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EQUATIONS IN *p*-CURVATURE AND INTERTWINERS

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Abstract

The equations in p-curvatures, which is a key to prove a stable equivalence of Jacobian problem and Dixmier conjecture in the author's previous paper, is provided an easier proof, related to the existence of 'intertwining operator'. In an appendix, we show that every symplectic morphism between nonsingular symplectic varieties are of Jacobian 1, regardless of the characteristics.

1. Introduction

In a recent paper [6], the author derived a set of differential equations which leads to a stable equivalence of Jacobian problem and Dixmier conjecture.

The aim of the present paper is to provide an easier derivation of the equations, relating them to the existence of 'intertwining operator' G. It makes the argument much clearer, and may introduce a new insight into the theory.

As the first appendix, we show that every symplectic morphism between non singular symplectic varieties are of Jacobian 1, regardless of the characteristics. This is an elementary fact, but may help some people.

As the second appendix, for the sake of being self-containedness, we prove an important formula in the theory of p-curvatures (Proposition A2.2), which already appears in an argument of Katz [3, Proposition 7.1.2]

2. Simplification

2.1. Integrability of connections. A fundamental tool of this paper is the following

Proposition 2.1 (A very special case of [2, Theorem 5.1]). *Let k be a field of characteristic $p \neq 0$. Let X be a smooth scheme over* $\operatorname{Spec}(k)$. *Let $\mathcal{F}$ be a coherent sheaf with a connection ∇. Then $\mathcal{F}$ is locally spanned by parallel sections if and only if the curvature and the p-curvature of ∇ are both zero.*

Proof. We only prove the "only if" part. (The "if" part is conceptually important but is not used in the proofs of our results.) Let $x_1, \dots, x_n$ be a local coordinates.

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Assume $\mathcal{F}$ is spanned by parallel sections $s_1, \dots, s_N$. Then we have

$$[\nabla_{\partial/\partial x_i}, \nabla_{\partial/\partial x_j}]s_k = 0, \quad \nabla^p_{\partial/\partial x_i} s_k = 0.$$

So the curvature tensor and p-curvature tensor are both zero. □

2.2. A review. In this subsection we review and summarize some results obtained by the author [5], [6].

DEFINITION 2.2. Let k be a field of characteristic $p \neq 0$. Let n be a positive integer. A Weyl algebra $A_n(k)$ over a commutative ring k is an algebra over k generated by $2n$ elements $\{\gamma_1, \gamma_2, \dots, \gamma_{2n}\}$ with the "canonical commutation relations"

$$\text{(CCR)} \qquad [\gamma_i, \gamma_j] \, (= \gamma_i\gamma_j - \gamma_j\gamma_i) = h_{ij} \quad (1 \le i, j \le 2n),$$

where h is a non-degenerate anti-Hermitian $2n \times 2n$ matrix of the following form:

$$(h_{ij}) = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

Throughout this paper, the letter h will always represent the matrix above. We denote by $\bar{h}$ the inverse matrix of h.

Let us define operators (matrices) $\{\mu_i\}_{i=1}^{2n}$ acting on p^n-dimensional vector space

$$V_n = k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots x_n^p)$$

by

$$\left.\begin{aligned} \mu_i &= \text{multiplication by } x_i \\ \mu_{i+n} &= \frac{\partial}{\partial x_i}. \end{aligned}\right\} \quad i = 1, 2, \dots, n.$$

Let

$$S_n = k[T_1, T_2, \dots, T_{2n-1}, T_{2n}]$$

be a polynomial ring of $2n$-variables over k. Then we have a faithful representation

$$\Phi \colon A_n \to M_{p^n}(S_n)$$

of the Weyl algebra A_n by putting

$$\Phi(\gamma_i) = T_i \cdot 1_{p^n} + \mu_i.$$

Let $\phi\colon A_n \to A_n$ be a k-algebra endomorphism of A_n. Then we have a matrix valued function $G \in \mathrm{GL}_{p^n}(S_n)$ and a morphism $f\colon \mathrm{Spec}(S_n) \to \mathrm{Spec}(S_n)$ which enables the following diagram commute;

$$\begin{array}{ccc} A_n & \xrightarrow{\ \phi\ } & A_n \\ \Phi\big\downarrow & & \big\downarrow\Phi \\ M_{p^n}(S_n) & \xrightarrow{\ \bar{\phi}\ } & M_{p^n}(S_n), \end{array}$$

where $\bar{\phi}$ is defined as

$$\bar{\phi}(x) = Gf^*(x)G^{-1}.$$

We may write down the commutative diagram above as the following equation.

$$\Phi(\phi(a)) = Gf^*(\Phi(a))G^{-1}.$$

2.3. Proof of the main result. In the author's paper [6, Proposition 3.2], we obtained a result which is essentially equivalent to our main Proposition 2.3. It plays an essential role concerning the stable equivalence of the Jacobian problem and the Dixmier conjecture.

In this section we give an improved proof of the proposition, by focusing the relation between the existence of differential equations and the existence of the intertwiner G.

To do the task, we follow a general theory on connections on bundles. Readers who are not familiar with this kind of argument may as well check the following computations by direct calculations.

In what follows, we employ the following notation.

$$\mathcal{A}_n = \mathcal{O}_{X_n} \otimes_{Z_n} A_n,$$
$$\mathcal{V}_n = \mathcal{O}_{X_n} \otimes_{S_n} V_n.$$

$\mathcal{A}_n$ is an algebra-bundle on X_n whose generic fiber is M_{p^n}. In other words, it is a PGL_{p^n}-bundle on X_n. In our case it has a flat connection $\nabla^{(0)}$ on $\mathcal{A}_n$. To introduce it let us put a matrix valued function F on $\mathrm{Spec}(S_n)$ by

$$F = -\sum_{ij=1}^{n} \bar{h}_{ij}\mu_j T_i,$$

where $\bar{h}_{ij}$ is the ij-component of the inverse matrix $\bar{h}$ of h, as we stated in Section 2.2. Then we define

$$\nabla^{(0)} = d + \mathrm{ad}(dF).$$

It is the unique connection on $M_{p^n}(\mathcal{O}_{X_n})$ which vanishes on the image $\Phi(A_n)$. It follows that the connection $\nabla^{(0)}$ is a flat connection and that it is preserved under algebra endomorphisms of A_n.

We would like to consider a lift of the connection $\nabla^{(0)}$ of a PGL_n-bundle to that of a GL_n-bundle $\mathcal{V}$. Such thing may or may not exist in general, but in our particular case, it does.

$$\nabla = d + dF.$$

Since it is a lift of a flat connection, a general theory tells us that it is a projectively flat connection.

The matrix valued function G may be interpreted as an intertwiner

$$G \in W = \mathrm{Hom}_{\mathcal{O}_{X_n}}(f^*\mathcal{V}, \mathcal{V}).$$

Again from a general theory of connection, we know that there exists an connection on $\mathcal{W} = \mathcal{H}\mathrm{om}_{\mathcal{O}_{X_n}}(f^*\mathcal{V}, \mathcal{V})$

$$\nabla^{\mathrm{gauge}} = d + \lambda(dF) - \varrho(d(f^*F)),$$

where λ (respectively, ϱ) is the "multiplication from the left (respectively, from the right)". That means,

$$\nabla^{\mathrm{gauge}}\sigma = d\sigma + dF\cdot\sigma - \sigma\, d(f^*F)$$

for any local section σ of $\mathcal{W}$.

∇^{gauge} is compatible with the connection ∇ on $\mathcal{V}_n$ in the sense that it satisfies the following "Leibniz rule".

$$\nabla(\sigma\,.\,f^*v) = \nabla^{\mathrm{gauge}}(\sigma)\,.\,f^*v + \sigma\,.\,(f^*(\nabla v)).$$

for all local section σ of $\mathcal{W}$ and v of $\mathcal{V}$.

Since ∇ is projectively flat, ∇^{gauge} is also projectively flat. Now, let us consider an operator

$$\eta = G^{-1}\nabla^{\mathrm{gauge}}G.$$

For any $a \in A_n$, let us put $b = f^*(\Phi(a))$. Then we have

$$\begin{aligned} 0 &= \nabla^{(0)}(\Phi(\phi(a))) \\ &= \nabla^{(0)}(Gf^*\Phi(a)G^{-1}) \\ &= (\nabla^{\mathrm{gauge}}G)bG^{-1} + Gf^*(\nabla^{(0)})(b)G^{-1} + Gb\cdot(-G^{-1}(\nabla^{\mathrm{gauge}}G)G^{-1}) \\ &= G[G^{-1}(\nabla^{\mathrm{gauge}}G), b]G^{-1} = G[\eta, b]G^{-1}. \end{aligned}$$

Therefore, η commutes with all the μ_i's. So it is a scalar valued 1-form. Under these conditions, we have the following proposition.

Proposition 2.3. *$\eta = \sum_i \eta_i \, dT_i$ satisfies the following equations*:
(1)

$$d(f^*\rho - \rho + \eta) = 0$$

where ρ is a one-form on $X_n = \mathrm{Spec}(S_n)$ defined as

$$\rho = \sum_{i=1}^{n} T_i \, dT_{i+n},$$

and
(2)

$$\frac{\partial^{p-1}\eta_i}{\partial T_i^{p-1}} + \eta_i^p + (\partial_{T_i})^{p-1} \sum_{j=1}^{n} \left(\bar{T}_j \frac{\partial \bar{T}_{j+n}}{\partial T_i} \right) = 0 \quad (i = 1, 2, 3, \ldots, 2n)$$

where we put $\bar{T}_j = f^*(T_j)$ $(j = 1, 2, 3, \ldots, 2n)$.

Proof.

$$\nabla^\eta = \nabla^{\mathrm{gauge}} - \eta$$

has G as a parallel section. (η is designed to be so.) It is also easy to verify that ∇^η satisfies the following compatibility condition.

$$\nabla^\eta(ax) = \nabla^{(0)}(a)x + a\nabla^\eta(x) \quad (\forall a \in \mathrm{End}(\mathcal{V}),\ \forall x \in \mathcal{W}).$$

Then we see that a set

$$W_0 = A_n \cdot G \subset W$$

consists of ∇^η-flat connections. On the other hand, from the definition of Φ we see that A_n generates $M_{p^n}(S_n)$ as a S_n-module. (This is also a result of Lemma 2.5 of [6].) Since G is invertible, we see that $\mathcal{W}$ is generated by W_0. Thus $\mathcal{W}$ is generated by flat connections W_0. (One may also see that W_0 coincides with the set of parallel sections of $\mathcal{W}$.)

Therefore, by Proposition 2.1 we conclude that the curvature and the p-curvature of ∇^η should be 0.

Let us put

$$\tilde{\eta} = \eta - \rho + f^*\rho$$

and define the following subsidiary connections on $\mathcal{W}$.

$$\nabla^{\mathrm{I}} = d + \lambda(d(F+\rho) + \tilde{\eta}),$$
$$\nabla^{\mathrm{II}} = d - \varrho(f^*(F+\rho)) = f^*(d - \varrho(F+\rho)).$$

These connections are created by a connection $\nabla^\circ = \nabla + d\rho$ in the following manner:

$$\nabla^{\mathrm{I}} = \nabla^\circ + \tilde{\eta},$$
$$\nabla^{\mathrm{II}} = f^*((\nabla^\circ)^{\mathrm{dual}}).$$

Then we may easily verify that for any $a, b \in \mathcal{W} \cong M_{p^n}(\mathcal{O}_{X_n})$, the following formula holds:

$$\nabla^\eta(a \cdot b) = (\nabla^{\mathrm{I}} a) \cdot b + a \cdot (\nabla^{\mathrm{II}} b).$$

Now, for any vector fields D_1, D_2 on X_n, we have

$$[\nabla^\eta_{D_1}, \nabla^\eta_{D_2}](a \cdot b) = ([\nabla^{\mathrm{I}}_{D_1}, \nabla^{\mathrm{I}}_{D_2}]a) \cdot b + a \cdot ([\nabla^{\mathrm{II}}_{D_1}, \nabla^{\mathrm{II}}_{D_2}]b),$$
$$\mathrm{curv}(\nabla^\eta)(D_1 \wedge D_2)(a \cdot b) = (\mathrm{curv}(\nabla^{\mathrm{I}})(D_1 \wedge D_2)a) \cdot b + a \cdot (\mathrm{curv}(\nabla^{\mathrm{II}})(D_1 \wedge D_2)b),$$
$$\mathrm{curv}(\nabla^{\mathrm{I}}) = d\tilde{\eta}, \qquad \mathrm{curv}(\nabla^{\mathrm{II}}) = 0.$$

Thus we see that

$$\mathrm{curv}(\nabla^\eta) = 0 \iff \mathrm{curv}(\nabla^{\mathrm{I}}) = 0 \iff d\tilde{\eta} = 0.$$

Let us now assume that $d\tilde{\eta} = 0$ and proceed to the computation of p-curvatures of these connections. We first note that for any vector field D on X_n, we have

$$(\nabla^\eta_D)^k(ab) = \sum_{l=0}^{k} \binom{k}{l} ((\nabla^{\mathrm{I}}_D)^l a)((\nabla^{\mathrm{II}}_D)^{k-l} b).$$

In particular, we have

$$(\nabla^\eta_D)^p(ab) = ((\nabla^{\mathrm{I}}_D)^p a)b + a((\nabla^{\mathrm{II}}_D)^p b).$$

So we have the following relation of the p-curvatures.

$$\mathrm{curv}_p(\nabla^\eta)(D)(ab) = (\mathrm{curv}_p(\nabla^{\mathrm{I}})(D)a)b + a(\mathrm{curv}_p(\nabla^{\mathrm{II}})(D)b).$$

By taking p-th power of the equation

$$\nabla^{\mathrm{I}}_D = \nabla^\circ_D + \langle \tilde{\eta}, D \rangle$$

and using Proposition A2.2 in Appendix 2, we obtain

$$(\nabla^{\mathrm{I}}_D)^p = (\nabla^{\circ}_D)^p + \langle\tilde{\eta}, D\rangle^p + D^{p-1}(\langle\tilde{\eta}, D\rangle).$$

So for any vector field D with $D^p = 0$, we have

$$\mathrm{curv}_p(\nabla^{\mathrm{I}})(D) = \langle\rho, D\rangle^p + \langle\tilde{\eta}, D\rangle^p + D^{p-1}(\langle\tilde{\eta}, D\rangle).$$

Similarly, for any vector field D, we have

$$\mathrm{curv}_p((\nabla^{\circ})^{\mathrm{dual}})(D) = -\langle\rho, D\rangle^p.$$

So we see

$$\mathrm{curv}_p(\nabla^{\mathrm{II}})(D) = -\langle f^*\rho, D\rangle^p. \qquad \square$$

One may rewrite the above formula in terms of Cartier operator ([7, Proposition 5.4]). It is also possible to prove the “formal scheme version” of the proposition above (as in [7, Proposition 5.4]) in a similar manner.

Appendix 1. Symplectic implies non-zero Jacobian

In this section we prove the following proposition which is elementary but may supplement/help reading our previous paper [7].

Proposition A1.1. *Let d be a positive integer. Let (X, ω_X), (Y, ω_Y) be smooth symplectic algebraic varieties of dimension $2n$ over a field k. Let $\phi\colon X \to Y$ be a symplectic morphism. That means, it is a morphism which preserves the symplectic structure*:

$$\phi^*(\omega_Y) = \omega_X.$$

Then the tangent map of ϕ is of full rank at every point P on X.

We make an effective use of the theory of the Pfaffian $\mathrm{Pf}(M)$ of a given matrix M. A good reference is in [4, XV, §9]. Especially important theorem we need to know is the following lemma.

Lemma A1.2 ([4, XV, Theorem 9.1]). *Let R be a commutative ring. Let $(m_{ij}) = M$ be an alternating matrix with $g_{ij} \in R$. Then*

$$\det(M) = (\mathrm{Pf}(M))^2.$$

Furthermore, if C is an $n \times n$ matrix in R, then

$$\mathrm{Pf}(CM^tC) = \det(C)\,\mathrm{Pf}(M).$$

Proof of Proposition A1.1. We may assume that k is algebraically closed and that P is a k-valued point. Let us represent the tangent map of f at P by $T_P f$. One may choose a local coordinate system $x_1, \dots, x_{2n}$ on X around P such that the symplectic form ω_X at P is represented by the matrix h when expressed in terms of $dx_1, \dots, dx_{2n}$. Likewise one may choose a local coordinate system $y_1, \dots, y_{2n}$ around $f(P)$ such that the symplectic form ω_Y at $f(P)$ is represented by the matrix h when expressed in terms of $dy_1, \dots, dy_{2n}$. Then using the base $\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_{2n}$ $\partial/\partial y_1, \partial/\partial y_2, \dots, \partial/\partial y_{2n}$, $T_P f$ may be identified with a $2n \times 2n$-matrix. Since by hypothesis $T_P f$ reserves the symplectic form, we have

$$({}^t T_P f) h (T_P f) = h.$$

Let us compare the Pfaffian of the both hand sides.

$$\det(T_P f)^2 \cdot \mathrm{Pf}(h) = \mathrm{Pf}(({}^t T_P f) h T_P f) = \mathrm{Pf}(h).$$

Since $\mathrm{Pf}(h) = 1$, we conclude that the determinant of $T_P f$ should be equal to 1 or (-1). □

NOTE. It goes without saying that when X is connected, and if the coordinate systems $x_1, \dots, x_{2n}$ and $y_1, \dots, y_{2n}$ are able to chosen globally (for example if X, Y are affine space $\mathbb{A}^{2n} = \operatorname{Spec} k[T_1, T_2, T_3, \dots, T_{2n}]$ with

$$\omega = dT_1 \wedge dT_{n+1} + dT_2 \wedge dT_{n+2} + dT_3 \wedge dT_{n+3} + \cdots + dT_n \wedge dT_{2n}$$

as the symplectic form), then the Jacobian of f should either be 1 on the whole of X or be -1 on the whole of X.

Appendix 2. A formula on p-curvature

In this section we prove a formula on p-curvature (Proposition A2.2). The treatment here is based on an argument which appears in [3, Proposition 7.1.2].

We first cite a formula of Jacobson [1]

Proposition A2.1. *Let p be a prime number. Let A be an algebra over $\mathbb{F}_p$ (which is not necessarily commutative, but unital, associative as we always assume.) Then for any elements $a, b \in A$, we have*

$$(a+b)^p = a^p + b^p + \sum_{j=1}^{p-1} s_j(a,b)$$

where s_j is a universal polynomial in a, b given by the following manner.

$$s_j(a,b) = \frac{1}{j}\,\mathrm{coeff}(\mathrm{ad}(Ta+b)^{p-1}a,\ T^{j-1}).$$

(Here, $\mathrm{coeff}(\bullet, T^j)$ *denotes the coefficient of* T^j *in* $\bullet$.*) In particular,* $s_j(a, b)$ *belongs to the Lie algebra generated by* a, b.

Proposition A2.2. *Let* p *be a prime number. Let* D *be a derivation on a commutative algebra* C *of characteristic* p. *Assume that there exists a non commutative algebra* A *which contains* C *as a subalgebra and that there exists an element* $x \in A$ *such that*

$$[x, f] = D(f)$$

holds for all $f \in C$. *Then for any element* f *of* C *we have*

$$(x + f)^p = x^p + D^{p-1}(f) + f^p.$$

Proof. We substitute $a = f$ and $b = x$ in the Proposition A2.1. We need to know $\mathrm{ad}(Tf + x)^{p-1} f$. To do that, first we see by induction that

$$\mathrm{ad}(Tf + x)^k f = D^k f$$

holds for any $k \in \mathbb{N}$. In particular,

$$\mathrm{ad}(Tf + x)^{p-1} f = D^{p-1} f.$$

So

$$s_j(f, x) = \frac{1}{j} \mathrm{coeff}(\mathrm{ad}(Tf + x)^{p-1} f, T^{j-1}) = \begin{cases} D^{p-1} f & \text{if } \ j = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

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