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ON THE UNIPOTENT SUPPORT OF CHARACTER SHEAVES

Dedicated to Professors Ken-ichi Shinoda and Toshiaki Shoji on their 60th birthday

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Abstract

Let G be a connected reductive group over $\mathbb{F}_q$, where q is large enough and the center of G is connected. We are concerned with Lusztig's theory of *character sheaves,* a geometric version of the classical character theory of the finite group $G(\mathbb{F}_q)$. We show that under a certain technical condition, the restriction of a character sheaf to its *unipotent support* (as defined by Lusztig) is either zero or an irreducible local system. As an application, the generalized Gelfand-Graev characters are shown to form a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module of unipotently supported virtual characters of $G(\mathbb{F}_q)$ (Kawanaka's conjecture).

1. Introduction

Let G be a connected reductive algebraic group over $\overline{\mathbb{F}}_p$, an algebraic closure of the finite field with p elements where p is a prime. Let q be a power of p and assume that G is defined over the finite field $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_p$, with corresponding Frobenius map $F: G \to G$. Then it is an important problem to determine and to understand the values of the irreducible characters (in the sense of Frobenius) of the finite group G^F. For this purpose, Lusztig [12] has developed the theory of *character sheaves*; see [15] for a general overview. This theory produces some geometric objects over G (provided by intersection cohomology with coefficients in $\overline{\mathbb{Q}}_l$, where $l \neq p$ is a prime) from which the irreducible characters of G^F can be deduced for any q. In this way, the rather complicated patterns involved in the values of the irreducible characters of G^F are seen to be governed by geometric principles.

In this paper, we discuss an example of this interrelation between geometric principles and properties of character values. On the geometric side, we will be concerned with the restriction of a character sheaf A to the unipotent variety of G. Under some restriction on p, Lusztig [14] has associated to A a well-defined unipotent class $\mathcal{O}_A$ of G, called its *unipotent support*. We will be interested in the restriction of A to $\mathcal{O}_A$. Under a certain technical condition (formulated in [4], following a suggestion of Lusztig) the restriction of A to $\mathcal{O}_A$ is either zero or an irreducible G-equivariant local system on $\mathcal{O}_A$ (up to shift); see Section 3. The verification of that technical condition

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can be reduced to a purely combinatorial problem, involving the induction of characters of Weyl groups, the Springer correspondence and the data on families of characters in Chapter 4 of Lusztig's book [10]. The details of the somewhat lengthy case-by-case verification are worked out in the second author's thesis [6]; the main ingredients will be explained in Section 2.

On the character-theoretic side, we will consider the *generalized Gelfand-Graev representations* (GGGR's for short) introduced by Kawanaka [7], [8]. In Section 4, assuming that p, q are large and the center of G is connected, we deduce that Kawanaka's conjecture [9] holds, that is, the characters of the various GGGR's of G^F form a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module of unipotently supported virtual characters of G^F. As a further application, in Proposition 4.6, we obtain a new characterisation of GGGR's in terms of vanishing properties of their character values.

2. The Springer correspondence, families and induction

In this section, we deal with the combinatorial basis for the discussion of the unipotent support of character sheaves. We keep the basic assumptions of the introduction: G is a connected reductive algebraic group over $\overline{\mathbb{F}}_p$; we assume throughout that p is a good prime for G and that the center of G is connected. Let $B \subseteq G$ be a Borel subgroup and $T \subseteq B$ a maximal torus. Let $W = \mathrm{N}_G(T)/T$ be the Weyl group of G, with set of generators S determined by the choice of $T \subseteq B$.

Let $\mathrm{Irr}(W)$ be the set of irreducible characters of W (over an algebraically closed field of characteristic 0). The Springer correspondence associates with each $E \in \mathrm{Irr}(W)$ a pair (u, ψ) where $u \in G$ is unipotent (up to G-conjugacy) and ψ is an irreducible character of the group of components $\mathrm{A}_G(u) = \mathrm{C}_G(u)/\mathrm{C}_G(u)^\circ$; see [10, §13.1]. We write this correspondence as $E \leftrightarrow (u, \psi)$.

Now we can define three invariants a_E, b_E and d_E for $E \in \mathrm{Irr}(W)$.

- b_E is the smallest $i \geqslant 0$ such that E appears with non-zero multiplicity in the i-th symmetric power of the reflection representation of W; see [10, (4.1.2)].
- a_E is the largest $i \geqslant 0$ such that u^i divides the generic degree $D_E(u) \in \mathbb{Q}[u]$ defined in terms of the generic Iwahori-Hecke algebra over $\mathbb{Q}[u^{1/2}, u^{-1/2}]$; see [10, (4.1.1)].
- d_E is $\dim \mathfrak{B}_u$ where $\mathfrak{B}_u$ is the variety of Borel subgroups containing a unipotent $u \in G$ such that $E \leftrightarrow (u, \psi)$ for some $\psi \in \mathrm{Irr}(\mathrm{A}_G(u))$; see [10, §13.1].

We will be interested in several compatibility properties of these invariants.

Lemma 2.1. *We have $a_E \leqslant d_E \leqslant b_E$ for all $E \in \mathrm{Irr}(W)$.*

Proof. See [14, Corollary 10.9] for the first inequality and [18, §1.1] for the second. The inequality $a_E \leqslant b_E$ was first observed by Lusztig; see [10, 4.1.3]. □

Recall that $\mathrm{Irr}(W)$ is partitioned into *families* and that each family contains a unique *special* $E \in \mathrm{Irr}(W)$, that is, a character such that $a_E = b_E$; see [10, 4.1.4]. Furthermore, in [10, Chapter 4], Lusztig associates with any family $\mathcal{F} \subseteq \mathrm{Irr}(W)$ a finite group $\mathcal{G}_{\mathcal{F}}$, case-by-case for each type of finite Weyl group. (The groups $\mathcal{G}_{\mathcal{F}}$ form a crucial ingredient in the statement of the Main Theorem 4.23 of [10].) If G is simple modulo its center, then $\mathcal{G}_{\mathcal{F}} \cong \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5$ or $(\mathbb{Z}/2\mathbb{Z})^e$ for some $e \geqslant 0$.

Now let G^* be the Langlands dual of G, with Borel subgroup B^* and maximal torus $T^* \subseteq G^*$. Let $W^* = \mathrm{N}_{G^*}(T^*)/T^*$ be the Weyl group of G^*, with generating set S^* determined by $T^* \subseteq B^*$. We can naturally identify W and W^*. Note that a_E and b_E are independent of whether we regard E as a representation of W or of W^*. However, it does make a difference as far as d_E is concerned.

Let $s \in G^*$ be semisimple and W_s be the Weyl group of $\mathrm{C}_{G^*}(s)$. (Note that $\mathrm{C}_{G^*}(s)$ is a connected reductive group since the center of G is connected.) Replacing s by a conjugate, we may assume that $s \in T^*$. Then W_s is a subgroup of W^* and, hence, may be identified with a subgroup of W. So we can consider the induction of characters from W_s to W.

Proposition 2.2. *Let $s \in G^*$ be semisimple and $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family. If E_0 is the special character in $\mathcal{F}$, then we have*

$$\mathrm{Ind}_{W_s}^{W}(E_0) = E_0' + \textit{a combination of } \tilde{E} \in \mathrm{Irr}(W) \quad \textit{with} \quad b_{\tilde{E}} > d_{\tilde{E}} \geqslant b_{E_0},$$

where $E_0' \in \mathrm{Irr}(W)$ is such that $b_{E_0'} = d_{E_0'} = b_{E_0}$; furthermore, $E_0' \leftrightarrow (u, 1)$ under the Springer correspondence, where 1 stands for the trivial character.

Proof. See [14, §10] and [10, §13.1]. □

We are now looking for a condition which guarantees that all $\tilde{E} \neq E_0'$ occurring in the decomposition of $\mathrm{Ind}_{W_s}^{W}(E_0)$ have $d_{\tilde{E}} > b_{E_0}$. Following a suggestion of Lusztig, such a condition has been formulated in [4, 4.4]. In order to state it, we introduce the following notation.

Let $\mathcal{S}_G$ be the set of all pairs $(s, \mathcal{F})$ where $s \in G^*$ is semisimple (up to G^*-conjugacy) and $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ is a family. Following [10, §13.3], we define a map

$$\Phi_G : \mathcal{S}_G \to \{\text{unipotent classes of } G\},$$

as follows. Let $(s, \mathcal{F}) \in \mathcal{S}_G$ and $E_0 \in \mathcal{F}$ be special. Then consider the induction $\mathrm{Ind}_{W_s}^{W}(E_0)$ and let E_0' be as in Proposition 2.2. Now define $\mathcal{O} = \Phi_G(s, \mathcal{F})$ to be the unipotent class containing u where $E_0' \leftrightarrow (u, 1)$ under the Springer correspondence.

Proposition 2.3 (Hézard [6]). *Assume that $s \in G^*$ is semisimple and isolated, that is, $\mathrm{C}_{G^*}(s)$ is not contained in a Levi complement of any proper parabolic subgroup of G^*. Let $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family and assume that*

$$(*) \qquad |\mathcal{G}_{s,\mathcal{F}}| = |\mathrm{A}_G(u)| \quad \textit{where} \quad u \in \mathcal{O} = \Phi_G(s, \mathcal{F}).$$

Then the following sharper version of Proposition 2.2 *holds*: *If E_0 is the special character in $\mathcal{F}$, then we have*

$$\mathrm{Ind}_{W_s}^W(E_0) = E_0' + \textit{a combination of } \tilde{E} \in \mathrm{Irr}(W) \quad \textit{with} \quad d_{\tilde{E}} > b_{E_0}.$$

Proof. In the setting of Proposition 2.2, let us write

$$\mathrm{Ind}_{W_s}^W(E_0) = E_0' + E_0'' + \text{a combination of } \tilde{E} \in \mathrm{Irr}(W) \quad \text{with} \quad d_{\tilde{E}} > b_{E_0}$$

where E_0'' is the sum of all $\tilde{E} \in \mathrm{Irr}(W)$ such that $d_{\tilde{E}} = b_{E_0}$, $\tilde{E} \neq E_0'$ and $\tilde{E}$ appears in $\mathrm{Ind}_{W_s}^W(E_0)$. Thus, we must show that $E_0'' = 0$ if $(*)$ holds. By standard arguments, this can be reduced to the case where G is simple modulo its center.

The reflection subgroups of W which can possibly arise as W_s for some semisimple element $s \in G^*$ are classified by a standard algorithm; see [2].

Now, if G is of exceptional type, E_0'' can be computed in all cases using explicit tables for the Springer correspondence [18] and induce/restrict matrices for the characters of Weyl groups; see [6, §2.6] where tables specifying E_0'' can be found for each type of G. By inspection of these tables, one checks that if $(*)$ holds, then $E_0'' = 0$.

If G is of classical type, the induction of characters of Weyl groups and the Springer correspondence can be described in purely combinatorial terms, involving manipulations with various kinds of symbols ([11, §13]). The condition $(*)$ can also be formulated in purely combinatorial terms. Using this information, it is then possible to check that, if $(*)$ holds, then $E_0'' = 0$. For the details of this verification, see [6, Chapter 3].

We remark that, for G of type B_n, Lusztig [13, 4.10] has shown that $E_0'' = 0$ even without assuming that $(*)$ holds. □

Finally, the following result settles the question of when condition $(*)$ is actually satisfied.

Proposition 2.4 (Lusztig [10, 13.3, 13.4][1]; see also Hézard [6]). *Let $\mathcal{O}$ be a unipotent class. Then*

$$|\mathcal{G}_{s,\mathcal{F}}| \leqslant |\mathrm{A}_G(u)| \quad \textit{for all} \quad (s, \mathcal{F}) \in \mathcal{S}_G \quad \textit{such that} \quad u \in \mathcal{O} = \Phi_G(s, \mathcal{F}).$$

Furthermore, there exists some $(s, \mathcal{F})$ where s is isolated and we have equality. If $\mathcal{O}$ is F-stable (where F is a Frobenius map on G), then such a pair $(s, \mathcal{F})$ can be chosen to be F-stable, too.

[1]Note added January 2008: A new recent preprint by Lusztig [16] provides a detailed proof of the statements in [10, 13.3, 13.4].

Proof. Again, this can be reduced to the case where G is simple modulo its center, where the assertion is checked case-by-case along the lines of the proof of Proposition 2.3. The existence of suitable semisimple elements $s \in G^*$ with centralisers of the required type is checked using the tables in [1], [2] (for G of exceptional type) or using explicit computations with suitable matrix representations (for G of classical type). Again, see [6] for more details. $\square$

It would be interesting to find proofs of Propositions 2.3 and 2.4 which do not rely on a case-by-case argument.

3. Unipotent support

Recall that G is assumed to have a connected center and that we are working over a field of good characteristic. Now let $\hat{G}$ be the set of character sheaves on G (up to isomorphism) over $\overline{\mathbb{Q}}_l$ where l is a prime, $l \neq p$. By Lusztig [12, §17], we have a natural partition

$$\hat{G} = \coprod_{(s,\mathcal{F}) \in \mathcal{S}_G} \hat{G}_{s,\mathcal{F}} \quad \text{where} \quad \hat{G}_{s,\mathcal{F}} \xleftrightarrow{\text{1-1}} \mathcal{M}(\mathcal{G}_{\mathcal{F}}).$$

Here, as in Section 2, $\mathcal{G}_{\mathcal{F}}$ is the finite group associated to a family $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ as in [10, Chapter 4]. Furthermore, for any finite group Γ, the set $\mathcal{M}(\Gamma)$ consists of all pairs (x, σ) (up to conjugacy) where $x \in \Gamma$ and $\sigma \in \mathrm{Irr}(\mathrm{C}_\Gamma(x))$.

Also recall that we have a natural map $\Phi_G\colon \mathcal{S}_G \to \{\text{unipotent classes of } G\}$, defined as in [10, §3.3]. From now on, we assume that p is large enough, so that the main results of Lusztig [14] hold. (Here, "large enough" means that we can operate with the Lie algebra of G as if we were in characteristic 0, e.g., we can use exp to define a morphism from the nilpotent variety in the Lie algebra to the unipotent variety of G.)

Theorem 3.1 (Lusztig [14, Theorem 10.7]). *Let $(s, \mathcal{F}) \in \mathcal{S}_G$ and $\mathcal{O} = \Phi_G(s, \mathcal{F})$ be the associated unipotent class. Then the following hold.*
(a) *There exists some $A \in \hat{G}_{s,\mathcal{F}}$ and an element $g \in G$ with Jordan decomposition $g = g_s g_u = g_s g_u$ (where g_s is semisimple and $g_u \in \mathcal{O}$) such that $A|_{\{g\}} \neq 0$.*
(b) *For any $A \in \hat{G}_{s,\mathcal{F}}$, any unipotent class $\mathcal{O}' \neq \mathcal{O}$ with $\dim \mathcal{O}' \geqslant \dim \mathcal{O}$, and any $g' \in G$ with unipotent part in $\mathcal{O}'$, we have $A|_{\{g'\}} = 0$.*

Consequently, the class $\mathcal{O}$ is called the *unipotent support* for the character sheaves in $\hat{G}_{s,\mathcal{F}}$. Note that it may actually happen that $A|_{\mathcal{O}} = 0$ for $A \in \hat{G}_{s,\mathcal{F}}$.

Given a unipotent class $\mathcal{O}$, we denote by $\mathcal{I}_{\mathcal{O}}$ the set of irreducible G-equivariant $\overline{\mathbb{Q}}_l$-local systems on $\mathcal{O}$ (up to isomorphism).

Theorem 3.2 (Geck [4, Theorem 4.5]; see also the remarks in Lusztig [13, 1.6]). *Let $s \in G^*$ be semisimple and $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family. Let $\mathcal{O} = \Phi_G(s, \mathcal{F})$ be the associated unipotent class and assume that condition* $(*)$ *in* Proposition 2.3 *is satisfied. Then, for any $A \in \hat{G}_{s,\mathcal{F}}$, the restriction $A|_{\mathcal{O}}$ is either zero or an irreducible G-equivariant local system (up to shift). Furthermore, the map $A \mapsto A|_{\mathcal{O}}$ defines a bijection from the set of all $A \in \hat{G}_{s,\mathcal{F}}$ with $A|_{\mathcal{O}} \neq 0$ onto $\mathcal{I}_{\mathcal{O}}$.*

(Note: In [4, Theorem 4.5], the conclusion of Proposition 2.3, i.e., the validity of the sharper version of Proposition 2.2, was added as an additional hypothesis; this can now be omitted.)

Now let q be a power of p and assume that G is defined over $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_p$, with corresponding Frobenius map $F\colon G \to G$. We translate the above results to class functions on the finite group G^F.

If A is a character sheaf on G then its inverse image F^*A under F is again a character sheaf. There are only finitely many A such that F^*A is isomorphic to A; such a character sheaf will be called F-stable. Let $\hat{G}^F$ be the set of F-stable character sheaves. For any $A \in \hat{G}^F$ we choose an isomorphism $\phi\colon F^*A \xrightarrow{\sim} A$ and we form the characteristic function $\chi_{A,\phi}$. This is a class function $G^F \to \overline{\mathbb{Q}}_l$ whose value at g is the alternating sum of traces of ϕ on the stalks at g of the cohomology sheaves of A. Now ϕ is unique up to scalar hence $\chi_{A,\phi}$ is unique up to scalar. Lusztig [12, §25] has shown that

$$\{\chi_{A,\phi} \mid A \in \hat{G}^F\} \quad \text{is a basis of the vector space of class functions} \quad G^F \to \overline{\mathbb{Q}}_l.$$

Let $\mathcal{O}$ be an F-stable unipotent class of G. We denote by $\mathcal{I}_{\mathcal{O}}^F$ the set of all $\mathcal{E} \in \mathcal{I}_{\mathcal{O}}$ such that $\mathcal{E}$ is isomorphic to its inverse image $F^*\mathcal{E}$ under F. For any such $\mathcal{E}$, we can define a class function $Y_{\mathcal{E}}\colon G^F \to \overline{\mathbb{Q}}_l$ as in [12, (24.2.2)–(24.2.4)]. We have $Y_{\mathcal{E}}(g) = 0$ for $g \notin \mathcal{O}^F$ and $Y_{\mathcal{E}}(g) = \mathrm{Trace}(\psi, \mathcal{E}_g)$ for $g \in \mathcal{O}^F$, where $\psi\colon F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is a suitably chosen isomorphism. On the level of characteristic functions, Theorem 3.2 translates to the following statement (see [13, §2, §3], where such a translation is discussed in a more general setting):

Corollary 3.3. *Let $(s, \mathcal{F}) \in \mathcal{S}_G$ be F-stable and $\mathcal{O} = \Phi_G(s, \mathcal{F})$ be the associated unipotent class (which is F-stable). Assume that condition* $(*)$ *in* Proposition 2.3 *holds. Then, for any F-stable $A \in \hat{G}_{s,\mathcal{F}}$, we have either $\chi_{A,\phi}(g) = 0$ for all $g \in \mathcal{O}^F$ or ϕ can be normalized such that $\chi_{A,\phi}(g) = Y_{\mathcal{E}}(g)$ for all $g \in \mathcal{O}^F$ where $\mathcal{E} = A|_{\mathcal{O}}$.*

Now let us consider the irreducible characters of G^F. Lusztig [10] has shown that we have a natural partition

$$\mathrm{Irr}(G^F) = \coprod_{(s,\mathcal{F}) \in \mathcal{S}_G^F} \mathrm{Irr}_{s,\mathcal{F}}(G^F).$$

Furthermore, each piece $\mathrm{Irr}_{s,\mathcal{F}}(G^F)$ in this partition is parametrized by a "twisted" version of the set $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$; see [10, Chapter 4]. Lusztig [12] gave a precise conjecture about the expression of the characteristic functions of F-stable character sheaves as linear combinations of the irreducible characters of G^F. Since we are assuming that G has a connected center (and p is large), this conjecture is known to hold by Shoji [17]. In particular, the following statement holds:

Proposition 3.4 (Shoji [17]). *Let* $(s,\mathcal{F}) \in \mathcal{S}_G^F$ *and* $A \in \hat{G}_{s,\mathcal{F}}$ *be* F*-stable. Then* $\chi_{A,\phi}$ *is a linear combination of the irreducible characters in* $\mathrm{Irr}_{s,\mathcal{F}}(G^F)$.

We can now deduce the following result, whose statement only involves the values of the irreducible characters of G^F, but whose proof relies in an essential way on the above results on character sheaves.

Corollary 3.5. *Let* $\mathcal{O}$ *be an* F*-stable unipotent class and* $u_1,\ldots,u_d$ *be representatives for the* G^F*-conjugacy classes contained in* $\mathcal{O}$*. Let* $(s,\mathcal{F}) \in \mathcal{S}_G$ *be* F*-stable such that* $\mathcal{O} = \Phi_G(s,\mathcal{F})$ *and condition* $(*)$ *in* Proposition 2.3 *holds. Then there exist* $\rho_1,\ldots,\rho_d \in \mathrm{Irr}_{s,\mathcal{F}}(G^F)$ *such that the matrix* $(\rho_i(u_j))_{1\leqslant i,j\leqslant d}$ *has a non-zero determinant.*

Proof. By the proof of [12, 24.2.7], there are precisely d irreducible G-equivariant local systems $\mathcal{E}_1,\ldots,\mathcal{E}_d$ on $\mathcal{O}$ (up to isomorphism) which are isomorphic to their inverse image under F; furthermore, the matrix $(Y_{\mathcal{E}_i}(u_j))_{1\leqslant i,j\leqslant d}$ is non-singular.

By Theorem 3.2, we can find $A_1,\ldots,A_d \in \hat{G}_{s,\mathcal{F}}$ such that $A_i|_{\mathcal{O}} = \mathcal{E}_i$ for all i. Since each $\mathcal{E}_i$ is isomorphic to its inverse image under F, the same is true for A_i as well. (Indeed, since $(s,\mathcal{F})$ is F-stable, we have $F^*A_i \in \hat{G}_{s,\mathcal{F}}$ for all i; furthermore, $F^*A_i|_{\mathcal{O}} \cong F^*\mathcal{E}_i \cong \mathcal{E}_i$. So we must have $F^*A_i \cong A_i$ by Theorem 3.2.) By Corollary 3.3, we have $\chi_{A_i,\phi_i} = Y_{\mathcal{E}_i}$ for all i (where ϕ_i is normalized suitably). It follows that the matrix $(\chi_{A_i,\phi_i}(u_j))_{1\leqslant i,j\leqslant d}$ has a non-zero determinant.

By Proposition 3.4, every χ_{A_i,ϕ_i} can be expressed as a linear combination of the characters in $\mathrm{Irr}_{s,\mathcal{F}}(G^F)$. Hence there must exist $\rho_1,\ldots,\rho_d \in \mathrm{Irr}_{s,\mathcal{F}}(G^F)$ such that the matrix $(\rho_i(u_j))_{1\leqslant i,j\leqslant d}$ has a non-zero determinant. □

4. Kawanaka's conjecture

Kawanaka [8] has shown that, assuming we are in good characteristic, one can associate with every unipotent element $u \in G^F$ a so-called *generalized Gelfand-Graev representation* Γ_u (GGGR for short). They are obtained by inducing certain irreducible representations from unipotent radicals of parabolic subgroups of G^F. At the extreme cases when u is trivial or a regular unipotent element we obtain the regular representation of G^F or an ordinary Gelfand-Graev representation, respectively. Subsequently, assuming that p, q are large, Lusztig [14] gave a geometric interpretation of GGGR's in the framework of the theory of character sheaves.

Conjecture 4.1 (Kawanaka [7, (3.3.1)]). *The characters of the various GGGR's of G^F form a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module of unipotently supported virtual characters of G^F.*

By Kawanaka [9, Theorem 2.4.3], the conjecture holds if the center of G is connected and G is of type A_n or of exceptional type. In this section, assuming that p, q are large enough, we will show that it also holds for G of classical type.

Given a unipotent element $u \in G^F$, denote by γ_u the character of the GGGR Γ_u. The usual hermitian scalar product for class functions on G^F will be denoted by $\langle\ ,\ \rangle$. The following (easy) result provides an effective method for verifying that the above conjecture holds.

Lemma 4.2. *Let $u_1, \ldots, u_n$ be representatives for the conjugacy classes of unipotent elements in G^F. Assume that there exist virtual characters $\rho_1, \ldots, \rho_n$ of G^F such that the matrix of scalar products $(\langle \rho_i, \gamma_{u_j} \rangle)_{1 \leqslant i, j \leqslant n}$ is invertible over $\mathbb{Z}$. Then* Conjecture 4.1 *holds.*

Proof. Since the above matrix of scalar products is invertible, $\gamma_{u_1}, \ldots, \gamma_{u_n}$ are linearly independent class functions on G^F. Consequently, they form a basis of the $\overline{\mathbb{Q}}_l$-vectorspace of unipotently supported class functions on G^F. In particular, given any unipotently supported virtual character χ of G^F, we can write $\chi = \sum_{i=1}^n a_j \gamma_j$ where $a_j \in \overline{\mathbb{Q}}_l$, and it remains to show that $a_j \in \mathbb{Z}$ for all j.

To see this, consider the scalar products of χ with the virtual characters ρ_i. We obtain $\sum_j a_j \langle \rho_i, \gamma_j \rangle = \langle \rho_i, \chi \rangle \in \mathbb{Z}$ for all $i = 1, \ldots, n$. Since the matrix of scalar products $(\langle \rho_i, \gamma_j \rangle)$ is invertible over $\mathbb{Z}$, we can invert these equations and conclude that $a_j \in \mathbb{Z}$ for all j, as desired. $\square$

Let D_G be the Alvis-Curtis-Kawanaka duality operation on the character ring of G^F. For any $\rho \in \mathrm{Irr}(G^F)$, there is a sign $\varepsilon_\rho = \{\pm 1\}$ such that

$$\rho^* := \varepsilon_\rho \mathrm{D}_G(\rho) \in \mathrm{Irr}(G^F).$$

The following result will be crucial for dealing with groups of classical type. We assume from now on that the center of G is connected and that p, q are large, so that the results in Section 3 can be applied.

Proposition 4.3. *Let $\mathcal{O}$ be an F-stable unipotent class and $u_1, \ldots, u_d$ be representatives for the G^F-conjugacy classes contained in $\mathcal{O}$. Let $(s, \mathcal{F}) \in \mathcal{S}_G$ be F-stable such that $\mathcal{O} = \Phi_G(s, \mathcal{F})$ and condition $(*)$ in* Proposition 2.3 *holds.*

Assume that $\mathcal{G}_{\mathcal{F}}$ is abelian. Then there exist $\rho_1, \ldots, \rho_d \in \mathrm{Irr}_{s,\mathcal{F}}(G^F)$ such that $\langle \rho_i^*, \gamma_{u_j} \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant d$.

Proof. The following argument is inspired by the proof of [3, Proposition 5.6]. By [14, Theorem 11.2] and the discussion in [5, Remark 3.8], we have

$$\sum_{i=1}^{d}[\mathrm{A}_G(u_i) : \mathrm{A}_G(u_i)^F]\langle \rho^*, \gamma_{u_i}\rangle = \frac{|\mathrm{A}_G(u_1)|}{n_\rho} \quad \text{for any} \quad \rho \in \mathrm{Irr}_{s,\mathcal{F}}(G^F),$$

where $n_\rho \geqslant 1$ is an integer determined as follows; see [10, 4.26.3]. Let $E_0 \in \mathrm{Irr}(W_s)$ be the special character in $\mathcal{F}$. Then

$$\rho(1) = \pm n_\rho^{-1} q^{a_{E_0}} N \quad \text{where} \quad N \text{ is an integer, } N \equiv 1 \bmod q;$$

note also that n_ρ is divisible by bad primes only.

Now, Lusztig [10, 4.26.3] actually gives a precise formula for the integer n_ρ, in terms of a certain Fourier coefficient. In the case where $\mathcal{G}_\mathcal{F}$ is abelian, this Fourier coefficient evaluates to $|\mathcal{G}_\mathcal{F}|^{-1}$. Thus, we have $n_\rho = |\mathcal{G}_\mathcal{F}|^{-1}$. So, since $(*)$ is assumed to hold, we obtain

$$\sum_{i=1}^{d}[\mathrm{A}_G(u_i) : \mathrm{A}_G(u_i)^F]\langle \rho^*, \gamma_{u_i}\rangle = 1 \quad \text{for any} \quad \rho \in \mathrm{Irr}_{s,\mathcal{F}}(G^F).$$

Now note that each term $[\mathrm{A}_G(u_i) : \mathrm{A}_G(u_i)^F]$ is a positive integer and each term $\langle \rho^*, \gamma_{u_i}\rangle$ is a non-negative integer. It follows that, given $\rho \in \mathrm{Irr}_{s,\mathcal{F}}(G^F)$, there exists a unique $i \in \{1, \ldots, d\}$ such that $\langle \rho^*, \gamma_{u_i}\rangle = 1$ and $\langle \rho^*, \gamma_{i'}\rangle = 0$ for $i' \in \{1, \ldots, d\} \setminus \{i\}$. Thus, we have a partition $\mathrm{Irr}_{s,\mathcal{F}}(G^F) = I_1 \amalg I_2 \amalg \cdots \amalg I_d$ such that

$$\langle \rho^*, \gamma_{u_i}\rangle = \begin{cases} 1 & \text{if} \quad \rho \in I_i, \\ 0 & \text{if} \quad \rho \in I_j \quad \text{where} \quad j \neq i. \end{cases}$$

Assume, if possible, that $I_r = \varnothing$ for some $r \in \{1, \ldots, d\}$. This means that $\langle \rho, \mathrm{D}_G(\gamma_{u_r})\rangle = \langle \mathrm{D}_G(\rho), \gamma_{u_r}\rangle = 0$ for all $\rho \in \mathrm{Irr}_{s,\mathcal{F}}(G^F)$. Thus, by the definition of the scalar product, we have

$$0 = \frac{1}{|G^F|}\sum_{g \in G^F} \overline{\rho(g)} D_G(\gamma_{u_r})(g) \quad \text{for all} \quad \rho \in \mathrm{Irr}_{s,\mathcal{F}}(G^F).$$

Let $g \in G^F$ and assume that the corresponding term in the above sum is non-zero. First of all, since $D_G(\gamma_{u_r})$ is unipotently supported, g must be unipotent. Let $\mathcal{O}'$ be the conjugacy class of g. By [14, 6.13 (i) and 8.6], we have $D_G(\gamma_{u_r})(g) = 0$ unless $\mathcal{O}$ is contained in the closure of $\mathcal{O}'$. Furthermore, by [14, Theorem 11.2], we have $\rho(g) = 0$ unless $\mathcal{O}' = \mathcal{O}$ or $\dim \mathcal{O}' < \dim \mathcal{O}$. Hence, to evaluate the above sum, we only need to let g run over all elements in $\mathcal{O}^F$. Thus, we have

$$0 = \sum_{j=1}^{d} \frac{1}{|\mathrm{C}_{G^F}(u_j)|}\overline{\rho(u_j)}\mathrm{D}_G(\gamma_{u_r}(u_j)) \quad \text{for all} \quad \rho \in \mathrm{Irr}_{s,\mathcal{F}}(G^F).$$

In particular, this holds for the characters $\rho_1, \dots, \rho_d$ in Corollary 3.5. The invertibility of the matrix of values in Corollary 3.5 then implies that $\mathrm{D}_G(\gamma_{u_r})(u_j) = 0$ for $1 \leqslant j \leqslant d$. Thus, the restriction of $\mathrm{D}_G(\gamma_{u_r})$ to $\mathcal{O}^F$ is zero. Now, the relations in [4, (2.4a)] (which are formally deduced from the main results in [14]) imply that $\langle \mathrm{D}_G(\gamma_{u_r}), Y_{\mathcal{E}} \rangle$ equals $\overline{Y_{\mathcal{E}}(u_r)}$ times a non-zero scalar, for any $\mathcal{E} \in \mathcal{I}_{\mathcal{O}}^F$. Hence, we have $Y_{\mathcal{E}}(u_r) = 0$ for any $\mathcal{E} \in \mathcal{I}_{\mathcal{O}}^F$. However, this contradicts the fact that the matrix of values $(Y_{\mathcal{E}}(u_j))$ is invertible (see the remarks at the beginning of the proof of Corollary 3.5). This contradiction shows that we have $I_i \neq \varnothing$ for all i. Now choose $\rho_i \in I_i$ for $1 \leqslant i \leqslant d$. Then we have $\langle \rho_i^*, \gamma_{u_j} \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant d$, as desired. □

REMARK 4.4. In the setting of Proposition 4.3, let us drop the assumption that $\mathcal{G}_{\mathcal{F}}$ is abelian and assume instead that $\mathcal{G}_{\mathcal{F}}$ is isomorphic to $\mathfrak{S}_3$, $\mathfrak{S}_4$ or $\mathfrak{S}_5$. (These cases occur when G is simple modulo its center and of exceptional type.) Then, by the Main Theorem 4.23 of [10], we have a bijection $\mathrm{Irr}_{s,\mathcal{F}}(G^F) \leftrightarrow \mathcal{M}(\mathcal{G}_{\mathcal{F}})$.

Let $u_1, \dots, u_d$ be representatives for the G^F-conjugacy classes contained in $\mathcal{O}^F$. Since condition $(*)$ in Proposition 2.3 is assumed to hold, we can identify $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$ with the set of all pairs (u_i, σ) where $1 \leqslant i \leqslant d$ and $\sigma \in \mathrm{Irr}(\mathrm{A}_G(u_i)^F)$. Thus, via the above-mentioned bijection, we have a parametrization

$$\mathrm{Irr}_{s,\mathcal{F}}(G^F) = \{\rho_{(u_i,\sigma)} \mid 1 \leqslant i \leqslant d,\ \sigma \in \mathrm{Irr}(\mathrm{A}_G(u_i)^F)\}.$$

On the other hand, Kawanaka [8], [9] obtained explicit formulas for the values of the characters of the GGGR's (for G of exceptional type). Using these formulas, one can check that

$$\langle \rho_{u_i,\sigma}^*, \gamma_{u_j} \rangle = \begin{cases} \sigma(1) & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, setting $\rho_i := \rho_{(u_i,1)}$ for $1 \leqslant i \leqslant d$ (where 1 stands for the trivial character), we see that the conclusion of Proposition 4.3 holds in these cases as well.

Theorem 4.5. *Recall our standing assumption that p, q are large enough and the center of G is connected. Then Kawanaka's* Conjecture 4.1 *holds.*

Proof. By standard reduction arguments, we can assume without loss of generality that G is simple modulo its center. If G is of type A_n or of exceptional type, the assertion has been proved by Kawanaka [9, Theorem 2.4.3], using his explicit formulas for the character values of GGGR's. The following argument covers these cases as well.

Let $\mathcal{O}_1, \dots, \mathcal{O}_N$ be the F-stable unipotent classes of G, where the numbering is chosen such that $\dim \mathcal{O}_1 \leqslant \cdots \leqslant \dim \mathcal{O}_N$. By Proposition 2.4, for each i, we can find an F-stable pair $(s_i, \mathcal{F}_i) \in \mathcal{S}_G$ such that $\mathcal{O}_i = \Phi_G(s_i, \mathcal{F}_i)$ and condition $(*)$ in Proposition 2.3 holds.

For each i, let $u_{i,1},\dots,u_{i,d_i}$ be a set of representatives for the G^F-conjugacy classes contained in $\mathcal{O}_i^F$. Let $\rho_{i,1},\dots,\rho_{i,d_i}$ be irreducible characters as in Proposition 4.3 (if G is of classical type) or as in Remark 4.4 (if G is of exceptional type). We claim that

$$\langle \rho^*_{i_1,j_1}, \gamma_{u_{i_2,j_2}} \rangle = 0 \quad \text{if} \quad i_1 < i_2.$$

This is seen as follows. We have $\langle \rho^*_{i_1,j_1}, \gamma_{u_{i_2,j_2}} \rangle = \pm\langle \rho_{i_1,j_1}, \mathrm{D}_G(\gamma_{u_{i_2,j_2}})\rangle$. By the definition of the scalar product, we have

$$\langle \rho_{i_1,j_1}, \mathrm{D}_G(\gamma_{u_{i_2,j_2}})\rangle = \frac{1}{|G^F|}\sum_{g\in G^F} \overline{\rho_{i_1,j_1}(g)}\mathrm{D}_G(\gamma_{u_{i_2,j_2}})(g).$$

We now argue as in the proof of Proposition 4.3 to evaluate this sum. First of all, it's enough to let g run over all unipotent elements of G^F. Now let $g \in G^F$ be unipotent and assume, if possible, that the corresponding term in the above sum is non-zero. The fact that $\rho_{i_1,j_1}(g) \neq 0$ implies that the class of g either equals $\mathcal{O}_{i_1}$ or has dimension $< \dim \mathcal{O}_{i_1}$. Furthermore, the fact that $\mathrm{D}_G(\gamma_{u_{i_2,j_2}})(g) \neq 0$ implies that $\mathcal{O}_{i_2}$ is contained in the closure of the class of g. Since we numbered the unipotent classes according to increasing dimension, we conclude that $\dim \mathcal{O}_{i_1} = \dim \mathcal{O}_{i_2}$; furthermore, $g \in \mathcal{O}_{i_1}$ and $\mathcal{O}_{i_2}$ is contained in the closure of the class of g, which finally shows that $\mathcal{O}_{i_1} = \mathcal{O}_{i_2}$, a contradiction. Thus, our assumption was wrong, and the above scalar product is zero.

Together with the relations in Proposition 4.3 (or Remark 4.4), we now see that the matrix of all scalar products

$$\langle \rho^*_{i_1,j_2}, \gamma_{u_{i_2,j_2}} \rangle_{1\leq i_1,i_2\leq N,\, 1\leq j_1\leq d_{i_1},\, 1\leq j_2 \leq d_{i_2}}$$

is a block triangular matrix where each diagonal block is an identity matrix. Hence that matrix of scalar products is invertible over $\mathbb{Z}$ and so Kawanaka's conjecture holds by Lemma 4.2. □

Proposition 4.6 (Characterisation of GGGR's). *Recall that p, q are large enough and the center of G is connected. Let $\mathcal{O}$ be an F-stable unipotent class in G and χ be a character of G^F. Then $\chi = \gamma_u$ for some $u \in \mathcal{O}^F$ if and only if the following three conditions are satisfied*:
(a) *If $\chi(g) \neq 0$ for some $g \in G^F$, then the conjugacy class of g is contained in the closure of $\mathcal{O}$.*
(b) *If $\mathrm{D}_G(\chi)(g) \neq 0$ for some $g \in G^F$, then $\mathcal{O}$ is contained in the closure of the conjugacy class of g.*
(c) *We have $\chi(1) = |G^F|q^{-\dim \mathcal{O}/2}$.*

Proof. If $\chi = \gamma_u$ for some $u \in \mathcal{O}^F$, then (a) and (c) are easily seen to hold by the construction of Γ_u; see Kawanaka [8]. Condition (b) is obtained as a consequence of [14, 6.13 (i) and 8.6]. To prove the converse, by standard reduction arguments, we can assume without loss of generality that G is simple modulo its center. Assume now that (a), (b) and (c) hold for χ. Since χ is unipotently supported, we can write χ as an integral linear combination of the characters of the various GGGR's of G^F; see Theorem 4.5.

Now, given any F-stable unipotent class $\mathcal{O}'$, the characters γ_u, where $u \in \mathcal{O}'^F$, satisfy (a) with respect to $\mathcal{O}'$. Hence, all characters γ_u, where u is contained in the closure of $\mathcal{O}$, satisfy (a). One easily deduces that *any* class function satisfying (a) is a linear combination of various γ_u where u is contained in the closure of $\mathcal{O}$. Similarly, *any* class function satisfying (b) is a linear combination of various $\mathrm{D}_G(\gamma_u)$ where $\mathcal{O}$ is contained in the closure of the class of u. Hence, a class function satisfying both (a) and (b) will be a linear combination of various γ_u such that $u \in \mathcal{O}^F$.

Let $u_1, \ldots, u_d$ be representatives for the G^F-conjugacy classes in $\mathcal{O}^F$. Then the above discussion shows that we can write $\chi = \sum_{j=1}^{d} a_j \gamma_{u_j}$ where $a_j \in \mathbb{Z}$ for all i.

Now consider the characters ρ_i in Proposition 4.3 (for G of classical type) or in Remark 4.4 (for G of exceptional type). Taking scalar products of χ with ρ_i^*, we find that $a_i \geqslant 0$ for all i and so χ is a positive sum of characters of various GGGR's associated with $\mathcal{O}^F$. All these GGGR's have dimension $|G^F|q^{-\dim \mathcal{O}/2}$. Hence $\chi(1)$ is a positive integer multiple of $|G^F|q^{-\dim \mathcal{O}/2}$. Condition (c) now forces that $\chi = \gamma_u$ for some $u \in \mathcal{O}^F$, as required. $\square$

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