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MOTIONS OF CURVES IN THE COMPLEX HYPERBOLA AND THE BURGERS HIERARCHY

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Abstract

We study theory of curves in the complex hyperbola and show that special motions of curves are linked with the Burgers hierarchy, which also leads to a Hamiltonian formulation of the hierarchy and to the difference Burgers equation via their discretization.

1. Introduction

It is widely recognized that a lot of differential equations in soliton theory arise from differential geometry especially theory of curves or surfaces ([1, 2, 3]). For example surfaces in the Euclidean 3-space with constant negative Gaussian or non-zero mean curvature are described by the sine or sinh Gordon equation respectively and proper affine spheres are described by the Tzitzéica equation. If we refer to theory of curves, the curvature of curves in the Euclidean 2-space evolves according to the mKdV equation under special motions. Pinkall ([4]) showed that the space of closed centroaffine curves in the centroaffine plane possesses a natural symplectic structure and the centroaffine curvature evolves according to the KdV equation when the flow is generated by a Hamiltonian given by the total centroaffine curvature. Chou and Qu ([5, 6]) showed that many soliton equations arise from special motions of plane or space curves. Moreover, Hoffmann and Kutz ([7]) showed that special motions of curves in the complex 1 or 2-space or the complex projective line whose curvature evolves according to the mKdV or KdV equation can be discretized by use of cross ratios.

In this paper we study theory of curves in the complex hyperbola which are determined by a certain curvature up to some symmetry. We shall show that special motions of curves are linked with the Burgers hierarchy, which can be formulated as a Hamiltonian system and also leads to the difference Burgers equation ([8]) via their discretization.

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2. Curves in the complex hyperbola

The complex hyperbola C is the subset of the complex 2-plane $\mathbf{C}^2$ given by

$$C = \{(z, w) \in \mathbf{C}^2 \mid zw = 1\}.$$

In this paper we call an immersion from an interval I to C a curve in C. We put $\gamma = (z, w)$ for a curve in C. Since

$$\frac{z'}{z} = -\frac{w'}{w},$$

we have $z', w' \neq 0$. Since

$$\det\begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = -2\frac{z'}{z} \neq 0,$$

γ and γ' are linearly independent over $\mathbf{C}$.

By a direct calculation, we have

$$\gamma'' = \tau^2\gamma + \sqrt{-1}\kappa\gamma', \tag{2.1}$$

where

$$\tau = \frac{z'}{z}, \quad \kappa = -\sqrt{-1}\frac{\tau'}{\tau}.$$

Changing γ to $\tilde{\gamma} = (\alpha z^{\beta}, \alpha^{-1}w^{\beta})$ with $\alpha, \beta \in \mathbf{C}^{\times}$, we have

$$\tilde{\gamma}'' = (\beta\tau)^2\tilde{\gamma} + \sqrt{-1}\kappa\tilde{\gamma}'.$$

Hence κ is invariant under the change. We call κ the curvature of γ.

REMARK 2.1. When a curve γ is given from an arclength parametrized curve $\hat{\gamma}$ in the Euclidean 2-space $\mathbf{R}^2$ identified with the complex plane $\mathbf{C}$ by $\gamma = (e^{\hat{\gamma}}, e^{-\hat{\gamma}})$, then κ coincides with the curvature of $\hat{\gamma}$, i.e., $\hat{\gamma}'' = \sqrt{-1}\kappa\hat{\gamma}'$ holds.

The fundamental theorem for curves in C is stated as follows:

Proposition 2.2. *For any function $\kappa\colon I \to \mathbf{C}$, there exists a curve $\gamma = (z, w)$ in C with curvature κ. If $\tilde{\gamma} = (\tilde{z}, \tilde{w})\colon I \to C$ is another curve with curvature κ, there exist constants $\alpha, \beta \in \mathbf{C}^{\times}$ such that $\tilde{z} = \alpha z^{\beta}$.*

REMARK 2.3. As in the theory of curves in the Euclidean 2-space, a curve γ in C with curvature κ can be given explicitly:

$$\gamma = (z, z^{-1}), \quad z = \exp\left(\int \exp\left(-\sqrt{-1}\int \kappa\, ds\right) ds\right).$$

A curve γ in C with zero curvature is given by $\gamma = (\alpha e^{\beta s}, \alpha^{-1} e^{-\beta s})$ with $\alpha, \beta \in \mathbf{C}^{\times}$.

3. Motions of curves and the Burgers hierarchy

A motion of a curve in C is given by a map $\gamma = \gamma(s, t)$ from the product of intervals I and J to C, where $t \in J$ is considered to be a time parameter and $s \in I$ is a parameter of a curve with fixed time. The time evolution of γ is given by

$$\gamma_t = \lambda\gamma + \mu\gamma_s, \tag{3.1}$$

where $\lambda, \mu\colon I \times J \to \mathbf{C}$. Since $(zw)_t = 0$, we have $\lambda = 0$ from (3.1). Hence we have

$$\gamma_t = \mu\gamma_s. \tag{3.2}$$

Differentiating (3.2) by s and using (2.1), we have

$$\gamma_{ts} = \tau^2\mu\gamma + (\mu_s + \sqrt{-1}\kappa\mu)\gamma_s. \tag{3.3}$$

Proposition 3.1. *The above time evolution exists if and only if $\tau \neq 0$ and*

$$\tau_t = \tau\mu_s + \tau_s\mu. \tag{3.4}$$

Proof. We have only to calculate the integrability condition:

$$\gamma_{sst} = \gamma_{tss}.$$

From (2.1) and (3.3), we have

$$\gamma_{sst} = (2\tau\tau_t + \sqrt{-1}\kappa\tau^2\mu)\gamma + (\tau^2\mu + \sqrt{-1}\kappa_t + \sqrt{-1}\kappa\mu_s - \kappa^2\mu)\gamma_s,$$
$$\gamma_{tss} = (2\tau\tau_s\mu + 2\tau^2\mu_s + \sqrt{-1}\kappa\tau^2\mu)\gamma + (\tau^2\mu + \mu_{ss} + \sqrt{-1}\kappa_s\mu + 2\sqrt{-1}\kappa\mu_s - \kappa^2\mu)\gamma_s.$$

Comparing the coefficients of γ and γ_s in the right-hand sides, we have (3.4) and

$$\sqrt{-1}\kappa_t = \mu_{ss} + \sqrt{-1}\kappa\mu_s + \sqrt{-1}\kappa_s\mu. \tag{3.5}$$

Since

$$\kappa = -\sqrt{-1}(\log\tau)_s, \tag{3.6}$$

(3.5) is also derived from (3.4). □

Note that (3.5) can be written as

$$\kappa_t = \Omega\mu_s,$$

where $\Omega = (-\sqrt{-1}D_s + \kappa + \kappa_s D_s^{-1})$ is the recursion operator of the Burgers equation:

$$\kappa_t = -\sqrt{-1}\kappa_{ss} + 2\kappa\kappa_s.$$

Moreover, it is not so hard to see by induction on $n \in \mathbf{N}$ that

$$D_s^{-1}\Omega^{n-1}\kappa_s = (-\sqrt{-1})^n \frac{D_s^n \tau}{\tau}. \tag{3.7}$$

Then we have the following:

Theorem 3.2. *The curvature of a curve in C associated to the time evolution*:

$$\gamma_t = (D_s^{-1}\Omega^{n-1}\kappa_s)\gamma_s \quad (n \in \mathbf{N})$$

evolves according to the Burgers hierarchy:

$$\kappa_t = \Omega^n \kappa_s. \tag{3.8}$$

In particular, κ and τ are related by the Cole-Hopf transformation (3.6) *and τ evolves according to the equation*:

$$\tau_t = (-\sqrt{-1})^n D_s^{n+1}\tau. \tag{3.9}$$

REMARK 3.3. From (3.4), $\log\tau$ evolves according to the equation:

$$(\log\tau)_t = \mu_s + \mu(\log\tau)_s. \tag{3.10}$$

When $\mu = \sqrt{-1}\kappa$, (3.10) becomes the potential Burgers equation:

$$(\log\tau)_t = (\log\tau)_{ss} + (\log\tau)_s^2.$$

When $\mu = \log\tau$, (3.10) becomes the equation:

$$(\log\tau)_t = (\log\tau)_s + (\log\tau)(\log\tau)_s,$$

whose solutions are given explicitly:

$$\log\tau + 1 = \varphi(s + t(\log\tau + 1)),$$

where φ is an arbitrary function.

REMARK 3.4. We can reduce a curve γ in C to a curve p in $\mathbf{C}^{\times}$ by putting $p = z$. Then τ plays a role of the curvature of p. If $\tilde{p}\colon I \to \mathbf{C}^{\times}$ is another curve with curvature τ, there exists a constant $\alpha \in \mathbf{C}^{\times}$ such that $\tilde{p} = \alpha p$. The time evolution of γ reduces to that of p:

$$p_t = (\tau\mu)p,$$

which satisfies (3.4).

4. A formulation as a Hamiltonian system

In this section, we give a formal Hamiltonian system describing the motion of closed curves in the complex hyperbola C given in Theorem 3.2.

We denote by $\mathcal{M}$ the space of all closed curves in C, that is, the set of all curves $\gamma : S^1 \to C$, where

$$S^1 = \mathbf{R}/2\pi\mathbf{Z} \cong \{z \in \mathbf{C} \mid |z| = 1\}.$$

From (3.2), the tangent space of $\mathcal{M}$ at γ can be identified with the set $\{\mu\gamma' \mid \mu\colon S^1 \to \mathbf{C}\}$. We note that any tangent vector $\mu\gamma'$ with $\mu\colon S^1 \to \mathbf{R}$ comes from an action of diffeomorphism group of the unit circle $\mathrm{Diff}(S^1)$ by

$$\varphi \cdot \gamma = \gamma \circ \varphi \quad (\varphi \in \mathrm{Diff}(S^1),\ \gamma \in \mathcal{M}).$$

Indeed, if φ_t is a curve in $\mathrm{Diff}(S^1)$ such that

$$\varphi_0 = \text{identity}\,, \quad \left.\frac{d}{dt}\right|_{t=0} \varphi_t = \mu\frac{\partial}{\partial s},$$

we have

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi_t \cdot \gamma) = \mu\gamma'.$$

We define a 2-form ω on $\mathcal{M}$ as follows:

$$\omega(\mu_1\gamma', \mu_2\gamma') = \mathrm{Im}\int_{S^1} \mu_1\overline{\mu_2}|\tau|^2\, ds \quad (\mu_1, \mu_2\colon S^1 \to \mathbf{C}).$$

By a direct calculation, we can verify that ω is closed, hence it defines a ‘symplectic structure’ on $\mathcal{M}$. For $n \in \mathbf{N}$ we define a function on $\mathcal{M}$ by

$$H_n(\gamma) = \frac{(-\sqrt{-1})^{n-1}}{2}\int_{S^1} \tau^{(n-1)}\overline{\tau}\, ds \quad (\gamma \in \mathcal{M}).$$

Then we have the following:

Theorem 4.1. *The Hamiltonian vector field X_n for H_n with respect to ω is given by*

$$(X_n)_\gamma = (D_s^{-1}\Omega^{n-1}\kappa')\gamma' \quad (\gamma \in \mathcal{M}).$$

Hence H_n is the Hamiltonian of the motion of curves in C associated with (3.8).

Proof. For a one-parameter family $\gamma(\,\cdot\,, t) \in \mathcal{M}$ such that $\gamma_t = \mu\gamma_s$, we have

$$\begin{aligned}
\frac{d}{dt}H_n(\gamma) &= \frac{(-\sqrt{-1})^{n-1}}{2}\int_{S^1}\frac{\partial}{\partial t}\{(D_s^{n-1}\tau)\overline{\tau}\}\,ds \\
&= \frac{(-\sqrt{-1})^{n-1}}{2}\int_{S^1}\{(D_s^{n-1}\tau_t)\overline{\tau} + (D_s^{n-1}\tau)\overline{\tau_t}\}\,ds \\
&= \frac{(-\sqrt{-1})^{n-1}}{2}\int_{S^1}[\{D_s^{n}(\tau\mu)\}\overline{\tau} + (D_s^{n-1}\tau)D_s(\overline{\tau\mu})]\,ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Proposition 3.1)} \\
&= \frac{(-\sqrt{-1})^{n-1}}{2}\int_{S^1}\{(-1)^n(D_s^{n}\overline{\tau})\tau\mu - (D_s^{n}\tau)\overline{\tau\mu}\}\,ds \\
&= \operatorname{Im}\int_{S^1}(-\sqrt{-1})^n(D_s^{n}\tau)\overline{\tau\mu}\,ds.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\omega((D_s^{-1}\Omega^{n-1}\kappa_s)\gamma_s, \mu\gamma_s) &= \operatorname{Im}\int_{S^1}(D_s^{-1}\Omega^{n-1}\kappa_s)\overline{\mu}|\tau|^2\,ds \\
&= \operatorname{Im}\int_{S^1}(-\sqrt{-1})^n(D_s^{n}\tau)\overline{\tau\mu}\,ds
\end{aligned}$$

by (3.7). □

REMARK 4.2. The symplectic structure ω on $\mathcal{M}$ comes from a flat Kähler structure. In fact, $\mathcal{M}$ has a natural complex structure $T\mathcal{M} \ni \mu\gamma' \mapsto \sqrt{-1}\mu\gamma' \in T\mathcal{M}$, and ω is the fundamental 2-form of a Hermitian metric h defined by

$$h(\mu_1\gamma', \mu_2\gamma') = 2\int_{S^1}\mu_1\overline{\mu_2}|\tau|^2\,ds.$$

For a vector field $\tilde{\mu}\gamma_s$ along a path $\gamma(\,\cdot\,, t)$ of $\mathcal{M}$ with the velocity vector $\gamma_t = \mu\gamma_s$, we set

$$\nabla_{\mu\gamma_s}(\tilde{\mu}\gamma_s) = (\tilde{\mu}_t + (\log\tau)_t\tilde{\mu})\gamma_s.$$

Then, one can verify that ∇ is the Levi-Civita connection for h and $(\mathcal{M}, h)$ is flat as a Riemannian manifold.

5. Discretization

The Burgers equation can be discretized in terms of soliton theory ([8]). For a function $f(x)$, the advanced and the central difference operators Δ_{+x} and Δ_x are defined by

$$\Delta_{+x} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad \Delta_x f(x) = \frac{f(x+\Delta x/2) - f(x-\Delta x/2)}{\Delta x}.$$

Using these operators, we discretize (3.9) as

$$\Delta_{+t}\tau(s,t) = \begin{cases} (-\sqrt{-1})^n \Delta_s^{n+1}\tau(s,t) & (n = 2m-1,\ m \in \mathbf{N}), \\ (-\sqrt{-1})^n \Delta_{+s}\Delta_s^n \tau(s,t) & (n = 2m,\ m \in \mathbf{N}). \end{cases}$$

By setting $\tau_{i,j} = \tau(i\Delta s, j\Delta t)$ $(i \in \mathbf{Z},\ j \in \mathbf{N})$, we have a more explicit equation:

$$\tau_{i,j+1} - \tau_{i,j} = \begin{cases} \dfrac{(-\sqrt{-1})^n \Delta t}{(\Delta s)^{n+1}} \displaystyle\sum_{k=-m}^{m} (-1)^{m+k} \binom{n+1}{m+k} \tau_{i+k,j} & (n = 2m-1), \\ \dfrac{(-\sqrt{-1})^n \Delta t}{(\Delta s)^{n+1}} \displaystyle\sum_{k=-m}^{m+1} (-1)^{m+k+1} \binom{n+1}{m+k} \tau_{i+k,j} & (n = 2m). \end{cases}$$

If we put

$$\alpha = (-1)^m \frac{(-\sqrt{-1})^n \Delta t}{(\Delta s)^{n+1}}, \quad c_k = (-1)^{|k|} \binom{n+1}{m+k},$$

we have

$$\tau_{i,j+1} = \begin{cases} (1 + c_0\alpha)\tau_{i,j} + \alpha \displaystyle\sum_{k=-m}^{-1} c_k \tau_{i+k,j} + \alpha \sum_{k=1}^{m} c_k \tau_{i+k,j} & (n = 2m-1), \\ (1 - c_0\alpha)\tau_{i,j} - \alpha \displaystyle\sum_{k=-m}^{-1} c_k \tau_{i+k,j} - \alpha \sum_{k=1}^{m+1} c_k \tau_{i+k,j} & (n = 2m). \end{cases} \tag{5.1}$$

The difference Cole-Hopf transformation given by

$$\kappa_{i,j} = -\sqrt{-1}\frac{\tau_{i+1,j}}{\tau_{i,j}}$$

leads to the difference Burgers hierarchy:

$$\kappa_{i,j+1} = \kappa_{i,j} A_{i,j}, \tag{5.2}$$

where

$$A_{i,j} = \frac{1 + c_0\alpha + \alpha \sum_{k=1}^{m} c_k \prod_{l=1}^{k}(\sqrt{-1}\kappa_{i+l,j}) + \alpha \sum_{k=-m}^{-1} c_k \prod_{l=-k+1}^{0}(\sqrt{-1}\kappa_{i+l,j})^{-1}}{1 + c_0\alpha + \alpha \sum_{k=1}^{m} c_k \prod_{l=0}^{k-1}(\sqrt{-1}\kappa_{i+l,j}) + \alpha \sum_{k=-m}^{1} c_k \prod_{l=-k}^{-1}(\sqrt{-1}\kappa_{i+l,j})^{-1}}$$

when $n = 2m - 1$, and

$$A_{i,j} = \frac{1 - c_0\alpha - \alpha \sum_{k=1}^{m+1} c_k \prod_{l=1}^{k}(\sqrt{-1}\kappa_{i+l,j}) - \alpha \sum_{k=-m}^{-1} c_k \prod_{l=-k+1}^{0}(\sqrt{-1}\kappa_{i+l,j})^{-1}}{1 - c_0\alpha - \alpha \sum_{k=1}^{m+1} c_k \prod_{l=0}^{k-1}(\sqrt{-1}\kappa_{i+l,j}) + \alpha \sum_{k=-m}^{1} c_k \prod_{l=-k}^{-1}(\sqrt{-1}\kappa_{i+l,j})^{-1}}$$

when $n = 2m$.

A discrete curve in C is a map $\gamma_i = (z_i, w_i)$ $(i \in \mathbf{Z})$ from integers $\mathbf{Z}$ to C such that $\gamma_i \neq \gamma_{i+1}$. The discrete curvature κ_i is given by

$$\kappa_i = -\sqrt{-1}\frac{\log(z_{i+2}/z_{i+1})}{\log(z_{i+1}/z_i)} = -\sqrt{-1}\frac{\log(w_{i+2}/w_{i+1})}{\log(w_{i+1}/w_i)}.$$

We define the discrete time evolution of a discrete curve in C by

$$z_{i,j+1} = \begin{cases} z_{i,j} \displaystyle\prod_{k=-m}^{m} (z_{i+k,j})^{c_k\alpha} & (n = 2m - 1), \\ z_{i,j} \displaystyle\prod_{k=-m}^{m+1} (z_{i+k,j})^{-c_k\alpha} & (n = 2m). \end{cases} \tag{5.3}$$

The right-hand side is determined independently of a choice of branch of log because the sum of the exponents $c_k\alpha$ is equal to zero. We can easily verify that $\tau_{i,j} = \log(z_{i+1,j}/z_{i,j})$ of a curve evolving according to (5.3) satisfies (5.1), thereby we have the following:

Theorem 5.1. *The discrete curvature of a discrete curve in C associated to the discrete time evolution* (5.3) *evolves according to the difference Burgers hierarchy* (5.2), *while $\tau_{i,j}$ evolves according to the difference equation* (5.1).

REMARK 5.2. It is obvious to see that the discrete time evolution (5.3) keeps the periodicity of curves in C.

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