. Therefore

$$\Delta_\mathfrak{g} = \{\tilde{\alpha} \mid \alpha \in \mathcal{S}\}, \quad \mathfrak{g}_{\tilde{\alpha}} \subseteq \mathcal{L}(\sigma)_\alpha;\ \alpha \in \mathcal{S}, \quad \langle\tilde{\beta}, \check{\tilde{\alpha}}\rangle = \langle\beta, \check{\alpha}\rangle;\ \alpha, \beta \in \mathcal{S}^\times. \tag{2.41}$$

Now let $\alpha \in \mathcal{S}^\times$, we want to find an invertible element in $\tilde{\mathcal{L}}_\alpha^0$. We know from the finite dimensional theory that there exists $\tilde{h}_{\tilde{\alpha}} \in \mathfrak{g}_0$ with $[\mathfrak{g}_{+\tilde{\alpha}}, \mathfrak{g}_{-\tilde{\alpha}}] = \mathbb{F}\tilde{h}_{\tilde{\alpha}}$ and $\tilde{\beta}(\tilde{h}_{\tilde{\alpha}}) = \langle\tilde{\beta}, \check{\tilde{\alpha}}\rangle$ for all $\beta \in \mathcal{S}^\times$. This together with (2.40), (2.41) and (2.39) implies that

$$\begin{aligned} &\text{there exist} \quad \tilde{h}_{\tilde{\alpha}} \in \mathfrak{g}_0 \quad \text{and} \quad \tilde{e}_{\pm\tilde{\alpha}} \in \mathfrak{g}_{\pm\tilde{\alpha}} \subseteq \mathcal{L}(\sigma)_{\pm\alpha} \cap \tilde{\mathcal{L}}^0 = \tilde{\mathcal{L}}_{\pm\alpha}^0 \quad \text{with} \\ &\tilde{\beta}(\tilde{h}_{\tilde{\alpha}}) = \langle\beta, \check{\alpha}\rangle \quad \text{for all} \quad \beta \in \mathcal{S} \quad \text{and} \quad [\tilde{e}_{-\tilde{\alpha}}, \tilde{e}_{+\tilde{\alpha}}] = \tilde{h}_{\tilde{\alpha}}. \end{aligned} \tag{2.42}$$

Now since $\tilde{h}_{\tilde{\alpha}} \in \mathfrak{g}_0 \subseteq \mathcal{H}(\sigma) + Z(\mathcal{L}(\sigma))$, there exist $h_\alpha \in \mathcal{H}(\sigma)$ and $z_\alpha \in Z(\mathcal{L}(\sigma))$ such that $\tilde{h}_{\tilde{\alpha}} = h_\alpha + z_\alpha$. Also by (2.41), $\dot{e}_j \in \mathfrak{g}_{\tilde{\alpha}_j} \subseteq \mathcal{L}(\sigma)_{\alpha_j}$, $1 \le j \le m$, therefore by (2.24), (2.23) and (2.42), we have

$$\alpha_j(h_\alpha)\dot{e}_j = [h_\alpha + z_\alpha, \dot{e}_j] = [\tilde{h}_{\tilde{\alpha}}, \dot{e}_j] = \tilde{\alpha}_j(\tilde{h}_{\tilde{\alpha}})\dot{e}_j = \langle\alpha_j, \check{\alpha}\rangle\dot{e}_j.$$

This means that

$$\alpha_j(h_\alpha) = \langle\alpha_j, \check{\alpha}\rangle; \quad 1 \le j \le m. \tag{2.43}$$

Now if $\gamma = \sum_{i=1}^{m} r_i\alpha_i \in \mathcal{R}^\times$ and $x \in \tilde{\mathcal{L}}_\gamma = \mathcal{L}(\sigma)_\gamma$ (see (2.39)), then (2.43) implies that

$$[\tilde{h}_{\tilde{\alpha}}, x] = [h_\alpha + z_\alpha, x] = [h_\alpha, x] = \gamma(h_\alpha)x = \sum_{i=1}^{m} r_i\alpha_i(h_\alpha)x = \sum_{i=1}^{m} r_i\langle\alpha_i, \check{\alpha}\rangle x = \langle\gamma, \check{\alpha}\rangle x \tag{2.44}$$

which in turn implies that for $\gamma \in \mathcal{R}^\times$ and $x \in \tilde{\mathcal{L}}_\gamma$, $y \in \tilde{\mathcal{L}}_{-\gamma}$, we have

$$[\tilde{h}_{\tilde{\alpha}}, [x, y]] = [[\tilde{h}_{\tilde{\alpha}}, x], y] - [[\tilde{h}_{\tilde{\alpha}}, y], x] = \langle\gamma, \check{\alpha}\rangle[x, y] - \langle-\gamma, \check{\alpha}\rangle[y, x] = 0. \tag{2.45}$$

Now (2.42) together with (2.44), (2.45) and (2.39) implies that $\tilde{e}_{+\tilde{\alpha}} \in \tilde{\mathcal{L}}_\alpha^0$ is an invertible element. This completes the proof. □

Now we are ready to state our main theorem:

Theorem 2.46 (Main theorem). *Let R be a finite root system with a subsystem S satisfying* ($\star$) *and $\mathcal{L}$ be an (R, S, Λ)-graded Lie algebra with a fixed grading subalgebra $\mathcal{G}$. Let σ be an automorphism of $\mathcal{L}$ satisfying* (GC1)–(GC5). *Then*
(i) *$\mathcal{L}(\sigma)$ has compatible $\mathcal{Q}(R(\sigma))$- and Λ-gradings,*
(ii) *there are subsystems $\mathcal{R}, \mathcal{S}$ of $R(\sigma)$ with $\mathcal{S} \subseteq \mathcal{R}$ such that the subalgebra $\tilde{\mathcal{L}}$ of $\mathcal{L}(\sigma)$ generated by $\mathcal{L}(\sigma)_\alpha$, $\alpha \in \mathcal{R}^\times$, is an $(\mathcal{R}, \mathcal{S}, \Lambda)$-graded Lie algebra containing a maximal splitting subalgebra $\mathfrak{g}$,*
(iii) *$\mathcal{L}(\sigma)_c = \tilde{\mathcal{L}} + \mathcal{K}$ where $\mathcal{K} := \sum_{\alpha\in R(\sigma)^\times\setminus\mathcal{R}}[\mathcal{L}(\sigma)_\gamma, \mathcal{L}(\sigma)_{-\gamma}] + \sum_{\gamma\in R(\sigma)^\times\setminus\mathcal{R}} \mathcal{L}(\sigma)_\gamma$,*
(iv) *take $\mathcal{I}$ to be the isolated subspace of $\mathcal{L}(\sigma)$ with respect to the compatible gradings* (Definition 1.6), *then $[\mathcal{I}, \mathcal{L}(\sigma)_c] = \{0\}$ and $\mathcal{L}(\sigma)$ is decomposed into $\mathcal{L}(\sigma) = \mathcal{L}(\sigma)_c \oplus \mathcal{I} \oplus \mathcal{D} = (\tilde{\mathcal{L}} + \mathcal{K}) \oplus \mathcal{I} \oplus \mathcal{D}$ where and $\mathcal{D}$ is a subspace of $\mathcal{L}(\sigma)$ satisfying* $[\mathcal{D}, \mathfrak{g}] = \{0\}$.

Proof. Use Lemma 2.27 to get compatible $\mathcal{Q}(R(\sigma))$- and Λ-gradings on $\mathcal{L}(\sigma)$. Next we note that $[\mathcal{I}, \mathcal{L}(\sigma)_c] = \{0\}$ as $[\mathcal{I}, \mathcal{L}(\sigma)_\alpha^\mu] = \{0\}$ for all $\alpha \in R(\sigma)^\times$ and $\mu \in \Lambda$. Now fix a maximal σ-splitting subalgebra $\mathfrak{g}$ of $\mathcal{L}$. Consider the root system $\mathcal{R}$ as defined in (2.35) and its subsystem $\mathcal{S}$ as defined in Proposition 2.37. Then by Proposition 2.37, $\tilde{\mathcal{L}}$, the subalgebra of $\mathcal{L}(\sigma)$ generated by $\mathcal{L}(\sigma)_\alpha$, $\alpha \in \mathcal{R}^\times$, is an $(\mathcal{R}, \mathcal{S}, \Lambda)$-graded Lie algebra containing the maximal splitting subalgebra $\mathfrak{g}$. It is trivial that $\mathcal{L}(\sigma)_c = \tilde{\mathcal{L}} + \mathcal{K}$. Now one can find a subspace E of $\mathcal{L}(\sigma)$ such that $\mathcal{L}(\sigma) = \mathcal{L}(\sigma)_c \oplus \mathcal{I} \oplus E$. Let $x \in E$. Since $\mathcal{L}(\sigma)_c$ is an ideal of $\mathcal{L}(\sigma)$, the restriction of $\mathrm{ad}_{\mathcal{L}(\sigma)}\, x$ to $\mathcal{L}(\sigma)_c$ is a derivation of $\mathcal{L}(\sigma)_c$. Using the complete reducibility of $\mathcal{L}(\sigma)_c$ as a $\mathfrak{g}$-module and the first Whitehead lemma for $\mathfrak{g}$-modules, we can apply [6, Proposition 3.2] to each element of a basis of E and in this way construct a subspace $\mathcal{D}$ of $\mathcal{L}(\sigma)$ such that $\mathcal{L}(\sigma) = \mathcal{L}(\sigma)_c \oplus \mathcal{I} \oplus \mathcal{D} = (\tilde{\mathcal{L}} + \mathcal{K}) \oplus \mathcal{I} \oplus \mathcal{D}$ and $[\mathcal{D}, \mathfrak{g}] = \{0\}$. This completes the proof. □

3. Examples

In this section, we present several examples elaborating on the result obtained in Section 2. In each example, we start with an (R, S, Λ)-graded Lie algebra $\mathcal{L}$ and an automorphism σ of $\mathcal{L}$ satisfying (GC1)–(GC5). We illustrate how the terms $\tilde{\mathcal{L}}, \mathcal{K}$ (see Theorem 2.46) appear as the core of the f.p.s. of $\mathcal{L}$ under the automorphism σ. In Examples 3.4 and 3.23, $\mathcal{K} = 0$ and so $\mathcal{L}(\sigma)_c = \tilde{\mathcal{L}}$ is an $(\mathcal{R}, \mathcal{S}, \Lambda)$-graded Lie algebra for a finite root system $\mathcal{R}$ with a root system $\mathcal{S}$. In Examples 3.7 and 3.12, $\mathcal{K}$ is a nonzero subalgebra of $\mathcal{L}(\sigma)$ and in Example 3.22, $\mathcal{K}$ is a nonzero subspace of $\mathcal{L}(\sigma)_c$ that is not a subalgebra. Throughout this section for a star algebra $(\mathcal{A}, \bar{\ })$, we set $\mathcal{A}_\pm := \{a \in \mathcal{A} \mid \bar{a} = \pm a\}$. Also for an algebra $\mathcal{A}$ and natural numbers m, n, we mean by $\mathcal{A}^{m\times n}$, the set of all $m \times n$-matrices with entries in $\mathcal{A}$. For $A \in \mathcal{A}^{m\times n}$, we use A^t to denote the transposition of A and for $A \in \mathcal{A}^{n\times n}$, we mean by $\mathrm{tr}(A)$, the trace of A. If, in addition, $\mathcal{A}$ is unital, for $1 \le i, j \le n$, we take $e_{i,j}$ to be an element of $\mathcal{A}^{n\times n}$ with 1 in (i, j)-position and 0 elsewhere. We also keep the same notation as in the previous section. Our first four examples have the same nature, so we start with stating this common nature. Let R be a finite root system in an l-dimensional vector space over $\mathbb{F}$ with a base $\{\alpha_i \mid 1 \le i \le l\}$ and $\rho\colon \langle R\rangle \to \mathbb{F} \setminus \{0\}$ be any group homomorphism. One knows that ρ is uniquely determined by specifying $\rho(\alpha_i)$ for $1 \le i \le l$. Next let $\mathcal{L} = \bigoplus_{\alpha\in R} \mathcal{L}_\alpha$ be a $\mathcal{Q}(R)$-graded Lie algebra. The homomorphism ρ induces an automorphism ω_ρ of $\mathcal{L}$ by letting

$$\omega_{\rho|_{\mathcal{L}_\alpha}} = \rho(\alpha)\,\mathrm{id}_{\mathcal{L}_\alpha} \quad \text{for} \quad \alpha \in R. \tag{3.1}$$

We note that ω_ρ is of finite order if and only if $\rho(\alpha_i)$ is a root of unity for $1 \le i \le l$. We also note that the subalgebra $\mathcal{L}(\omega_\rho)$ of the fixed points of $\mathcal{L}$ under ω_ρ is $\bigoplus_{\substack{\alpha\in R,\\ \rho(\alpha)=1}} \mathcal{L}_\alpha$. Now as an especial case, consider the irreducible finite root system $R := \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i \le j \le 3\}$ of type BC_3 with base $\Delta := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\}$ in a 3-dimensional vector space over $\mathbb{F}$. Define the following group homomorphism:

$$\begin{gathered}\rho\colon \langle R\rangle \to \mathbb{F} \setminus \{0\},\\ \varepsilon_1 - \varepsilon_2 \mapsto 1, \quad \varepsilon_2 - \varepsilon_3 \mapsto -1, \quad \varepsilon_3 \mapsto -1\end{gathered} \tag{3.2}$$

and let $\mathcal{L}$ be a $\mathcal{Q}(R)$-graded Lie algebra. Next consider the automorphism ω_ρ of $\mathcal{L}$ defined as in (3.1), then

$$\mathcal{L}(\omega_\rho) = \sum_{\alpha\in R(\omega_\rho)} \mathcal{L}_\alpha \quad \text{where} \quad R(\omega_\rho) := \{\pm 2\varepsilon_3, \pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i \le j \le 2\}. \tag{3.3}$$

EXAMPLE 3.4. $\mathcal{L}$ be the derived algebra of the twisted affine Lie algebra of type $A_6^{(2)}$. Then $\mathcal{L}$ has a realization as

$$\mathcal{L} = (\mathcal{G} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 2}]) \oplus (\mathcal{M} \otimes_{\mathbb{C}} t\mathbb{C}[t^{\pm 2}]) \oplus \mathbb{C}z$$

where $\mathcal{G}$ and $\mathcal{M}$ are subspaces of a finite dimensional complex simple Lie algebra $(\dot{\mathcal{G}}, [\cdot, \cdot])$ of type A_6 such that $\mathcal{G}$ is a subalgebra of $\dot{\mathcal{G}}$ which is simple of type B_3 with a Cartan subalgebra $\mathcal{H}$ and the root system $S = \{0, \pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq 3\}$ and under the adjoint representation, $\mathcal{M}$ is an irreducible $\mathcal{G}$-module whose set of weights is $R = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i \leq j \leq 3\}$. The Lie bracket on $\mathcal{L}$ is defined by

$$
\begin{aligned}
&[x \otimes t^m, y \otimes t^n]\hat{} := ([x, y] \otimes t^{m+n}) + m\delta_{m,-n}\kappa(x, y)z, \quad [z, \mathcal{L}]\hat{} := \{0\} \\
&\text{for} \quad x \otimes t^n, y \otimes t^m \in \mathcal{L}
\end{aligned} \tag{3.5}
$$

where κ denotes Killing form of $\dot{\mathcal{G}}$. For $n \in \mathbb{Z}$, define

$$
\mathcal{L}^n := \begin{cases} \mathcal{G} \otimes t^n, & n \text{ is nonzero and even} \\ \mathcal{M} \otimes t^n, & n \text{ is odd} \\ (\mathcal{G} \otimes 1) \oplus \mathbb{C}z, & n = 0, \end{cases}
$$

then $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n$ is a $\mathbb{Z}$-graded Lie algebra. We also have $\mathcal{L} = \sum_{\alpha\in R} \mathcal{L}_\alpha$ where

$$
\mathcal{L}_\alpha := \begin{cases} (\mathcal{G}_\alpha \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 2}]) + (\mathcal{M}_\alpha \otimes_{\mathbb{C}} t\mathbb{C}[t^{\pm 2}]), & \alpha \in S^\times, \\ \mathcal{M}_\alpha \otimes_{\mathbb{C}} t\mathbb{C}[t^{\pm 2}], & \alpha \in R^\times \setminus S, \\ (\mathcal{G}_0 \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 2}]) + (\mathcal{M}_0 \otimes_{\mathbb{C}} t\mathbb{C}[t^{\pm 2}]) + \mathbb{C}z, & \alpha = 0. \end{cases}
$$

It is not difficult to see that $\mathcal{L}$ is an $(R, S, \mathbb{Z})$-graded Lie algebra with grading subalgebra $\mathcal{G}$. Now consider the group homomorphism ρ from $\langle R\rangle$ to $\mathbb{C} \setminus \{0\}$ defined by (3.2) and the automorphism ω_ρ of $\mathcal{L}$ defined as in (3.1). Then $\omega_\rho\colon \mathcal{L} \to \mathcal{L}$ is an automorphism satisfying (GC1)–(GC5). Contemplating (3.3), we have

$$
\mathcal{L}(\omega_\rho) = \sum_{\alpha\in R(\omega_\rho)} \mathcal{L}_\alpha \quad \text{where} \quad R(\omega_\rho) = \{\pm 2\varepsilon_3, \pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j),\ 1 \leq i \leq j \leq 2\},
$$

a weight space decomposition of $\mathcal{L}(\omega_\rho)$ with respect to $\mathcal{H} = \mathcal{H}(\omega_\rho)$ with $\mathcal{L}(\omega_\rho)_\alpha = \mathcal{L}_\alpha$ for $\alpha \in R(\omega_\rho)$. As $\mathcal{L}$ is the core of an extended affine Lie algebra (see [1] and [2, Theorem 2.32]), there are $\ddot{e} \in \mathcal{M}_{2\varepsilon_3} \otimes t$ and $\ddot{f} \in \mathcal{M}_{-2\varepsilon_3} \otimes t^{-1}$ such that $(\ddot{e}, \ddot{h} := [\ddot{e}, \ddot{f}], \ddot{f})$ is an $\mathfrak{sl}_2$-triple and so $\ddot{\mathcal{G}} := \mathbb{C}\ddot{e} + \mathbb{C}\ddot{f} + \mathbb{C}\ddot{h}$ is a 3-dimensional simple Lie subalgebra of $\mathcal{L}$. Identify $\mathcal{G}$ as a subset of $\mathcal{L}(\omega_\rho)$ with $\mathcal{G} = \mathcal{G} \otimes 1$, then $\mathcal{G}(\omega_\rho) = \mathcal{G} \cap \mathcal{L}(\omega_\rho)$ has a weight space decomposition $\mathcal{G}(\omega_\rho) = \bigoplus_{\alpha\in\Delta_{\mathcal{G}(\omega_\rho)}} \mathcal{G}(\omega_\rho)_\alpha$ with respect to $\mathcal{H} = \mathcal{H}(\omega_\rho)$ where $\Delta_{\mathcal{G}(\omega_\rho)} = R(\omega_\rho) \cap S = \{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm(\varepsilon_1 \pm \varepsilon_2)\}$.

Lemma 3.6. *Consider the derived algebra $\mathcal{G}'(\omega_\rho)$ of $\mathcal{G}(\omega_\rho)$ and set $\mathfrak{g} := \mathcal{G}'(\omega_\rho) \oplus \ddot{\mathcal{G}}$. Then $\mathfrak{g}$ is a maximal ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$.*

Proof. Since for $\alpha \in (\Delta_{\mathcal{G}(\omega_\rho)})^\times$, $\pm 2\varepsilon_3 + \alpha \notin R$, $[\mathcal{G}'(\omega_\rho), \ddot{\mathcal{G}}]\hat{} = \{0\}$, and so $\mathfrak{g}$ is a finite dimensional semisimple Lie subalgebra of $\mathcal{L}$ of type $B_2 \cup A_1$. Now noting that $\ddot{e} \in \mathcal{L}(\omega_\rho)^1_{2\varepsilon_3}$, $\ddot{f} \in \mathcal{L}(\omega_\rho)^{-1}_{-2\varepsilon_3}$ and for $\alpha \in \Delta_{g(\omega_\rho)}$, $\mathcal{G}_\alpha \subset \mathcal{L}(\omega_\rho)^0_\alpha$, one can easily see that $\mathfrak{g}$ is an ω_ρ-spitting subalgebra of $\mathcal{L}(\omega_\rho)$ with splitting Cartan subalgebra $\mathfrak{h} := (\mathcal{H} \cap \mathcal{G}'(\omega_\rho)) \oplus \mathbb{C}\ddot{h} \subseteq \mathcal{H}(\omega_\rho) + Z(\mathcal{L}(\omega_\rho))^0_0$ and $\mathcal{S}_\mathfrak{g} = \{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm 2\varepsilon_3, \pm(\varepsilon_1 \pm \varepsilon_2)\}$ that is a finite root system of type $B_2 \cup A_1$. Now let $\mathfrak{G}$ be an ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$ containing $\mathfrak{g}$ with ω_ρ-splitting Cartan subalgebra $\mathcal{C}$ and the root system $\Delta_\mathfrak{G}$. For each root $\alpha \in \Delta_\mathfrak{G}$, there exist $\beta_\alpha \in R(\omega_\rho)$ and $n_\alpha \in \mathbb{Z}$ such that $\mathfrak{G}_\alpha \subseteq \mathcal{L}(\sigma)^{n_\alpha}_{\beta_\alpha}$ and $\mathcal{S}_\mathfrak{G} = \{\beta_\alpha \in R(\omega_\rho) \mid \alpha \in \Delta_\mathfrak{G}\}$ is a subsystem of $R(\omega_\rho)$ isomorphic to $\Delta_\mathfrak{G}$. Since $\mathfrak{g} \subset \mathfrak{G}$, Lemma 2.31 implies that $\mathcal{S}_\mathfrak{g} = \{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm 2\varepsilon_3, \pm(\varepsilon_1 \pm \varepsilon_2)\} \subseteq \mathcal{S}_\mathfrak{G}$, but $\mathcal{S}_\mathfrak{G}$ is a subsystem of $R(\omega_\rho)$ which is isomorphic to the root system of a semisimple Lie algebra and so two times of a root of $\mathcal{S}_\mathfrak{G}$ cannot be a root. Therefore $\mathcal{S}_\mathfrak{g} = \mathcal{S}_\mathfrak{G}$ which implies that $\mathfrak{g} = \mathfrak{G}$. This completes the proof. □

Now take $\mathcal{S} := \mathcal{S}_\mathfrak{g}$, then $\mathcal{R} := \mathrm{span}_\mathbb{Z} \mathcal{S} \cap R(\omega_\rho) = R(\omega_\rho)$. So $\tilde{\mathcal{L}}$, the Lie subalgebra of $\mathcal{L}(\omega_\rho)$ generated by $\mathcal{L}(\omega_\rho)_\alpha$ with $\alpha \in \mathcal{R}^\times$, coincides with $\mathcal{L}(\omega_\rho)_c$. Therefore $\mathcal{L}(\omega_\rho)_c = \tilde{\mathcal{L}}$ is an $(\mathcal{R}, \mathcal{S}, \mathbb{Z})$-graded Lie algebra using Proposition 2.37. Moreover, since there is no isolated root for $\mathcal{L}(\omega_\rho)$, Theorem 2.46 implies that $\mathcal{L}(\omega_\rho) = \mathcal{L}(\omega_\rho)_c \oplus \mathcal{D}$, a decomposition of $\mathcal{L}(\omega_\rho)$ into an $(\mathcal{R}, \mathcal{S}, \mathbb{Z})$-graded Lie algebra and a subspace $\mathcal{D}$ of $\mathcal{L}(\omega_\rho)$ satisfying $[\mathcal{D}, \mathfrak{g}] = \{0\}$.

EXAMPLE 3.7. Suppose that l is a positive integer greater than 3. Let $\mathcal{V}$ be a $2l$-dimensional vector space over the field $\mathbb{F}$ and I be the identity matrix of rank l. Take $(\cdot, \cdot)$ to be the non-degenerate skew-symmetric bilinear form on $\mathcal{V}$ whose matrix is $s := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Then there exists a basis $\{u_i, v_i \mid 1 \le i \le l\}$ for $\mathcal{V}$ such that

$$(u_i, v_j) = -(v_j, u_i) = \delta_{i,j}, \quad (u_i, u_j) = (v_i, v_j) = 0; \quad 1 \le i, j \le l.$$

The algebra $\mathcal{G}$, consisting of all endomorphisms X of $\mathcal{V}$ which are skew relative to the $(\cdot, \cdot)$ i.e., $(X(v), w) = -(v, X(w))$ for $v, w \in \mathcal{V}$, is a finite dimensional split simple Lie algebra of type C_l [9, Theorem IV.6.8]. Also by [9, §IV.6], we have that

$$\text{(3.8)} \qquad \mathcal{H} := \bigoplus_{i=1}^{l} \mathbb{F}\dot{h}_i \quad \text{where} \quad \dot{h}_i := e_{i,i} - e_{l+i,l+i}; \quad 1 \le i \le l$$

is a splitting Cartan subalgebra of $\mathcal{G}$. For $1 \le i \le l$, define $\varepsilon_i \in H^\star$ to be such that $\varepsilon_i(\dot{h}_j) = \delta_{i,j}$ for $1 \le j \le l$ and set

$$\text{(3.9)} \qquad h_{\pm 2\varepsilon i} := \pm\dot{h}_i, \ 1 \le i \le l \quad \text{and} \quad h_{\pm(\varepsilon i \pm \varepsilon_j)} := \pm(\dot{h}_i \pm \dot{h}_j), \ 1 \le i \ne j \le l.$$

One knows that $\mathcal{V}$ and $\mathcal{G}$ are $\mathcal{H}$-modules having the following weight space decompositions with respect to $\mathcal{H}$:

$$(3.10)\qquad \begin{aligned} \mathcal{V} &= \bigoplus_{i=1}^{l} \mathcal{V}_{\pm\varepsilon_i} \quad \text{in which} \quad \mathcal{V}_{\varepsilon_i} = \mathbb{F}u_i,\ \mathcal{V}_{-\varepsilon_i} = \mathbb{F}v_i,\ 1 \leq i \leq l,\\ \mathcal{G} &= \bigoplus_{\alpha\in S} \mathcal{G}_\alpha \quad \text{where} \quad S = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i \leq j \leq l\}. \end{aligned}$$

Now let z be a symbol and take $\mathcal{A} := \mathcal{V} \oplus \mathbb{F}z$ to be the Heisenberg Lie algebra with the multiplication $[\cdot, \cdot]^{\cdot}$ given by $[z, \mathcal{A}]^{\cdot} = \{0\}$ and $[u, v]^{\cdot} = (u, v)z$ for $u, v \in \mathcal{V}$. We know that the set of derivations of $\mathcal{A}$, $\mathrm{Der}(\mathcal{A})$, is a subalgebra of the Lie algebra $\mathfrak{gl}(\mathcal{A})$ whose Lie bracket will be denoted by $[\cdot, \cdot]^{\sim}$. Now define

$$\mathfrak{d}\colon \mathcal{A} \to \mathcal{A};\quad u_i \mapsto 0,\ v_i \mapsto v_i,\ z \mapsto z \quad \text{for} \quad 1 \leq i \leq l.$$

It is easily checked that $\mathfrak{d}$ belongs to $\mathrm{Der}(\mathcal{A})$. Next we extend an element $f \in \mathcal{G} \subseteq \mathrm{End}(\mathcal{V})$ to an element of $\mathrm{End}(\mathcal{A})$ by $f(z) = 0$, then $D := \mathcal{G} \oplus \mathbb{F}\mathfrak{d}$ is a subalgebra of $\mathrm{Der}(\mathcal{A})$. Set $\mathfrak{L} := \mathcal{A} \rtimes D$, then $\mathfrak{L}$ is a Lie algebra with the bracket defined by

$$[a_1 + d_1, a_2 + d_2] := [a_1, a_2]^{\cdot} - d_2(a_1) + d_1(a_2) + [d_1, d_2]^{\sim};\quad a_1, a_2 \in \mathcal{A},\ d_1, d_2 \in D.$$

Take $\mathcal{L}$ to be the derived algebra of $\mathfrak{L}$, then $\mathcal{L} = \mathcal{A} \rtimes \mathcal{G}$. Also $\mathcal{H}$, the splitting Cartan subalgebra of $\mathcal{G}$, is an abelian subalgebra of $\mathcal{L}$ with respect to which $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha\in R} \mathcal{L}_\alpha$ where $R := S \cup \{\pm\varepsilon_i \mid 1 \leq i \leq l\}$ and

$$(3.11)\qquad \mathcal{L}_\alpha = \begin{cases} \mathcal{G}_\alpha, & \alpha \in S^\times,\\ \mathcal{V}_{\pm\varepsilon_i}, & \alpha = \pm\varepsilon_i;\ 1 \leq i \leq l,\\ \mathbb{F}z + \mathcal{G}_0, & \alpha = 0. \end{cases}$$

Since $\mathcal{G}$ is a finite dimensional split simple Lie subalgebra of $\mathcal{L}$, for each $\alpha \in S^\times$, there exist $e_{\pm\alpha} \in \mathcal{G}_{\pm\alpha}$ such that $[e_{+\alpha}, e_{-\alpha}] = h_\alpha$ (see (3.9)). Now it is easy to see that $e_{+\alpha} \in \mathcal{G}_\alpha \subseteq \mathcal{L}_\alpha$ is an invertible element. This together with the fact that $\mathcal{L}_0 = \sum_{\alpha\in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$ implies that $\mathcal{L}$ is an (R, S)-graded Lie algebra with grading subalgebra $\mathcal{G}$. Now let $l = 3$ and $\rho\colon \langle R\rangle \to \mathbb{F} \setminus \{0\}$ be the group homomorphism defined by (3.2) and consider the automorphism ω_ρ of $\mathcal{L}$ defined as in (3.1). One can see that ω_ρ satisfies (GC1)–(GC5). Considering (3.3), we have

$$\mathcal{L}(\omega_\rho) = \sum_{\alpha\in R(\omega_\rho)} \mathcal{L}_\alpha, \quad \text{where} \quad R(\omega_\rho) = \{\pm 2\varepsilon_3, \pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i \leq j \leq 2\},$$

a weight space decomposition of $\mathcal{L}(\omega_\rho)$ with respect to $\mathcal{H}$ with $\mathcal{L}(\omega_\rho)_\alpha = \mathcal{L}_\alpha$ for $\alpha \in R(\omega_\rho)$. Therefore we have

$$\mathfrak{g} := \mathcal{G}(\omega_\rho) = \mathcal{G} \cap \mathcal{L}(\omega_\rho) = \mathcal{H} \oplus \sum_{i=1}^{3} \mathcal{G}_{\pm 2\varepsilon_3} \oplus \mathcal{G}_{\varepsilon_1-\varepsilon_2} \oplus \mathcal{G}_{\varepsilon_1+\varepsilon_2} \oplus \mathcal{G}_{-\varepsilon_1-\varepsilon_2} \oplus \mathcal{G}_{-\varepsilon_1+\varepsilon_2}$$

which is a perfect Lie subalgebra of $\mathcal{L}(\omega_\rho)$. Now using Proposition 2.32, one concludes that $\mathfrak{g}$ is an ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$ with ω_ρ-splitting Cartan subalgebra $\mathcal{H}$. Now since

$$\left[\sum_{i=1}^{2} \mathcal{L}(\omega_\rho)_{\pm\varepsilon_i}, \sum_{i=1}^{2} \mathcal{L}(\omega_\rho)_{\pm\varepsilon_i}\right] = \left[\sum_{i=1}^{2} \mathcal{L}_{\pm\varepsilon_i}, \sum_{i=1}^{2} \mathcal{L}_{\pm\varepsilon_i}\right] = \left[\sum_{i=1}^{2} \mathcal{V}_{\pm\varepsilon_i}, \sum_{i=1}^{2} \mathcal{V}_{\pm\varepsilon_i}\right] = \mathbb{F}z$$

and z is central, we have that $\mathfrak{g}$ is a maximal ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$. Now $\mathcal{S} := \mathcal{S}_\mathfrak{g} = \{0, \pm 2\varepsilon_i, \pm(\varepsilon_1 \pm \varepsilon_2) \mid 1 \leq i \leq 3\}$ is a finite root system of type $C_2 \cup A_1$. We also have that $\mathcal{R} = \text{span}_\mathbb{Z} \mathcal{S} \cap R(\omega_\rho) = \mathcal{S}$ and so by (3.11), $\tilde{\mathcal{L}}$, the subalgebra of $\mathcal{L}(\omega_\rho)$ generated by $\mathcal{L}(\omega_\rho)_\alpha$, $\alpha \in \mathcal{R}^\times$, coincides with $\mathfrak{g}$. Therefore the subalgebra $\tilde{\mathcal{L}} = \mathfrak{g}$ of $\mathcal{L}(\omega_\rho)$ is an $\mathcal{R}$-graded Lie algebra. Also $\mathcal{K} = \sum_{i=1}^{2} \mathcal{V}_{\pm\varepsilon_i} \oplus \mathbb{F}z$ is a subalgebra of $\mathcal{L}(\omega_\rho)$ and $\mathcal{L}(\omega_\rho) = \mathcal{L}(\omega_\rho)_c = \tilde{\mathcal{L}} \oplus \mathcal{K}$.

EXAMPLE 3.12. Let $\mathcal{A} := \mathbb{F}^{3\times 3}$, then $\mathcal{A}$ is a unital associative algebra. Define an involution on $\mathcal{A}$ as follows:

$$\bar{\ }: \mathcal{A} \to \mathcal{A}; \quad (a_{i,j})_{i,j} \mapsto (a_{4-j,4-i})_{i,j}.$$

Next let l be a positive integer and set $J := \begin{pmatrix} 0 & I_l & 0 \\ I_l & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where I_l is the identity matrix of rank l. Take $q := 2l + 1$ and define the following involution

$$^*: \mathcal{A}^{q\times q} \to \mathcal{A}^{q\times q}; \quad X \mapsto J^{-1}\bar{X}^t J.$$

Note that as $\mathcal{A}$ is unital, we can identify $\mathbb{F}$ as a subset of $\mathcal{A}$. Now set $\mathfrak{L} := \{X \in \mathcal{A}^{q\times q} \mid X^* = -X\}$ and $\mathcal{G} := \{X \in \mathbb{F}^{q\times q} \mid X^* = -X\}$. It is easy to see that $X \in \mathfrak{L}$ if and only if $X = \begin{pmatrix} A & B & M \\ D & -\bar{A}^t & N \\ -\bar{N}^t & -\bar{M}^t & p \end{pmatrix}$ where $A, B, D \in \mathcal{A}^{l\times l}$ with $\bar{B}^t = -B$, $\bar{D}^t = -D$, $M, N \in \mathcal{A}^{l\times 1}$ and $p \in \mathcal{A}$ with $\bar{p} = -p$. Set $h_i := e_{i,i} - e_{l+i,l+i}$ and $\mathcal{H} := \sum_{i=1}^{l} \mathbb{F}h_i$. We know that $\mathcal{G}$ is a finite dimensional split simple Lie algebra of type B_l with splitting Cartan subalgebra $\mathcal{H}$ [9, §IV.6]. For $1 \leq i \leq l$, define

$$\varepsilon_i \in \mathcal{H}^\star; \quad h_j \mapsto \delta_{i,j},\ 1 \leq j \leq l.$$

It is easy to check that $\mathfrak{L}$ has a weight space decomposition $\mathfrak{L} = \sum_{\alpha\in R} \mathfrak{L}_\alpha$ with respect to $\mathcal{H}$, where $R := \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i, j \leq l\}$, an irreducible finite root system of type BC_l. Take $\mathcal{L}$ to be the core of $\mathfrak{L}$, then $\mathcal{G}$ is a subalgebra of $\mathcal{L}$. One can see that $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \sum_{\alpha\in R} \mathcal{L}_\alpha$ with respect to $\mathcal{H}$, with

$$\begin{aligned}
&\mathcal{L}_{\epsilon_i} = \{ae_{i,2l+1} - \bar{a}e_{2l+1,l+i} \mid a \in \mathcal{A}\}, && \mathcal{L}_{2\epsilon_i} = \{ae_{i,l+i} \mid a \in \mathcal{A}_-\},\\
&\mathcal{L}_{-\epsilon_i} = \{ae_{l+i,2l+1} - \bar{a}e_{2l+1,i} \mid a \in \mathcal{A}\}, && \mathcal{L}_{-2\epsilon_i} = \{ae_{l+i,i} \mid a \in \mathcal{A}_-\},\\
&\mathcal{L}_{\epsilon_i+\epsilon_j} = \{ae_{i,l+j} - \bar{a}e_{j,l+i} \mid a \in \mathcal{A}\}, && \mathcal{L}_{\epsilon_i-\epsilon_j} = \{ae_{ij} - \bar{a}e_{l+j,l+i} \mid a \in \mathcal{A}\},\\
&\mathcal{L}_{-\epsilon_i-\epsilon_j} = \{ae_{l+i,j} - \bar{a}e_{l+j,i} \mid a \in \mathcal{A}\}, && \mathcal{L}_0 = \sum_{\alpha\in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}],
\end{aligned} \tag{3.13}$$

for $1 \le i \ne j \le l$. Now let $\alpha \in R^\times_{\text{ind}}$ and set $h_\alpha := 2t_\alpha/\kappa(t_\alpha, t_\alpha)$, then there exist $e_\alpha \in \mathcal{G}_\alpha$ and $f_\alpha \in \mathcal{G}_{-\alpha}$ such that $[f_\alpha, e_\alpha] = h_\alpha$. Now if $\beta \in R$, then $\beta(h_\alpha) = \langle\beta, \check{\alpha}\rangle$ and for $x \in \mathcal{L}_\beta$, we have

$$[h_\alpha, x] = \beta(h_\alpha)x = \langle\beta, \check{\alpha}\rangle x$$

which means that e_α is an invertible element of $\mathcal{L}$. Therefore $\mathcal{L}$ is an R-graded Lie algebra and $\mathcal{G}$ is a grading subalgebra of $\mathcal{L}$. Now let $l = 3$. Consider the group homomorphism ρ from $\langle R\rangle$ to $\mathbb{F}\backslash\{0\}$ defined by (3.2) and the automorphism ω_ρ of $\mathcal{L}$ defined as in (3.1). Then ω_ρ satisfies (GC1)–(GC5). Contemplating (3.3), we have

$$\mathcal{L}(\omega_\rho) = \sum_{\alpha\in R(\omega_\rho)} \mathcal{L}(\omega_\rho)_\alpha \quad \text{with} \quad R(\omega_\rho) = \{\pm 2\varepsilon_3, \pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j),\ 1 \le i \le j \le 2\}$$

$$\text{and} \quad \mathcal{L}(\omega_\rho)_\alpha := \mathcal{L}_\alpha \quad \text{for} \quad \alpha \in R(\omega_\rho).$$

Lemma 3.14. *The derived algebra $\mathcal{G}'(\omega_\rho)$ of $\mathcal{G}(\omega_\rho)$ is a maximal ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$.*

Proof. We carry out the proof in two steps:

STEP 1) There are not $A, B \in \mathcal{A}_-$ such that $ABC + CBA = 2C$ for all $C \in \mathcal{A}_-$: We first note that

$$\begin{aligned}
\mathcal{A}_- &= \{A \in \mathcal{A} \mid \bar{A} = -A\}\\
&= \left\{ A \in \mathbb{F}^{3\times 3} \;\middle|\; A = \begin{pmatrix} a & b & 0\\ c & 0 & -b\\ 0 & -c & -a\end{pmatrix} \text{ for some } a, b, c \in \mathbb{F} \right\}.
\end{aligned}$$

Now to the contrary, suppose that there exist $A, B \in \mathcal{A}_-$ such that $ABC + CBA = 2C$ for all $C \in \mathcal{A}_-$. Let $a, b, c, m, n, p \in \mathbb{F}$ such that $A = \begin{pmatrix} a & b & 0\\ c & 0 & -b\\ 0 & -c & -a\end{pmatrix}$ and $B = \begin{pmatrix} m & n & 0\\ p & 0 & -n\\ 0 & -p & -m\end{pmatrix}$. Since for $x, y, z \in \mathbb{F}$, $C_{x,y,z} := \begin{pmatrix} x & y & 0\\ z & 0 & -y\\ 0 & -z & -x\end{pmatrix} \in \mathcal{A}_-$, we have $ABC_{x,y,z} +$

$C_{x,y,z}BA = 2C_{x,y,z}$. This implies that

$$2amx + ypa + xnc + bpx + anz = 2x, \tag{3.15}$$

$$amy + 2bpy + bnz + xmb + nyc = 2y, \tag{3.16}$$

$$mcx + 2ncz + bpz + zma + ycp = 2z \tag{3.17}$$

for all $x, y, z \in \mathbb{F}$. Now let $x = a$, $y = b$ and $z = c$, then (3.15)–(3.17) imply

$$a(am + bp + nc) = a, \quad b(am + bp + nc) = b, \quad c(am + bp + nc) = c.$$

But $A \neq \mathbf{0}$, so

$$am + bp + nc = 1. \tag{3.18}$$

Also if $x = y = 0$, $z \neq 0$, or $x = z = 0$, $y \neq 0$, then (3.15) implies

$$an = 0 \quad \text{and} \quad ap = 0. \tag{3.19}$$

Similarly using (3.16) and (3.17), we get

$$mb = 0, \quad nb = 0, \quad cp = 0, \quad mc = 0. \tag{3.20}$$

Now note that since $A \neq \mathbf{0}$, one of a, b, c is not zero, say $a \neq 0$. Then (3.19) implies that $n = p = 0$ and so (3.18) implies that $m \neq 0$. Therefore by (3.20), we have $b = c = 0$. Using (3.18), we get $am = 1$. Contemplating this together with the fact that $n = p = b = c = 0$, (3.16) implies that $y = 2y$ for each $y \in \mathbb{F}$ which is a contradiction. The same contradictions arise if we consider the cases that $b \neq 0$ or $c \neq 0$. This completes the proof of this step.

STEP 2) $\mathcal{G}'(\omega_\rho)$ is a maximal splitting subalgebra: By Proposition 2.32, $\mathcal{G}'(\omega_\rho)$ is an ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$ with ω_ρ-splitting Cartan subalgebra $\mathbb{F}h_1 + \mathbb{F}h_2$ and the root system $\{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm(\varepsilon_1 \pm \varepsilon_2)\}$. Let $\mathcal{G}'(\omega_\rho)$ be not a maximal one, then there is an ω_ρ-splitting subalgebra $\mathfrak{g}$ of $\mathcal{L}(\omega_\rho)$ such that $\mathcal{G}'(\omega_\rho) \subsetneqq \mathfrak{g}$. Using Remark 2.30 (i) together with the fact that two times of a nonzero root of a finite dimensional split simple Lie algebra is not a root, one concludes that there exist $e_\pm \in \mathcal{L}_{\pm 2\varepsilon_3} \cap \mathfrak{g}$ such that $(e_+, [e_+, e_-], e_-)$ is an $\mathfrak{sl}_2$-triple and $h := [e_+, e_-] \in \mathcal{H} + Z(\mathcal{L}(\omega_\rho))$. Since $e_\pm \in \mathcal{L}_{\pm 2\varepsilon_3}$, there exist nonzero elements $a, b \in \mathcal{A}_-$ such that $e_+ = ae_{3,6}$ and $e_- = be_{6,3}$. Since $h \in \mathcal{H} + Z(\mathcal{L}(\omega_\rho))$, there exist $\{s_i \mid 1 \leq i \leq 3\} \subset \mathbb{F}$ and $z \in Z(\mathcal{L}(\omega_\rho))$ such that $h = \sum_{i=1}^{3} s_i h_i + z$. Also since $ae_{3,6} = e_+ \in \mathfrak{g} \subseteq \mathcal{L}(\omega_\rho)$ and for $i = 1, 2$, $[h_i, ae_{3,6}] = 0$, we have

$$\begin{aligned} 2ae_{3,6} = 2e_+ = [h, e_+] = [h, ae_{3,6}] = \left[\sum_{i=1}^{3} s_i h_i + z, ae_{3,6}\right] &= [s_3h_3, ae_{3,6}] \\ &= s_3a[h_3, e_{3,6}] \\ &= 2s_3ae_{3,6}, \end{aligned}$$

which implies $s_3 = 1$. Therefore we have

$$\begin{aligned}\sum_{i=1}^{2} s_i h_i + z = h - s_3 h_3 = h - h_3 &= [e_+, e_-] - h_3\\ &= [a e_{3,6}, b e_{6,3}] - h_3\\ &= ab e_{3,3} - ba e_{6,6} - e_{3,3} + e_{6,6}\\ &= (ab-1)e_{3,3} - (ba-1)e_{6,6}.\end{aligned} \tag{3.21}$$

Now let $d \in \mathcal{A}_-$, then $de_{3,6} \in \mathcal{L}(\omega_\rho)$ and so

$$\begin{aligned}0 = \left[\sum_{i=1}^{2} s_i h_i + z, d e_{3,6}\right] &= [(ab-1)e_{3,3} - (ba-1)e_{6,6}, d e_{3,6}]\\ &= (abd - d)e_{3,6} + (dba - d)e_{3,6},\end{aligned}$$

which means that

$$abd + dba = 2d; \quad d \in \mathcal{A}_-.$$

This makes a contradiction using Step 1). □

Put $\mathfrak{g} := \mathcal{G}'(\omega_\rho)$. Then $\mathfrak{g}$ is a maximal ω_ρ-splitting subalgebra and $\mathcal{S} := \mathcal{S}_\mathfrak{g} = \{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm(\varepsilon_1 \pm \varepsilon_2)\}$, an irreducible finite root system of type B_2. We also have $\mathcal{R} = R(\omega_\rho) \cap \mathrm{span}_\mathbb{Z} \mathcal{S} = \{0, \pm\varepsilon_1, \pm\varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm(\varepsilon_1 \pm \varepsilon_2)\}$, an irreducible finite root system of type BC_2. Now thanks to Proposition 2.37, $\tilde{\mathcal{L}} = \sum_{\alpha \in \mathcal{R}^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in \mathcal{R}^\times} \mathcal{L}_\alpha$, the Lie subalgebra of $\mathcal{L}(\omega_\rho)$ generated by $\mathcal{L}_\alpha$, $\alpha \in \mathcal{R}^\times$, is an $\mathcal{R}$-graded Lie algebra. Also $\mathcal{K} = [\mathcal{L}_{2\varepsilon_3}, \mathcal{L}_{-2\varepsilon_3}] \oplus \mathcal{L}_{2\varepsilon_3} \oplus \mathcal{L}_{-2\varepsilon_3}$ is a subalgebra of $\mathcal{L}(\omega_\rho)$ and we have $\mathcal{L}(\omega_\rho)_c = \tilde{\mathcal{L}} \oplus \mathcal{K}$. Moreover Theorem 2.46 implies that $\mathcal{L}(\omega_\rho) = \tilde{\mathcal{L}} \oplus \mathcal{K} \oplus \mathcal{D}$ in which $\mathcal{D}$ is a subspace of $\mathcal{L}(\omega_\rho)$ satisfying $[\mathfrak{g}, \mathcal{D}] = \{0\}$.

The following example is adopted from [3]. We ask the reader, who is not familiar with the notation, to consult [3, Example 3.18].

EXAMPLE 3.22. Let $\mathfrak{a}$ be a unital alternative algebra with involution η whose η-symmetric elements are in the nucleus, C be an associative left $\mathfrak{a}$-module and $\chi(\cdot,\cdot)$ be a nonzero skew-hermitian form on C. Then $\mathfrak{A} := \mathfrak{a}^{3\times 3}$ is an algebra with involution $*$ defined by $m^* := (m^\eta)^t$, $m \in \mathfrak{a}^{3\times 3}$. It is well-known that $J := H(\mathfrak{A}, *)$, the set of hermitian elements of $\mathfrak{A}$ under $*$, is a unital Jordan algebra under the product $m_1 \cdot m_2 := (1/2)(m_1 m_2 + m_2 m_1)$, $m_1, m_2 \in J$. The space $X := C^3$ of (3×1)-column matrices over C is an associative left $\mathfrak{A}$-module with action given by matrix multiplication. If $\psi\colon X \times X \to \mathfrak{A}$ is defined by

$$\psi((c_1, c_2, c_3)^t, (c_1', c_2', c_3')^t) := (-\chi(c_i, c_j'))_{i,j},$$

then ψ is a skew-hermitian form on X. Now define

$$a \bullet x := ax,$$
$$\{x, y\} := \psi(x, y) - \psi(y, x),$$
$$\{x, y, z\} := \psi(x, y)z + \psi(z, y)x + \psi(z, x)y$$

for $x, y, z \in X$ and $a \in J$. Then X is a J-*ternary* algebra. Set $N := X \oplus J$, the direct sum of two vector spaces X and J. An element of N is denoted by $\langle x \parallel a\rangle$ where $x \in X$ and $a \in J$. If $a = 0$ (resp. $x = 0$), we simply denote $\langle x \parallel a\rangle$ by x (resp. a). Consider the Lie algebra $\mathfrak{gl}(N)$ corresponding to the associative algebra $\mathrm{End}(N)$. For $a \in J$ and $x, y \in X$, define $\mathbb{L}_a, \mathbb{L}_{x,y} \in \mathfrak{gl}(N)$ as follows:

$$\mathbb{L}_a\langle z \parallel a'\rangle = \langle (1/2)a \bullet z \parallel a \cdot a'\rangle,$$
$$\mathbb{L}_{x,y}\langle z \parallel a'\rangle = \langle \{x, y, z\} \parallel \{x, a' \bullet y\}\rangle,$$
$$a' \in J,\ z \in X.$$

Put $\mathfrak{Instr}(J, X) := \mathbb{L}_J + [\mathbb{L}_J, \mathbb{L}_J] + \mathbb{L}_{X,X}$. Then $\mathfrak{Instr}(J, X)$ is a Lie subalgebra of $\mathfrak{gl}(N)$. The Lie algebra $\mathfrak{Instr}(J, X)$ has an automorphism ε of period 2 defined by $T^{\varepsilon} := T - 2\mathbb{L}_{T(1)}$. Define

$$\mathcal{L} := \tilde{N} \oplus \mathfrak{Instr}(J, X) \oplus N$$

where $\tilde{N} := \{\tilde{n} \mid n \in N\}$ is a copy of N. Extend the Lie bracket on $\mathfrak{Instr}(J, X)$ to an anti-commutative product $[\,\cdot\,,\,\cdot\,]$ on $\mathcal{L}$ as follows:

$$[T, n] := T(n), \quad [T, \tilde{n}] := (T^{\varepsilon}(n))^{\sim},$$
$$[\langle x \parallel a\rangle, \langle x' \parallel a'\rangle] := \langle 0 \parallel -\{x, x'\}\rangle, \quad [\langle x \parallel a\rangle^{\sim}, \langle x' \parallel a'\rangle^{\sim}] := \langle 0 \parallel \{x, x'\}\rangle^{\sim},$$
$$[\langle x \parallel a\rangle, \langle x' \parallel a'\rangle^{\sim}] := \langle -a' \bullet x \parallel 0\rangle^{\sim} + \mathbb{L}_{x,x'} + 2(\mathbb{L}_{a\cdot a'} + [\mathbb{L}_a, \mathbb{L}_{a'}]) + \langle a \bullet x' \parallel 0\rangle$$

for $T \in \mathfrak{Instr}(J, X)$, $n \in N$, $x, x' \in X$, $a, a' \in J$. Relative to this product, $\mathcal{L}$ is a Lie algebra. For $1 \le i \le 3$, define $h_i := \mathbb{L}_{e_{i,i}}$ and $\mathcal{H} := \sum_{i=1}^{3} \mathbb{F}h_i$. Take $\mathcal{G}$ to be the *Tits-Kantor-Koecher Lie algebra* of $\{A \in \mathbb{F}^{3\times 3} \mid A^t = A\}$. Then $\mathcal{G}$ is a finite dimensional split simple Lie algebra of type C_3 with splitting Cartan subalgebra $\mathcal{H}$ and the root system $S := \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i, j \le 3\}$. For $1 \le i \le 3$, define ε_i by:

$$\varepsilon_i(h_j) := \frac{1}{2}\delta_{i,j}; \quad 1 \le j \le 3.$$

Then $\mathcal{L}$ has a weight space decomposition with respect to $\mathcal{H}$:

$$\mathcal{L} = \bigoplus_{\alpha\in R} \mathcal{L}_\alpha \quad \text{where} \quad R = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \colon 1 \leq i,\ j \leq 3\}$$

and

$$\mathcal{L}_0 + \sum_{1\leq i\neq j\leq 3} \mathcal{L}_{\varepsilon_i-\varepsilon_j} = \mathfrak{Instr}(J, X),$$

$$\sum_{i=1}^{3} \mathcal{L}_{\varepsilon_i} = \langle C^3 \parallel 0\rangle, \quad \sum_{i=1}^{3} \mathcal{L}_{2\varepsilon_i} + \sum_{1\leq i<j\leq 3} \mathcal{L}_{\varepsilon_i+\varepsilon_j} = \langle 0\parallel J\rangle,$$

$$\sum_{i=1}^{3} \mathcal{L}_{-\varepsilon_i} = \langle C^3 \parallel 0\rangle^{\sim}, \quad \sum_{i=1}^{3} \mathcal{L}_{-2\varepsilon_i} + \sum_{1\leq i<j\leq 3} \mathcal{L}_{-\varepsilon_i-\varepsilon_j} = \langle 0 \parallel J\rangle^{\sim}.$$

One can see that $\mathcal{L}$ is an (R, S)-graded Lie algebra for which $\mathcal{G}$ is a grading subalgebra. Now let $\rho\colon \langle R\rangle \to \mathbb{F}\setminus\{0\}$ be the group homomorphism defined by (3.2) and consider the automorphism ω_ρ of $\mathcal{L}$ defined as in (3.1). Considering (3.3) together with (3.11), we have

$$\mathcal{L}(\omega_\rho) = \sum_{\alpha\in R(\omega_\rho)} \mathcal{L}_\alpha \quad \text{where} \quad R(\omega_\rho) = \{\pm 2\varepsilon_3, \pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i \leq j \leq 2\}.$$

It follows from Proposition 2.32 that the perfect Lie algebra $\mathcal{G}(\omega_\rho)$ is an ω_ρ-splitting subalgebra of $\mathcal{L}(\omega_\rho)$ with ω_ρ-splitting Cartan subalgebra $\mathcal{H}$ and the root system $\{0, \pm 2\varepsilon_i, \pm(\varepsilon_1 \pm \varepsilon_2) \mid 1 \leq i \leq 3\}$ which is a finite root system of type $A_1 \cup C_2$. Now as two times of a root of a finite dimensional split semisimple Lie algebra is not a root, one concludes that $\mathcal{G}(\omega_\rho)$ is a maximal splitting subalgebra. Now we have $\mathcal{S} := \mathcal{S}_{\mathcal{G}(\omega_\rho)} = \{0, \pm 2\varepsilon_i, \pm(\varepsilon_1 \pm \varepsilon_2) \mid 1 \leq i \leq 3\}$ and $\mathcal{R} = \mathrm{span}_{\mathbb{Z}}\, \mathcal{S} \cap R(\omega_\rho) = \mathcal{S}$. Now $\mathcal{L}(\omega_\rho)_c = \tilde{\mathcal{L}} + \mathcal{K}$ where $\tilde{\mathcal{L}}$ is the Lie subalgebra of $\mathcal{L}(\omega_\rho)$ generated by $\mathcal{L}(\omega_\rho)_\alpha = \mathcal{L}_\alpha$, $\alpha \in \mathcal{R}^\times$ and

$$\mathcal{K} = \sum_{i=1}^{2} \mathcal{L}_{\varepsilon_i} + \sum_{i=1}^{2} \mathcal{L}_{-\varepsilon_i} + \sum_{i=1}^{2} [\mathcal{L}_{\varepsilon_i}, \mathcal{L}_{-\varepsilon_i}].$$

Moreover $\tilde{\mathcal{L}}$ is an $\mathcal{R}$-graded Lie algebra, also by Theorem 2.46, there is a subspace $\mathcal{D}$ of $\mathcal{L}(\omega_\rho)$ satisfying $[\mathcal{D}, \mathcal{G}(\omega_\rho)] = \{0\}$ and $\mathcal{L}(\omega_\rho) = (\tilde{\mathcal{L}} + \mathcal{K}) \oplus \mathcal{D}$. Now note that since χ is nonzero, there exist $c, d \in \mathfrak{a}$ such that $\chi(c, d) \neq 0$. Therefore for $x := (c, 0, 0)^t$, $x' := (d, 0, 0)^t$, $\{x, x'\}$ is a 3×3 matrix whose $(1, 1)$-entry is the nonzero element $-\chi(c, d)$. So we have

$$0 \neq \langle 0 \parallel -\{x, x'\}\rangle = [\langle (c, 0, 0)^t \parallel 0\rangle, \langle (0, d, 0)^t \parallel 0\rangle] \in [\mathcal{L}_{\varepsilon_1}, \mathcal{L}_{\varepsilon_1}] \subseteq \mathcal{L}_{2\varepsilon_1}.$$

This implies that $\mathcal{K}$ is not a subalgebra.

EXAMPLE 3.23. Let $(\mathcal{A}, \bar{\ })$ be a unital associative star algebra and l be a positive integer. We know that $\mathcal{A}^{(l+1)\times(l+1)}$ is a unital associative algebra, so under the commutator product, it is a Lie algebra denoted by $\mathfrak{gl}_{l+1}(\mathcal{A})$. The subalgebra $\mathcal{L} := \mathfrak{e}_{l+1}(\mathcal{A})$ of $\mathfrak{gl}_{l+1}(\mathcal{A})$ generated by $\{ae_{i,j} \mid a \in \mathcal{A},\ 1 \leq i \neq j \leq l+1\}$ is a perfect ideal of $\mathfrak{gl}_{l+1}(\mathcal{A})$. For $1 \leq i \leq l$, set $h_i := e_{i,i} - e_{i+1,i+1}$ and take $\mathcal{H} := \sum_{i=1}^{l} \mathbb{F}h_i$. Define

$$\varepsilon_j \in \mathcal{H}^\star; \quad h_i \mapsto \delta_{i,j} - \delta_{i+1,j}; \quad 1 \leq j \leq l+1,\ 1 \leq i \leq l.$$

Now $\mathcal{L}$ has a weight space decomposition with respect to $\mathcal{H}$. In fact $\mathcal{L} = \sum_{\alpha\in R} \mathcal{L}_\alpha$ where $R := \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i \leq j \leq l+1\}$ and

$$\mathcal{L}_{\varepsilon_i-\varepsilon_j} = \mathcal{A}e_{i,j}; \quad 1 \leq i \neq j \leq l+1 \quad \text{and} \quad \mathcal{L}_0 = \sum_{\alpha\in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}].$$

One can easily see that $\mathcal{L}$ is an R-graded Lie algebra with a grading subalgebra $\mathcal{G} := \mathcal{L} \cap \mathbb{F}^{(l+1)\times(l+1)}$. Now define an involution on the associative algebra $\mathcal{A}^{(l+1)\times(l+1)}$ as follows:

$$*: \mathcal{A}^{(l+1)\times(l+1)} \to \mathcal{A}^{(l+1)\times(l+1)}; \quad (a_{i,j}) \mapsto (\overline{a}_{l+2-j,l+2-i})$$

and then define

$$\sigma : \mathfrak{gl}_{l+1}(\mathcal{A}) \to \mathfrak{gl}_{l+1}(\mathcal{A}); \quad X \mapsto -X^*.$$

which is an automorphism of $\mathfrak{gl}_{l+1}(\mathcal{A})$ of period 2. Since * maps the generating set of $\mathcal{L} = \mathfrak{e}_{l+1}(\mathcal{A})$ to itself, the restriction of σ to $\mathcal{L}$ is an automorphism of $\mathcal{L}$ of period 2 which we denote it again by σ. So $\mathcal{L}$ satisfies (GC1). One can easily check that

$$\mathcal{H}(\sigma) = \begin{cases} \displaystyle\sum_{i=1}^{(l-1)/2} \mathbb{F}(h_i + h_{l-i+1}) + \mathbb{F}h_{(l+1)/2}, & \text{if } l \text{ is odd,} \\ \displaystyle\sum_{i=1}^{l/2} \mathbb{F}(h_i + h_{l-i+1}), & \text{if } l \text{ is even.} \end{cases}$$

Now for $1 \leq i \leq [(l+1)/2]$, put $\varepsilon_i' := (1/2)(\varepsilon_i - \varepsilon_{l-i+2})$, then

$$R(\sigma) := \begin{cases} \left\{ \pm(\varepsilon_i' \pm \varepsilon_j') \;\middle|\; 1 \leq i,\ j \leq \dfrac{l+1}{2} \right\}, & \text{if } l \text{ is odd,} \\ \left\{ \pm\varepsilon_i',\ \pm(\varepsilon_i' \pm \varepsilon_j') \;\middle|\; 1 \leq i,\ j \leq \dfrac{l}{2} \right\}, & \text{if } l \text{ is even} \end{cases}$$

is an irreducible finite root system of type $C_{(l+1)/2}$ if l is odd and of type $BC_{l/2}$ if l is even. Also it is easy to see that $\mathcal{L}(\sigma)$ has a weight space decomposition $\mathcal{L}(\sigma) =$

$\sum_{\alpha\in R(\sigma)}\mathcal{L}(\sigma)_\alpha$ with respect to $\mathcal{H}(\sigma)$ in which for $1\le i\ne j\le[(l+1)/2]$,

$$\begin{aligned}
\mathcal{L}(\sigma)_{\varepsilon_i'-\varepsilon_j'} &= \{ae_{i,j}-\bar{a}e_{l+2-j,l+2-i}\mid a\in\mathcal{A}\},\\
\mathcal{L}(\sigma)_{\varepsilon_i'+\varepsilon_j'} &= \{ae_{i,l+2-j}-\bar{a}e_{j,l+2-i}\mid a\in\mathcal{A}\},\\
\mathcal{L}(\sigma)_{-\varepsilon_i'-\varepsilon_j'} &= \{ae_{l+2-i,j}-\bar{a}e_{l+2-j,i}\mid a\in\mathcal{A}\},\\
\mathcal{L}(\sigma)_{2\varepsilon_i'} &= \{ae_{i,l+2-i}\mid a\in\mathcal{A}_-\},\\
\mathcal{L}(\sigma)_{-2\varepsilon_i'} &= \{ae_{l+2-i,i}\mid a\in\mathcal{A}_-\},
\end{aligned}$$

and if l is even,

$$\begin{aligned}
\mathcal{L}(\sigma)_{\varepsilon_i'} &= \{ae_{i,(l+2)/2}-\bar{a}e_{(l+2)/2,l+2-i},\ a\in\mathcal{A}\},\\
\mathcal{L}(\sigma)_{-\varepsilon_i'} &= \{ae_{l+2-i,(l+2)/2}-\bar{a}e_{(l+2)/2,i},\ a\in\mathcal{A}\}.
\end{aligned}$$

Now set

$$\mathcal{S}:=\begin{cases}\left\{0,\ \pm(\varepsilon_i'\pm\varepsilon_j')\ \middle|\ 1\le i\ne j\le\frac{l+1}{2}\right\}, & \text{if } l \text{ is odd},\\ \left\{0,\ \pm\varepsilon_i',\ \pm(\varepsilon_i'\pm\varepsilon_j')\ \middle|\ 1\le i\ne j\le\frac{l}{2}\right\}, & \text{if } l \text{ is even}.\end{cases}\tag{3.24}$$

Then $\mathcal{G}(\sigma)=\mathcal{L}(\sigma)\cap\mathcal{G}$ has a weight space decomposition $\mathcal{G}(\sigma)=\sum_{\alpha\in\mathcal{S}}\mathcal{G}(\sigma)_\alpha$ with respect to $\mathcal{H}(\sigma)$ where for $1\le i\ne j\le[(l+1)/2]$,

$$\begin{aligned}
&\mathcal{G}(\sigma)_{\varepsilon_i'-\varepsilon_j'}=\mathbb{F}(e_{i,j}-e_{l+2-j,l+2-i}), \quad \mathcal{G}(\sigma)_{-\varepsilon_i'-\varepsilon_j'}=\mathbb{F}(e_{l+2-i,j}-e_{l+2-j,i}),\\
&\mathcal{G}(\sigma)_{\varepsilon_i'+\varepsilon_j'}=\mathbb{F}(e_{i,l+2-j}-e_{j,l+2-i}), \quad \mathcal{G}(\sigma)_0=\mathcal{H}(\sigma)
\end{aligned}$$

and if l is even,

$$\begin{aligned}
\mathcal{G}(\sigma)_{\varepsilon_i'} &= \mathbb{F}(e_{i,(l+2)/2}-e_{(l+2)/2,l+2-i}),\\
\mathcal{G}(\sigma)_{-\varepsilon_i'} &= \mathbb{F}(e_{l+2-i,(l+2)/2}-e_{(l+2)/2,i}),
\end{aligned}$$

in particular the centralizer of $\mathcal{H}(\sigma)$ in $\mathcal{G}(\sigma)$ coincides with $\mathcal{H}(\sigma)$ which means that (GC2) is satisfied. Next we note that as σ maps $\mathcal{G}$ to $\mathcal{G}$ and $\sigma(\mathcal{H})\subseteq\mathcal{H}$, the restriction of the Killing form of $\mathcal{G}$ to $\mathcal{G}(\sigma)$ is non-degenerate. Therefore (GC3) is also satisfied. Also since the element of $\mathcal{H}$ representing an element α of $\mathcal{H}(\sigma)^\star$ (identifying as a subset of $\mathcal{H}^\star$) through the Killing form is the same as the element of $\mathcal{H}(\sigma)$ representing α through the $\kappa(\cdot,\cdot)|_{\mathcal{H}(\sigma)\times\mathcal{H}(\sigma)}$, (GC4) is also satisfied. Therefore σ satisfies (GC1)–(GC5).

Proposition 3.25. *The derived algebra $\mathcal{G}'(\sigma)$ of $\mathcal{G}(\sigma)$ is a maximal σ-splitting subalgebra of $\mathcal{L}(\sigma)$.*

Proof. By Proposition 2.32, $\mathcal{G}'(\sigma)$ is a σ-splitting subalgebra of $\mathcal{L}(\sigma)$. If it is not a maximal one, then there exist a σ-splitting subalgebra $\mathfrak{g}$ of $\mathcal{L}(\sigma)$ with splitting Cartan subalgebra $\mathcal{C}$ and the root system $\Delta_{\mathfrak{g}}$ such that $\mathcal{G}'(\sigma) \subsetneq \mathfrak{g}$. So by Lemma 2.31, $\mathcal{S} = \mathcal{S}_{\mathcal{G}'(\sigma)} \subsetneq \mathcal{S}_{\mathfrak{g}}$, so there exists $1 \leq t \leq [(l+1)/2]$ such that $2\varepsilon'_t \in \mathcal{S}_{\mathfrak{g}}$. If l is even, $\varepsilon'_t \in \mathcal{S} \subseteq \mathcal{S}_{\mathfrak{g}}$ which makes a contradiction as $2\varepsilon'_t \in \mathcal{S}_{\mathfrak{g}}$ and $\mathcal{S}_{\mathfrak{g}}$ is an indivisible finite root system. Now let l be odd, since $2\varepsilon'_t \notin \mathcal{S}$, for $1 \leq i \leq (l+1)/2$, $[\mathcal{G}'(\sigma) \cap \mathcal{L}(\sigma)_{\varepsilon'_t-\varepsilon'_i}, \mathcal{G}'(\sigma) \cap \mathcal{L}(\sigma)_{\varepsilon'_t-\varepsilon'_i}] = \{0\}$. Also since $\mathcal{G}'(\sigma) \subseteq \mathfrak{g}$, Remark 2.30 (iii) implies that for $\alpha \in \mathcal{S}$, $\{0\} \neq \mathcal{G}'(\sigma) \cap \mathcal{L}(\sigma)_\alpha = \mathfrak{g} \cap \mathcal{L}(\sigma)_\alpha$. We recall that $2\varepsilon'_t \in \mathcal{S}_{\mathfrak{g}}$ and use Remark 2.30 (iii), then we conclude that for $1 \leq i \leq (l+1)/2$,

$$\begin{aligned}\{0\} &= [\mathcal{G}'(\sigma) \cap \mathcal{L}(\sigma)_{\varepsilon'_t-\varepsilon'_i}, \mathcal{G}'(\sigma) \cap \mathcal{L}(\sigma)_{\varepsilon'_t-\varepsilon'_i}] \\ &= [\mathfrak{g} \cap \mathcal{L}(\sigma)_{\varepsilon'_t-\varepsilon'_i}, \mathfrak{g} \cap \mathcal{L}(\sigma)_{\varepsilon'_t-\varepsilon'_i}] \\ &= \mathfrak{g} \cap \mathcal{L}(\sigma)_{2\varepsilon'_t} \neq \{0\}\end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 3.26. *$\mathcal{L}(\sigma)_c$ is an $(R(\sigma), \mathcal{S})$-graded Lie algebra and $\mathcal{L}(\sigma)$ is decomposed into $\mathcal{L}(\sigma) = \mathcal{L}(\sigma)_c \oplus \mathcal{D}$, where $\mathcal{D}$ is a subspace of $\mathcal{L}(\sigma)$ satisfying $[\mathcal{D}, \mathcal{G}'(\sigma)] = \{0\}$.*

Proof. We note that $\text{span}_{\mathbb{Z}} \mathcal{S} \cap R(\sigma) = R(\sigma)$. Therefore we are done, using Proposition 2.37 together with Theorem 2.46. □

ACKNOWLEDGEMENTS. The author would like to thank Professor Saeid Azam for some helpful discussion. She also wishes to thank the referee of the paper for his/her suggestions.

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