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MINIMAL PENCILS ON SMOOTH SURFACES IN $\mathbb{P}^3$

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Abstract

Pencils of curves of minimal genus and slope are determined for smooth surfaces of degree at least seven in the projective 3-space.

Introduction

One of the most important approaches in the study of projective varieties is to find rational functions with an extremal property which reflects well the geometry of the variety. In the theory of curves, the minimum value among degrees of pencils on a smooth projective curve is called the *gonality* and plays a very important rôle. It is not greater than $(g+3)/2$ for a curve of genus g by the Brill-Noether theory. For a smooth plane curve of degree $n \geq 3$, Namba [12, Theorem 2.3.1] showed that the gonality depends only on n and is in fact given by $n-1$. Furthermore, every pencil of minimal degree $n-1$ is obtained as the linear projection from a point on it.

The present article is a trial to extend the notion of "gonality" to surfaces. To be more precise, let S be a smooth projective algebraic surface and consider a non-constant rational function Φ on it, regarded as a dominant rational map to $\mathbb{P}^1$. Then, in a canonical way, we can transform it to a relatively minimal fibration $f\colon X \to B$, where X is a smooth surface birationally equivalent to S and B a smooth curve with a particular morphism $\tau\colon B \to \mathbb{P}^1$ such that the original Φ can be identified, in the birational sense, with the composite $\tau \circ f$. It allows us to regard various numerical invariants of f as those of Φ. For example, if a general fibre of f is of genus g, we say that Φ is of genus g. When $g \geq 2$ and f is not a fibre bundle, the slope of f is a well-defined positive rational number [15]. Then the slope of Φ is defined as that of f. Furthermore, we can consider their minimums when Φ moves in the rational function field of S. The birational invariants thus obtained are our candidates for the "gonality". It should be noticed that a rational function of the smallest genus does not necessarily give us a fibration with the smallest slope, and vice versa.

In this paper, we shall study how those invariants behave for smooth surfaces in $\mathbb{P}^3$, expecting a result similar to Namba's theorem for plane curves referred above. One should notice, however, that the smallest genus for rational functions may vary even if

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we fix the degree of surfaces, unlike the case of plane curves. A result along such a line is already found in [3]. Recall that the smallest possible value for genera of non-hyperelliptic curves is 3 and that a generic quintic surface does not contain a line by a classical theorem of Max Noether. What we showed in [3, Proposition 2.5] is that a quintic surface has a pencil of curves of genus 3 if and only if it contains a line. Thus for a generic quintic surface the smallest genus is strictly bigger than 3. One of our main results, Theorem 2.1, is exactly an extension of this fact and states that the same phenomena happens also when $n \geq 6$. Furthermore, we observe in Theorem 3.1 the same is true for the smallest slope of functions. We hope that our results give sufficient evidence of these invariants being right candidates for the "gonality".

The organization of the paper is as follows. In §1, we introduce the minimal genus and slope for surfaces, and discuss how they relate to the geometry of surfaces. It will show that these two invariants have different flavor in general, though both seem equally fundamental. The rest is devoted to smooth surfaces of degree n in $\mathbb{P}^3$. We study the minimal genus in §2 and the minimal slope in §3, and show Theorems 2.1 and 3.1. Since a general member of a pencil can be considered as a space curve in the present case, its degree is an important invariant. We show that it can be $n-1$, n but the next value jumps to $2n-4$ by using Castelnuovo's bound, though it also follows from the known result for plane curves if we cut the surface with a general hyperplane. Using such information on degrees, one can estimate the genus as well as the slope without much difficulty. The last section, §4, treats some extra pencils which may be minimal for some quintic and sextic surfaces. As is naturally expected, the presence of a special pencil gives us a particular description of the defining equation of the surface itself. See, Propositions 4.1 and 4.4 for the detail.

1. Some invariants

Let S be a smooth projective algebraic surface defined over $\mathbb{C}$. In this section, we introduce some birational invariants for S detected by rational functions, which seem to be basic and need further explorations. We use the standard notation. We denote by K_S the canonical bundle (or a canonical divisor) on S. For a sheaf $\mathcal{F}$, we put $\chi(\mathcal{F}) = h^0(S, \mathcal{F}) - h^1(S, \mathcal{F}) + h^2(S, \mathcal{F})$, $h^i(S, \mathcal{F}) = \dim H^i(S, \mathcal{F})$, and, when $\mathcal{F}$ is invertible, $\Phi_{\mathcal{F}}$ denotes the rational map associated with the complete linear system $|\mathcal{F}|$. The irregularity and the geometric genus are respectively defined by $q(S) := h^1(S, \mathcal{O}_S)$ and $p_g(S) := h^2(S, \mathcal{O}_S)$.

A rational function on S is geometrically a dominant rational map from S to $\mathbb{P}^1$. In other words, it gives us a pencil Λ without fixed components but possibly with base points, and vice versa. Let $\sigma : \tilde{S} \to S$ be a minimal succession of blowing-ups which eliminates the base points of Λ. The fibres of the induced morphism $\tilde{S} \to \mathbb{P}^1$ may well be disconnected. So we transform it by the Stein factorization to a more acceptable form: there exist a finite (ramified) covering $\tau : B \to \mathbb{P}^1$ and a morphism $\tilde{f} : \tilde{S} \to B$ with connected fibres such that the original rational function is essentially the composite $\tau \circ \tilde{f}$. We denote by g the genus of a general fibre of $\tilde{f}$. We further take a relatively minimal model $f: X \to B$ of $\tilde{f}$, which is unique when $g > 0$. In this way, the study of rational functions on S can be reduced to that of relatively minimal fibrations plus functions on curves (corresponding to $\tau: B \to \mathbb{P}^1$).

Numerical invariants of f can be regarded as those of Λ. Here we focus on two invariants which seem to be most important and basic. The first one may be obvious. We put $g(\Lambda) = g$ and call it the *genus* of Λ (or the rational function). When $g = 0$, f is a $\mathbb{P}^1$-bundle and we can regard such a case as known, since S is then a ruled surface whose structure is well understood. When $g = 1$, that is, f is an elliptic surface, we may apply the beautiful theory due to Kodaira. We may also ignore the case that f is a fibre bundle even when $g \geq 2$. Our second invariant is introduced for non-trivial cases. Put $K_f = K_X - f^*K_B$ and

$$\chi_f := \deg f_*\mathcal{O}_X(K_f) = \chi(\mathcal{O}_X) - (g-1)(g(B)-1).$$

By Arakelov's theorem [1], K_f^2 is a non-negative integer and $K_f^2 = 0$ holds only if f is isotrivial. It is known that χ_f is a non-negative integer and $\chi_f = 0$ holds if only if f is an algebraic fibre bundle. For these facts, see [5]. We put $s(\Lambda) := K_f^2/\chi_f$ and call it the slope of Λ, when f is not a fibre bundle. Recall that we have $4 - 4/g \leq s(\Lambda) \leq 12$ by the slope inequality [15] and Noether's formula. Known results show that the smaller the slope is, the simpler the structure of f becomes.

We now put

$$\mu_g(S) := \min_{\Lambda}\{g(\Lambda)\}$$

where Λ runs over the set of all pencils on S without fixed components, and call it the *minimal genus* of S. When $\mu_g(S) \geq 2$ and S is not birationally equivalent to a fibre bundle, we put

$$\mu_s(S) := \inf_{\Lambda}\{s(\Lambda)\}$$

and call it the *minimal slope* of S. Obviously, these are birational invariants of S.

The minimal genus has been used, consciously or not, in the classification of surfaces as follows. Surfaces with $\mu_g = 0$ are exactly ruled surfaces, while those with $\mu_g = 1$ are non-ruled elliptic surfaces. The class of surfaces with $\mu_g \geq 2$ consists of surfaces of general type and, possibly, some abelian or K3 surfaces. Another remark is that $\mu_g(S) = 2$ forces the index of S to be non-positive, i.e., $K_S^2 \leq 8\chi(\mathcal{O}_S)$, by a result of Xiao [13] and Ueno.

One can also introduce the minimal gonality (resp. Clifford index) of S by means of the gonality (resp. Clifford index) of a general fibre of f. These invariants may be closely related to the degree of irrationality introduced in [11] and developed for surfaces in [16].

REMARK 1.1. (1) Since we start from a rational function, a pencil in the above discussion strictly corresponds to a linear subspace of projective dimension one in a complete linear system. But a pencil often means an algebraic family of curves parametrized by an irreducible curve in the literature. The latter usage allows us to say that any general member of a pencil without fixed components is irreducible. We adopt harmlessly this new interpretation in the sequel.
(2) An abelian surface does not have a pencil of genus two. A K3 surface with a pencil of genus two is a double covering of $\mathbb{P}^2$ branched along a sextic. For these facts, see [13, Théorème 4.5].

How to find pencils of small invariants is another problem. We close the section with a remark on surfaces of general type, which concerns how the Albanese and the canonical maps relate to small pencils.

Let S be a surface of general type. Assume that $p_g(S) \geq 2$. If the canonical map of S is not birational onto its image, then it often shows up a particular pencil (e.g., [9]). We consider the extremal case that the canonical map is composed of a pencil; we call the pencil the canonical pencil and denote it by Λ_{can}. It is known [14] that the base curve of Λ_{can} is either $\mathbb{P}^1$ or an elliptic curve.

Proposition 1.2. *Let S be a surface of general type whose canonical map is composed of a pencil. If $\chi(\mathcal{O}_S) \geq 9$, then the canonical pencil is the unique pencil of minimal genus on S; in particular $2 \leq \mu_g(S) \leq 7$.*

Proof. Let Λ be a pencil on S different from the canonical pencil. We shall show that $g(\Lambda) \geq p_g(S)$. We move to a birational model X of S such that the canonical map and the rational map induced by Λ are both morphisms on X. Then K_X is numerically equivalent to $aF + Z$, where F is a member of the canonical pencil, Z is an effective divisor and a is an integer with $a \geq p_g(S) - 1$. If D denotes the irreducible curve coming from a general member of Λ, then $D^2 = 0$ and $2g(\Lambda) - 2 = K_X D = aFD + DZ \geq aFD$. We have $FD \geq 2$, because Λ is not Λ_{can}, $g(\Lambda) \geq 2$ and the base curve of Λ_{can} is of genus at most one. Then $g(\Lambda) \geq a + 1 \geq p_g(S)$ as wished.

Assume now that $\chi(\mathcal{O}_S) \geq 9$. Similarly as in the proof of [4, Proposition 2.1], we can show that $g(\Lambda_{\mathrm{can}}) \leq 7$. Since $\chi(\mathcal{O}_S) \leq p_g(S) + 1$, we have $g(\Lambda_{\mathrm{can}}) < p_g(S)$. □

We next assume that $q(S) > 0$ and the image of the Albanese map is a curve. For ruled surfaces, the Albanese pencil is the only pencil of minimal genus 0 as is well known. In analogy, one may expect that the Albanese map gives us a pencil of minimal genus. However, it is not true in general. There exist irregular surfaces of general type with a pencil whose genus is strictly smaller than that of the Albanese pencil, as we shall see below. Nevertheless, the Albanese map is so natural that we have the following at least for surfaces with small K^2:

Proposition 1.3. *Let S be a minimal, irregular surface of general type with $K_S^2 < 4p_g(S)$ whose Albanese image is a curve. Then the Albanese pencil is the unique pencil of minimal slope.*

Proof. Let g be the fibre genus of the Albanese pencil $\alpha\colon S \to C \subseteq \mathrm{Alb}(S)$, where C is a non-singular projective curve of genus $q(S)$. Then $g \geq 2$, because S is of general type. If $K_S^2 < 4p_g(S)$, then $K_S^2 < 4p_g(S) + 4(g-2)(q(S)-1)$ which is equivalent to $K_\alpha^2 < 4\chi_\alpha$. Hence the Albanese pencil has slope less than 4. Let $f\colon X \to B$ be the relatively minimal fibration associated with a pencil Λ on S. If Λ is not the Albanese pencil, then $g(B) < q(S)$ and it follows from [15] that $s(\Lambda) \geq 4$. □

A simple example explains the situation. Let E be an elliptic curve and put $\Sigma = \mathbb{P}^1 \times E$. Let g and h be integers not less than 2 and consider a double covering S of Σ branched along a smooth curve of bi-degree $(2g+2, 2h-2)$. Then $q(S) = 1$ and $K_S^2 = (4 - 4/g)p_g(S)$. Furthermore, the Albanese pencil is hyperelliptic of genus g and slope $4 - 4/g$, while S has a linear pencil of genus h and slope 4 induced by the projection $\Sigma \to \mathbb{P}^1$. When $g \gg h$, this shows that there is a big difference between the minimal genus and the minimal slope.

2. Minimal genus

From now on, S is a smooth surface in $\mathbb{P}^3$ of degree $n \geq 2$. We are going to find a pencil of minimal genus. At a first glance, the problem seems almost trivial, since the Néron-Severi group is generated by the class of hyperplane-sections when S is a generic surface of degree ≥ 4; so the minimal pencil should be a subpencil of $|\mathcal{O}_S(1)|$ at least when S is generic. The purpose of the section is to justify such a naive feeling and clarify what "generic" means. Namely, we shall show the following theorem with several lemmas.

Theorem 2.1. *Let S be a smooth surface of degree $n \geq 2$ in $\mathbb{P}^3$. Then*

$$\mu_g(S) \geq \frac{(n-2)(n-3)}{2}.$$

Furthermore, when $n \geq 5$, the equality sign holds if and only if S contains a line. If $n \geq 7$, then

$$\mu_g(S) = \begin{cases} \dfrac{(n-2)(n-3)}{2} & \text{if} \quad S \text{ contains a line,} \\ \dfrac{(n-1)(n-2)}{2} & \text{otherwise} \end{cases}$$

and every pencil of minimal genus can be obtained as the projection from a line; in the former case the line is on S.

Let L be the hyperplane bundle on S. Then $K_S = (n-4)L$, $L^2 = n$ and $h^0(S, L) = 4$. Let Λ be a pencil on S (without fixed components) and $D \in \Lambda$ a general member. We may assume that D is irreducible. We denote by g the geometric genus of D and put $d := LD$. Let $\sigma\colon \tilde{S} \to S$ be a minimal succession of blowing-ups which eliminates all the base points of Λ. By the adjunction formula, we have

$$2p_a(D) - 2 = K_S D + D^2 = (n-4)d + D^2. \tag{2.1}$$

If m_i denotes the multiplicity of the i-th center of σ as a base point of the pencil induced by Λ, then $D^2 = \sum m_i^2$ and

$$2g - 2 = (n-4)d + \sum_i m_i. \tag{2.2}$$

Let us consider the restriction map $H^0(S, L) \to H^0(D, L)$ and put

$$r = \operatorname{rank}\{H^0(S, L) \to H^0(D, L)\}.$$

Since L is very ample, Φ_L maps D isomorphically onto an irreducible non-degenerate curve in $\mathbb{P}^{r-1}$. We in particular have $d \geq r-1$, $2 \leq r \leq 4$.

Lemma 2.2. *If $r = 2$, then $LD = 1$, $D^2 = 0$, $D \simeq \mathbb{P}^1$ and $n = 2$.*

Proof. Since $r = 2$, we have $D \simeq \mathbb{P}^1$. Then $-2 = 2p_a(D) - 2 = (n-4)d + D^2$. Since $d > 0$ and $D^2 \geq 0$, we get $n \leq 3$. Note that we have $h^0(S, L - D) = 2$. If we take $D' \in |L - D|$, then $L \sim D + D'$. Since $n = L^2 = LD + LD' > LD = d$, we get $(d, D^2) = (1, 0)$ when $n = 2$, and $(d, D^2) = (2, 0)$ when $n = 3$.

We exclude the possibility that $n = 3$. Assume that $n = 3$. A general member $C \in |L|$ is an elliptic curve being a smooth plane curve of degree three. We have either $LD = 1$ or $LD' = 1$ by $LD + LD' = 3$. Then one of the rational maps induced by Λ, $|D'|$ would map C onto $\mathbb{P}^1$ isomorphically, which is impossible. Therefore, $n \neq 3$. $\square$

When $n = 2$, we have $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and Λ as above corresponds to one of the natural projections.

Lemma 2.3. *Assume that $r = 3$. Then* $\max\{n-1, 2\} \leq d \leq n$ *and the rational map induced by Λ can be identified with a projection from a line in $\mathbb{P}^3$. Furthermore,*

$$g = \begin{cases} \dfrac{(n-2)(n-3)}{2}, & \text{if} \quad d = n-1, \\[2mm] \dfrac{(n-1)(n-2)}{2}, & \text{if} \quad d = n. \end{cases}$$

Proof. Since $r = 3$, D is mapped by Φ_L isomorphically onto a non-degenerate plane curve of degree d. In particular, $d \geq 2$ and we have $p_a(D) = (d-1)(d-2)/2$. We get $0 \leq D^2 = d^2 - (n-1)d$ from (2.1), and it follows that $d \geq n-1$. On the other hand, we have $H^0(S, L-D) \neq 0$. Hence there exists an effective divisor Z such that $Z + D \in |L|$. Then $n = L^2 = L(D+Z) \geq LD$. Therefore, we have either $d = n-1$ or $d = n$. Since Λ is a subpencil of $|L|$, its module V is a two dimensional linear subspace of $H^0(S, L)$. Therefore, the rational map induced by Λ can be identified with the projection with center the line l corresponding to the quotient $H^0(S, L)/V$. The effective divisor Z above is nothing more than the divisorial part of the inverse image of l by Φ_L.

Suppose that $d = n-1$. Then we have $D^2 = 0$ implying that Λ is free from base points and, therefore, D is smooth with $g = p_a(D) = (n-2)(n-3)/2$. We have $LZ = 1$. Since L is very ample, we conclude that Z is an irreducible curve mapped isomorphically onto l, that is, $Z \simeq \mathbb{P}^1$. In other words, S as a hypersurface in $\mathbb{P}^3$ contains l.

Suppose that $d = n$. Then $D^2 = n$. Furthermore, we have $LZ = 0$ which implies $Z = 0$. Therefore, $l \not\subset S$ and the base locus of Λ is exactly the intersection 0-cycle given by l on S. We shall compute g. Let $\tilde{\sigma}\colon W \to \mathbb{P}^3$ be the blowing-up along l. Then W has a $\mathbb{P}^2$-bundle structure over $\mathbb{P}^1$ and we in fact have $W \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$. If $\tilde{D}$ denotes the proper transform of D by σ, then we have an exceptional divisor E for σ such that $\tilde{D} + E \in |\sigma^* L|$. Note that $E \neq 0$, since $D^2 = n$ and $\tilde{D}^2 = 0$. Since E is exactly the inverse image of l by $\Phi_{\sigma^* L}$, we can lift $\Phi_{\sigma^* L}\colon \tilde{S} \to \mathbb{P}^3$ to a morphism $\Phi\colon \tilde{S} \to W$. Let $f\colon \tilde{S} \to \mathbb{P}^1$ be the fibration induced by Λ. Then Φ can be identified with the morphism defined by $|\sigma^* L + f^* \mathfrak{d}|$ for a sufficiently ample divisor $\mathfrak{d}$ on $\mathbb{P}^1$. The image of Φ is nothing but the proper transform of $S \subset \mathbb{P}^3$ by $\tilde{\sigma}$. Then it has at most isolated singular points arizing from possible vertical components of E with respect to f. Since $\tilde{D}$ is a general fibre of f, we see that Φ maps $\tilde{D}$ isomorphically onto a plane curve of degree $n = \tilde{D}\sigma^* L$. In particular, we have $g = (n-1)(n-2)/2$. Recall that Φ_L maps D isomorphically onto an irreducible plane curve of degree $n = LD$. Hence $p_a(D) = (n-1)(n-2)/2 = g$, which shows that D is smooth. $\square$

A smooth cubic surface has exactly 27 lines. If we choose one of them, then the projection from it gives a pencil of minimal genus zero.

Lemma 2.4. *If $r = 4$, then $d \geq \max\{2n-4, 3\}$.*

Proof. In this case, D is isomorphic to a non-degenerate space curve of degree d. In particular, $d \geq 3$. If we denote by m the integer part of $(d-1)/2$, then Castelnuovo's bound (e.g., [2]) shows

$$p_a(D) \leq m(m-1) + m(d-1-2m) = m(d-m-2). \tag{2.3}$$

From this and (2.1), we get

$$2D^2 \leq d^2 - 2(n-2)d - \begin{cases} 0, & \text{if } d \text{ is even}, \\ 1, & \text{if } d \text{ is odd}. \end{cases}$$

Since $D^2 \geq 0$, we get $d \geq 2n - 4$. □

Note that, when $d = 2n - 4$, we have $g = p_a(D) = (n-3)^2$.

Lemma 2.5. *If $g < (n-3)^2$, then $r = 3$ and $d = n-1, n$. If $(n-3)^2 \leq g < (n-2)(2n-7)/2$, then $d = 2n-4$ and $g = (n-3)^2$.*

Proof. If $d \geq 2n-4$, then (2.2) shows $2g-2 = (n-4)LD + \sum m_i \geq 2(n-2)(n-4)$, that is, $g \geq (n-3)^2$. Hence $d < 2n-4$ when $g < (n-3)^2$, and we get the first assertion by the above lemmas. Similarly, we get $g \geq (n-2)(2n-7)/2$ when $d \geq 2n-3$. □

Lemma 2.6. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $n \geq 2$. Then $\mu_g(S) \geq (n-2)(n-3)/2$ and the equality sign holds for $n \geq 5$ if and only if S contains a line.*

Proof. The inequality is clear when $n = 2, 3$. When $n \geq 4$, we have $2g - 2 = (n-4)d + \sum m_i \geq (n-4)d \geq (n-4)(n-1)$ by (2.2). Hence $g \geq (n-2)(n-3)/2$ with equality holding only when $\sum m_i = 0$ and either $n = 4$ or $n \geq 5$, $d = n-1$. □

Now, the first half of Theorem 2.1 is nothing more than Lemma 2.6. The last half follows from Lemmas 2.5 and 2.3.

REMARK 2.7. When $n \geq 4$, the locus of surfaces containing a line is of codimension $n-3$ in the moduli space of surfaces of degree n in $\mathbb{P}^3$. If S contains a line, then the defining equation can be standardized as $Z_0\Psi_1 = Z_1\Psi_0$, where the Ψ_i's are homogeneous forms of degree $n-1$ in (Z_0, Z_1, Z_2, Z_3).

3. Minimal slope

In this section, we focus on the minimal slope, another candidate for the "gonality", by computing the slope of the corresponding relatively minimal fibration $f\colon \tilde{S} \to \mathbb{P}^1$ when $n \geq 5$.

We denote by Λ_d an irreducible pencil on S with $d = LD$ for $D \in \Lambda_d$. Let ν be the number of blowing-ups appearing in $\sigma\colon \tilde{S} \to S$ and put $\mu = \sum m_i$. Then

$$\begin{cases} 6\chi_f = (n-1)(n-2)(n-3) + 3(n-4)d + 3\mu + 6, \\ K_f^2 = n(n-4)^2 + 4(n-4)d + 4\mu - \nu, \end{cases} \tag{3.1}$$

since $K_S^2 = n(n-4)^2$, $\chi(\mathcal{O}_S) = (n-1)(n-2)(n-3)/6 + 1$ and $2g - 2 = (n-4)d + \mu$. For the convenience of readers, we exhibit the genus and the slope of Λ_d for the first three possible values of d detected in the previous section:

$$g(\Lambda_{n-1}) = \frac{(n-2)(n-3)}{2},\ s(\Lambda_{n-1}) = 6\frac{n-4}{n-3} \quad (\mu = \nu = 0),$$
$$g(\Lambda_n) = \frac{(n-1)(n-2)}{2},\ s(\Lambda_n) = 6\frac{n-3}{n-2} \quad (\mu = \nu = n),$$
$$g(\Lambda_{2n-4}) = (n-3)^2,\ s(\Lambda_{2n-4}) = 6\frac{(n-4)(n^2+4n-16)}{(n-3)(n^2+3n-16)} \quad (\mu = \nu = 0).$$

The following may show that the minimal slope behaves more nicely than the minimal genus when $n = 6$.

Theorem 3.1. *Let S be a smooth surface of degree $n \geq 5$ in $\mathbb{P}^3$. Then*

$$\mu_s(S) \geq 6\frac{(n-4)}{(n-3)}$$

and the equality sign holds if and only if S contains a line. If $n \geq 6$, then

$$\mu_s(S) = \begin{cases} 6\dfrac{(n-4)}{(n-3)} & \textit{if} \quad S \textit{ contains a line,} \\ 6\dfrac{(n-3)}{(n-2)} & \textit{otherwise} \end{cases}$$

and every pencil of minimal slope can be obtained as the projection from a line.

Proof. We have $s(\Lambda_{n-1}) < s(\Lambda_n)$. Since a Λ_n always exists, we compare $s(\Lambda_d)$ with $s(\Lambda_n)$ for $d > n$. By (3.1), we have

$$K_f^2 - 6\frac{n-3}{n-2}\chi_f = \frac{(n+1)(n-4)}{n-2}(d-n-1) + \left(\frac{n+1}{n-2}\mu - \nu\right) + \frac{n^2-6n-4}{n-2}.$$

Recall that we have $d \geq 2n-4$ if $d > n$.

We first assume that $n \geq 6$. We clearly have $\mu \geq \nu \geq 0$. Since $d \geq 2n-4$, the right hand side of the above equality is not less than

$$\frac{(n+1)(n-4)(n-5)}{n-2} + \frac{3}{n-2}\nu + \frac{n^2-6n-4}{n-2}.$$

Therefore, when $d > n \geq 6$, we have $s(\Lambda_d) > 6(n-3)/(n-2) = s(\Lambda_n)$.

We consider quintic surfaces. When $d \geq 8$, it is easy to see that $4 = s(\Lambda_5) < s(\Lambda_d)$. When $d = 7$, noting that μ must be a positive odd integer by (2.2), we get $s(\Lambda_7) \geq 4$ with equality holding only if $\mu = \nu = 1$. If we denote by $\Lambda_{7,1}$ such a pencil with $d = 7$ and $\mu = \nu = 1$, then we have $s(\Lambda_4) < s(\Lambda_6) < s(\Lambda_{7,1}) = s(\Lambda_5)$. $\square$

4. Some special pencils

We have found so far a satisfactory answer when $n \geq 7$ with both invariants. But we need a further study when $n = 5, 6$. The purpose of the section is to clarify the pencils which satisfy the following:

- $n = 5$. (a) $g = 4$, $LD = 6$, $D^2 = 0$, (b) $g = 5$, $LD = 7$, $D^2 = 1$.
- $n = 6$. (a) $g = 9$, $LD = 8$, $D^2 = 0$, (b) $g = 10$, $LD = 9$, $D^2 = 0$.

The list exhausts the possible unknown pencils for the smallest genus when $n = 6$, and for the smallest slope when $n = 5$.

Proposition 4.1. *Let S be a smooth surface of degree $n \geq 4$ in $\mathbb{P}^3$ and k an integer satisfying $2 \leq k \leq n - k$. Assume that S has an irreducible pencil Λ satisfying $LD = k(n - k)$, $D^2 = 0$ and $H^0(S, L - D) = 0$ for $D \in \Lambda$. If a general member $D \in \Lambda$ is projectively normal as a space curve, then the equation of S is of the form*

$$\Phi_0\Psi_1 = \Phi_1\Psi_0, \tag{4.1}$$

where the Φ_i's and Ψ_j's are homogeneous forms of respective degrees k and $n - k$ in four variables Z_0, Z_1, Z_2, Z_3. Furthermore, Λ is induced by the rational function Φ_0/Φ_1 on $\mathbb{P}^3$.

Proof. By the assumption, D is a smooth irreducible curve which is projectively normal in $\mathbb{P}^3$. Since $K_S = (n - 4)L$ and $D^2 = 0$, we have $\omega_D \simeq \mathcal{O}_D((n - 4)L)$ by the adjunction formula. Since D is projectively normal, it follows from G. Gherardelli's theorem (see [2, p.147]) that $D \subset \mathbb{P}^3$ is a complete intersection of two surfaces. Let a, b be their respective degrees ($a \leq b$). Then we have $b = n - a$ by $\omega_D \simeq \mathcal{O}_D(a+b-4)$. Since $\deg D = ab = a(n - a)$, we get $k(n - k) = a(n - a)$. It follows from $k \leq n - k$ and $a \leq n - a$ that $a = k$. Therefore, D is a complete intersection of type $(k, n - k)$. Then one can compute $h^0(D, mL)$, $m \in \mathbb{Z}$, by using the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-n) \to \mathcal{O}_{\mathbb{P}^3}(-k) \oplus \mathcal{O}_{\mathbb{P}^3}(-n + k) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_D \to 0$$

obtained by the Koszul resolution of the ideal sheaf of D. If we put $N_i := h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(i))$ for $i \in \mathbb{Z}$, we in particular have

$$h^0(D, kL) = N_k - 1, \quad h^0(D, (n - k)L) = N_{n-k} - N_{n-2k} - 1$$

when $k < n - k$, and $h^0(D, kL) = N_k - 2$ when $n = 2k$.

We consider the multiplication map $\psi_m\colon H^0(S, D) \otimes H^0(S, mL) \to H^0(S, mL + D)$ for $m \in \mathbb{N}$. Since S and D are both projectively normal in $\mathbb{P}^3$, the restriction maps $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(S, mL)$ and $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(D, mL)$ are both surjective. Then $H^0(S, mL) \to H^0(D, mL)$ is also surjective. Since $H^1(S, mL) = 0$, the cohomology long exact sequence for

$$0 \to \mathcal{O}_S(mL - D) \to \mathcal{O}_S(mL) \to \mathcal{O}_D(mL) \to 0$$

shows that $H^1(S, mL - D) = 0$. Since $|D|$ is a free pencil, we have an exact sequence

$$0 \to \mathcal{O}_S(mL - D) \to H^0(S, D) \otimes \mathcal{O}_S(mL) \to \mathcal{O}_S(mL + D) \to 0.$$

From this and $h^1(S, mL - D) = 0$, we see that ψ_m is surjective. Furthermore, $h^0(S, mL + D) = h^0(S, mL) + h^0(D, mL)$.

We regard S as a subvariety of $W = \mathbb{P}^3 \times \mathbb{P}^1$ by the embedding $(\Phi_L, f)\colon S \to W$, where as always $f\colon S \to \mathbb{P}^1$ denotes the fibration defined by Λ. Let H_i be the pull-back to W of the hyperplane bundle on $\mathbb{P}^i$ $(i = 1, 3)$. Then $H^0(W, aH_3 + bH_1) \simeq H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$.

We shall show that S is a complete intersection of two hypersurfaces in W. Consider the commutative diagram:

$$\begin{tikzcd} H^0(S, D) \otimes H^0(S, mL) \arrow[r, "\psi_m"] & H^0(S, mL + D) \\ H^0(W, H_1) \otimes H^0(W, mH_3) \arrow[u] \arrow[r, "\simeq"] & H^0(W, mH_3 + H_1) \arrow[u] \end{tikzcd}$$

Since ψ_m and the left vertical map are surjective, we see that $H^0(W, mH_3 + H_1) \to H^0(S, mL + D)$ is also surjective. Assume first that $n = 2k$. Then we have $h^0(W, kH_3 + H_1) - h^0(S, kL + D) = 2N_k - (N_k + N_k - 2) = 2$. This implies that there are two independent hypersurfaces $G, G' \in |kH_3 + H_1|$ through S. We next assume that $k < n - k$. Then we have $h^0(W, kH_3 + H_1) - h^0(S, kL + D) = 1$. So, we can find the unique hypersurface $G \in |kH_3 + H_1|$ through S. We have $h^0(W, (n-k)H_3 + H_1) - h^0(S, (n-k)L + D) = 2N_{n-k} - (N_{n-k} + N_{n-k} - N_{n-2k} - 1) = N_{n-2k} + 1$. Since $h^0(W, (n-k)H_3 + H_1 - G) = h^0(W, (n-2k)H_3) = N_{n-2k}$, we see that there exists a hypersurface $G' \in |(n-k)H_3 + H_1|$ through S but not G. In both cases, it is not so hard to see that S is obtained as the complete intersection $G \cap G'$ scheme theoretically.

Let $(Z_0 : Z_1 : Z_2 : Z_3)$ and $(t_0 : t_1)$ be systems of homogeneous coordinates on $\mathbb{P}^3$ and $\mathbb{P}^1$, respectively. The equation of $G \in |kH_3 + H_1|$ can be written as $\Phi_0 t_1 = \Phi_1 t_0$, where the Φ_i's are homogeneous forms of degree k in the Z_i's. Similarly, the equation of $G' \in |(n-k)H_3 + H_1|$ is of the form $\Psi_0 t_1 = \Psi_1 t_0$, where the Ψ_j's are homogeneous forms of degree $n - k$ in the Z_i's. Hence S is defined in W by the simultaneous equation: $\Phi_0 t_1 = \Phi_1 t_0$, $\Psi_0 t_1 = \Psi_1 t_0$. Then, by eliminating t_0, t_1, we obtain $\Phi_0 \Psi_1 = \Phi_1 \Psi_0$ which is the equation of S in $\mathbb{P}^3$. $\square$

Unfortunately, the above proposition assumes the projective normality of D. We give a numerical sufficient condition for D to be projectively normal, though it seems rather crude.

Lemma 4.2. *Let S be a smooth surface of degree $n \geq 4$ in $\mathbb{P}^3$ and k an integer satisfying $2 \leq k \leq n-k$. Assume that S has an irreducible pencil Λ satisfying $LD = k(n-k)$, $D^2 = 0$ and $H^0(S, L-D) = 0$ for $D \in \Lambda$. If either $n > k^2$ or $(n, k) = (4, 2)$, $(6, 3)$, then any general member of Λ is projectively normal.*

Proof. The assertion is clear when $(n,k) = (4,2)$, since then D is an elliptic curve of degree 4 in $\mathbb{P}^3$. So we assume that $n \geq 5$. Assume that $(n,k) = (6,3)$. Then D is a curve of genus 10 and $\mathcal{O}_D(L)$ is a half-canonical bundle with $h^0(D, L) = 4$ which is very ample. It is an easy exercise to see then that D does not lie on a quadric surface and is a complete intersection of two cubics.

Let C and δ be general hyperplane sections of S and D, respectively. Then C is a smooth plane curve of degree n. Note that δ moves in $\Lambda|_C = g^1_{k(n-k)}$ and is a set of non-degenerate points in uniform position being a general hyperplane-section of an irreducible non-degenerate curve. Since $\deg \delta = k(n-k) \leq g(C) = (n-1)(n-2)/2$ holds when $n \geq 5$, $\Lambda|_C$ is induced by a pencil of plane curves of degree $\leq n-3$, that is, it is given as the restriction of a rational function on $\mathbb{P}^2$ of the form P/P', where P and P' are homogeneous forms in three variables without common factors and $\deg P = \deg P' \leq n-3$ (see, e.g., [12]). Let k_0 be the minimum degree such that $\Lambda|_C$ is induced by a pencil of plane curves of degree k_0. Then $k_0 \leq n-3$. As in [12, p.82], we have $k(n-k) = \deg \Lambda|_C \geq k_0 n - k_0^2$, that is,

$$(k_0 - k)(n - k - k_0) \leq 0. \tag{4.2}$$

Observe that the restriction map $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)) \to H^0(C, aL)$ is an isomorphism for $a < n$. We shall show that $h^0(C, kL - \delta) \neq 0$ by using an argument in [6]. Assume not. Then $k < k_0$ and the multiplication map $V \otimes H^0(C, kL) \to H^0(C, kL + \delta)$ is injective by the free-pencil-trick, where V denotes the module of $\Lambda|_C$. It follows that $h^0(C, kL+\delta) \geq 2h^0(C, kL) = (k+1)(k+2)$. By the duality theorem, we have $h^0(C, kL+\delta) = h^1(C, (n-3-k)L - \delta)$. Then, by the Riemann-Roch theorem, $h^0(C, (n-3-k)L - \delta) = h^0(C, kL+\delta) + (n-3-k)n - k(n-k) - n(n-3)/2 \geq (n-2k-1)(n-2k-2)/2 + 1 > 0$ for $2k \leq n$. This implies that $V \otimes H^0(C, (n-3-k)L) \to H^0(C, (n-3-k)L + \delta)$ is not injective, which is sufficient to see that $\Lambda|_C$ is induced by a pencil of plane curves of degree $n-3-k$ or less. Hence $k < k_0 \leq n-3-k$. It is, however, impossible by (4.2). Therefore, $H^0(C, kL - \delta) \neq 0$ and $k \geq k_0$.

For the degree reason, we must have $k(n-k) \leq k_0 n$. If $n > k^2$, then $k_0 \geq k - k^2/n > k-1$ and we get $k_0 = k$. In particular, we have shown that the smallest degree of plane curves on which δ lies is exactly k. Since δ is in uniform position, such a curve of degree k must be irreducible. Recall that the bound due to Harris [8] states that the geometric genus of an irreducible non-degenerate space curve of degree $d >$

$k(k-1)$ is not greater than

$$\pi_k(d) = \frac{d^2}{2k} + \frac{d(k-4)}{2} + 1 - \frac{\epsilon}{2}\left(k - \epsilon - 1 + \frac{\epsilon}{k}\right)$$

provided that k is the smallest degree of plane curves on which a general hyperplane section of the curve lies, where ϵ is the integer satisfying $\epsilon + d \equiv 0 \pmod{k}$ and $0 \leq \epsilon \leq k-1$. Since D achieves the bound, we see that D is projectively normal (and, in fact, it is a complete intersection of type $(k, n-k)$). $\square$

The above covers completely the case $d = 2n-4$ and most part of $d = 3n-9$. In particular, the formerly unknown pencils, $n = 5$, (a) and $n = 6$, (a), (b) listed at the beginning of the section, are now understood. We remark that when $d = 2n-4$ there is an interesting (possibly singular) elliptic curve on S given by $\Phi_0 = \Phi_1 = 0$. We do not know, however, whether a surface as in Proposition 4.1 contains a line or not. Here we remark the following:

Lemma 4.3. *Let S be a surface as in* Proposition 4.1. *If S is generic, then* $\mathrm{Pic}(S)$ *is a free abelian group generated by L and D. In particular, generic S does not contain a line.*

Proof. As we have seen, S is defined in $W = \mathbb{P}^3 \times \mathbb{P}^1$ by a section of $\mathcal{F} = \mathcal{O}_W(kH_3 + H_1) \oplus \mathcal{O}_W((n-k)H_3 + H_1)$. By a long but standard calculation, we can show that the natural restriction map $\mathrm{Pic}(W) \to \mathrm{Pic}(S)$ is an isomorphism along the same line as in [7, Theorem 2.4] provided that S is generic. Indeed, the outline goes as follows. Note that $H^1(S, N_{S/W}) = 0$, where $N_{S/W}$ denotes the normal bundle of S in W. Let $U \subset H^1(S, \Theta_S)$ be the image of the Kodaira-Spencer map with respect to the maximal family of displacements of S in W, that is, the image of $H^0(S, N_{S/W}) \to H^1(S, \Theta_S)$. We can show that $H^1(W, \Omega^1_W) \to H^1(S, \Omega^1_W|_S)$ is an isomorphism. The key point in [7] is to show that the cup-product map $U \otimes H^0(S, K_S) \to (H^1(S, \Omega^1_S)/H^1(W, \Omega^1_W))^\vee$ is surjective, in order to see that a line bundle on S extends to the whole family only when it is the restriction to S of a line bundle on W. Using the standard exact sequence $H^1(\Omega^1_W) \simeq H^1(\Omega^1_W|_S) \to H^1(\Omega^1_S) \to H^2(N^\vee_{S/W})$ and the Serre duality, it is reduced to showing that $H^0(S, N_{S/W}) \otimes H^0(S, K_S) \to H^0(S, K_S \otimes N_{S/W})$ is surjective. Consider the commutative diagram:

$$\begin{array}{ccc}
H^0(S, N_{S/W}) \otimes H^0(S, K_S) & \longrightarrow & H^0(S, K_S \otimes N_{S/W}) \\
\uparrow & & \uparrow \\
H^0(W, \mathcal{F}) \otimes H^0\left(W, K_W \otimes \bigwedge^2 \mathcal{F}\right) & \longrightarrow & H^0\left(W, K_W \otimes \bigwedge^2 \mathcal{F} \otimes \mathcal{F}\right).
\end{array}$$

Since it can be checked that the multiplication map at the bottom and the restriction map $H^0\big(W, K_W \otimes \bigwedge^2 \mathcal{F} \otimes \mathcal{F}\big) \to H^0(S, K_S \otimes N_{S/W})$ are both surjective, we see that the map in question is surjective.

Suppose that S is generic and let Λ' be an irreducible pencil on S such that $(D')^2 = 0$ for $D' \in \Lambda'$. Put $D' \sim \alpha L - \beta D$ with two integers α, β. By $(D')^2 = 0$, we get $\alpha(n\alpha - 2k(n-k)\beta) = 0$. We have $0 \le DD' = k(n-k)\alpha$. Hence if $\alpha = 0$, then $D' \sim D$ and $LD' = k(n-k)$. If $\alpha > 0$, then $n\alpha = 2k(n-k)\beta$ and we get $LD' = n\alpha - k(n-k)\beta = k(n-k)\beta$. In any case, LD' is a positive multiple of $k(n-k)$. In particular, we cannot have $LD' = n-1$, since $k \ge 2$ and $n \ge 4$. This is sufficient to see that S does not contain a line. □

Recall that, when $n \ge 5$, S does not have a pencil of hyperelliptic curves, because the canonical map is birational onto the image. The following treats the case $n = 5$, (b).

Proposition 4.4. *Suppose that S is a smooth quintic surface with an irreducible pencil Λ satisfying $LD = 7$ and $D^2 = 1$ for $D \in \Lambda$. Then with a suitable system of homogeneous coordinates $(Z_0 : Z_1 : Z_2 : Z_3)$ on $\mathbb{P}^3$ such that the base point of Λ is* $(0:0:0:1)$, *the equation of S is of the form*

$$(Q_{1,0}Q_{2,1} - Q_{1,1}Q_{2,0})Z_0 + (Q_{0,1}Q_{2,0} - Q_{0,0}Q_{2,1})Z_1 + (Q_{0,0}Q_{1,1} - Q_{0,1}Q_{1,0})Z_2 = 0$$

where the $Q_{i,j}$'s are quadratic forms. Furthermore, Λ is generated by two curves defined respectively by

$$\operatorname{rk}\begin{pmatrix} Q_{0,0} & Q_{1,0} & Q_{2,0} \\ Z_0 & Z_1 & Z_2 \end{pmatrix} < 2, \quad \operatorname{rk}\begin{pmatrix} Q_{0,1} & Q_{1,1} & Q_{2,1} \\ Z_0 & Z_1 & Z_2 \end{pmatrix} < 2.$$

Proof. In this case, D is a non-hyperelliptic curve of genus 5 and $h^0(D, L) = 4$. $\Phi_L|_D$ is identified with the canonical map of D followed by the projection with center the point on D corresponding to $\mathrm{Bs}\Lambda$. Since $\Phi_L|_D$ is an embedding, we see that D is tetragonal, because if it were trigonal the projection would produce a double point. Then $D \subset \mathbb{P}^3$ is projectively normal (e.g., [10]).

Let $\sigma : X \to S$ be the blowing-up at the base point of Λ. If E denotes the exceptional (-1)-curve, the canonical bundle of X is given by $K_X = \sigma^*L + [E]$. The relatively minimal fibration $f : X \to \mathbb{P}^1$ induced by Λ is a tetragonal fibration of genus 5. We let F denote a general fibre of f and identify it with the proper transform of D by σ.

Let $\{s_0, s_1, s_2, s_3\}$ be a basis for $H^0(X, \sigma^*L)$ and let $e \in H^0(X, [E])$ define E. Since $|\sigma^*L|$ is free from base point and $\mathcal{O}_E(\sigma^*L) \simeq \mathcal{O}_E$, we can assume that $s_0 = s_1 = s_2 = 0$ on E but $s_3|_E$ is a non-zero constant.

We claim that $|K_X + F|$ is free from base points. Note that $K_X + F = \sigma^*(L + D)$. So if $|L + D|$ were not free, then every member should pass through the base point of Λ. On the other hand, since $H^1(S, L) = 0$, the restriction map $H^0(S, L + D) \to H^0(D, K_D)$ is surjective and we can find an element of $H^0(S, L + D)$ which does not vanish at that point since $|K_D|$ is free. It follows that we can find an element $\eta \in H^0(X, K_X + F)$ which is a non-zero constant on E. If $\{t_0, t_1\}$ denotes a basis for $H^0(X, F)$, then the 9 elements $es_i t_j$, $0 \le i \le 3$, $0 \le j \le 1$, and η give us a basis for $H^0(X, K_X + F)$.

Since $\mathrm{Sym}^2 H^0(X, \sigma^*L) \to H^0(X, 2\sigma^*L)$ is isomorphic, the 10 elements $s_i s_j$ are independent. Recall that $H^0(X, 2\sigma^*L) \to H^0(F, 2\sigma^*L)$ is a surjection between vector spaces of the same dimension 10. It follows that $H^q(X, 2\sigma^*L - F) = 0$ for $q = 0, 1$, which in turn implies that the multiplication $H^0(X,F)\otimes H^0(X,2\sigma^*L) \to H^0(X,2\sigma^*L+F)$ is isomorphic. We consider $H^0(X, \sigma^*L + K_X + F)$ which is of dimension 21. Here we have 20 independent elements of the form $es_i s_j t_k$. These together with $s_3\eta$ form a basis, because $s_3\eta$ is not zero on E. Hence $s_i\eta$, $0 \le i \le 2$, can be expressed as linear combinations of them:

$$s_i\eta = e(Q_{i,0}(s)t_1 - Q_{i,1}(s)t_0) + a_i s_3 \eta, \quad (i = 0, 1, 2)$$

where the $Q_{i,j}$'s are quadratic forms and the a_k's are constants. By restricting the above relations to E, we have $a_0 = a_1 = a_2 = 0$. Therefore, by eliminating η and e, we get

$$\frac{Q_{0,0}(s)t_1 - Q_{0,1}(s)t_0}{s_0} = \frac{Q_{1,0}(s)t_1 - Q_{1,1}(s)t_0}{s_1} = \frac{Q_{2,0}(s)t_1 - Q_{2,1}(s)t_0}{s_2}.$$

Then

$$\frac{t_1}{t_0} = \frac{s_0 Q_{1,1}(s) - s_1 Q_{0,1}(s)}{s_0 Q_{1,0}(s) - s_1 Q_{0,0}(s)} = \frac{s_0 Q_{2,1}(s) - s_2 Q_{0,1}(s)}{s_0 Q_{2,0}(s) - s_2 Q_{0,0}(s)} = \frac{s_1 Q_{2,1}(s) - s_2 Q_{1,1}(s)}{s_1 Q_{2,0}(s) - s_2 Q_{1,0}(s)}.$$

Now, if we put

$$\Xi(s) = (Q_{1,0}Q_{2,1} - Q_{1,1}Q_{2,0})s_0 + (Q_{0,1}Q_{2,0} - Q_{0,0}Q_{2,1})s_1 + (Q_{0,0}Q_{1,1} - Q_{0,1}Q_{1,0})s_2,$$

then

$$(s_0Q_{1,0} - s_1Q_{0,0})(s_0Q_{2,1} - s_2Q_{0,1}) - (s_0Q_{1,1} - s_1Q_{0,1})(s_0Q_{2,0} - s_2Q_{0,0}) = s_0\,\Xi(s)$$
$$(s_0Q_{1,0} - s_1Q_{0,0})(s_1Q_{2,1} - s_2Q_{1,1}) - (s_0Q_{1,1} - s_1Q_{0,1})(s_1Q_{2,0} - s_2Q_{1,0}) = s_1\,\Xi(s)$$
$$(s_0Q_{2,0} - s_2Q_{0,0})(s_1Q_{2,1} - s_2Q_{1,1}) - (s_0Q_{2,1} - s_2Q_{0,1})(s_1Q_{2,0} - s_2Q_{1,0}) = s_2\,\Xi(s)$$

and it follows that there exists a non-trivial quintic relation $\Xi(s) = 0$ in the s_i's. This gives us the equation of S in $\mathbb{P}^3$, that is, S is defined by $\Xi(Z_0, Z_1, Z_2, Z_3) = 0$. $\square$

REMARK 4.5. Let S and X be as above. We have three relations $s_i\eta = e(Q_{i,0}(s)t_1 - Q_{i,1}(s)t_0)$. By using $(t_0, t_1; s_0, s_1, s_2, s_3; \eta, e)$, we can embed X into the total space of the $\mathbb{P}^1$-bundle $\varpi\colon \tilde{W} = \mathbb{P}(\mathcal{O}_W \oplus \mathcal{O}_W(H_3+H_1)) \to W = \mathbb{P}^3 \times \mathbb{P}^1$ as a complete intersection of three hypersurfaces in $|H + \varpi^* H_3|$, where H denotes the tautological line bundle. Using this expression, we can show as in [7, Theorem 2.4] that $\mathrm{Pic}(X) \simeq \mathrm{Pic}(\tilde{W})$ when X is generic. This implies that $\mathrm{Pic}(S)$ is freely generated by L and D for S generic, since $\mathrm{Pic}(X) \simeq \sigma^* \mathrm{Pic}(S) \oplus \mathbb{Z}[E]$. Then we can check that S has neither a Λ_4 nor a Λ_6 as in Lemma 4.3.

When $n = 5, 6$, we have Tables 1 and 2, respectively, for pencils with small invariants, where "codim" means the codimension of the locus in the moduli space of surfaces of respective degree, which can be computed by using the explicit expressions given in Propositions 4.1, 4.4, is (at least) the indicated value. In Table 1, Λ_7 means $\Lambda_{7,1}$ in the proof of Theorem 3.1.

Theorem 4.6. *If* $n = 5$, *then* $\mu_g = 3, 4, 5, 6$ *and* $\mu_s = 3, 29/8, 4$. *If* $n = 6$, *then* $\mu_g = 6, 9, 10$. *Furthermore, there exists a surface with a pencil attaining each of such minimal values.*

Table 1. quintic surfaces

d	$g(\Lambda_d)$	$s(\Lambda_d)$	$\mathrm{gon}(\Lambda_d)$	$\mathrm{cliff}(\Lambda_d)$	codim
4	3	3	3	1	2
5	6	4	4	1	0
6	4	29/8	3	1	4
7	5	4	4	2	4

Table 2. sextic surfaces

d	$g(\Lambda_d)$	$s(\Lambda_d)$	$\mathrm{gon}(\Lambda_d)$	$\mathrm{cliff}(\Lambda_d)$	codim
5	6	4	4	1	3
6	10	9/2	5	2	0
8	9	88/19	4	2	8
9	10	24/5	6	3	10

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