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THE BOUNDARY OF THE MILNOR FIBER FOR SOME NON-ISOLATED SINGULARITIES OF COMPLEX SURFACES

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Abstract

We study the boundary L_t of the Milnor fiber for the non-isolated singularities in $\mathbf{C}^3$ with equation $z^m - g(x, y) = 0$ where $m \geq 2$ and $g(x, y) = 0$ is a non-reduced plane curve germ. We give a complete proof that L_t is a Waldhausen graph manifold and we provide the tools to construct its plumbing graph. As an example, we give the plumbing graph associated to the germs $z^2 - (x^2 - y^3)y^l = 0$ with l odd and $l \geq 3$. We prove that the boundary of the Milnor fiber is a Waldhausen manifold new in complex geometry, as it cannot be the boundary of a normal surface singularity.

1. Introduction

In [16] the authors state with a sketch of proof that the boundary L_t of the Milnor fiber of a non-isolated surface singularity in $\mathbf{C}^3$ is a Waldhausen graph manifold (non-necessarily "reduziert"). These manifolds are conveniently described by a plumbing graph. The present paper is devoted to the study of germs with equation $z^m - g(x, y) = 0$ where $m \geq 2$ and $g(x, y) = 0$ is a non-reduced plane curve germ. For them:
1) We prove in details that L_t is indeed a Waldhausen manifold (Section 4). The Waldhausen decomposition for L_t is obtained by gluing two specific Waldhausen submanifolds along boundary tori: the trunk and the (non-necessarily connected) vanishing zone.
2) We prove that the vanishing zone is in fact a Seifert manifold and we elucidate its structure (Section 5).
3) We show how to obtain the trunk (Section 2) and how to determine the gluing between the two submanifolds (Section 4).

In particular, we elucidate in this paper, for singularities with equations $z^m - g(x, y) = 0$, the following points which are not treated in [16]:
a) We prove that the vanishing zone is a Seifert manifold. As stated in the erratum [17], the vanishing zone is in general a Waldhausen manifold but non-necessarily a Seifert one.

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b) The explicit description of the vanishing zone given below in Section 5 enables us to give an explicit description of the plumbing graph of L_t, whereas the Waldhausen structure was not explicitly described in [16]. We need this explicit description to obtain the following new results:

In Section 6, we expound when L_t is a lens space for the germs under consideration in this paper. The reason why lens spaces come up is explained at the end of Section 2.

In Section 7, the plumbing graph is given for the singularities $z^2 - (x^2 - y^3)y^l = 0$ with l odd and $l \geq 3$. In [15], D. Massey computes the homology of the Milnor fiber F_t for these examples, but he doesn't study the topology of its boundary L_t. Here, we prove that the boundaries of their Milnor fibers are Waldhausen manifolds new in complex geometry, as they cannot be the boundary of a normal surface singularity.

Information about the homology of L_t is given in Section 8. In [18] we determine the plumbing graph for the boundary of the Milnor fiber of Hirzebruch singularities $z^m - x^k y^l = 0$. Here we obtain the following result.

Theorem 8.1. *Let* $f(x, y, z) = z^m - x^k y^l = 0$ *be the equation of a Hirzebruch singularity. Assume that* $\gcd(m, k, l) = 1$, *that* $1 \leq k < l$ *and that* $m \geq 2$. *Let* $d = \gcd(k, l)$ *and write* $\bar{k} = k/d$ *and* $\bar{l} = l/d$. *Then* $H_1(L_t, \mathbf{Z})$ *is isomorphic to the direct sum of a free abelian group of rank* $2(m-1)(d-1)$ *and a torsion group. The torsion subgroup is the direct sum of* $(m-1)$ *cyclic factors. One of them is of order* $m\bar{k}\bar{l}$ *and the other* $(m-2)$ *factors are of order* $\bar{k}\bar{l}$.

Necessary results about Seifert and Waldhausen manifolds are recalled in Section 3. The dictionary which translates Waldhausen decompositions into plumbing graphs provided by [22] can then be used to obtain the canonical plumbing graph for L_t.

There are two questions which need clarification in our use of 3-manifold theory: irreducibility and normal forms for plumbing graphs. Since the relevant statements are somewhat scattered through the literature, we group them in an appendix (Section 9).

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2. Definitions and main results

We consider germs $f(x, y, z) \in \mathbf{C}\{x, y, z\}$ such that $f(0, 0, 0) = 0$. We deal with germs f such that the dimension of the singular locus $\Sigma(f)$ is equal to 1. Hence f is reduced.

We denote by B_r^{2n} the $2n$-ball with radius $r > 0$ centered at the origin of $\mathbf{C}^n$ and by S_r^{2n-1} the boundary of B_r^{2n}. We set $F_0 = B_\epsilon^6 \cap f^{-1}(0)$ and $L_0 = S_\epsilon^5 \cap f^{-1}(0)$. According to the theory of Milnor [19], extended by Burghelea and Verona [3] in the non-isolated case, the homeomorphism classes of the pairs (B_ϵ^6, F_0) and (S_ϵ^5, L_0) do not depend on $\epsilon > 0$ if it is sufficiently small. As a consequence, we shall usually remove "ϵ" from our notations.

The restriction $f|B^6_\epsilon \cap f^{-1}(B^2_\eta - \{0\}) \to (B^2_\eta - \{0\})$ is a locally trivial differentiable fibration whose isomorphism class does not depend on $\eta > 0$ provided that η is sufficiently small ($0 < \eta \ll \epsilon$). See Milnor [19] and also Hamm-Lê [6]. Therefore, the diffeomorphism classes of the manifolds $F_t = B^6_\epsilon \cap f^{-1}(t)$ and $L_t = S^5_\epsilon \cap f^{-1}(t)$ do not depend on t if $0 < |t| \leq \eta$. We say that F_t is the Milnor fiber of f and that L_t is the boundary of the Milnor fiber. F_t is oriented by its complex structure and L_t is oriented as the boundary of F_t.

We denote by $n\colon \tilde{F}_0 \to F_0$ the normalisation. It follows from the arguments in Durfee [4] that the boundary $\tilde{L}_0$ of an algebraic neighbourhood of $n^{-1}(0)$ is well defined. We shall call $\tilde{L}_0$ the boundary of the normalisation.

The strategy used to obtain the boundary of the Milnor fiber for non-isolated singularities is the following. Let $\Sigma(f)$ be the singular locus of f. By hypothesis $\Sigma(f)$ is a curve. Let $K_0 = L_0 \cap \Sigma(f)$ be the link of the singular locus in L_0. Let $\tilde{K}_0 = n^{-1}(K_0)$ be the pull-back of K_0 in $\tilde{L}_0$. A good resolution of $\tilde{F}_0$ provides a Waldhausen decomposition for $\tilde{L}_0$ as a union of Seifert manifolds such that $\tilde{K}_0$ is a union of Seifert leaves. Let $\tilde{M}_0$ be a tubular neighbourhood of $\tilde{K}_0$ in $\tilde{L}_0$. The closure $\tilde{N}_0$ of $(\tilde{L}_0 - \tilde{M}_0)$ is called the *trunk* of L_t.

From now on, let us suppose that f is of the form $f(x, y, z) = z^m - g(x, y)$. In 4.6 we define a submanifold M_t of L_t called the *vanishing zone* around K_0. A slightly less general version of Theorem 4.7 can be easily stated as follows.

Theorem. (1) *The closure N_t of $L_t \setminus M_t$ is homeomorphic to the trunk $\tilde{N}_0$.*
(2) *The manifold M_t is a Seifert manifold.*

The construction (see 4.6) of the vanishing zone is so precise that it gives rise to a very explicit description of M_t. To each irreducible component σ_i of the singular locus of f corresponds a connected component $M_t(i)$ of M_t. A hyperplane section argument provides a plane curve germ $(z^m - y^{n_i})$ and an integer k. Let $d = \gcd(n_i, k)$. In Section 5 we prove the following result.

Theorem 5.4. *The vanishing zone $M_t(i)$ is the mapping torus of a diffeomorphism $h\colon G_t \to G_t$ such that*:
(1) *G_t is diffeomorphic to the Milnor fiber of the plane curve germ $z^m - y^{n_i}$.*
(2) *The diffeomorphism h is finite of order n_i/d.*
(3) *If $d < n_i$, the diffeomorphism h has exactly m fixed points and the action of h has order n_i/d on all other points.*
(4) *Around a fixed point h is a rotation of angle $-2\pi k/n_i$.*

Following the terminology introduced by D. Siersma in [26], we call the above diffeomorphism h the *vertical monodromy* for σ_i.

REMARK. We prove the above theorem without assuming that f is irreducible. The number of connected components of $\tilde{F}_0$ and $\tilde{L}_0$ is equal to the number of irreducible components of f. The intersection between two irreducible components of $f = 0$ furnishes at least one irreducible component of the singular locus $\Sigma(f)$ and a corresponding connected component of the vanishing zone. Hence, the constructions given here show that after the gluing of all connected components of the vanishing zone with the trunk, we obtain a connected manifold L_t. This implies that the Milnor fiber F_t is connected. As the singular locus of f has dimension 1, F_t is connected by a much more general result of M. Kato and Y. Matsumoto in [12].

It is stated in [16] that L_t is never homeomorphic to $\tilde{L}_0$. But the particular case when both the trunk is a solid torus and L_t is a lens space is rather delicate. Indeed, when the trunk is a solid torus the complexity of the Waldhausen manifolds, defined in [16] p. 2309, could vanish. In [17], p. 310, we state (without proof) that when both the trunk is a solid torus and L_t is a lens space, then f is analytically equivalent to $z^2 - xy^l$ for some $l \geq 2$. This statement corrects the point (2) p. 2310, in [16], and completes the proof that L_t is never homeomorphic to $\tilde{L}_0$. To produce in a forthcoming paper a complete proof that L_t is never homeomorphic to $\tilde{L}_0$, the first two authors need a characterization of the germs $z^m - g(x, y)$ for which L_t is a lens space. Theorem 6.3 solves the problem.

Theorem 6.3. *The boundary of the Milnor fiber of an irreducible germ $f(x,y,z) = z^m - g(x, y)$, where $m \geq 2$ and $g(x, y) = 0$ is non-reduced, is a lens space if and only if f is analytically equivalent to $z^2 - xy^l$ for some $l \geq 2$.*

REMARK. For our purpose lens spaces are defined as graph manifolds obtained from a plumbing graph which is a "bamboo" with genus zero vertices.

For technical reasons, we use in this paper a polydisc

$$B(\alpha) = B^2_\alpha \times B^2_\beta \times B^2_\gamma = \{(x, y, z) \in B^6_\epsilon,\ |x| \leq \alpha,\ |y| \leq \beta,\ |z| \leq \gamma\}$$

where $0 < \alpha < \beta < \gamma < \epsilon/3$ in place of a standard ball B^6_ϵ.

DEFINITION. The polydisc $B(\alpha)$ is a Milnor polydisc for f if:

i) For each α' with $0 < \alpha' \leq \alpha$ the pair $(B(\alpha'),\ f^{-1}(0) \cap B(\alpha'))$ is homeomorphic to the pair $(B^6_\epsilon,\ f^{-1}(0) \cap B^6_\epsilon)$.

ii) For each α' with $0 < \alpha' \leq \alpha$ there exists η with $0 < \eta \ll \alpha'$ such that:

1) the restriction of f to $W(\alpha', \eta) = B(\alpha') \cap f^{-1}(B^2_\eta - \{0\})$ is a locally trivial differentiable fibration on $(B^2_\eta - \{0\})$,

2) this fibration does not depend on α' (when $0 < \alpha' \leq \alpha$) up to isomorphism.

3. Three-dimensional manifolds

In this section, we recall some facts pertaining to 3-dimensional manifolds in a setting appropriate to our needs.

We consider differentiable, compact (usually connected) 3-manifolds M possibly with boundary. When the boundary ∂M is non-empty, we assume that it is a disjoint union of tori. Manifolds are oriented. Classifications are done up to orientation preserving diffeomorphism. In the situations we meet, M is quite often the boundary of a complex surface V. The complex structure gives rise to an orientation of V and $M = \partial V$ receives an orientation via the boundary homomorphism $\partial\colon H_4(V \bmod \partial V; \mathbf{Z}) \to H_3(\partial V; \mathbf{Z})$.

3.1. Seifert foliations. In this paper, we only need to consider orientable Seifert fibrations (to be called Seifert foliations, since we have too many fibrations present). As our manifolds are oriented and compact, we may define a Seifert foliation on M as an orientable foliation by circles. Thanks to a theorem of Epstein [5], this is equivalent to requiring that there exists a fixed point free S^1-action on M such that the leaves coincide with the orbits.

An exceptional orbit (leaf) is one such that the isotropy subgroup is non-trivial. It is a finite cyclic subgroup of order $\alpha \geq 2$. The slice theorem (and orientability of M) imply that for each exceptional leaf there exist:

i) a tubular neighbourhood which is a union of leaves,

ii) an orientation preserving diffeomorphism of this neighbourhood with the mapping torus of a rotation of order α on an oriented 2-disc, sending leaves to leaves.

A Seifert invariant for an exceptional leaf is defined as follows. Suppose that the rotation angle on the 2-disc is equal to $2\pi\beta^*/\alpha$. We need the orientation of the 2-disc to get the correct sign for the angle. We have $\gcd(\alpha, \beta^*) = 1$ and we choose β^* such that $0 < \beta^* < \alpha$. Now let β be any integer such that $\beta\beta^* \equiv 1 \pmod{\alpha}$. The pair (α, β) is a Seifert invariant of the exceptional leaf. See [20] pp. 135–140. The choice of a β in its residue class (mod α) is related to the choice of a section of the foliation near the exceptional leaf. The Seifert invariant (α, β) is called *normalised* if a section is chosen in such a way that $0 < \beta < \alpha$.

Let $r \geq 0$ be the number of boundary components of M. The space of leaves is a compact connected orientable surface of genus $g \geq 0$ with r boundary components.

Suppose now that sections of the foliation are chosen on each boundary component of M and that they are kept fixed during the following discussion. We then choose a β for each exceptional leaf. Once these choices have been made, the Euler number $e \in \mathbf{Z}$ is defined. See [20] for details. Essentially it is the obstruction to extend the section already defined on some part of the orbit space. The integer e depends on the choice of the β's, but the rational number $e_0 = e - \sum \beta_i/\alpha_i$ does not. Of course, if $r > 0$ the numbers e and e_0 still depend on the choice of a section on the boundary of M.

3.2. Waldhausen manifolds and plumbing graphs. The manifolds $\tilde{L}_0$ and L_t we study in this paper are graph manifolds in Waldhausen's sense [27]. They will appear in the following dress.

A Seifert manifold is a 3-dimensional compact oriented manifold given with a Seifert foliation.

A finite decomposition $M = \bigcup M_i$ of a 3-manifold M is Waldhausen if:
(1) Each M_i is a Seifert manifold.
(2) If $i \neq j$ the intersection $M_i \cap M_j$ is either empty or equal to a union of common boundary components.

A manifold is Waldhausen if it admits a Waldhausen decomposition. It is best described by a plumbing graph. To begin with, we consider oriented 3-manifolds which are circle bundles over a closed oriented surface (we only need here to consider these). Such a bundle is characterised by its Euler number and the genus of the base space. Two bundles may be glued together by an operation called plumbing. See [22] for details.

A 3-manifold constructed by plumbing is represented by a graph. The vertices represent the bundles. They carry two integral weights: the genus g of the base space and the Euler number e. An edge represents a plumbing operation. The dual graph of a good resolution for a normal surface singularity is also weighted like this. If understood as a plumbing graph, it describes the boundary of a semi-algebraic neighbourhood of the exceptional locus. See [22] for details. In [22] Neumann assigns a canonical plumbing graph to each Waldhausen manifold. Particularly useful are the bamboo o–o–$\cdots$–o with genus zero vertices and Euler numbers $e \leq -2$ for a lens space (see [22, Theorem 6.1]) and the star-shaped tree for the other Seifert spaces (see [22, Corollary 5.7]).

3.3. Mapping tori and Nielsen invariants. Let G be a compact, connected and oriented differential surface. Let $h\colon G \to G$ be an orientation preserving diffeomorphism of order $n \geq 2$. Let P be a point in the interior of G whose orbit under h is of cardinal $m < n$. Let λ be the integer defined as

$$\lambda = \frac{n}{m} \geq 2.$$

Consider the diffeomorphism h^m for which P is a fixed point. Choose a little disc D^2 with centre P invariant by h^m. "Little" means that, at the exception of P, all points in D^2 have an orbit of cardinal n. The disc D^2 is oriented by the orientation of G and its boundary ∂D^2 is oriented as the boundary of D^2. Then $h^m\colon D^2 \to D^2$ is conjugate to a rotation of angle ω/λ with $0 < \omega < \lambda$ and ω prime to λ. The orientation convention for D^2 and its boundary is essential to obtain a well-defined angle. Let σ be the integer such that $0 < \sigma < \lambda$ and $\omega\sigma \equiv 1 \bmod \lambda$.

DEFINITION. The pair (λ, σ) is the *Nielsen invariant* of h at P (or for the orbit of P) and the rational number σ/λ is the *Nielsen quotient*.

If G has a non-empty boundary, let $\hat{G}$ be the closed oriented surface obtained by attaching a 2-disc on each boundary component of G. Let $\hat{h}$ be the conical extension of h to $\hat{G}$. It may be that $\hat{h}$ is not quite differentiable at the centre of the new discs but this is unimportant.

RULE (Nielsen invariants for boundary components). We define the Nielsen invariant for boundary components of G as the Nielsen invariants for the centre of the attached discs. It is important to notice that the boundary components of G are oriented as the boundary of the attached discs and not as the boundary of G.

If we follow this rule we can always imagine that the surfaces are closed.
Let $h\colon G \to G$ be an orientation preserving diffeomorphism of order n of an oriented surface as above. The *mapping torus* $T(h)$ of h is defined as follows.

DEFINITION. The mapping torus $T(h)$ is the quotient of the product $G \times \mathbf{R}$ by the equivalence relation $(x, t+1) \sim (h(x), t)$. It is oriented by the orientation of G followed by the usual orientation of $\mathbf{R}$ (the order is unimportant). *This is the definition adopted in complex geometry* (*often implicitly*) *as well as in foliation theory* (*holonomy*). *Beware that topologists* (*for instance in knot theory*) *often use the opposite equivalence relation* $(x, t) \sim (h(x), t+1)$.

Since h is of finite order, the mapping torus $T(h)$ is an oriented Seifert manifold.

Proposition 3.3.1. *The Nielsen invariant for an orbit of length* $< n$ *and the normalised Seifert invariant for the corresponding exceptional leaf of the mapping torus coincide.*

For a proof see [20] p. 145–150.

Proposition 3.3.2. *The sum of the Nielsen quotients* σ/λ *of* h *around all the short orbits* (*i.e. with cardinal* $< n$) *and boundary components is an integer.*

Proof. The lemma is an immediate consequence of Proposition 3.3.1 and the fact that $e_0 = 0$ for a mapping torus (next lemma). □

Proposition 3.3.3. *Suppose that the surface G is closed. Then the rational Euler number* e_0 *of the Seifert foliation on the mapping torus* $T(h)$ *vanishes.*

Proof. Let n be the order of h. The mapping torus $T(h^n)$ is a covering of $T(h)$ of order n and the covering map preserves the Seifert structures. Hence, by the functoriality of e_0, we have $e_0(T(h^n)) = ne_0(T(h))$. See [11] 3.3. or [23] 1.2. But $T(h^n) = G \times S^1$ because $h^n = id$ and hence $e_0(T(h^n)) = 0$. □

3.4. Comments. i) The plumbing graph for L_t can be obtained as follows. The plumbing graph for the trunk is part of the plumbing graph for the normalised surface. From the mapping torus of the vertical monodromy, we obtain the Seifert-Waldhausen invariants of the vanishing zone by the dictionary given in [25]. Then [22] gives the plumbing graph for the vanishing zone. The pasting of two Seifert pieces along a common boundary component is represented in the plumbing graph by a bamboo having vertices with $g = 0$.

ii) Neumann proves in [22] that the boundary of a normal surface singularity is an irreducible 3-manifold, i.e. each embedded 2-sphere bounds a 3-ball. In [16], Section 5, we show that the boundary L_t of the Milnor fiber of the germ $f(x, y, z) = z^2 - y^2$ is diffeomorphic to $S^1 \times S^2$, which is not an irreducible 3-manifold.

iii) Usually when lens spaces $L(n, q)$ are considered it is implicitly assumed that $n \geq 2$. In this paper we shall call generalised lens space an oriented 3-manifold which is orientation preserving diffeomorphic to $L(n, q)$ or S^3 or $S^1 \times S^2$. They are exactly the 3-manifolds which admit a genus one Heegaard decomposition. A beautiful result of F. Bonahon [1] says that such a Heegaard decomposition is unique up to isotopy.

iv) A manifold which has two Seifert structures (one of them non-orientable) is a frequent pebble in the shoe. Let h be "the" orientation-preserving involution of the annulus $S^1 \times [0, 1]$ which exchanges the two boundary components. The mapping torus of h is a Seifert manifold which has two exceptional leaves with $\alpha = 2$. This is the Seifert structure that Waldhausen calls Q. See [27]. We shall not meet the other Seifert structure, except in Fig. 1 since Neumann uses it in his normalisation process.

4. From the boundary of the normalisation to the boundary of the Milnor fiber

Let $g \in \mathbf{C}\{x, y\}$ be non-reduced and such that $g(0, 0) = 0$. Let $\prod_{i=1}^{l} g_i^{n_i}$ be the factorisation of g into a product of irreducible factors with g_i prime to g_j if $i \neq j$. We choose the indices in such a way that $n_i > 1$ if and only if $i \leq i_0$ for some i_0 with $1 \leq i_0 \leq l$. We choose the coordinate axis such that x is prime to g.

Now let $f(x, y, z) = z^m - g(x, y)$ and let $\Gamma = \{\partial g/\partial y = 0\} \cap \{f = 0\}$. The singular locus $\Sigma(f)$ of f is the intersection of $\{z = 0\}$ with $\{g'(x, y) = 0\}$ where $g'(x, y) = \prod_{i=1}^{i_0} g_i^{n_i - 1}$.

(4.1) As x is prime to g, for a sufficiently small ϵ, the hyperplanes $H_a = \{x = a\}$ intersect transversally the curve $\Sigma(f)$ at any point of $(B_\epsilon^6 - \{0\}) \cap \Sigma(f)$ and:

$$B_\epsilon^6 \cap \{z = 0\} \cap \left\{ \frac{\partial g}{\partial y} = 0 \right\} \cap \{f = 0\} \subset \Sigma(f).$$

Moreover, we choose a sufficiently general coordinate axis y in order to have:

$$B_\epsilon^6 \cap \{z = 0\} \cap \left\{ \frac{\partial g}{\partial x} = 0 \right\} \cap \{ f = 0\} \subset \Sigma(f).$$

Let S be the boundary of the polydisc $B(\alpha) = B_\alpha^2 \times B_\beta^2 \times B_\gamma^2$ where $0 < \alpha < \beta < \gamma < \epsilon/3$ and let $S(\alpha) = S_\alpha^1 \times \operatorname{Int} B_\beta^2 \times \operatorname{Int} B_\gamma^2$. We take a sufficiently small β such that:

1) $$L_0 = (\{ f = 0\} \cap S) \subset \{|z| < \gamma\}.$$

We take a sufficiently small α with $0 < \alpha < \beta$ such that:

2) $$(\{g = 0\} \cap \{z = 0\} \cap S) \subset S(\alpha).$$

Using D.T. Lê, Section 1 in [14], the above conditions imply that $B(\alpha)$ is a Milnor polydisc for f as defined at the end of Section 2. We will use this polydisc in place of a standard ball B_ϵ^6.

Let $F_0 = f^{-1}(0) \cap B(\alpha)$. Then $L_0 = S \cap F_0$ is the boundary of F_0. The link K_0 of the singular locus $\Sigma(f)$ of f is by definition $K_0 = \Sigma(f) \cap L_0$.

Now let $n\colon \tilde{F}_0 \to F_0$ be the normalisation of F_0. We have seen in Section 2 that $\tilde{L}_0 = n^{-1}(L_0)$ can be identified with the boundary of the normalisation. Finally let $\tilde{K}_0 = n^{-1}(K_0)$ be the pull-back of K_0 by the normalisation.

REMARK 4.2. The resolution theory implies that there exists a decomposition of $\tilde{L}_0$ as a gluing of Seifert manifolds such that $\tilde{K}_0$ is a union of Seifert leaves.

Let $\varphi\colon \mathbf{C}^3 \to \mathbf{C}^2$ be the projection defined by $\varphi(x, y, z) = (x, z)$. For a small θ with $0 < \theta \ll \alpha$ we denote by M_0 the union of the connected components of $(\varphi^{-1}(S_\alpha^1 \times B_\theta^2)) \cap F_0$ which meet K_0.

Proposition 4.3. *There exists a sufficiently small θ such that*:

(1) $M_0 \subset S(\alpha)$,

(2) $M_0 \cap \{z = 0\} = K_0$,

(3) $n^{-1}(M_0) = \tilde{M}_0$ *is a tubular neighbourhood of* $\tilde{K}_0$ *in* $\tilde{L}_0$*. Moreover* $\tilde{K}_0$ *is the ramification locus of* $\varphi \circ n$ *restricted to* $\tilde{M}_0$.

Corollary 4.4. *The closure N_0 of $(L_0 - M_0)$ in L_0 is a Waldhausen manifold.*

Proof of Corollary 4.4. The restriction of the normalisation n to the closure $\tilde{N}_0$ of $(\tilde{L}_0 - \tilde{M}_0)$ in $\tilde{L}_0$ is a diffeomorphism onto N_0. But $\tilde{N}_0$ is a Waldhausen manifold by Remark 4.2. □

Proof of Proposition 4.3. From (4.1) fact 2) we have $K_0 \subset S(\alpha)$. Then there exists θ such that $M_0 \subset S(\alpha)$. We can choose θ small enough such that $L_0 \cap \{|z| \leq \theta\}$ is a tubular neighbourhood of $\{g = 0\} \cap L_0$ in L_0. This proves (2). The singular locus of φ restricted to F_0 is the curve $\Gamma \cap B(\alpha)$. Let $\Delta = \varphi(\Gamma)$. We can choose θ still smaller in order that $\Delta \cap (S^1_\alpha \times B^2_\theta) = S^1_\alpha \times \{0\}$. As $\Sigma(f) = \{z = 0\} \cap \Gamma$ this proves (3). □

(4.5) From the definition of $B(\alpha)$ given at the end of Section 2, there exists a very small η with $0 < \eta \ll \theta < \alpha$ such that f restricted to $W(\alpha, \eta) = B(\alpha) \cap f^{-1}(B^2_\eta - \{0\})$ is a locally trivial fibration on $(B^2_\eta - \{0\})$. When $0 < |t| \leq \eta$ we say that $F_t = W(\alpha, \eta) \cap f^{-1}(t)$ is "the" Milnor fiber of f and that $L_t = F_t \cap S$ is the boundary of the Milnor fiber of f.

In S we consider $\bar{S}(\alpha) = S^1_\alpha \times B^2_\beta \times \mathrm{Int}\, B^2_\gamma$ and $\bar{S}(\beta) = B^2_\alpha \times S^1_\beta \times \mathrm{Int}\, B^2_\gamma$. As α, β, γ have been chosen such that $L_0 = (f^{-1}(0) \cap S) \subset (\bar{S}(\alpha) \cup \bar{S}(\beta))$ (see (4.1) fact 1)) there exists η with $0 < \eta \ll \alpha$ such that $L_t \subset (\bar{S}(\alpha) \cup \bar{S}(\beta))$ for all t with $0 \leq |t| \leq \eta$.

(4.6) Let $M(\eta)$ be the union of the connected components of $S \cap \{|f| \leq \eta\} \cap \{|z| \leq \theta\}$ which meet K_0. Let $N(\eta)$ be the closure of $(W(\alpha, \eta) \cap S) - M(\eta)$ in S. For any t with $0 \leq |t| \leq \eta$ let $M_t = L_t \cap M(\eta)$ and let $N_t = L_t \cap N(\eta)$ be the closure of $(L_t - M_t)$ in L_t.

Theorem 4.7. *There exists a sufficiently small η such that for any t with $0 < |t| \leq \eta$ we have*
(1) *$M_t \subset S(\alpha)$,*
(2) *f restricted to $N(\eta)$ is a fibration on B^2_η with fiber N_t for $0 \leq |t| \leq \eta$,*
(3) *M_t has a Seifert structure such that the restriction of z on any Seifert leaf is constant.*

REMARK 4.8. Theorem 4.7 enables us to describe L_t as the union of the Seifert manifold M_t with the manifold N_t which is diffeomorphic to the Waldhausen submanifold $\tilde{N}_0$ of $\tilde{L}_0$ defined in the proof of Corollary 4.4. Moreover, the intersection $M_t \cap N_t$ is equal to $\partial M_t = \partial N_t$ which is a disjoint union of tori. Hence we have:

Corollary 4.9. *L_t is a Waldhausen manifold.*

Proof of Theorem 4.7. Proposition 4.3 implies that $M_0 = M(\eta) \cap f^{-1}(0)$ is included in $S(\alpha)$. As $S(\alpha)$ is open in S, we may choose η sufficiently small in order that $M(\eta) \subset S(\alpha)$. Thus point (1) is proved.

As noticed in (4.5), for a sufficiently small η and for t such that $0 \leq |t| \leq \eta$ we have $L_t \subset \bar{S}(\alpha) \cup \bar{S}(\beta)$. Let $L(\eta) = N(\eta) \cup M(\eta)$. We restrict η to have

$$\left(L(\eta) \cap \{z = 0\} \cap \left\{\frac{\partial g}{\partial y} = 0\right\} \cap \{|x| = \alpha\}\right) \subset K_0,$$
$$\left(L(\eta) \cap \{z = 0\} \cap \left\{\frac{\partial g}{\partial x} = 0\right\} \cap \{|y| = \beta\}\right) \subset K_0.$$

Moreover, K_0 is included in the interior of $M(\eta)$. Hence the restriction of f to $N(\eta)$ is a submersion. The intersection $N(\eta) \cap M(\eta)$ is included in $S(\alpha)$ and in $\{|z| = \theta\}$. In Proposition 4.3 we have chosen θ such that $\partial N_0 = \partial M_0$ does not meet $\{\partial g/\partial y = 0\}$. Hence, for a sufficiently small η the intersection $N(\eta) \cap M(\eta)$ does not meet $\{\partial g/\partial y = 0\}$ either. This proves (2).

We consider the projection φ defined in 4.3. For $0 < |t| \leq \eta$ let us denote by φ_t the restriction of φ to M_t. The singular locus of φ_t is $M_t \cap \{g' = 0\} = M_t \cap \{z^m = t\}$. For each c with $0 \leq |c| \leq \theta$ we have

$$\varphi_t^{-1}(S^1_\alpha \times \{c\}) = M_t \cap \{z = c\}.$$

We saturate the solid torus $S^1_\alpha \times B^2_\theta$ with the circles $S^1_\alpha \times \{c\}$. We pull-back this foliation by φ_t. As φ_t is a branched cover whose ramification locus consists of the m leaves $S^1_\alpha \times \{c\}$ with $c^m = t$, this gives a foliation in circles on M_t with leaves defined by $M_t \cap \{z = c\}$. □

4.10. Comments. At any point P of K_0, we consider the plane curve germ $f(a, y, z) = 0$. the Milnor theory applied to this plane curve germ, implies that the connected component of $M(\eta) \cap \{x = a\}$ which contains P is homeomorphic to a ball. Hence there exists a deformation retraction from $M(\eta)$ onto the link K_0. We say that M_t is the *vanishing zone* around K_0. Up to a diffeomorphism, N_t is a common Waldhausen submanifold of L_t, L_0 and $\tilde{L}_0$. This is why we say that N_t (resp. $\tilde{N}_0$) is the *trunk* of L_t (resp. $\tilde{L}_0$).

4.11. The gluing. We will explain now how L_t can be constructed as a gluing of $\tilde{N}_0$ with M_t. Let Φ be the restriction of (x, z, f) on the intersection $N(\eta) \cap M(\eta)$. For any $(a, c, t) \in S^1_\alpha \times S^1_\theta \times S^1_\eta$, we consider the arc $I(a, c, t) = \{(a, c, st) \in S^1_\alpha \times S^1_\theta \times B^2_\eta, s \in [0, 1]\}$ and we denote by $C(a, c, t)$ the inverse image $\Phi^{-1}(I(a, c, t))$ of the arc $I(a, c, t)$.

As Φ is a finite covering onto $S^1_\alpha \times S^1_\theta \times B^2_\eta$, $C(a, c, t)$ is a union of disjoint oriented embedded arcs parametrised by s. These arcs provide a homeomorphism r from $M_0 \cap N_0$ to $M_t \cap N_t$. By Theorem 4.7, the restriction of f on $N(\eta)$ is a fibration. As in the Milnor theory, any pull back by f of the radial vector field on B^2_η has integral curves which provide a homeomorphism R from N_0 to N_t. But on $N(\eta) \cap M(\eta)$, the vector field tangent to the arcs $C(a, c, t)$ is a pull back by f of the radial vector field on B^2_η. Hence we can construct R such that its restriction on $\partial N_0 = N_0 \cap M_0$ is r. We use $r \circ n$ to glue $\tilde{N}_0$ with M_t, and we denote by $\tilde{L}$ the result of this gluing. The identity map on M_t and $R \circ n$ on $\tilde{N}_0$ induce a homeomorphism from the gluing $\tilde{L}$ onto L_t. We also consider the manifold L obtained by the gluing of N_0 with M_t (constructed with the help of r). Then the identity map on M_t and R on N_0 induce a homeomorphism from L onto L_t. Let T be the boundary of N_0.

4.12. Comments. As r identifies each point P of T with $r(P) \in \partial M_t$, T is also (in L) the boundary of M_t. On the other hand, for each $t \in B^2_\eta$ the intersection $M_t \cap N_t$ (which is equal to $\partial M_t = \partial N_t$) is a disjoint union of tori saturated by two transversal foliations in circles. One of these foliations is given by the intersections with $\{x = a\}$ and the other by the intersections with $\{z = c\}$. By construction, the homeomorphism r used for the gluing preserves these two foliations. Hence, the disjoint union of tori T embedded in L is also saturated by these two foliations in circles.

Proposition 4.13. *The trunk and the vanishing zone are both irreducible* 3-*manifolds*.

Proof. The definition of irreducible 3-manifolds is given in the appendix of this paper. In the appendix (9.1), it is recalled that a Seifert manifold with non empty boundary is irreducible. Then, Theorem 4.7 implies that the vanishing zone is irreducible. As the trunk has as many connected components as $\tilde{L}_0$, it is sufficient to prove that the connected component W of the trunk contained in a connected component $\tilde{W}$ of $\tilde{L}_0$ is irreducible. But W is obtained by removing an open tubular neighbourhood of the components K_W of $\tilde{K}_0$. By 9.2 (Corollary J) such a Waldhausen manifold is irreducible. □

5. The vertical monodromy

With the notations of (4.1), the link K_0 of the singular locus of f has i_0 connected components. We choose i with $1 \le i \le i_0$ and we denote by K_i the component of K_0 which corresponds to the irreducible factor g_i of g. More precisely:

$$K_i = (S \cap \{z = 0\} \cap \{g_i(x, y) = 0\}).$$

Let $M(i)$ be the connected component of the vanishing zone $M(\eta)$ (see (4.6)) which contains K_i. Let $\pi\colon M(\eta) \to S^1_\alpha$ be the projection on the x-axis. Let $M_t(i) = M_t \cap M(i)$. Let π_t be π restricted to $M_t(i)$ with $0 < |t| \le \eta$.

Lemma 5.1. *The projection π_t is a fibration. Moreover the Seifert leaves constructed in* Theorem 4.7 *are transverse to the fibers of π_t.*

Proof. The equation of the singular locus of π_t is $\{z = 0\} \cap \{\partial g/\partial y = 0\}$. This curve does not meet $M_t(i)$ when $t \neq 0$. □

We now choose a with $|a| = \alpha$ and $P \in K_i \cap \{x = a\}$. Let $U(P)$ be the connected component of $\pi^{-1}(a) \cap M(i)$ which contains the point P. Let f_P denote f restricted to $U(P)$. Then f_P is a plane curve germ with an isolated singular point at P and $G_t = U(P) \cap M_t(i)$ is its Milnor fiber.

DEFINITION 5.2. The vertical monodromy around K_i is the first return diffeomorphism $h\colon G_t \to G_t$ along the Seifert leaves of $M_t(i)$.

The conjugacy class of h does not depend on the choices of P and a.

REMARK 5.3. Let $(s^r, w(s))$ be a Puiseux expansion of the branch $g_i(x, y) = 0$. Then $G'_t = M_t(i) \cap \pi^{-1}(a)$ has r connected components. There exists a monodromy $h'\colon G'_t \to G'_t$ for the fibration π_t such that $(h'|_{G_t})^r$ is the vertical monodromy h.

Consider the following decomposition $g = g_i^{n_i} \cdot g''$ in $\mathbf{C}\{x, y\}$ with g'' prime to g_i. Let k be the intersection multiplicity at the origin between g_i and g''. Let $d = \gcd(n_i, k)$.

Theorem 5.4. *The vanishing zone $M_t(i)$ around K_i is the mapping torus of $h\colon G_t \to G_t$ and we have*:

(1) *G_t is diffeomorphic to the Milnor fiber of the plane curve germ $z^m - y^{n_i}$.*

(2) *The vertical monodromy h is finite of order n_i/d.*

(3) *If $d < n_i$ the vertical monodromy h has exactly m fixed points and the action of h has order n_i/d on all other points.*

(4) *Around a fixed point h is a rotation of angle $-2\pi k/n_i$.*

Proof. The fact that the vanishing zone is the mapping torus of h is an immediate consequence of Lemma 5.1 and Definition 5.2.

We first prove statements (1) to (4) when $g_i(x, y) = y$. In this case, G_t is the Milnor fiber of $f(a, y, z) = z^m - y^{n_i} g''(a, y)$ with g'' prime to y. Hence $f(a, y, z)$ has at $P = (a, 0, 0)$ the topological type of $z^m - y^{n_i}$. Thus point (1) is proved. A Seifert leaf of $M_t(i)$ is in the hyperplane $\{z = c\}$ with $0 \leq |c| \leq \theta$. It is parametrised by $x = ae^{iv}$ with $v \in [0, 2\pi]$. Moreover, there exists a unity $u(a)$ in $\mathbf{C}\{a\}$ such that $g''(a, y) = a^k u(a) + y(\cdots)$. Hence, the intersection points (a, y, c) of G_t with this Seifert leaf satisfy an equation of the following type:

$$(\star) \qquad y^{n_i} = (a^k u(a) + y(\cdots))^{-1}(c^m - t).$$

As $y = 0$ if and only if $c^m = t$, we have m fixed points for h when z is equal to each m-th root of t. But $|y| \ll |a|$, then the equation $(\star)$ implies that h is conjugate to a rotation of angle $-2k\pi/n_i$ around each of the m fixed points. As $d = \gcd(n_i, k)$, the generic order of h is n_i/d. If $d = n_i$, then h is the identity. If $d < n_i$, then h has exactly m fixed points.

In the general case, we consider the Puiseux expansion $(s^r, w(s))$ of $g_i(x, y)$. If we make the substitution of variables $x = s^r$, $y' = y - w(s)$ and $f'(s, y', z) = f(s^r, y' + w(s), z)$ we are back to the preceeding case with f replaced by f'. $\square$

6. When is the boundary of the Milnor fiber a lens space?

In this section, we assume that f is irreducible. In particular, this implies that $\tilde{L}_0$ and N_0 are both connected. In 4.11 we have described the boundary L_t of the Milnor fiber by gluing the vanishing zone M_t to the trunk N_0.

Proposition 6.1. (1) *A connected component of M_t is never a solid torus.*
(2) *When $m > 2$ a connected component of M_t has m exceptional leaves or has a basis with non-zero genus or both.*

Proof. In Theorem 5.4 we have described a connected component $M_t(i)$ of M_t as the mapping torus of the vertical monodromy h acting on a differentiable surface G_t which is diffeomorphic to the Milnor fiber of the plane curve germ $z^m - y^{n_i}$ with $n_i \geq 2$. As $m \geq 2$, G_t is always connected and never diffeomorphic to a disc. As a consequence $M_t(i)$ is never a solid torus.

When $m > 2$ the surface G_t has non-zero genus. Then:

i) If h is the identity, the basis of $M_t(i)$ is G_t itself which has non-zero genus.

ii) If h is not the identity, we have proved in 5.4 that h has exactly m fixed points and hence $M_t(i)$ has m exceptional leaves. □

Proposition 6.2. *If L_t is a lens space, then the trunk N_0 is a solid torus, M_t is connected with a connected boundary and $\Sigma(f)$ is irreducible.*

Proof. Let T be a connected component of $\partial N_0 = \partial M_t$. As the connected components of M_t are irreducible Seifert manifolds none of them being a solid torus (see Proposition 4.13 and 6.1), Proposition D in 9.1 implies that T is incompressible in M_t. Proposition 4.13 shows also that the trunk is irreducible. If the trunk were not a solid torus, T would also be incompressible in N_0 (see again Proposition D in 9.1). Then, van Kampen's theorem and Dehn's lemma would imply that T is incompressible in L_t. But a torus embedded in a lens space is always compressible. Hence N_0 is a solid torus.

As the trunk is a solid torus, the vanishing zone M_t is connected with a connected boundary because $\partial N_0 = \partial M_t$. By construction of the vanishing zone, the number of connected components of M_t is equal to the number of irreducible components of the singular locus $\Sigma(f)$ of f. □

Theorem 6.3. *The boundary of the Milnor fiber of an irreducible germ $f(x,y,z) = z^m - g(x,y)$, where $m \geq 2$ and $g(x,y) = 0$ is non-reduced, is a lens space if and only if f is analytically equivalent to $z^2 - xy^l$ for some $l \geq 2$.*

Proof. In [18] Section 4 it is proved that the lens space $L(2l, 1)$ is indeed the boundary of the Milnor fiber of $z^2 - xy^l$.

Conversely, when L_t is a lens space, Proposition 6.2 implies that N_0 is a solid torus and that $\Sigma(f)$ is irreducible. We have a decomposition of L_t as a gluing of the Seifert manifold M_t and a solid torus. But a lens space is never a non-trivial connected sum of two 3-manifolds (i.e. a lens space is prime). Hence by Waldhausen classical argument (see [27] p. 90–91), the Seifert foliation on M_t extends in a Seifert foliation on L_t. Now, we have a Seifert foliation on L_t which has at most m exceptional leaves or a basis with non-zero genus or both (see Proposition 6.1). Thanks to A. Hatcher (see [7] p. 31) the only possible case is $m = 2$ and the genus of the basis is 0.

Therefore, we can write $g(x, y) = g_1(x, y)^l \cdot g''(x, y)$ with g_1 irreducible, $l = n_1 \geq 2$, g'' being either reduced and prime to g_1 or a unity.

Let $\psi : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ be the projection defined by $\psi(x, y, z) = (x, y)$. Let S_1 be the boundary of the polydisc $B_1 = B^2_\alpha \times B^2_\beta$ with $0 < \alpha \leq \beta$ such that B_1 is a Milnor polydisc for g. Let $K_1 = S_1 \cap \{g_1 = 0\}$. By construction $\psi(M_0)$ is a tubular neighbourhood of K_1 in S_1 and the closure W of its complement in S_1 is $\psi(N_0)$.

Let us consider the Milnor fibration $\rho = g_1/|g_1|\colon W \to S^1$ for the plane curve germ g_1. Let G_1 be the Milnor fiber of this fibration. Then $\rho \circ \psi : N_0 \to S^1$ is a fibration with fiber G_1' which is a ramified covering of G_1 induced by ψ. The ramification values of this covering are $G_1 \cap \{g'' = 0\}$. Hence the cardinality of the set of ramification values is equal to the intersection multiplicity $m_0(g_1, g'')$ of g_1 and g'' at the origin of $\mathbf{C}^2$.

As $m = 2$ this covering has degree 2. Hence

$$\chi(G_1) = 1 - \mu(g_1),$$
$$\chi(G_1') = 2(1 - \mu(g_1)) - m_0(g_1, g''),$$

where $\chi(\ )$ is the Euler characteristic and $\mu(\ )$ is the Milnor number.

As N_0 is a solid torus, G_1' is a disjoint union of discs. The only solution for the second equation just above is $\mu(g_1) = 0$ and $m_0(g_1, g'')$ equals either 1 or 0.

When g'' is not a unity, i.e. $m_0(g_1, g'') = 1$, then we can choose the axis in such a way that $g_1(x, y) = y$ and $g''(x, y) = x$. As a consequence we obtain that $f(x, y, z) = z^2 - xy^l$.

Otherwise, we can choose the second axis in such a way that $g_1(x, y) = y$. Then, $f(x, y, z) = z^2 - y^l$ with $l \geq 3$ as f is irreducible. Then the vertical monodromy is the identity on a surface which has non-zero genus. Hence the vanishing zone is a Seifert manifold whose basis has non-zero genus. Then, we can use [7] as above (or compute the rank of the first homology group of L_t) to see that we never get a lens space. □

REMARK. The reducible case $z^2 - y^2$ is treated in [16]. It is proved that L_t is then diffeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$.

7. Examples

In this section we apply the method presented above to the singularities with equation $z^2 - (x^2 - y^3)y^l = 0$ $(l \geq 3)$ with l odd. The ingredients necessary to get the Waldhausen structure are stated in Proposition 7.1.

Proposition 7.1. *Write* $l = 2\overline{l} + 1$ $(\overline{l} \geq 1)$. *Then*:
(1) *The trunk is the Waldhausen manifold* Q.
(2) *The vanishing zone is connected with one boundary component. More precisely, it is the mapping torus of an orientation preserving diffeomorphism* h *of order* l *acting on the Milnor fiber of the plane curve singularity* $z^2 - y^l = 0$. *It has two fixed points. The rotation angle at the fixed points is equal to* $(-2/l)2\pi$. *On the complement of the fixed points the diffeomorphism* h *induces a free action of a cyclic group of order* l.
(3) *The Waldhausen* (α, β) *for the gluing between the trunk and the vanishing zone is equal to* $(l+3, l+2)$.

Indications for the computations. 1. The trunk can be obtained by computing explicitly the normalisation. Since l is odd, the normalisation of the surface $z^2 - (x^2 - y^3)y^l = 0$ has equation $z^2 - (x^2 - y^3)y = 0$. As the singularity is of multiplicity two, the method of Laufer [13] is very efficient. The result is as follows. The boundary $\tilde{L}_0$ of the normalisation is a Seifert manifold with a base space of genus $g = 0$, Euler number $e' = +1$ and with 3 exceptional leaves. Their invariants (α, β) are respectively $(2, 1)$, $(2, 1)$ and $(3, 1)$. The knot $\tilde{K}_0$ of the singular locus of f is represented by the exceptional leaf with invariants $(3, 1)$. Hence the trunk $\tilde{N}_0$ is the complement of a small open tubular neighbourhood of this knot in $\tilde{L}_0$. We denote by T the single boundary component of $\tilde{N}_0$.

The referee has remarked that Orlik-Wagreich [24] can also be used to get the Seifert structure of the boundary of the normalisation, since the equation is weighted homogeneous.

2. As above in Section 5 we denote by $h\colon G_t \to G_t$ the vertical monodromy. The order l of h and the two fixed points with angles $(-2/l)2\pi$ are obtained from Section 5. Hence the vanishing zone M_t has two exceptional Seifert leaves with normalised Seifert invariants $(l, \overline{l})$ (see Section 3.1).

3. We now compute Waldhausen's invariant (α, β) which characterises the gluing between the trunk $\tilde{N}_0$ and the vanishing zone M_t. See [27] p. 109 or [25] p. 342.2 and p. 366 for the definitions.

We consider the manifold $\tilde{L}$ homeomorphic to L_t obtained in (4.11) as the gluing of the trunk $\tilde{N}_0$ and the vanishing zone M_t. Let T be the common boundary of $\tilde{N}_0$ and M_t in $\tilde{L}$. The torus T is oriented as the boundary of M_t. Curves on T which come from M_t will be denoted by a subscript $+$ and those which come from $\tilde{N}_0$ by a subscript $-$. As in the classical Seifert's notations, "H" denotes a regular Seifert leaf while "Q" denotes a section. Since sections will always be written with a subscript + or $-$, there should be no confusion with Waldhausen manifold Q. Following Waldhausen we set in $H_1(T, \mathbf{Z})$:

$$H_- = \alpha Q_+ + \beta H_+.$$

For the above formula to make sense, one has to orient the Seifert leaves H_+ and H_- and the section Q_+. Here we orient ∂G_t as the boundary of the Milnor fiber of the plane curve germ $f(a, y, z) = 0$. We orient H_+ such that $\partial G_t \bullet H_+ > 0$ where $\bullet$ denotes the intersection pairing in $H_1(T, \mathbf{Z})$. A section Q_+ is oriented from the orientation chosen on H_+ by the rule $Q_+ \bullet H_+ = +1$. From an orientation of H_+ one traditionally obtains an orientation of H_- by requiring that $H_- \bullet H_+ = \alpha > 0$.

Claim 1. *With the above orientations, we have $\alpha = l + 3$ and $\partial G_t \bullet H_+ = l$.*

Proof. The leaf H_+ has for equation $z = c$ with $c \in S^1_\theta$. On the other hand the intersection $T \cap \{x^2 - uy^3 = 0\}$ for some $u \in \mathbf{C}^*$ provides two leaves H_- of the Seifert structure of the trunk N_0. Indeed there exist two non-zero complex numbers v, w such that one of these leaves is parametrised by (s^3, vs^2, ws^{l+3}) with $s^3 \in S^1_\alpha$. *Of course this subscript α has nothing in common with Seifert's or Waldhausen's α.* We have to prove that $H_- \bullet H_+ = l + 3$. With the chosen orientations $H_- \bullet H_+$ is positive. From the equation of H_+ and the given parametrisation of H_- we directly obtain that the absolute value of α is $l + 3$. With the chosen orientations the intersection $\partial G_t \bullet H_+$ is positive and the order of the vertical monodromy being l, its absolute value is l. $\square$

Lemma 7.2. *There exists a section Q_+ such that $\partial G_t = l\ Q_+ - H_+$.*

Proof. Let $\overline{G_t}$ be the closed surface obtained from G_t by gluing a disk along its boundary and let $\overline{h}\colon \overline{G_t} \to \overline{G_t}$ be the periodic extension of h. As the sum of the Nielsen quotients of $\overline{h}$ is an integer (Proposition 3.3.2), the Nielsen quotient of h on $-\partial G_t$ (*i.e.* ∂G_t with the opposite orientation) equals $1/l$. Applying Proposition 3.3.1, there then exists a section Q_+ on T (i.e. $Q_+ \bullet H_+ = +1$ in $H_1(T, \mathbb{Z})$) such that

$$\partial G_t = l\ Q_+ - H_+. \qquad \square$$

Claim 2. *We have $\beta = -1$.*

Proof. As ∂G_t is the intersection of $x = a$ with T, from the parametrisation of H_- given in Claim 1 we obtain that the absolute value of $(\partial G_t) \bullet H_-$ is 3. We write $(\partial G_t) \bullet H_- = 3\epsilon$ where $\epsilon \in \{+1, -1\}$. On the other hand, by Lemma 7.2 we have $lQ_+ = \partial G_t + H_+$. Then:

$$l\beta = lQ_+ \bullet H_- = 3\epsilon - l - 3 = 3(\epsilon - 1) - l.$$

As l is odd, when $l > 3$ the only solution of the above equality (in the integers) is $\epsilon = +1$ and $\beta = -1$. When $l = 3$ the equation implies $\beta = (\epsilon - 2)$. Then, we still have the solution $\epsilon = +1$ and $\beta = -1$. Again it is the only one because we can exclude $\beta = -3$ by the following argument: there exists a section Q'_+ (for example one of the two connected components of $T \cap \{y = b\}$, where b is any suitable constant) such that there is a unique intersection point between H_- and Q'_+. Of course this last argument can be used (for any $l = 2\overline{l} + 1$ ($\overline{l} \geq 1$)) to show that β is +1 or -1. □

4. Now, let us compute the Euler numbers of M_t and $\tilde{N}_0$ corresponding to the choices of sections on their common boundary component T and around their exceptional Seifert leaves.

As the rational Euler number of the mapping-torus of $\overline{h}$ is zero (see Proposition 3.3.3), the Euler number e of M_t corresponding to the above choices of section Q_+ on T and around the two exceptional leaves is given by: $e_0 = 0 = e - (1/l + \overline{l}/l + \overline{l}/l)$. Then $e = 1$. As this choice of sections leads to the non-normalized Waldhausen pair $(\alpha, \beta) = (l + 3, -1)$, one has to replace the section Q_+ by $Q'_+ = Q_+ - H_+$ in order to obtain the normalized Waldhausen pair $(l + 3, l + 2)$. For M_t, the corresponding Euler number is then $e_{\text{norm}} = e + 1 = 2$.

Now, we compute the invariants of $\tilde{N}_0$. The equality $H_- = (l+3)Q_+ - H_+$ leads to $Q_+ \bullet H_- = -1$ (always with T oriented as the boundary of M_t). Therefore, $-Q_+$ can be used as a section on T, and we set $Q_- = -Q_+$. Hence $H_+ = -(l+3)Q_- - H_-$. By Lemma 7.2, we also have $H_+ = -lQ_- - \partial G_T$, then $\partial G_t = 3Q_- + H_-$. Recall that ∂G_t is glued along the meridian m of the knot $\tilde{K}_0$ in $\tilde{L}_0$. Hence we have $m = 3Q_- + H_-$. As the orientation on T coincides with the orientation of the boundary of the oriented tubular neighbourhood $\tilde{M}_0$ of $\tilde{K}_0$, our choice of section Q_- gives the normalized Seifert invariant (3, 1) in $\tilde{L}_0$. Hence the corresponding Euler number e' is the same as in $\tilde{L}_0$ i.e. $e' = 1$. To compute the normalized Waldhausen pair for the trunk $\tilde{N}_0$, we must consider the torus T with the opposite orientation, we denote it by T'. Then T' is oriented as the boundary of $\tilde{N}_0$. But as $H_+ = -(l+3)Q_- - H_-$, we have to replace the section Q_- by $Q'_- = -Q_- - H_-$ in order to obtain $H_+ = (l+3)Q'_- + (l+2)H_-$. This section Q'_- on T', gives the normalized Waldhausen pair $(l+3, l+2)$ for $\tilde{N}_0$. The corresponding normalized Euler number for $\tilde{N}_0$ is $e'_{\text{norm}} = e' + 1 = 2$.

5. Summary. The Waldhausen graph for the boundary of the Milnor fibre is the following. There are two vertices (each representing a Seifert manifold) joined by an edge. Each vertex has Seifert weights $g = 0$ and $e = +2$. At one vertex there are two exceptional leaves with invariants $(l, \overline{l})$; at the other vertex there are two exceptional leaves with invariants (2, 1). Along the edge the Waldhausen gluing invariant is $(l + 3, l + 2)$ (there is no need to put an orientation on the edge, since the inverse of -1 is -1).

From the process described in ([22], Theorem 5.6) (see also Sections 3.2 and 3.4 i) of the present paper) we can compute a plumbing graph of L_t from this Waldhausen graph. The resulting plumbing graph is shown in Fig. 1. The plumbing graph of the

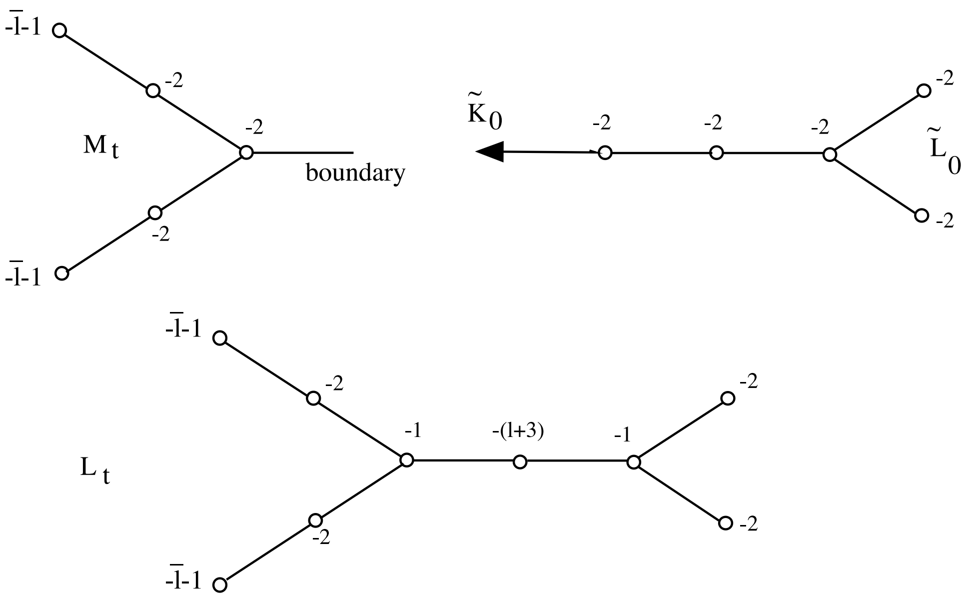

Fig. 1.

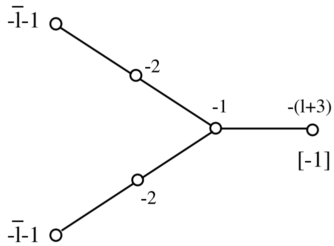

Fig. 2.

mapping torus $M_t = T(h)$ and that of the pair $(\tilde{L}_0, \tilde{K}_0)$ are also shown in Fig. 1. When we proceed from the graphs of M_t and $(\tilde{L}_0, \tilde{K}_0)$ to the graph of L_t, the Euler numbers at the rupture vertices have to be changed in order to take into account the normalisation of the Waldhausen pair (α, β).

Theorem 7.3. *The boundary L_t of the Milnor fiber of the non-isolated singularity with equation $z^2 - (x^2 - y^3)y^l$ $(l \geq 3)$ with l odd is not orientation preserving diffeomorphic to the boundary of a normal surface singularity.*

Proof. The normalized form of the graph of L_t is given in Fig. 2 above, where the genus on the vertex on the right is < 0. This contradicts Neumann's theorem [22], 8.2 p. 335. □

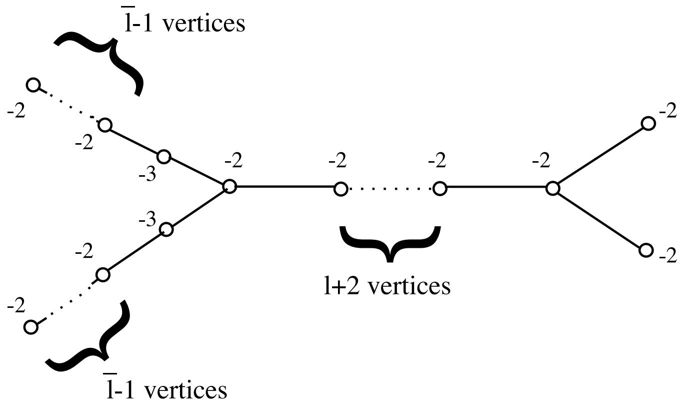

Fig. 3.

REMARK 7.4. If we reverse the orientation of L_t, we obtain a Waldhausen 3-manifold which is orientation preserving diffeomorphic to the boundary of a normal surface singularity. Indeed, if we apply the recipe given by [22] in bottom p. 310 and top p. 311 we have to reverse the sign of the Euler numbers. Observe that we do not need to worry about edge signs, since the graphs we consider are trees. Then we apply the procedure of p. 313 to obtain a graph which satisfies N2. From this graph the normalisation process of Neumann's Section 3 (especially N3) produces the graph in normal form of Fig. 3. The intersection form of this graph is negative definite.

8. The homology of the boundary of the Milnor fiber

Theorem 8.1. *Let $f(x, y, z) = z^m - x^k y^l = 0$ be the equation of a Hirzebruch singularity. Assume that* $\gcd(m, k, l) = 1$, *that* $1 \leq k < l$ *and that* $m \geq 2$. *Let* $d = \gcd(k, l)$ *and write* $\bar{k} = k/d$ *and* $\bar{l} = l/d$. *Then* $H_1(L_t, \mathbf{Z})$ *is isomorphic to the direct sum of a free abelian group of rank* $2(m-1)(d-1)$ *and a torsion group. The torsion subgroup is the direct sum of* $(m-1)$ *cyclic factors. One of them is of order* $m\overline{kl}$ *and the other* $(m-2)$ *factors are of order* $\overline{kl}$.

The proof is a consequence of the description we give for L_t in [18]. The main ingredient is the determination of the monodromy $\mathbf{Z}[t, t^{-1}]$ module associated to the vanishing zone. As we proved in [18] that L_t is in fact a Seifert manifold, one can check that the result fits with [2].

Theorem 8.2. *When l is odd, the group* $H_1(L_t, \mathbf{Z})$ *for the singularity* $z^2 - (x^2 - y^3)y^l$ *is cyclic of order* $4l$.

9. Appendix

There are two questions which need clarification in our use of 3-manifold theory: irreducibility and normal forms for plumbing graphs. Since the relevant statements are somewhat scattered through the literature, we group them in this appendix. Recall that 3-manifolds are supposed to be compact, oriented, with boundary a disjoint union of tori. We work in the differentiable category.

9.1. Irreducible 3-manifolds.

DEFINITION. A 3-manifold M is *irreducible* if every (embedded) 2-sphere in M bounds a 3-ball in M.

Alexander proved that $\mathbf{R}^3$ is irreducible and that a 2-sphere in S^3 bounds two 3-balls.

Lemma A. *Suppose that the* 3*-manifold M is fibered over the circle S^1 with a connected and orientable fibre F not diffeomorphic to the* 2*-sphere. Then M is irreducible.*

Proof. Consider the universal cover $\tilde{M}$ of M. Observe that its interior is diffeomorphic to $\mathbf{R}^3$, which is irreducible by Alexander. By the "going down argument" we deduce that M is irreducible. □

Theorem B (Going up and down). *Let $p\colon \hat{M} \to M$ be a covering map. Then M is irreducible if and only if $\hat{M}$ is.*

Comments. $\hat{M}$ irreducible $\Rightarrow$ M irreducible is classical. It is proved in Hatcher's notes [7, Proposition 1.6]. The reverse implication M irreducible $\Rightarrow$ $\hat{M}$ irreducible is much harder. It is proved in [7, Theorem 3.15]. See also the introduction to [7].

Waldhausen's Satz (1.8). Let $\mathcal{F}$ be a system of incompressible surfaces in M. Let U be a small neighbourhood of $\mathcal{F}$. Let $\breve{M} = M \setminus \mathring{U}$. Then M is irreducible if and only if $\breve{M}$ is.

Theorem C. *Except $S^1 \times S^2$ and $P^3(\mathbf{R}) \sharp P^3(\mathbf{R})$ every Seifert manifold is irreducible. In particular, every Seifert manifold with boundary is irreducible.*

For a proof see Jaco's book [10] p. 88 and Hatcher's notes [7] p. 18.

We denote by ∂M the boundary of M.

Proposition D. *Let M be an irreducible and connected* 3*-manifold such that a component Σ of ∂M is compressible. Then M is a solid torus. In particular, ∂M is connected.*

Proof. Let D be a compressing disc for Σ. Let U be a small tubular neighbourhood of D. Consider $\breve{M} = M \setminus \mathring{U}$. By construction the boundary $\partial \breve{M}$ contains a 2-sphere. Hence it bounds a 3-ball B in M. Now $V = B \cup U$ is a solid torus since U is attached along ∂B as a 1-handle. Being compact, V is closed in M and it is easy to see that it is also open. Since M is connected, we have $M = V$. □

Corollary E. *Let M be an irreducible and connected* 3*-manifold and suppose that ∂M is not connected. Then each connected component of ∂M is incompressible.*

General Principle. Let $\{M_i\}$ $i = 1, \dots, k$ be a finite collection of Seifert manifolds with non-empty boundary, none of them being a solid torus. Let M be constructed by gluing the M_i's along boundary tori. Then M is irreducible.

Comments. 1) The proof is an easy consequence of Waldhausen's Satz (1.8) in [27] and from the fact that a Seifert manifold with boundary is irreducible and with incompressible boundary if it is distinct from a solid torus.

2) We wish the manifolds M constructed this way to be orientable (even oriented). This requirement is achieved as follows: Each M_i is oriented; hence each of its boundary component is also oriented. For the gluing it is required that the diffeomorphisms are orientation reversing.

3) It is allowed that two boundary components of the same M_i are glued together.

4) The image of a boundary component of a submanifold M_i is an incompressible torus in M (a short proof uses van Kampen's theorem and Dehn's lemma).

9.2. Irreducibility and plumbing graphs. We consider plumbed 3-manifolds as in Neumann [22] with oriented bases. To begin with, we consider plumbing graphs which produce closed manifolds (there are no arrows).

DEFINITION. The *valency* (also called the degree) of a vertex is the number of edges attached to the vertex (a loop counts for 2). A *rupture vertex* is a vertex which has genus ≥ 1 or valency ≥ 3 (or both as Waldhausen says). A *dead branch* is a bamboo (always of curves of genus 0) attached to a rupture vertex and ending at a vertex of valency 1 and genus 0. A *long edge* is a bamboo attached to rupture vertices (which may coincide). In Neumann [22], dead branches and long edges are called maximal chains.

Let M be the 3-manifold produced by a plumbing graph. The submanifold produced by a dead branch is a solid torus, while the one produced by a long edge is a thickened torus. In both cases there is an invariant essentially defined by Waldhausen [27] on p. 109 (and earlier by Seifert) and denoted by α. We propose to define it as the absolute value of the intersection number on some torus between two canonical curves. In the case of a solid torus α is the intersection number on its boundary between a meridian curve and a Seifert leaf from the rupture vertex. In the case of a thickened torus α is the intersection number on an intermediate torus between the Seifert leaves which come from both sides. Note that $\alpha = 0$ means in both cases that the two curves are isotopic.

Consider a dead branch (i.e. the submanifold is a solid torus). Then: 1) $\alpha = 0$ means that the core of the solid torus is a "singular leaf" (we can't extend the Seifert foliation inside the solid torus); 2) $\alpha = 1$ means that the Seifert foliation of the rupture vertex can be extended through the solid torus with the core as a regular leaf; 3) $\alpha \geq 2$ means that the Seifert foliation can be extended through the solid torus with the core as an exceptional leaf.

For each kind of bamboo there is a formula which computes the invariant α (it is a continued fraction). The formula is given in Neumann [22] top p.318 and bottom p. 323.

By Conditions N1,...,N6, we refer to the Neumann's conditions stated in [22] p. 311 and p. 312.

Neumann's Condition N2. A plumbing graph satisfies Condition N2 if each vertex on a bamboo has Euler number ≤ -2. Equivalently: each vertex of genus 0 and valency ≤ 2 has Euler number ≤ -2.

Fact. If a plumbing graph satisfies condition N2, then all α's are ≥ 2. This is an easy consequence of the continued fraction formula.

Neumann's Corollary 5.7. Closed Seifert manifolds have a plumbing graph satisfying N2 which is either a bamboo or is star-shaped. For a given Seifert manifold this plumbing graph is unique if we require that graphs are tree-like and satisfy N2. It is Neumann's canonical star-shaped plumbing graph for Seifert manifolds.

Conversely a star-shaped plumbing graph satisfying N2 produces a Seifert manifold. At the rupture vertex, there is no condition on the genus (it may be ≤ -1 in general; however we consider here only oriented bases) or on the Euler number. Dead branches are in bijection with exceptional leaves. This is important: Condition N2 implies that $\alpha \geq 2$ for a dead branch. Recall also that 3-manifolds are oriented. A change of orientation induces a change in the canonical graph.

Let us now turn towards Seifert manifolds with boundary and plumbing graphs which produce manifolds with boundary. Let Γ be some plumbing graph (always connected with oriented base spaces, but we do not suppose that Γ satisfies other conditions). Suppose that we select a finite number of fibres in the S^1-bundles represented by the vertices of Γ and that we remove the interior of a small tubular neighbourhood of each fibre (without loss of generality, we may assume that the neighbourhood is a union of S^1 fibres). We represent each selected fibre (indeed its little tubular neighbourhood) by an arrow attached to the corresponding vertex. This graph with arrows contains all the instructions to construct a plumbed manifold with boundary.

Our former vocabulary can easily be modified to cope with this new situation. The valency of a vertex is now the sum of the number of edges and of the number of arrows attached to it. A graph with arrows satisfies Condition N2 if each vertex with genus 0 and valency ≤ 2 has Euler number ≤ -2. A rupture vertex has a basis of genus ≥ 1 or has valency ≥ 3 (or both!).

Lemma F. *Let Γ be a connected plumbing graph* (*possibly with arrows*) *satisfying Condition N*2. *Suppose that Γ is star-shaped* (*i.e. tree-like with exactly one rupture vertex*). *Then Γ produces a Seifert manifold which is not a solid torus*.

Proof. Without loss of generality, we may assume that the number of arrows is 1. Since Γ is star-shaped it represents a Seifert manifold with one boundary component. Fundamental group considerations imply that the genus of the rupture vertex is 0. Since there is a rupture vertex V and only one arrow, the number of dead branches at V is at least 2. Since Γ satisfies Condition N2, each dead branch produces an exceptional leaf ($\alpha \geq 2$). But a Seifert foliation of a solid torus has at most one exceptional leaf. For a proof of this last assertion see Hatcher's notes [7] bottom p. 19. □

Theorem G. *Let Γ be a connected plumbing graph* (*possibly with arrows*) *with dead branches satisfying Condition N*2. *Then Γ produces an irreducible manifold*.

Proof. If Γ contains no rupture vertex, it is either a bamboo or a circuit. Bamboos with no arrows produce lens spaces which are irreducible by the going down theorem. Note that $S^1 \times S^2$ is excluded because Γ satisfies N2. Bamboos with possibly one arrow at extremities clearly produce irreducible manifolds. Circuits produce torus bundles over S^1 which we already know to be irreducible (see the first part of this addendum).

We may assume therefore that Γ contains rupture vertices. Split each long edge in the middle of some of its edges. Add an arrowhead at each open extremity of the split edges. We obtain a disjoint union of star-shaped plumbing graphs which satisfy Condition N2. By Lemma F each of them produces a Seifert manifold which is not a solid torus. Each such manifold is irreducible with incompressible boundary. The General Principle achieves the proof. □

Corollary H ([22], Theorem 1). *The boundary of a normal surface singularity is an irreducible* 3*-manifold.*

Corollary J. Let (Σ, P) be a normal surface singularity at P and let γ be a germ of analytic curve at P in Σ. Let M be the boundary of (Σ, P) and let $\breve{M}$ be equal to M minus a small open tubular neighbourhood of $M \cap \gamma$. Then $\breve{M}$ is irreducible.

Proof of Corollaries H and J. We accept that γ may be empty; this covers the first corollary. Consider a resolution $\pi\colon \tilde{\Sigma} \to \Sigma$ for which the total transform of γ has normal crossings. By the du Val-Mumford theorem all Euler numbers are ≤ -1. After possibly some blowing downs, we may further assume that each irreducible (!) component of the exceptional locus with genus 0 and with valency ≤ 2 has self-intersection ≤ -2. Condition N2 is satisfied everywhere in the plumbing graph. $\square$

References

[1] F. Bonahon: *Difféotopies des espaces lenticulaires*, Topology **22** (1983), 305–314.

[2] J. Bryden, T. Lawson, B. Pigott and P. Zvengrowski: *The integral homology of orientable Seifert manifolds*, Topology Appl. **127** (2003), 259–275.

[3] D. Burghelea and A. Verona: *Local homological properties of analytic sets*, Manuscripta Math. **7** (1972), 55–66.

[4] A.H. Durfee: *Neighborhoods of algebraic sets*, Trans. Amer. Math. Soc. **276** (1983), 517–530.

[5] D.B.A. Epstein: *Periodic flows on three-manifolds*, Ann. of Math. (2) **95** (1972), 66–82.

[6] H.A. Hamm and D.T. Lê: *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 317–355.

[7] A. Hatcher: *Notes on basic 3-dimensional topology*, available on the Net at `http://www.math.cornell.edu/~hatcher`.

[8] F. Hirzebruch: *Über Singularitäten komplexer Flächen*, Rend. Mat. e Appl. (5) **25** (1966), 213–232.

[9] F. Hirzebruch, W.D. Neumann and S.S. Koh: Differentiable Manifolds and Quadratic Forms, Lecture Notes in Pure and Applied Mathematics **4**, Dekker, New York, 1971.

[10] W. Jaco: Lectures on Three-Manifold Theory, AMS Regional Conference Series in Mathematics **43**, Amer. Math. Soc., Providence, R.I., 1980.

[11] M. Jankins and W.D. Neumann: Lectures on Seifert Manifolds, Brandeis Lecture Notes **2**, Brandeis Univ., Waltham, MA, 1983.

[12] M. Kato and Y. Matsumoto: *On the connectivity of the Milnor fiber of a holomorphic function at a critical point*; Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, 131–136.

[13] H.B. Laufer: *On normal two-dimensional double point singularities*, Israel J. Math. **31** (1978), 315–334.

[14] D.T. Lê: *La monodromie n'a pas de points fixes*, J. Fac. Sci. Univ. Tokyo, Sec. 1A **22** (1979), 409–427.

[15] D.B. Massey: *Semi-simple carrousels and the monodromy*, Ann. Inst. Fourier (Grenoble) **56** (2006), 85–100.

[16] F. Michel and A. Pichon: *On the boundary of the Milnor fibre of nonisolated singularities*, Int. Math. Res. Not. **43** (2003), 2305–2311.

[17] F. Michel and A. Pichon: *On the boundary of the Milnor fibre of nonisolated singularities* (Erratum), Int. Math. Res. Not. **6** (2004), 309–310.

[18] F. Michel, A. Pichon and C. Weber: *The boundary of the Milnor fiber of Hirzebruch surface singularities*; in Singularity Theory, World Sci. Publ., Hackensack, NJ, 2007, 745–760.
[19] J. Milnor: Singular Points of Complex Hypersurfaces, Ann. of Math. Stud. **61**, Princeton Univ. Press, Princeton, N.J., 1968.
[20] J.M. Montesinos: Classical Tessellations and Three-Manifolds, Springer, Berlin, 1987.
[21] D. Mumford: *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.
[22] W.D. Neumann: *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), 299–344.
[23] W.D. Neumann and F. Raymond: *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*; in Algebraic and Geometric Topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977), Lecture Notes in Math. **664**, Springer, Berlin, 1987, 163–196.
[24] P. Orlik and P. Wagreich: *Isolated singularities of algebraic surfaces with* $\mathbf{C}^*$ *action*, Ann. of Math. (2) **93** (1971), 205–228.
[25] A. Pichon: *Fibrations sur le cercle et surfaces complexes*, Ann. Inst. Fourier (Grenoble) **51** (2001), 337–374.
[26] D. Siersma: *The vanishing topology of non isolated singularities*; in New Developments in Singularity Theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem. **21**, Kluwer Acad. Publ., Dordrecht, 2001, 447–472.
[27] F. Waldhausen: *Eine Klasse von* 3*-dimensionalen Mannigfaltigkeiten*, I, II, Invent. Math. **3** (1967), 308–333, **4** (1967), 87–117.

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