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ALEXANDER POLYNOMIALS OF ALTERNATING KNOTS OF GENUS TWO

Dedicated to Professor Akio Kawauchi for his 60th birthday

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Abstract
We confirm R.H. Fox's trapezoidal conjecture for alternating knots of genus two by a method different from P. Ozsváth and Z. Szabó's one. As an application, we determine the alternating knots of genus two whose Alexander polynomials have minimal coefficients equal to one or two.

1. Introduction

An integer polynomial $f(t) = \sum_{n=0}^{m} a_n t^n$ is *trapezoidal* if it has the following four properties.
(i) The coefficients $a_0, a_1, \dots, a_m$ are nonzero and have the same sign.
(ii) $t^m f(t^{-1}) = f(t)$.
(iii) $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_{[m/2]}|$.
(iv) If $a_i = a_{i+1}$ for some i, then $a_i = a_j$ for every $j = i, i+1, \dots, [m/2]$.
Let $[f(t)]_\alpha$ be the coefficient of t^α in a polynomial $f(t)$. Let

$$\begin{aligned}
&\operatorname{maxdeg} f(t) = \max\{\alpha \mid [f(t)]_\alpha \neq 0\},\\
&\operatorname{mindeg} f(t) = \min\{\alpha \mid [f(t)]_\alpha \neq 0\},\\
&\operatorname{span} f(t) = \operatorname{maxdeg} f(t) - \operatorname{mindeg} f(t).
\end{aligned}$$

Throughout this paper, we suppose that every link is oriented. Let $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$ be the Alexander polynomial of a link L in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. We suppose that L is non-split and alternating. Then the coefficients of the polynomial $\Delta_L(-t)$ are nonzero and have the same sign, and $\operatorname{span} \Delta_L(t) = 2g(L) + \mu(L) - 1$ [3], [12]. Here $\mu(L)$ is the number of the components of L and $g(L)$ is the genus of L. In this paper, we adopt the normalization for the Alexander polynomial $\Delta_L(t)$ so that $\operatorname{mindeg} \Delta_L(t) = 0$ and $[\Delta_L]_0$ is positive. R.H. Fox conjectured the following.

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Trapezoidal conjecture ([5]). *If L is a non-split alternating link, then the normalized Alexander polynomial $\Delta_L(-t)$ is trapezoidal.*

Trapezoidal conjecture is true for a non-split two-bridge link [6] and for a larger class of algebraic alternating links [15]. Note that a two-bridge link is alternating and algebraic.

Theorem 1.1. *Trapezoidal conjecture is true for every alternating knot of genus ≤ 2.*

P. Ozsváth and Z. Szabó found out another property on the Alexander polynomials of alternating knots by using Heegaard Floer homology [17]: Given a knot K, let b_k be the k-th coefficients of the symmetrized Alexander polynomial of K, that is in the form $\Delta_K(t) = b_0 + \sum_{k>0} b_k(t^k + t^{-k})$ and satisfies $\Delta_K(1) = 1$. Let $\sigma = \sigma(K)$ be the signature of K, and $\delta(p, q) = \max(0, \lceil(|p| - 2|q|)/4\rceil)$ for integers p and q. Let $t_s(K)$ denote the torsion coefficients defined by $t_s(K) = \sum_{j=1}^{\infty} j b_{|s|+j}$, where s is an integer. If K is an alternating knot, then the inequality

$$(1) \qquad (-1)^{s+\sigma/2}(t_s(K) - \delta(\sigma, s)) \leq 0$$

holds for any integer s. By the inequality (1), they confirmed that the trapezoidal conjecture is true for alternating knots of genus two.

In this paper, we prove the trapezoidal conjecture for alternating knots of genus two in a combinatorial way. Our tools for the proof are the method for calculating the Alexander polynomial by using graphs due to R.H. Crowell [3] and the generators for knots of canonical genus two due to A. Stoimenow [18]. We explain these tools in the sections 2 and 3. As an application of our combinatorial way, we determine the alternating knots of genus two which possess the Alexander polynomials $\Delta(t)$ with $[\Delta(t)]_0 = 1$, or $[\Delta(t)]_0 = 2$. Then we give examples of the Alexander polynomials which satisfy the trapezoidal property and Ozsváth-Szabó's inequality (1) but are never realized by alternating knots.

One of our interests is a characterization of the Alexander polynomials of alternating knots. Our argument may allow to calculate the Alexander polynomials of alternating knots of genus two in an accessible way (see the formula (3) in Lemma 4.5 and the generators in Lemma 3.1).

2. Combinatorial method for calculating the Alexander polynomial

In this section, we review Crowell's method for calculating the Alexander polynomial of an alternating link by using certain planar graphs. The method is derived by applying the matrix-tree theorem [1] to the Alexander matrix obtained from an alternating link diagram. (For details, we refer the reader to [3].)

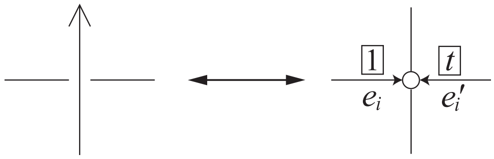

Fig. 1. The alternating orientation and the weights of edges.

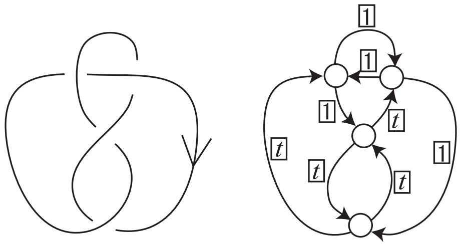

Fig. 2.

Let L be an alternating link. Let D be an alternating diagram of L with m crossings $c_1, \ldots, c_m$. Suppose that $m \geq 2$. We consider the underlying immersed graph of a diagram D. We denote by $V(D)$ the set of vertices, and by $E(D)$ the set of edges. We define orientations on the edges and a weight map of the edges as follows: For $i = 1, \ldots, m$, let e_i (resp. e_i') be the edge which is on the left (resp. right) side of the crossing c_i when one is going along the overpath in the original orientation of L (see Fig. 1). We define the orientation on e_i and e_i' so that the vertex c_i is the terminal point of both e_1 and e_2. We call this orientation *alternating orientation*. This is distinct from the original orientation of the link L. We define the weight map $\Phi\colon E(D) \to \{1, t\}$ by $\Phi(e_i) = 1$ and $\Phi(e_i') = t$. The alternating orientation and the weight map Φ are well-defined since the diagram D is alternating. We denote the graph with the alternating orientation and the weight map Φ by the same symbol D.

EXAMPLE 2.1. Let D be an alternating diagram depicted in Fig. 2, which represents the figure-eight knot. The graph D with the alternating orientation and the weights is drawn in Fig. 2. The weights of edges are drawn in a rectangle respectively.

We choose a vertex, which we call root vertex and denote it by c_0. A tree $T \subset D$ is a *maximal rooted tree with root vertex* c_0 if $V(T) = V(D)$, $\#E(T) = \#V(T) - 1$, and every vertex without the root vertex c_0 has a single incoming edge. Let $\mathcal{T}(D; c_0)$ be the set of all maximal rooted trees in D with root vertex c_0 and $W(T)$ the weight of

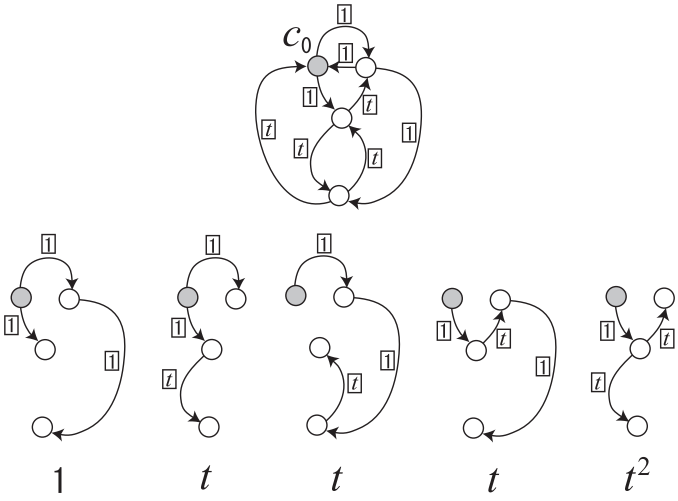

Fig. 3.

a tree T defined by $W(T) = \prod_{e \in E(T)} \Phi(e)$. We define a polynomial $P_{(D;c_0)}(t)$ by

$$P_{(D;c_0)}(t) = \sum_{T \in \mathcal{T}(D;c_0)} W(T).$$

Then we obtain the following lemma.

Lemma 2.2 ([3]). *Let $\Delta_L(t)$ be the normalized Alexander polynomial of an alternating link L. Then we have*

$$\Delta_L(-t) \doteq P_{(D;c_0)}(t), \tag{2}$$

where "$\doteq$" means "is equal to, up to multiplications by units of the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$".

Note that the polynomial $P = P_{(D;c_0)}(t)$ is independent of choices of a diagram and a root vertex up to multiplications by units of $\mathbb{Z}[t, t^{-1}]$. For a disconnected diagram D, we have $P_{(D;c_0)}(t) = 0$. We define the Alexander polynomial Δ_D of a diagram D as that of the link L represented by D: $\Delta_D = \Delta_L$.

EXAMPLE 2.3. The normalized Alexander polynomial of the figure-eight knot, substituted $-t$, is equal to $t^2 + 3t + 1$. We choose the root vertex $c_0 \in V(D)$ as in Fig. 3. The trees in $\mathcal{T}(D; c_0)$ are drawn in Fig. 3. The monomial below a tree in Fig. 3 indicates the weight of the tree.

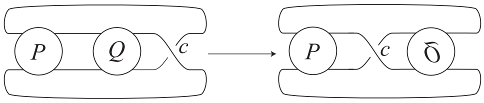

Fig. 4. A flype near the crossing c.

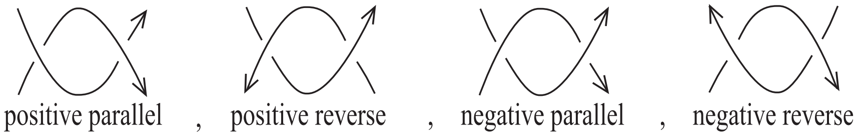

Fig. 5.

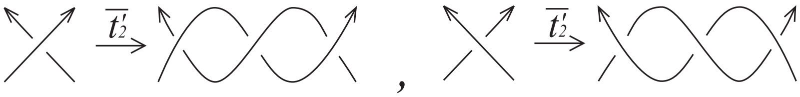

Fig. 6. $\overline{t_2'}$ move.

3. Generators for alternating knots of genus two

In 1992, W.W. Menasco and M.B. Thistlethwaite proved the statement, that reduced alternating diagrams of the same link must be transformable by flypes [11]. Here a *flype* is the local move of a diagram shown in Fig. 4. A *clasp* is a tangle of the form in the Fig. 5. We have four types of clasps: *positive parallel*, *positive reverse*, *negative parallel*, and *negative reverse* as shown in Fig. 5. A $\overline{t_2'}$ *move* [18] is a local operation on a diagram applied in a neighborhood of a crossing as shown in Fig. 6, which adds a reverse clasp with the same sign as the crossing.

The following lemma was proved by A. Stoimenow.

Lemma 3.1 ([18], see also [2], pp. 112–113). *Any alternating prime knot of genus two possesses a diagram which is obtained by* $\overline{t_2'}$ *moves and flypes from one of the diagram in* Fig. 7 *up to taking the mirror image.*

We denote by G_2 the set of these knot diagrams and their mirror images. Special alternating knots are drawn in the upper area divided by the broken line.

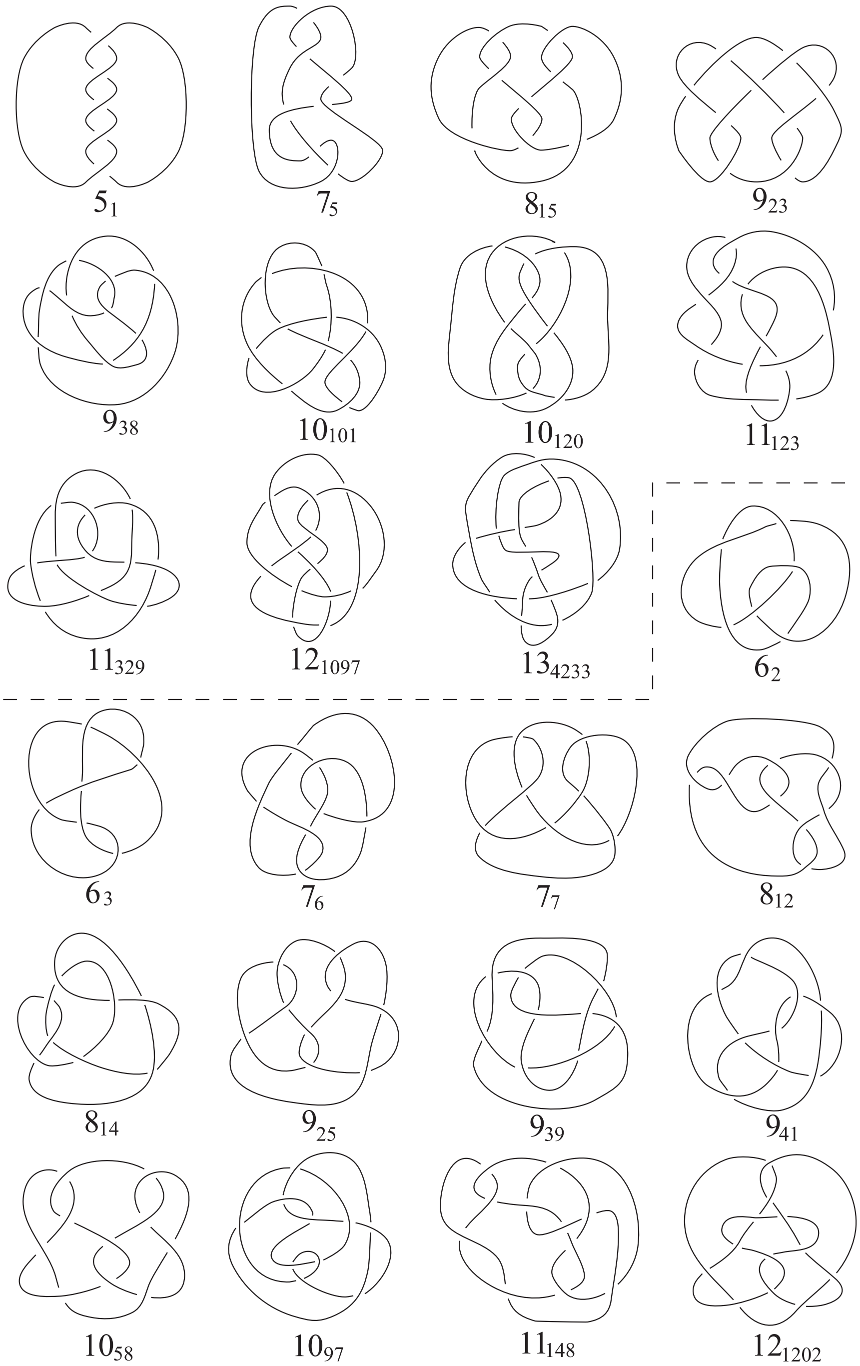
5_1
7_5
8_{15}
9_{23}
9_{38}
10_{101}
10_{120}
11_{123}
11_{329}
12_{1097}
13_{4233}
6_2
6_3
7_6
7_7
8_{12}
8_{14}
9_{25}
9_{39}
9_{41}
10_{58}
10_{97}
11_{148}
12_{1202}

Fig. 7.

4. The trapezoidal conjecture for alternating knots of genus two

In this section, we give the proof of Theorem 1.1. A. Stoimenow conjectured a restriction of the Alexander polynomials of alternating links [19]. We also confirm that Stoimenow's conjecture is true for alternating knots of genus ≤ 2. We begin with the following elementary lemma without a proof.

Lemma 4.1. *Let $f(t)$ and $g(t)$ be positive trapezoidal polynomials. Then we have the followings.*
(i) *The polynomial $f(t)g(t)$ is positive trapezoidal.*
(ii) *If* $\operatorname{maxdeg} f(t) = \operatorname{maxdeg} g(t)$ *and* $\operatorname{mindeg} f(t) = \operatorname{mindeg} g(t)$, *then the polynomial $f(t) + g(t)$ is positive trapezoidal.*

4.1. Proof of Theorem 1.1 for a knot of genus one and a composite knot of genus two.

First, we prove Theorem 1.1 for two easy cases.

4.1.1. For a knot of genus one. The Alexander polynomial of a genus one knot K is of the form $\Delta_K(t) = a_0 + a_1 t + a_0 t^2$ $(a_0, a_1 \in \mathbb{Z})$. By $\Delta_K(1) = \pm 1$, we have $a_1 = -2a_0 \pm 1$. Hence $\Delta_K(-t) = a_0 + (2a_0 \mp 1)t + a_0 t^2$ is always trapezoidal.

4.1.2. For a composite knot of genus two. Let $K_1 \sharp K_2$ be a composite knot of genus two. Notice that the genera of K_1 and K_2 are equal to one and $\Delta_{K_1 \sharp K_2} = \Delta_{K_1}\Delta_{K_2}$ holds. By Lemma 4.1, $\Delta_{K_1 \sharp K_2}(-t)$ is always trapezoidal.

4.2. Proof of Theorem 1.1 for a prime knot of genus two.

First, we discuss a relationship between the degree of the Alexander polynomial and a smoothing. A smoothing is a local operation on a link diagram applied in a neighborhood of a crossing as shown in Fig. 8.

A Seifert surface F is *flattened* if it lies in $\mathbb{R}^2$ except in small neighborhoods of the crossings where it is the surface show in Fig. 9. A diagram is *special* if its canonical Seifert surface is flattened. We choose a checkerboard coloring of a special diagram D such that the black regions coincide the regions obtained by the canonical Seifert surface for D. Then, for a flattened Seifert surface F and the special diagram $D = \partial F$, we have $\operatorname{rank} H_1(F; \mathbb{Z}) = \#\{\text{bounded white regions of } D\}$. Here $H_1(F; \mathbb{Z})$ is the first integral homology group of F.

The following lemma was proved by M. Hirasawa [7].

Lemma 4.2 ([7]). *Let D be a diagram of a link L, and F the canonical surface for D. Then F is isotopic to a flattened Seifert surface for L. In addition, we can take the isotopy which fixes neighborhoods of all crossings of D.*

The following lemma show a relationship between the degree of the Alexander polynomial and a smoothing.

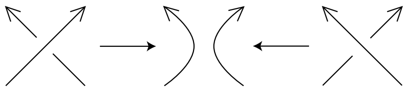

Fig. 8. Smoothing.

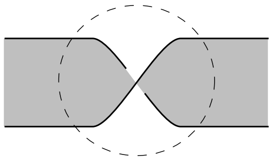

Fig. 9.

Lemma 4.3. *Let D be a reduced alternating diagram of a link L and c a crossing of D. Let D/c be the diagram obtained from D by smoothing the crossing c. Then we have*

$$\operatorname{span} \Delta_{D/c} = \operatorname{span} \Delta_D - 1.$$

Proof. Let F be the canonical Seifert surface for D. The canonical Seifert surface for an alternating diagram has the minimal genus of the link L [3], [12]. So we obtain

$$\begin{aligned} \operatorname{span} \Delta_D &= 2g(F) + \mu(L) - 1 \\ &= \operatorname{rank} H_1(F; \mathbb{Z}). \end{aligned}$$

Let $\tilde{F}$ be the flattened surface obtained by an isotopy φ which fixes the neighborhood of the crossing c (cf. Lemma 4.2). Then we have

$$\operatorname{rank} H_1(F; \mathbb{Z}) = \operatorname{rank} H_1(\tilde{F}; \mathbb{Z}).$$

Let $\tilde{D}$ be the special diagram determined by the surface $\tilde{F}$. Then we have

$$\operatorname{rank} H_1(\tilde{F}; \mathbb{Z}) = \#\{\text{bounded white regions of } \tilde{D}\}.$$

Let F' be the canonical Seifert surface for D/c. Set $\tilde{F}' = \varphi(F')$, and $\tilde{D}' = \partial \tilde{F}'$. By the same argument for F, we obtain

$$\operatorname{span} \Delta_{D/c} = \#\{\text{bounded white regions of } \widetilde{D/c}\}.$$

Since the isotopy φ fixes the neighborhood of c, the following diagram commutes.

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[d, "\text{smooth } c"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{D} \arrow[d, dotted, "\text{smooth } c"] \\ D/c \arrow[r, "\varphi"] & \widetilde{D/c} \end{tikzcd}$$

In a special diagram, a smoothing connect two white regions. Hence we have

$$\#\{\text{bounded white regions of } \tilde{D}\} - 1 = \#\{\text{bounded white regions of } \widetilde{D/c}\}.$$

Consequently, we obtain

$$\begin{aligned} \operatorname{span} \Delta_{D/c} &= \#\{\text{bounded white regions of } \widetilde{D/c}\} \\ &= \#\{\text{bounded white regions of } \tilde{D}\} - 1 \\ &= \operatorname{span} \Delta_D - 1. \end{aligned}$$

□

We prepare notations for $\overline{t'_2}$ moves and smoothings on a diagram. Let D be a link diagram and $c_1, c_2, \dots, c_m$ the crossings of D. We denote by $D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}$ the diagram obtained by applying k_i-times $\overline{t'_2}$ moves at c_i for $i = 1, 2, \dots, m$. We often omit the symbol $c_i^{k_i}$ if k_i is equal to zero. We denote by $D/c_{i_1} \cdots c_{i_l}$ the diagram obtained by smoothing at $c_{i_1}, \dots, c_{i_l}$.

By Lemma 3.1, for a proof of Theorem 1.1, it is sufficient to prove the following lemma.

Lemma 4.4. *Let D be a diagram in G_2 and $c_1, \dots, c_m$ the crossings of the diagram D. Then $\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}}(-t)$ is trapezoidal for any non-negative integers $k_1, \dots, k_m$.*

The following lemma is a key to prove Theorem 1.1, and then it provides an important fact to show our applications in Section 5.

Lemma 4.5. *Let L be a non-split alternating link, D a reduced alternating diagram of L, and c a crossing of the diagram D. Then we have the following formula*:

$$\Delta_{D^c}(-t) = \Delta_D(-t) + (1+t)\Delta_{D/c}(-t). \tag{3}$$

Proof. We can assume that the crossing c is positive since L^* possesses the same Alexander polynomial of the link L, where L^* means the mirror image of the link L. We denote by $c_t, c', c_1 \in V(D^c)$ the three vertices in shown in Fig. 10 and denote by $e_t, e', e_1 \in E(D^c)$ the three edges in shown in the figure. We take a vertex $c_t \in V(D^c)$

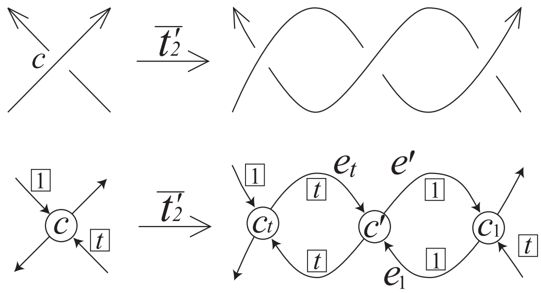

Fig. 10.

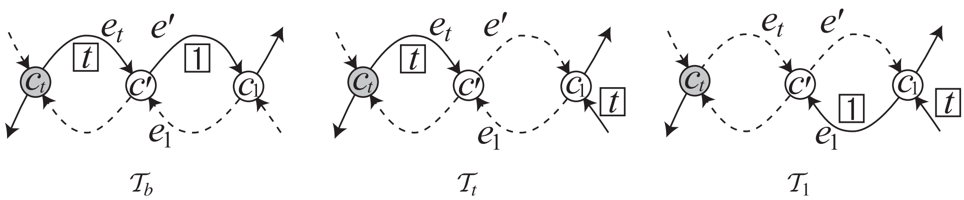

Fig. 11. The edges drawn by broken line are not contained.

as a root vertex and classify the maximal rooted trees in D^c into three types whether each of the edges e_t, e' and e_1 is contained or not (see Fig. 11):

$$\mathcal{T}_b = \{T \in \mathcal{T}_{(D^c;c_t)} \mid e_t, e' \in E(T)\},$$
$$\mathcal{T}_t = \{T \in \mathcal{T}_{(D^c;c_t)} \mid e_t \in E(T),\ e' \notin E(T)\},$$
$$\mathcal{T}_1 = \{T \in \mathcal{T}_{(D^c;c_t)} \mid e_t \notin E(T),\ e' \in E(T)\}.$$

Note that $\mathcal{T}(D^c; c_t) = \mathcal{T}_b \sqcup \mathcal{T}_t \sqcup \mathcal{T}_1$. We can regard the trees in $\mathcal{T}_b$ as the trees in $\mathcal{T}_{(D;c)}$, whose weights are multiplied by t. Hence we obtain

$$\sum_{T \in \mathcal{T}_b} W(T) = t P_{(D,c)}. \tag{4}$$

Next we consider the polynomials obtained from $\mathcal{T}_t$ and $\mathcal{T}_1$. We can regard the trees in $\mathcal{T}_t$ as the trees in $\mathcal{T}_{(D/c';c_t)}$, whose weights are multiplied by t (see Fig. 12). So we obtain

$$\sum_{T \in \mathcal{T}_t} W(T) = t P_{(D^c/c')}. \tag{5}$$

By the same argument, we obtain

$$\sum_{T \in \mathcal{T}_1} W(T) = P_{(D^c/c')} \tag{6}$$

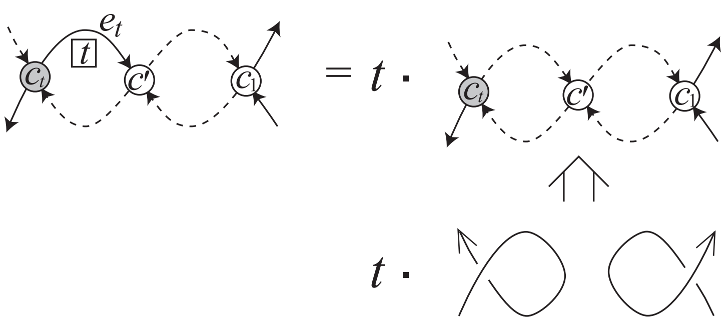

Fig. 12.

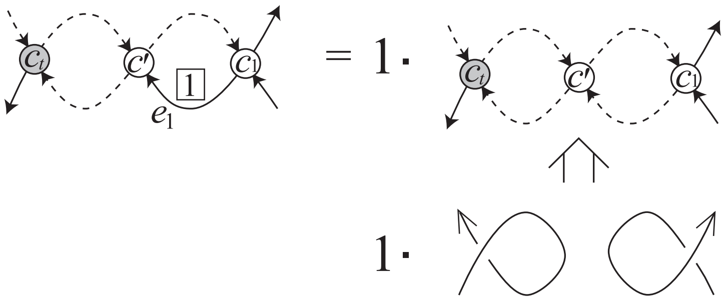

Fig. 13.

(see Fig. 13). By the equations (4), (5) and (6), we have

$$\begin{aligned}
P_{(D^c;c_t)} &= \sum_{T\in\mathcal{T}(D^c;c_t)} W(T) \\
&= \sum_{T\in\mathcal{T}_b} W(T) + \sum_{T\in\mathcal{T}_t} W(T) + \sum_{T\in\mathcal{T}_1} W(T) \\
&= tP_{(D;c)} + tP_{(D^c/c';c_t)} + P_{(D^c/c';c_t)} \\
&= tP_{(D;c)} + (1+t)P_{(D^c/c';c_t)}.
\end{aligned}$$

$\overline{t_2'}$ move preserves the genus of a canonical Seifert surface. So we obtain

$$\operatorname{span} P_{(D^c;c_t)} = \operatorname{span} tP_{(D;c)}.$$

Lemma 4.3 implies that

$$\operatorname{span} P_{(D^c/c';c_t)} = \operatorname{span}(1+t)P_{(D;c)}.$$

By Lemma 2.2, we have

$$\Delta_{D^c}(-t) \doteq P_{(D^c;c_t)},$$
$$\Delta_D(-t) \doteq P_{(D;c)},$$
$$\Delta_{D/c}(-t) \doteq P_{(D^c/c';c_t)}.$$

Consequently, we obtain the formula (3). □

We have the following corollary which is obtained from Lemma 4.5 immediately.

Corollary 4.6. *Let D be a non-split reduced alternating diagram, c a crossing of D, and $\Delta_D(-t)$ the normalized Alexander polynomial of D. Then $[\Delta_D(-t)]_i < [\Delta_{D^c}(-t)]_i$ for every $i = 0, 1, \ldots, \operatorname{maxdeg} \Delta_{(D;c)}$.*

Note that the formula (3) holds for a reducible alternating diagram since if c is reducible crossing, then we have $P_{(D^c/c';c_t)} = 0$.

Let $k = \sum_{i=0}^{m} k_i$. We start the proof of Lemma 4.4 by induction on k. If $k = 0$, that is, $k_1 = k_2 = \cdots = k_m = 0$, then we can confirm that $\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}}(-t)$ is trapezoidal for any $D \in G_2$:

$$\Delta_{5_1}(-t) = 1 + t + t^2 + t^3 + t^4,$$
$$\Delta_{6_2}(-t) = 1 + 3t + 3t^2 + 3t^3 + t^4,$$
$$\Delta_{6_3}(-t) = 1 + 3t + 5t^2 + 3t^3 + t^4,$$
$$\Delta_{7_5}(-t) = 2 + 4t + 5t^2 + 4t^3 + 2t^4,$$
$$\Delta_{7_6}(-t) = 1 + 5t + 7t^2 + 5t^3 + t^4,$$
$$\Delta_{7_7}(-t) = 1 + 5t + 9t^2 + 5t^3 + t^4,$$
$$\Delta_{8_{12}}(-t) = 1 + 7t + 13t^2 + 7t^3 + t^4,$$
$$\Delta_{8_{14}}(-t) = 2 + 8t + 11t^2 + 8t^3 + 2t^4,$$
$$\Delta_{8_{15}}(-t) = 3 + 8t + 11t^2 + 8t^3 + 3t^4,$$
$$\Delta_{9_{23}}(-t) = 4 + 11t + 15t^2 + 11t^3 + 4t^4,$$
$$\Delta_{9_{25}}(-t) = 3 + 12t + 17t^2 + 12t^3 + 3t^4,$$
$$\Delta_{9_{38}}(-t) = 5 + 14t + 19t^2 + 14t^3 + 5t^4,$$
$$\Delta_{9_{39}}(-t) = 3 + 14t + 21t^2 + 14t^3 + 3t^4,$$
$$\Delta_{9_{41}}(-t) = 3 + 12t + 19t^2 + 12t^3 + 3t^4,$$
$$\Delta_{10_{58}}(-t) = 3 + 16t + 27t^2 + 16t^3 + 3t^4,$$
$$\Delta_{10_{97}}(-t) = 5 + 22t + 33t^2 + 22t^3 + 5t^4,$$

$$\Delta_{10_{101}}(-t) = 7 + 21t + 29t^2 + 21t^3 + 7t^4,$$
$$\Delta_{10_{120}}(-t) = 8 + 26t + 37t^2 + 26t^3 + 8t^4,$$
$$\Delta_{11_{123}}(-t) = 9 + 29t + 41t^2 + 29t^3 + 9t^4,$$
$$\Delta_{11_{148}}(-t) = 7 + 29t + 43t^2 + 29t^3 + 7t^4,$$
$$\Delta_{11_{329}}(-t) = 11 + 36t + 51t^2 + 36t^3 + 11t^4,$$
$$\Delta_{12_{1097}}(-t) = 16 + 54t + 77t^2 + 54t^3 + 16t^4,$$
$$\Delta_{12_{1202}}(-t) = 9 + 42t + 67t^2 + 42t^3 + 9t^4,$$
$$\Delta_{13_{4233}}(-t) = 21 + 74t + 107t^2 + 74t^3 + 21t^4.$$

Assuming that the claim is true for polynomials with $k < n$, we prove it for polynomials with $k = n$.

By Lemma 4.5, for an integer j such that $k_j \neq 0$,

$$\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}}(-t) = \Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_j^{k_j - 1} \cdots c_m^{k_m}}}(-t) + (1 + t)\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}/c_j}(-t).$$

By the assumption, the polynomial $\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_j^{k_j - 1} \cdots c_m^{k_m}}}(-t)$ is trapezoidal. By Lemma 4.3, we have

$$\deg \Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}/c_j}(-t) = 3.$$

Notice that $\deg \Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_j^{k_j - 1} \cdots c_m^{k_m}}}(-t) = 4$, $\deg(1 + t) = 1$, and the polynomial $1 + t$ is trapezoidal. By Lemma 4.1, it is sufficient to complete the proof that the polynomial $\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}/c_j}(-t)$ is trapezoidal.

An integer polynomial $f(t) = a_0 + a_1 t + a_1 t^2 + a_0 t^3$ is trapezoidal if and only if a_0 and a_1 have the same sign and $0 < |a_0| \leq |a_1|$. The following lemma completes the proof of Lemma 4.4 for the eleven special alternating diagrams in G_2.

Lemma 4.7 ([16]). *Let $\Delta_L(-t)$ be the normalized Alexander polynomial of a non-trivial, non-split special alternating link L. Then $0 < [\Delta_L(-t)]_0 \leq [\Delta_L(-t)]_1$.*

It remains that we show the polynomial $\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}/c_j}(-t)$ is trapezoidal for thirteen non-special diagrams in G_2.

For every crossing c of the diagram D in G_2, the Alexander polynomial of D is of the form $\Delta_{D/c}(t) = a_0 - a_1 t + a_1 t^2 - a_0 t^3$ $(a_0, a_1 \in \mathbb{Z})$.

Here, we consider another normalization of $\Delta_{D/c}(t)$ via the Conway polynomial $\nabla(z) \in \mathbb{Z}[z]$: $t^{-3/2}\Delta_{D/c}(t) = \nabla_{D/c}(t^{-1/2} - t^{1/2}) = \nabla_{D/c}(z)|_{z=t^{-1/2}-t^{1/2}}$.

Then we have

$$a_1 - a_0 = 2a_0 - \mathrm{lk}(D/c) \tag{7}$$

since $(d/dt)\Delta_{D/c}(t)|_{t=1} = -\operatorname{lk}(D/c)$ (see [9], pp. 83), where $\operatorname{lk}(D/c)$ is the linking number of D/c.

The coefficients a_0 and a_1 are nonzero and have the same sign since the two component link diagram D/c is alternating. We can easily confirm that $|a_0| \leq |a_1|$ for each crossing of every non-special generator. By applying single $\overline{t_2'}$ move, the absolute value of the linking number is added no more than one and $|a_0|$ is added at least one. Considering the equation (7), this fact guarantees that $\overline{t_2'}$ moves preserve the inequality $|a_0| \leq |a_1|$. Consequently, we have the polynomial $\Delta_{D^{c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m}}/c_j}(-t)$ is trapezoidal.

Now we have completed the proof of Lemma 4.4.

4.3. Weakly Newton-like polynomial. A polynomial $f(t)$ is *weakly Newton-like* if $[f(t)]_{j-1}[f(t)]_{j+1} \leq [f(t)]_j{}^2$ for $j = \operatorname{mindeg} f + 1, \ldots, \operatorname{maxdeg} f - 1$. A. Stoimenow proposed the following conjecture.

Conjecture 4.8 ([19]). *The Alexander polynomial of an alternating link is weakly Newton-like.*

Note that if the Alexander polynomial of an alternating link $\Delta(t)$ is weakly Newton-like, then the polynomial $\Delta(-t)$ satisfies the condition (iv) in the trapezoidal property. In this sense, Stoimenow's conjecture is a natural strengthening of the trapezoidal conjecture. We confirm that Conjecture 4.8 is true for alternating knots of genus two. Let K be an alternating knot of genus two. Then the normalized Alexander polynomial of K is of the form $\Delta_K(t) = a_0 - a_1 t + a_2 t^2 - a_1 t^3 + a_0 t^4$ $(a_0, a_1, a_2 \in \mathbb{N})$. By $\Delta(1) = \pm 1$, we have $a_2 - 2(a_1 - a_0) = \pm 1$. Then we obtain

$$\begin{aligned} a_1{}^2 - a_0 a_2 &= a_1{}^2 - a_0(2(a_1 - a_0) \pm 1) \\ &= a_1{}^2 - 2a_0 a_1 + 2a_0{}^2 \mp a_0 \\ &= (a_1 - a_0)^2 + a_0(a_0 \mp 1) \\ &\geq 0. \end{aligned}$$

The inequality $a_1^2 \leq a_2^2$ is a consequence of Theorem 1.1.

5. Applications

In this section, we give the two complete lists of the alternating knots of genus two with $[\Delta(t)]_0 = 1$, and with $[\Delta(t)]_0 = 2$.

5.1. Alternating fibered knots of genus two. The knots in G_2 which possess monic Alexander polynomials are just 5_1, 5_1^*, 6_2, 6_2^*, 6_3, 7_6, 7_6^*, 7_7, 7_7^*, and 8_{12}. Note that the composite knot $K_1 \sharp K_2$ is alternating if and only if K_1 and K_2 are alternating

[10]. Composite alternating knots of genus two which possess monic Alexander polynomials are only $3_1 \sharp 3_1$, $3_1 \sharp 3_1^*$, $3_1^* \sharp 3_1^*$, $3_1 \sharp 4_1$, $3_1^* \sharp 4_1$, and $4_1 \sharp 4_1$. By this fact and Corollary 4.6, we obtain the complete list of the alternating knots of genus two with $[\Delta(t)]_0 = 1$.

Theorem 5.1. *The alternating knots of genus two with* $[\Delta(t)]_0 = 1$ *are just* 5_1, 5_1^*, 6_2, 6_2^*, 6_3, 7_6, 7_6^*, 7_7, 7_7^*, 8_{12}, $3_1 \sharp 3_1$, $3_1 \sharp 3_1^*$, $3_1^* \sharp 3_1^*$, $3_1 \sharp 4_1$, $3_1^* \sharp 4_1$, *and* $4_1 \sharp 4_1$.

The normalized Alexander polynomials of these composite knots are as follows:

$$\begin{aligned}
\Delta_{3_1 \sharp 3_1}(t) &= 1 - 2t + 3t^2 - 2t^3 + t^4, \\
\Delta_{3_1 \sharp 4_1}(t) &= 1 - 4t + 5t^2 - 4t^3 + t^4, \\
\Delta_{4_1 \sharp 4_1}(t) &= 1 - 6t + 11t^2 - 6t^3 + t^4.
\end{aligned}$$

Therefore we obtain the following corollary.

Corollary 5.2. *The normalized Alexander polynomials which satisfy the trapezoidal property*

$$\begin{aligned}
&1 - n_1 t + (2n_1 - 1)t^2 - n_1 t^3 + t^4 \quad \textit{for} \quad n_1 = 4 \textit{ or } n_1 \geq 8, \\
&1 - n_2 t + (2n_2 - 3)t^2 - n_2 t^3 + t^4 \quad \textit{for} \quad n_2 \geq 6
\end{aligned}$$

are never realized by those of an alternating knot.

The following example shows that the trapezoidal property and Ozsváth-Szabó's inequality (1) are not enough to characterize the Alexander polynomials of alternating knots.

EXAMPLE 5.3. The polynomial $\Delta(t) = 1-4t+7t^2-4t^3+t^4$ is the Alexander polynomial of a knot which has the trapezoidal property. The solutions of the equation $1-4t+7t^2-4t^3+t^4 = 0$ are $t = \left(2\sqrt{2}+\sqrt{\sqrt{17}-1}\right)/(2\sqrt{2})+\sqrt{-1}\left(\left(\sqrt{2}+\sqrt{\sqrt{17}+1}\right)/(2\sqrt{2})\right)$, $\left(2\sqrt{2}-\sqrt{\sqrt{17}-1}\right)/(2\sqrt{2})+\sqrt{-1}\left(\left(\sqrt{2}-\sqrt{\sqrt{17}+1}\right)/(2\sqrt{2})\right)$, and their conjugates. We denote them by α_1, α_2, $\overline{\alpha_1}$, and $\overline{\alpha_2}$ respectively. For a knot K, the number of zeros of $\Delta_K(t)$ in $\{z \in \mathbb{C} \setminus \mathbb{R} \mid |z| = 1\}$, counted with multiplicity, is greater than or equal to $|\sigma(K)|$ (see [8], pp. 161–162). Notice that $|\alpha_1| > 1$ and $|\alpha_2| < 1$. Hence the signature of any knot which possesses this Alexander polynomial is equal to zero. The polynomial $\Delta(t) = 1 - 4t + 7t^2 - 4t^3 + t^4$ satisfies Ozsváth-Szabó's inequality (1). However, this polynomial is never realized by an alternating knot. Incidentally, the non-alternating knot 9_{44} possesses this polynomial.

An alternating link is fibered if and only if the Alexander polynomial of the link is monic [14]. By Theorem 5.2, we obtain the following corollary.

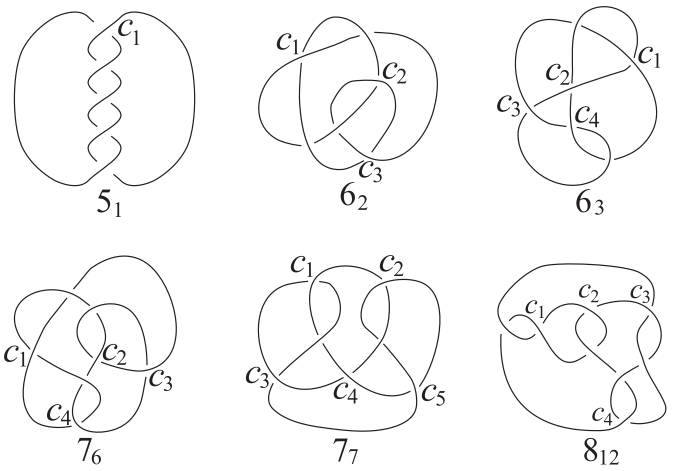

Fig. 14.

Corollary 5.4. *The alternating fibered knots of genus two are just* 5_1, 5_1^*, 6_2, 6_2^*, 6_3, 7_6, 7_6^*, 7_7, 7_7^*, 8_{12}, $3_1 \sharp 3_1$, $3_1 \sharp 3_1^*$, $3_1^* \sharp 3_1^*$, $3_1 \sharp 4_1$, $3_1^* \sharp 4_1$ *and* $4_1 \sharp 4_1$.

We denote the set of the alternating fibered knots of genus two by AF_2.

K. Murasugi showed that the alternating prime knots of genus two, whose Alexander polynomials are monic admit only the following knots: 5_1, 6_2, 6_3, 7_6, 7_7, 8_{12} in [13] (see also the review in AMS MathSciNet mathematical reviews on the Web written by R.H. Fox [4]). Our argument gives an alternative proof for this claim.

5.2. Alternating knots of genus two with $[\Delta(t)]_0 = 2$. Next we discuss the normalized Alexander polynomials of alternating knots of genus two with $[\Delta(t)]_0 = 2$. The knots in G_2 which possess the normalized Alexander polynomial with $[\Delta(t)]_0 = 2$ are just 7_5, 7_5^*, 8_{14}, and 8_{14}^*.

We name the crossings of the diagram in AF_2 as shown in Fig. 14. Then, by Corollary 4.6, we obtain each of the other alternating knots of genus two with $[\Delta]_0 = 2$ by applying once $\overline{t_2'}$ move at a crossing of a diagram in AF_2 as follows: $5_1^{c_1} = 7_3$, $6_2^{c_1} = 8_{11}$, $6_2^{c_2} = 8_4$, $6_2^{c_3} = 8_6$, $6_3^{c_1} = 8_{13}$, $6_3^{c_2} = 8_8$, $6_3^{c_3} = 8_8$, $6_3^{c_4} = 8_{13}$, $7_6^{c_1} = 9_8$, $7_6^{c_2} = 9_{21}$, $7_6^{c_3} = 9_{15}$, $7_6^{c_4} = 9_{12}$, $7_7^{c_1} = 9_{14}$, $7_7^{c_2} = 9_{14}$, $7_7^{c_3} = 9_{19}$, $7_7^{c_4} = 9_{37}$, $7_7^{c_5} = 9_{19}$, $8_{12}^{c_1} = 10_{35}$, $8_{12}^{c_2} = 10_{13}$, $8_{12}^{c_3} = 10_{35}$, $8_{12}^{c_4} = 10_{13}$.

Then we obtain the following theorem.

Theorem 5.5. *The alternating prime knots of genus two with* $a_0 = 2$ *are just the following knots*: 7_3, 7_5, 8_4, 8_6, 8_8, 8_{11}, 8_{13}, 8_{14}, 9_8, 9_{12}, 9_{14}, 9_{15}, 9_{19}, 9_{21}, 9_{37}, 10_{13}, 10_{35}, *and their mirror images. The alternating composite knots of genus two with* $a_0 = 2$ *are just the following knots*: $3_1 \sharp 5_2$, $3_1^* \sharp 5_2$, $3_1 \sharp 5_2^*$, $3_1^* \sharp 5_2^*$, $3_1 \sharp 6_1$,

$3_1^* \sharp 6_1,\ 3_1 \sharp 6_1^*,\ 3_1^* \sharp 6_1^*,\ 4_1 \sharp 5_2,\ 4_1 \sharp 5_2^*,\ 4_1 \sharp 6_1,$ *and* $4_1 \sharp 6_1^*$.

The Alexander polynomials of these knots are given below:

$$\begin{aligned}
\Delta_{7_3}(t) &= 2 - 3t + 3t^2 - 3t^3 + 2t^4,\\
\Delta_{7_5}(t) &= 2 - 4t + 5t^2 - 4t^3 + 2t^4,\\
\Delta_{8_4}(t) &= 2 - 5t + 5t^2 - 5t^3 + 2t^4,\\
\Delta_{8_6}(t) &= 2 - 6t + 7t^2 - 6t^3 + 2t^4,\\
\Delta_{8_8}(t) &= 2 - 6t + 9t^2 - 6t^3 + 2t^4,\\
\Delta_{8_{11}}(t) &= 2 - 7t + 9t^2 - 7t^3 + 2t^4,\\
\Delta_{8_{13}}(t) &= 2 - 7t + 11t^2 - 7t^3 + 2t^4,\\
\Delta_{8_{14}}(t) &= 2 - 8t + 11t^2 - 8t^3 + 2t^4,\\
\Delta_{9_8}(t) &= 2 - 8t + 11t^2 - 8t^3 + 2t^4,\\
\Delta_{9_{12}}(t) &= 2 - 9t + 13t^2 - 9t^3 + 2t^4,\\
\Delta_{9_{14}}(t) &= 2 - 9t + 15t^2 - 9t^3 + 2t^4,\\
\Delta_{9_{15}}(t) &= 2 - 10t + 15t^2 - 10t^3 + 2t^4,\\
\Delta_{9_{19}}(t) &= 2 - 10t + 17t^2 - 10t^3 + 2t^4,\\
\Delta_{9_{21}}(t) &= 2 - 11t + 17t^2 - 11t^3 + 2t^4,\\
\Delta_{9_{37}}(t) &= 2 - 11t + 19t^2 - 11t^3 + 2t^4,\\
\Delta_{10_{13}}(t) &= 2 - 13t + 23t^2 - 13t^3 + 2t^4,\\
\Delta_{10_{35}}(t) &= 2 - 12t + 21t^2 - 12t^3 + 2t^4,\\
\Delta_{3_1 \sharp 5_2}(t) &= 2 - 5t + 7t^2 - 5t^3 + 2t^4,\\
\Delta_{3_1 \sharp 6_1}(t) &= 2 - 7t + 9t^2 - 7t^3 + 2t^4,\\
\Delta_{4_1 \sharp 5_2}(t) &= 2 - 9t + 13t^2 - 9t^3 + 2t^4,\\
\Delta_{4_1 \sharp 6_1}(t) &= 2 - 11t + 19t^2 - 11t^3 + 2t^4.
\end{aligned}$$

Then we obtain the following corollary.

Corollary 5.6. *The Alexander polynomials which satisfy the trapezoidal property*

$$\begin{aligned}
&2 - m_1 t + (2m_1 - 3)t^2 - m_1 t^3 + 2t^4 \quad \textit{for} \quad m_1 = 8 \textit{ or } m_1 \geq 14,\\
&2 - m_2 t + (2m_2 - 5)t^2 - m_2 t^3 + 2t^4 \quad \textit{for} \quad m_2 \geq 12
\end{aligned}$$

are never realized by those of an alternating knot.

EXAMPLE 5.7. The polynomial $\Delta(t) = 2 - 8t + 13t^2 - 8t^3 + 2t^4$ is the Alexander polynomial of a knot which has the trapezoidal property. The solutions of the equation $2-8t+13t^2-8t^3+2t^4 = 0$ are $t = \left(4+\sqrt{\sqrt{33}-1}\right)/4 + \sqrt{-1}\left(\left(\sqrt{2}+\sqrt{\sqrt{33}+1}\right)/4\right)$, $\left(4-\sqrt{\sqrt{33}-1}\right)/4 + \sqrt{-1}\left(\left(\sqrt{2}-\sqrt{\sqrt{33}+1}\right)/4\right)$, and its conjugates. By the same argument as Example 5.3, the signature of a knot which possesses this Alexander polynomial is equal to zero. This polynomial satisfies Ozsváth-Szabó's inequality (1). However, this polynomial is never realized by an alternating knot. Incidentally, the non-alternating knot 10_{146} possesses this polynomial.

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