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THE GLAUBERMAN CORRESPONDENT OF A NILPOTENT BLOCK OF A FINITE GROUP

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Abstract

We prove that a nilpotent block of a finite group and its Glauberman correspondent are basic Morita equivalent.

1. Introduction

Let A and G be finite groups such that A is solvable, A acts on G by automorphisms and that the orders $|A|$ and $|G|$ are coprime. Let $C = C_G(A)$. Let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system such that $\mathcal{K}$ contains the $|G|$-th roots of unity and k is algebraically closed. We denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G and by $\mathrm{Irr}(G)^A$ the set of A-invariant elements of $\mathrm{Irr}(G)$. Then there is a natural one-to-one and onto map $\pi(G, A)\colon \mathrm{Irr}(G)^A \to \mathrm{Irr}(C)$ which we call the Glauberman character correspondence ([5] and [10], Chapter 13).

Let B be a (p-)block of G. B is a block algebra of $\mathcal{O}G$. By [19], Proposition 1 and Theorem 1, if B is A-invariant and a defect group D of B is centralized by A, then any $\chi \in \mathrm{Irr}(B)$, the set of ordinary irreducible characters of B, is A-invariant and there exists a block b of C such that $\pi(G, A)(\mathrm{Irr}(B)) = \mathrm{Irr}(b)$ (see [12], Theorem 4.3 for another proof of [19], Proposition 1). Moreover D is a defect group of b, and $\pi(G, A)$ gives an isotypy between B and b in the sense of [3], by [19], Proposition 5. Then we call b the Glauberman correspondent of B. In [8], Theorem 1.1, it is shown that if G is p-solvable, then there is a basic Morita equivalence, in the sense of [15], Chapter 7, between B and b such that the bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$ induced by it is $\pi(G, A)_{|\mathrm{Irr}(B)}$. Moreover in [7], Theorem 2 and [17], Theorem 4.9, it is shown that if the defect group D is normal in G, then there is a splendid Morita equivalence between B and b such that the bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$ induced by it is $\pi(G, A)_{|\mathrm{Irr}(B)}$ (see also [11], [9] and [16]). In this paper we prove the following.

Theorem 1. *With the above notations, let B be an A-invariant block of G such that a defect group D of B is contained in C, and let b be the Glauberman correspondent of B. Assume that B is nilpotent. Then b is nilpotent and there is a basic*

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Morita equivalence between B and b such that the bijection $\mathrm{Irr}(B)$ *and* $\mathrm{Irr}(b)$ *induced by it is* $\pi(G, A)_{|\mathrm{Irr}(B)}$.

For notations and terminologies in this paper, we follow [18]. A module is a left module unless stated explicitly.

We remark that in [6], it is shown that a block of a finite group and its Isaacs correspondent are basic Morita equivalent.

2. Proof of Theorem 1

Let G be a finite group and B be a block of G. We denote by 1_B the identity element of B. Let D_γ be a maximal local pointed group contained in the pointed group $G_{\{1_B\}}$ on $\mathcal{O}G$ and let $i \in \gamma$, then i is called a D-source idempotent of B, and iBi is called a source algebra of B. If L is a left (resp. right) iBi-module, then L is regarded as a left (resp. right) $\mathcal{O}D$-module via the $\mathcal{O}$-algebra homomorphism from $\mathcal{O}D$ to iBi.

Let G and G' be finite groups. If M is an $(\mathcal{O}G', \mathcal{O}G)$-bimodule, M is considered as an $\mathcal{O}(G' \times G)$-module through $(x', x)m = x'mx^{-1}$ for any $x' \in G'$, $x \in G$ and $m \in M$, which we denote by ${}_{G'\times G}M$. Similarly, if M is an $\mathcal{O}(G' \times G)$-module, then M is considered as an $(\mathcal{O}G', \mathcal{O}G)$-bimodule. Moreover, for $(\mathcal{O}G', \mathcal{O}G)$-bimodules M_1, M_2, those are isomorphic if and only if ${}_{G'\times G}M_1$ and ${}_{G'\times G}M_2$ are so. Let D and D' be subgroups of G and G' respectively and N be an $(\mathcal{O}D', \mathcal{O}D)$-bimodule. We have an isomorphism of $\mathcal{O}(G' \times G)$-modules

$$N_{D'\times D}^{G'\times G} \cong \mathcal{O}G' \otimes_{\mathcal{O}D'} N \otimes_{\mathcal{O}D} \mathcal{O}G$$

mapping $(x', x) \otimes n$ to $x' \otimes n \otimes x^{-1}$ for any $x' \in G'$, $x \in G$ and $n \in N$ where $N_{D'\times D}^{G'\times G}$ is the induced $\mathcal{O}(G' \times G)$-module from the $\mathcal{O}(D' \times D)$-module N.

Lemma 2. *Let G and G' be finite groups, and let B and B' be blocks of G and G' with defect groups D and D' respectively. Let i (resp. i') be a D (resp. D')-source idempotent of B (resp. B'). If N is an $(i'B'i', iBi)$-bimodule, then as $\mathcal{O}(G' \times G)$-modules*

$$B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB \mid N_{D'\times D}^{G'\times G}$$

where $N_{D'\times D}^{G'\times G}$ is the induced $\mathcal{O}(G' \times G)$-module from the $\mathcal{O}(D' \times D)$-module N.

Proof. By the above it suffices to show that as $(\mathcal{O}G', \mathcal{O}G)$-bimodules, $B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB$ is a component of $\mathcal{O}G' \otimes_{\mathcal{O}D'} N \otimes_{\mathcal{O}D} \mathcal{O}G$. By [13], Theorem 1.2, there exist $c, d \in (\mathcal{O}G)^D$ such that $1_B = \sum_{u\in U} ucidu^{-1}$ where U is the set of representatives of G/D. Here $(\mathcal{O}G)^D$ denotes the set of D-fixed elements of $\mathcal{O}G$. Similarly there exist $c', d' \in (\mathcal{O}G')^{D'}$ such that $1_{B'} = \sum_{u'\in U'} u'c'i'd'u'^{-1}$ where U' is the set of representatives of G'/D'. We can construct a homomorphism of $(\mathcal{O}G', \mathcal{O}G)$-bimodules

$$\varphi\colon B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB \to \mathcal{O}G' \otimes_{\mathcal{O}D'} N \otimes_{\mathcal{O}D} \mathcal{O}G$$

which satisfies

$$\varphi(a' \otimes n \otimes a) = \sum_{u'\in U'} \sum_{u\in U} a'u'c' \otimes (i'd'u'^{-1}i')n(iuci) \otimes du^{-1}a$$

for any $a' \in B'i'$, $n \in N$ and $a \in iB$. On the other hand there is a homomorphism of $(\mathcal{O}G', \mathcal{O}G)$-bimodules

$$\psi : \mathcal{O}G' \otimes_{\mathcal{O}D'} N \otimes_{\mathcal{O}D} \mathcal{O}G \to B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB$$

which satisfies

$$\psi(x' \otimes n \otimes x) = x'i' \otimes n \otimes ix$$

for any $x' \in \mathcal{O}G'$, $n \in N$ and $x \in \mathcal{O}G$. For $a' \otimes n \otimes a \in B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB$ where $a' \in B'i'$, $n \in N$ and $a \in iB$, we have

$$\begin{aligned}
(\psi \circ \varphi)(a' \otimes n \otimes a) &= \sum_{u'\in U'} \sum_{u\in U} a'u'c'i' \otimes (i'd'u'^{-1}i')n(iuci) \otimes idu^{-1}a \\
&= \sum_{u'\in U'} \sum_{u\in U} a'i'u'c'i' \otimes (i'd'u'^{-1}i')n(iuci) \otimes idu^{-1}ia \\
&= a' \otimes i'ni \otimes a \\
&= a' \otimes n \otimes a.
\end{aligned}$$

This implies that $B'i' \otimes_{i'B'i'} N \otimes_{iBi} iB$ is a component of $\mathcal{O}G' \otimes_{\mathcal{O}D'} N \otimes_{\mathcal{O}D} \mathcal{O}G$ as $(\mathcal{O}G', \mathcal{O}G)$-bimodules. This completes the proof. $\square$

Proof of Theorem 1. Firstly we recall the arguments in [19], §3. Let (D, B_D) be an A-invariant maximal B-Brauer pair. For $Q \leq D$, let (Q, B_Q) be a unique B-Brauer pair contained in (D, B_D). Then B_Q is A-invariant and a defect group of B_Q is centralized by A. Let b_Q be the Glauberman correspondent of B_Q. Then b_Q is associated with b and $(Q, b_Q) \subset (D, b_D)$. In particular (D, b_D) is a maximal b-Brauer pair. Moreover, by [19], Proposition 4, the Brauer categories of B and b are equivalent, and $N_G(Q, B_Q) = C_G(Q)N_C(Q, b_Q)$. Therefore $N_C(Q, b_Q)/C_C(Q)$ is a p-group for any $Q \leq D$, and hence b is a nilpotent block.

Let U be a set of representatives for the conjugacy classes of D and for each $u \in U$, set $B_u = B_{\langle u\rangle}$ and $b_u = b_{\langle u\rangle}$. By [2], Theorem 1.2 or [14], 1.7, $\{(u, B_u) \mid u \in U\}$ is a set of representatives for the G-conjugacy classes of B-Brauer elements and $\{(u, b_u) \mid u \in U\}$ is a set of representatives for the C-conjugacy classes of b-Brauer elements. By [2], Theorem 1.2 or [14], 1.9, B_u (resp. b_u) has a unique irreducible Brauer character $\varphi^{(u)}$ (resp. $\varphi^{*(u)}$). For $\chi \in \mathrm{Irr}(B)$, we denote by $d(\chi, u, \varphi^{(u)})$ the generalized decomposition number of χ with respect to $\varphi^{(u)}$. Also we set $\chi^* = \pi(G, A)(\chi)$. Since there is an isotypy between B and b given by the Glauberman character correspondence by [19], Proposition 5, for any $\chi \in \mathrm{Irr}(B)$ and $u \in U$

$$d(\chi, u, \varphi^{(u)}) = \epsilon_\chi \tilde{\omega}(u)\, d(\chi^*, u, \varphi^{*(u)}) \tag{1}$$

where $\epsilon_\chi = \pm 1$ and $\tilde{\omega}(u) = \pm 1$ because B_u and b_u has the same defect.

Let i (resp. j) be a D-source idempotent of B (resp. b). By [13], Theorem 3.5, there is a Morita equivalence between the source algebra iBi of B and the block algebra B realized by the (B, iBi)-bimodule Bi, and similarly there is a Morita equivalence between jbj and b realized by the (b, jbj)-bimodule bj. Moreover by [14], 1.6 and 1.8, there exist $\mathcal{O}$-simple interior D-algebras S and T such that

$$iBi \cong S \otimes_{\mathcal{O}} \mathcal{O}D \quad \text{and} \quad jbj \cong T \otimes_{\mathcal{O}} \mathcal{O}D$$

as interior D-algebras. In fact S and T are primitive Dade D-algebras. Suppose that

$$S \cong \mathrm{End}_{\mathcal{O}}(V) \quad \text{and} \quad T \cong \mathrm{End}_{\mathcal{O}}(W)$$

for some $\mathcal{O}$-free modules V and W. Thus V and W become indecomposable endo-permutation $\mathcal{O}D$-modules with vertex D. Now there is a Morita equivalence between the group algebra $\mathcal{O}D$ and iBi (resp. jbj) realized by the $(iBi, \mathcal{O}D)$ (resp. $(jbj, \mathcal{O}D)$)-bimodule $V \otimes_{\mathcal{O}} \mathcal{O}D$ (resp. $W \otimes \mathcal{O}D$). Hence we obtain the equivalences $\Psi_B\colon \mathrm{mod}(\mathcal{O}D) \to \mathrm{mod}(B)$ and $\Psi_b\colon \mathrm{mod}(\mathcal{O}D) \to \mathrm{mod}(b)$ where $\mathrm{mod}(B)$ denotes the category of finitely generated B-modules. Thus Ψ_B is realized by $Bi \otimes_{iBi} (V \otimes_{\mathcal{O}} \mathcal{O}D)$, and Ψ_b is realized by $bj \otimes_{jbj} (W \otimes_{\mathcal{O}} \mathcal{O}D)$.

For $\lambda \in \mathrm{Irr}(D)$, let L_λ be an $\mathcal{O}D$-lattice with the character λ. Also set $M_\lambda = Bi \otimes_{iBi} (V \otimes_{\mathcal{O}} L_\lambda)$ and $N_\lambda = bj \otimes_{jbj} (W \otimes_{\mathcal{O}} L_\lambda)$. Then $\lambda \in \mathrm{Irr}(D) \leftrightarrow \chi_\lambda \in \mathrm{Irr}(B)$ is a bijection induced by the equivalence Ψ_B, where χ_λ is the character of M_λ. By [14], 1.12, we have for any $u \in U$

$$d(\chi_\lambda, u, \varphi^{(u)}) = \omega(u)\lambda(u) \tag{2}$$

where $\omega(u) = \pm 1$. Similarly $\lambda \in \mathrm{Irr}(D) \leftrightarrow \zeta_\lambda \in \mathrm{Irr}(b)$ is a bijection induced by the equivalence Ψ_b, where ζ_λ is the character of N_λ. We have also for any $u \in U$

$$d(\zeta_\lambda, u, \varphi^{*(u)}) = \omega^*(u)\lambda(u) \tag{3}$$

where $\omega^*(u) = \pm 1$.

From (2) and (3) we have $\chi_\nu * \lambda = \chi_{\nu\lambda}$ and $\zeta_\nu * \lambda = \zeta_{\nu\lambda}$ for a linear character ν of D and $\lambda \in \mathrm{Irr}(D)$ where $\chi_\nu * \lambda$ is a Broué-Puig's generalized character defined in [1] (see [2], Theorem 1.2). Let 1_D be the trivial character of D. From (1)–(3), if we set $(\chi_{1_D})^* = \zeta_\eta$ for some $\eta \in \mathrm{Irr}(D)$, then η is linear. Note that if p is odd, then $\eta = 1_D$. Now, since there is an isotypy between B and b given by the Glauberman character correspondence, we can see

$$(\chi_\lambda)^* = (\chi_{1_D} * \lambda)^* = (\chi_{1_D})^* * \lambda = \zeta_\eta * \lambda = \zeta_{\eta\lambda}. \tag{4}$$

Now since η is linear, the $\mathcal{O}D$-bimodule $L_\eta \otimes_{\mathcal{O}} \mathcal{O}D$ realizes an equivalence of $\mathrm{mod}(\mathcal{O}D)$, which we denote by Ψ. Let $\Pi = \Psi_b \Psi \Psi_B^{-1}$: $\mathrm{mod}(B) \to \mathrm{mod}(b)$ where Ψ_B^{-1} is the equivalence from $\mathrm{mod}(B)$ to $\mathrm{mod}(\mathcal{O}D)$ realized by the $(\mathcal{O}D, B)$-bimodule $(V^* \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{iBi} iB$ where V^* is the dual module. Note that V^* is a right S-module. Thus Π is realized by the (b, B)-bimodule

$$bj \otimes_{jbj} (W \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (L_\eta \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (V^* \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{iBi} iB.$$

Moreover we see that the bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$ induced by Π is $\pi(G, A)_{|\mathrm{Irr}(B)}$ from (4).

Now we have a (b, B)-bimodule isomorphism

$$\begin{aligned} &bj \otimes_{jbj} (W \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (L_\eta \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{\mathcal{O}D} (V^* \otimes_{\mathcal{O}} \mathcal{O}D) \otimes_{iBi} iB \\ &\cong bj \otimes_{jbj} (W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*) \otimes_{iBi} iB. \end{aligned}$$

Here $W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*$ is regarded as a (jbj, iBi)-bimodule through

$$(t \otimes d_1)(w \otimes l \otimes a \otimes v^*)(s \otimes d_2) = tw \otimes d_1 l \otimes d_1 a d_2 \otimes v^* s$$

for any $t \in T$, $d_1, d_2 \in D$, $w \in W$, $l \in L_\eta$, $a \in \mathcal{O}D$, $v^* \in V^*$ and $s \in S$ identifying iBi (resp. jbj) with $S \otimes_{\mathcal{O}} \mathcal{O}D$ (resp. $T \otimes_{\mathcal{O}} \mathcal{O}D$). Let $\Delta D = \{(d, d) \in C \times G \mid d \in D\}$ and let

$$M = bj \otimes_{jbj} (W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*) \otimes_{iBi} iB.$$

In order to complete the proof, it suffices to show that M as an $\mathcal{O}(C \times G)$-module has ΔD as a vertex by [15], Corollary 7.4.

Let

$$X = W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} V^*.$$

We regard X as an $\mathcal{O}(\Delta D)$-module by the following action.

$$(d, d)(w \otimes l \otimes v^*) = dw \otimes dl \otimes v^* d^{-1}$$

where $d \in D$, $w \in W$, $l \in L_\eta$ and $v^* \in V^*$. We show jMi and $X_{\Delta D}^{D \times D}$ are isomorphic as $\mathcal{O}(D \times D)$-modules. (Note that jMi is a (jbj, iBi)-bimodule, and hence this is an

$\mathcal{O}D$-bimodule.) Here jMi is identified with $W \otimes_{\mathcal{O}} L_\eta \otimes_{\mathcal{O}} \mathcal{O}D \otimes_{\mathcal{O}} V^*$. Now we have an $\mathcal{O}$-linear map

$$f\colon X_{\Delta D}^{D\times D} \to jMi$$

defined by

$$f((d_1, d_2) \otimes (w \otimes l \otimes v^*)) = d_1 w \otimes d_1 l \otimes d_1 d_2^{-1} \otimes v^* d_2^{-1}$$

for any $d_1, d_2 \in D$, $w \in W$, $l \in L_\eta$ and $v^* \in V^*$. Then f is an $\mathcal{O}(D \times D)$-homomorphism. On the other hand we have an $\mathcal{O}$-linear map

$$g\colon jMi \to X_{\Delta D}^{D\times D}$$

defined by

$$g(w \otimes l \otimes d \otimes v^*) = (d, 1) \otimes (d^{-1} w \otimes d^{-1} l \otimes v^*)$$

for any $w \in W$, $l \in L_\eta$, $d \in D$ and $v^* \in V^*$. We see that g is an $\mathcal{O}$-isomorphism with the inverse f. Thus jMi and $X_{\Delta D}^{D\times D}$ are isomorphic as $\mathcal{O}(D \times D)$-modules.

By Lemma 2, M is a component of the induced module $(jMi)_{D\times D}^{C\times G}$ because $M \simeq \mathcal{O}C \otimes_{jbj} jMi \otimes_{iBi} \mathcal{O}G$. Hence by the above M is a component of $X_{\Delta D}^{C\times G}$. Hence M is ΔD-projective. Now jMi is a component of M as $\mathcal{O}(D \times D)$-modules. Since $jMi \cong X_{\Delta D}^{D\times D}$, X is a component of jMi as $\mathcal{O}(\Delta D)$-modules, and hence X is a component of M as $\mathcal{O}(\Delta D)$-modules. Since $p \nmid \operatorname{rank}_{\mathcal{O}} X$ from [14], 1.6 and 1.8 or [18], Corollary 28.11, an indecomposable component of X has ΔD as a vertex, and hence an indecomposable component of M as an $\mathcal{O}(\Delta D)$-module has ΔD as a vertex. This implies that ΔD is a vertex of M. Hence B and b are basic Morita equivalent. This completes the proof.

Here we give a direct proof of the fact that M has an endo-permutation $\mathcal{O}(\Delta D)$-module as a source (see [15], Corollary 7.4). Since W, L_η and V are endo-permutation $\mathcal{O}D$-modules, X is an endo-permutation $\mathcal{O}(\Delta D)$-module, in fact X is capped in the sense of [4]. Now, since $M \mid X_{\Delta D}^{C\times G}$, there exists an indecomposable component Y of X such that $M \mid Y_{\Delta D}^{C\times G}$. Then Y is an endo-permutation $\mathcal{O}(\Delta D)$-module and it is a source module of M. We note that $Y = \operatorname{cap}(X)$. $\square$

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