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A LENGTH CHARACTERIZATION OF $*$-SPREAD

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Abstract

The $*$-spread of an ideal is defined as the minimal number of generators of an ideal which is minimal with respect to having the same tight closure as the original ideal. We prove an asymptotic length formula for the $*$-spread.

1. Introduction

Several closure operations for ideals in a commutative Noetherian rings have been studied by numerous authors; among those closures, we mention integral closure, tight closure, Frobenius closure, and plus closure.

For each of the above-mentioned closure operations, a corresponding notion of spread can be defined as the minimal number of generators of a minimal reduction with respect to that operation. The fact that the minimal number of generators is independent of the choice of the reduction is well-known in the case of the integral closure ([6]), easy to see in the case of Frobenius closure, and recently proved ([2]) in the case of tight closure.

Note that in most cases, these spreads can be characterized asymptotically in terms of length, and without reference to corresponding reductions of the ideal. In the case of integral closure, bar-spread is equal to analytic spread (provided the residue field is infinite):

$$l(I) = l^{-}(I) = \deg_n \dim_k \lambda(I^n/\mathfrak{m}I^n) + 1 = \deg_n \dim_k \mu(I^n) + 1.^{1}$$

The F-spread of I is the eventual minimal number of generators of high Frobenius powers of I. That is,

$$l^F(I) = \lim_{q\to\infty} \lambda(I^{[q]}/\mathfrak{m}I^{[q]}) = \lim_{q\to\infty} \mu(I^{[q]}).$$

Finally, the $+$-spread of an ideal I in a henselian local domain R is the eventual mini-

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[1]Here, λ stands for length, and μ is the minimal-number-of-generators function.

mal number of generators of I expanded to domains which are integral extensions of R:

$$l^+(I) = \operatorname*{colim}_{\substack{(R,\mathfrak{m})\subseteq(S,\mathfrak{n})\\ \text{integral ext. domain}}} \lambda(IS/\mathfrak{n}IS) = \operatorname*{colim}_{\substack{(R,\mathfrak{m})\subseteq(S,\mathfrak{n})\\ \text{integral ext. domain}}} \mu(IS).$$

The main result of this paper is an asymptotic characterization of $*$-spread (the spread corresponding to tight closure) in terms of length[2]:

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be an analytically irreducible excellent local ring of characteristic $p > 0$ and Krull dimension d such that $k = \kappa(\overline{R})$. Let J be a proper ideal, and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then for $q_0 \gg 0$,*

$$l^*(J) = \frac{1}{\mathrm{e}_{\mathrm{HK}}(\mathfrak{a})} \lim_{q\to\infty} \frac{\lambda(J^{[qq_0]}/\mathfrak{a}^{[q]}J^{[qq_0]})}{q^d}. \tag{1}$$

In particular, if $\lambda(R/J) < \infty$, then

$$l^*(J) = \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{a}J^{[q_0]}) - \mathrm{e}_{\mathrm{HK}}(J^{[q_0]})}{\mathrm{e}_{\mathrm{HK}}(\mathfrak{a})}. \tag{2}$$

Here, $\kappa(\overline{R})$ stands for the residue field of the normalization $\overline{R}$ of R (which is a local domain, due to the analytic irreducibility of R; a proof of this fact can be found in [2, Lemma 4.3], although it has been known as folklore before).

As an application, we get a result which connects the rationality of the Hilbert-Kunz multiplicity for the ideals I, J, and $IJ^{[q]}$, where I and J are $\mathfrak{m}$-primary ideals (Proposition 3). We also prove a change of base formula for $*$-spread under flat local homomorphisms (Proposition 5).

2. Preliminaries

Throughout this paper, $(R, \mathfrak{m})$ denotes a Noetherian local ring of positive characteristic $p > 0$.

We review some of the notions and results that are used in the proof of our main result. We always use p for the characteristic of R, and q, q', q_0, q_1, q_2, etc. for various powers of p.

NOTATION 1. If $I \subset R$ is an ideal, and q is a power of p, $I^{[q]}$ denotes the ideal $(i^q \mid i \in I)$.

If $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of elements in R, and $t \geq 1$ an integer, $\mathbf{x}^t$ denotes the sequence $x_1^t, \ldots, x_n^t$.

[2]If one assumes that J is $\mathfrak{m}$-primary and $*$-independent, then the theorem is proved in [7, Theorem 3.5 (a)]

DEFINITION 1 (Tight closure and test elements, [3]). Let I be an ideal of a Noetherian ring R of characteristic $p > 0$, and $x \in R$. We say that x is in the *tight closure* of I, written $x \in I^*$, if there is some c not in any minimal prime of R such that for all $q \gg 0$, $cx^q \in I^{[q]}$.

If $c \in R$ is not in any minimal prime of R, and if there exists some q_0 such that for *all* pairs (x, I) with $x \in I^*$, we have $cx^q \in I^{[q]}$ for all $q \geq q_0$, we say that c is a *weak test element* of R. If $q_0 = 1$ works, then c is a *test element* of R.

In [3], Hochster and Huneke prove the remarkable fact that every excellent local R contains a weak test element, and that if R is also reduced, it has a test element. Throughout this paper, we tend to assume that R has a weak test element.

DEFINITION 2 ($*$-independence). Let R be a Noetherian local ring of characteristic $p > 0$, let $f_1, \dots, f_l \in R$. We say that $f_1, \dots, f_l$ are *$*$-independent* if $f_i \notin (f_1, \dots, \hat{f_i}, \dots, f_l)^*$ for all $i = 1, \dots, l$.

We say that an ideal $I \subset R$ is *$*$-independent* if can be generated by $*$-independent elements. If R is local, excellent, and analytically irreducible, this is equivalent to every minimal system of generators being $*$-independent [7, Proposition 3.3]. When this is the case, we say that I is *strongly $*$-independent*.

DEFINITION 3 ($*$-reductions). Let R be a Noetherian local ring of characteristic $p > 0$, $I, K \subset R$ ideals. We say that K is a *$*$-reduction of I* if $K \subseteq I \subseteq K^*$. We say that K is a *minimal $*$-reduction of I* if it is minimal with this property.

Note that, by [2, Propositions 2.1 and 2.3], K is a minimal $*$-reduction of I if and only if it is a $*$-reduction and strongly independent. Therefore, in the case when R is analytically irreducible, a minimal $*$-reduction is equivalent to a $*$-reduction generated by $*$-independent elements.

Also, by [2, Proposition 2.1 and Lemma 2.2], every ideal I has a minimal $*$-reduction.

DEFINITION 4 ($*$-spread). Let $(R, \mathfrak{m}, k)$ be an excellent analytically irreducible local domain of characteristic $p > 0$, $I \subset R$ an ideal. The *$*$-spread* of I, denoted $l^*(I)$, is the minimal number of generators of a minimal $*$-reduction of I. The fact that this number is independent of the choice of a minimal $*$-reduction is [2, Theorem 5.1].

DEFINITION 5 (Special tight closure, [7]). Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$, $x \in R$, $I \subset R$ an ideal. We say that $x \in I^{*\mathrm{sp}}$, the *special tight closure of I*, if there exists $q_0 = p^{e_0}$ such that $x^{q_0} \in (\mathfrak{m}I^{[q_0]})^*$.

Note that one can replace $\mathfrak{m}I^{[q_0]}$ by $\mathfrak{a}I^{[q_0]}$ in the above definition for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, by suitably increasing q_0.

The following result was proved in [2, Theorem 4.5]:

Theorem 2. *Let* $(R, \mathfrak{m}, k)$ *be an excellent analytically irreducible local domain of characteristic* $p > 0$. *Assume that* $k = \kappa(\overline{R})$. *Then for any proper ideal* I *of* R, *there exists a power* q' *of* p *such that*

$$(I^*)^{[q']} \subseteq I^{[q']} + (I^{[q']})^{*\mathrm{sp}}.$$

We will also use the following result of [1], which we will refer to as the *colon criterion*:

Proposition 1. *Let* $(R, \mathfrak{m})$ *be an excellent analytically irreducible local domain, let* $I \subset R$, *and* $x \notin I^*$. *Then there exists a* q_0 *such that* $I^{[q]} : x^q \subset \mathfrak{m}^{[q/q_0]}$ *for all* $q \geq q_0$.

We will also use the following result, from [2, Proposition 2.1]:

Proposition 2 (Nak*). *Let* R *be a Noetherian local ring possessing a weak test element* c. *Let* I, J *be ideals of* R *such that* $J \subseteq I \subseteq (J + \mathfrak{m}I)^*$. *Then* $I \subseteq J^*$.

DEFINITION 6. Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$, and $I \subset R$ an $\mathfrak{m}$-primary ideal. The *Hilbert-Kunz multiplicity* of I is

$$\mathrm{e}_{\mathrm{HK}}(I) = \lim_{q\to\infty} \frac{\lambda(R/I^{[q]})}{q^d},$$

where d is the Krull dimension of R.

The Hilbert-Kunz multiplicity $\mathrm{e}_{\mathrm{HK}}(I)$ of an $\mathfrak{m}$-primary ideal I was identified as a kind of growth rate for the Hilbert-Kunz function $\lambda(R/I^{[q]})$ by Monsky in [5][3], and it turned out to be an important tool in the study of tight closure, due to [3, Theorem 8.17], which asserts that two $\mathfrak{m}$-primary ideals $I \subseteq J$ have the same tight closure if and only if they have the same Hilbert-Kunz multiplicity.

3. Proof of the main result

Before we prove our main result, Theorem 1, we need some preliminary results.

Lemma 1. *Let* R *be a Noetherian local ring of characteristic* $p > 0$ *possessing a weak test element, let* $f_1, \ldots, f_l$ *be* $*$*-independent elements generating an ideal* K, *and let* $\mathbf{x} = x_1, \ldots, x_n$ *be parameters modulo* K. *Then there is some positive integer* t *such that* $f_1, \ldots, f_l, x_1^t, \ldots, x_n^t$ *are* $*$*-independent.*

[3] More precisely, $\mathrm{e}_{\mathrm{HK}}(I) := \lim_{q\to\infty} \lambda(R/I^{[q]})/q^d$. Monsky showed that it always exists.

Proof. First note that $x_j^t \notin (f_1, \dots, f_l, x_1^t, \dots, \hat{x}_j^t, \dots, x_n^t)^*$ for $1 \le j \le n$, $\forall t$ since the x_i are parameters mod K, so the heights of the latter ideal and $K + (\mathbf{x}^t)$ do not match modulo K.

Now pick some $1 \le i \le l$, and suppose $f_i \in (f_1, \dots, \hat{f}_i, \dots, f_l, x_1^t, \dots, x_n^t)^*$ for all t. Then since each of these ideals contains the next,

$$f_i \in \bigcap_{t \ge 1} (f_1, \dots, \hat{f}_i, \dots, f_l, x_1^t, \dots, x_n^t)^* = (f_1, \dots, \hat{f}_i, \dots, f_l)^*,$$

which contradicts the $*$-independence of the f_j. (The equality holds essentially because of the Krull intersection theorem.) Thus for each i with $1 \le i \le l$, there exists an integer t_i with

$$f_i \notin (f_1, \dots, \hat{f}_i, \dots, f_l, x_1^{t_i}, \dots, x_n^{t_i})^*.$$

Let $t = \max_i \{t_i\}$. Then $K + (\mathbf{x}^t)$ is a $*$-independent ideal. $\square$

Lemma 2. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ which has a weak test element c. Let $f_1, \dots, f_l \in R$ be $*$-independent, and $g_1, \dots, g_r \in (f_1, \dots, f_l)^{*\mathrm{sp}}$.*

Then $f_i \notin (f_1, \dots, \hat{f}_i, \dots, f_l, g_1, \dots, g_r)^*$ for all $i = 1, \dots, l$.

Proof. Fix some f_i, let $I := (f_1, \dots, f_l)$, $J := (f_1, \dots, \hat{f}_i, \dots, f_l)$, and $K := (g_1, \dots, g_r)$, and assume by contradiction that $f_i \in (J + K)^*$.

By assumption, we have $K \subseteq I^{*\mathrm{sp}}$, so there is some q_0 we have $cK^{[qq_0]} \subseteq \mathfrak{m}^{[q]} I^{[qq_0]}$ for all $q \gg 0$. Since $I = J + (f_i) \subseteq (J + K)^*$, we have

$$c2I^{[qq_0]} \subseteq cJ^{[qq_0]} + cK^{[qq_0]} \subseteq J^{[qq_0]} + \mathfrak{m}^{[q]} I^{[qq_0]} = (J^{[q_0]} + \mathfrak{m} I^{[q_0]})^{[q]}.$$

As the above containment holds for all $q \gg 0$ and since $J \subseteq I$, we have

$$J^{[q_0]} \subseteq I^{[q_0]} \subseteq (J^{[q_0]} + \mathfrak{m} I^{[q_0]})^*,$$

so that an application of Proposition 2 shows that $I^{[q_0]} \subseteq (J^{[q_0]})^*$, from which it follows easily that $I \subseteq J^*$, and thus $f_i \in J^*$. But this contradicts the $*$-independence of $f_1, \dots, f_l$. $\square$

Lemma 3. *Let $(R, \mathfrak{m}, k)$ be an excellent analytically irreducible local ring of characteristic $p > 0$ such that $k = \kappa(\overline{R})$. Let I be a proper ideal which is not $\mathfrak{m}$-primary, let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal, let L be a minimal $*$-reduction of I, and let z be a parameter modulo I such that $L + (z)$ is a $*$-independent ideal (note that such a z exists by* Lemma 1).

Then there is some power q_0 of p such that,

$$\lim_{q \to \infty} \frac{\lambda((I, z)^{[qq_0]} / \mathfrak{a}^{[q]} (I, z)^{[qq_0]})}{q^d} = \mathrm{e}_{\mathrm{HK}}(\mathfrak{a}) + \lim_{q \to \infty} \frac{\lambda(I^{[qq_0]} / \mathfrak{a}^{[q]} I^{[qq_0]})}{q^d}.$$

Proof. Let $L = (f_1, \dots, f_l)$ be a minimal generating set of L, and let $I = (f_1, \dots, f_l, g_1, \dots, g_r)$ be a minimal generating set of I. Such a minimal generating set exists by [2, Lemma 2.2].

Since $\mathfrak{a}$ is $\mathfrak{m}$-primary, there exists a q_2 such that $\mathfrak{m}^{[q_2]} \subseteq \mathfrak{a}$, and, since $z \notin I^*$, we can choose a q_1 such that $I^{[q]} : z^q \subseteq \mathfrak{m}^{[q/q_1]}$ by the colon criterion.

Since $f_1, \dots, f_l, z$ are $*$-independent, there exists a power q_0 of p such that

$$(L^{[qq_0]} : z^{qq_0}) + \sum_{i=1}^{l} (f_1, \dots, \hat{f_i}, \dots, f_l, z)^{[qq_0]} : f_i^{qq_0} \subseteq \mathfrak{a}^{[q]} \tag{3}$$

for all q, by the colon criterion (Proposition 1). We can moreover choose $q_0 \geq q_1 q_2$.

Now, consider the following short exact sequence:

$$0 \to \frac{\mathfrak{a}^{[q]} I^{[qq_0]} + (z^{qq_0})}{\mathfrak{a}^{[q]}(I, z)^{[qq_0]}} \to \frac{(I, z)^{[qq_0]}}{\mathfrak{a}^{[q]}(I, z)^{[qq_0]}} \to \frac{(I, z)^{[qq_0]}}{\mathfrak{a}^{[q]} I^{[qq_0]} + (z^{qq_0})} \to 0. \tag{4}$$

The first term is isomorphic to $R/((\mathfrak{a}^{[q]}(I, z)^{[qq_0]}) : z^{qq_0})$.

Let $u \in (\mathfrak{a}^{[q]}(I, z)^{[qq_0]}) : z^{qq_0}$. Then there is some $a \in \mathfrak{a}^{[q]}$ such that

$$u - a \in (\mathfrak{a}^{[q]} I^{[qq_0]}) : z^{qq_0} \subseteq I^{[qq_0]} : z^{qq_0} \subseteq \mathfrak{m}^{[qq_0/q_1]} \subseteq \mathfrak{a}^{[qq_0/(q_1q_2)]} \subseteq \mathfrak{a}^{[q]}.$$

Hence $u \in \mathfrak{a}^{[q]}$. The reverse containment is obvious, so

$$\mathfrak{a}^{[q]} = (\mathfrak{a}^{[q]}(I, z)^{[qq_0]}) : z^{qq_0}.$$

Hence, the first term of (4) has length $\lambda(R/\mathfrak{a}^{[q]})$.

For the third term of the sequence, we have:

$$\frac{(I, z)^{[qq_0]}}{\mathfrak{a}^{[q]} I^{[qq_0]} + (z^{qq_0})} \cong \frac{I^{[qq_0]}}{I^{[qq_0]} \cap (\mathfrak{a}^{[q]} I^{[qq_0]} + (z^{qq_0}))}. \tag{5}$$

Claim. *We can choose q_0 to be large enough so that for any q,*

$$\lim_{q \to \infty} \frac{\lambda((I^{[qq_0]} \cap (\mathfrak{a}^{[q]} I^{[qq_0]} + (z^{qq_0})))/\mathfrak{a}^{[q]} I^{[qq_0]})}{q^d} = 0.$$

Proof of Claim. First note that the numerator of the above quotient of ideals equals $\mathfrak{a}^{[q]} I^{[qq_0]} + I^{[qq_0]} \cap (z^{qq_0})$. Next, by Theorem 2, there is some q_3 such that

$$I^{[q_3]} \subseteq (L^*)^{[q_3]} \subseteq L^{[q_3]} + (L^{[q_3]})^{*\mathrm{sp}}$$

Hence, by replacing the f_i's, the g_j's, and z by their q_3 powers, we may assume that $I \subseteq L + L^{*\mathrm{sp}}$.

After this replacement, then, there exist $h_i \in L$ and $g_i' \in L^{*\mathrm{sp}}$ such that $g_i = g_i' + h_i$ for $1 \le i \le r$. We may replace the g_i with the g_i' and assume without loss of generality that $g_i \in L^{*\mathrm{sp}}$ for $1 \le i \le r$. By increasing q_0 if necessary, we may assume $g_i^{q_0} \in (\mathfrak{a}L^{[q_0]})^*$.

By Lemma 2 we have $f_i \notin (f_1, \dots, \hat{f_i}, \dots, f_l, g_1, \dots, g_r, z)^*$.

Let $H_j := (g_1, \dots, g_j)$, where $1 \le j \le r$ (so $I = L + H_r$), and $H_0 := (0)$. We show that $I^{[qq_0]} \cap (z^{qq_0}) \subseteq \mathfrak{a}^{[q]} I^{[qq_0]} + H_r^{[qq_0]}$. Let $x \in I^{[qq_0]} \cap (z^{qq_0})$.

Then

$$x = tz^{qq_0} = \sum_{i=1}^{l} u_i f_i^{qq_0} + \sum_{j=1}^{r} v_j g_j^{qq_0},$$

so that for each $1 \le i \le l$,

$$u_i \in (f_1, \dots, \hat{f_i}, \dots, f_l, g_1, \dots, g_r, z)^{[qq_0]} : f_i^{qq_0} \subseteq \mathfrak{a}^{[q]}$$

by the colon criterion (increasing q_0 if necessary). Thus, $x \in \mathfrak{a}^{[q]} I^{[qq_0]} + H_r^{[qq_0]}$, as claimed.

Note that

$$\lambda\left(\frac{I^{[qq_0]} \cap (\mathfrak{a}^{[q]} I^{[qq_0]} + (z^{qq_0}))}{\mathfrak{a}^{[q]} I^{[qq_0]}}\right) \le \lambda\left(\frac{\mathfrak{a}^{[q]} I^{[qq_0]} + H_r^{[qq_0]}}{\mathfrak{a}^{[q]} I^{[qq_0]}}\right)$$

by what we have shown immediately above. Let c be a test element. We have:

$$\lambda\left(\frac{\mathfrak{a}^{[q]} I^{[qq_0]} + H_r^{[qq_0]}}{\mathfrak{a}^{[q]} I^{[qq_0]}}\right) = \sum_{j=1}^{r} \lambda\left(\frac{\mathfrak{a}^{[q]} I^{[qq_0]} + H_j^{[qq_0]}}{\mathfrak{a}^{[q]} I^{[qq_0]} + H_{j-1}^{[qq_0]}}\right) = \sum_{j=1}^{r} \lambda\left(\frac{R}{(\mathfrak{a}^{[q]} I^{[qq_0]} + H_{j-1}^{[qq_0]}) : g_j^{qq_0}}\right)$$
$$\le \sum_{j=1}^{r} \lambda\left(\frac{R}{\mathfrak{a}^{[q]} + (c)}\right) = r\lambda\left(\frac{R}{\mathfrak{a}^{[q]} + (c)}\right) = r\lambda_{(R/c)}\left(\frac{R/c}{((\mathfrak{a}+(c))/(c))^{[q]}}\right).$$

The inequality is true because $cg_j^{qq_0} \in \mathfrak{a}^{[q]} L^{[qq_0]} \subseteq \mathfrak{a}^{[q]} I^{[qq_0]}$. Thus $\mathfrak{a}^{[q]} + (c) \subseteq \mathfrak{a}^{[q]} I^{[qq_0]} : g_j^{qq_0}$, which proves the inequality.

The last term is r times a Hilbert-Kunz function over the $d-1$ dimensional ring R/c, hence bounded by a constant times q^{d-1}, which proves the claim. $\square$

At this point, taking limits of lengths over q^d as $q \to \infty$ in (5) gives:

$$\lim_{q\to\infty} \frac{\lambda(I^{[qq_0]}/(I^{[qq_0]} \cap (\mathfrak{a}^{[q]} I^{[qq_0]} \cap (z^{qq_0}))))}{q^d} = \lim_{q\to\infty} \frac{\lambda(I^{[qq_0]}/\mathfrak{a}^{[q]} I^{[qq_0]})}{q^d},$$

so that the exact sequence (4) yields that

$$\lim_{q\to\infty} \frac{\lambda((I,z)^{[qq_0]}/\mathfrak{a}^{[q]}(I,z)^{[qq_0]})}{q^d} = \lim_{q\to\infty} \frac{\lambda(R/\mathfrak{a}^{[q]}) + \lambda(I^{[qq_0]}/\mathfrak{a}^{[q]}I^{[qq_0]})}{q^d}$$
$$= \mathrm{e}_{\mathrm{HK}}(\mathfrak{a}) + \lim_{q\to\infty} \frac{\lambda(I^{[qq_0]}/\mathfrak{a}^{[q]}I^{[qq_0]})}{q^d}. \qquad \square$$

Now we begin the proof of Theorem 1.

Proof. First suppose that $\lambda(R/J) < \infty$.

Let K be a minimal $*$-reduction of J. Consider the short exact sequences:

$$0 \to \frac{K^{[qq_0]}}{\mathfrak{a}^{[q]}K^{[qq_0]}} \to \frac{J^{[qq_0]}}{\mathfrak{a}^{[q]}K^{[qq_0]}} \to \frac{J^{[qq_0]}}{K^{[qq_0]}} \to 0 \tag{6}$$

and

$$0 \to \frac{(\mathfrak{a}J^{[q_0]})^{[q]}}{(\mathfrak{a}K^{[q_0]})^{[q]}} \to \frac{J^{[qq_0]}}{\mathfrak{a}^{[q]}K^{[qq_0]}} \to \frac{J^{[qq_0]}}{\mathfrak{a}^{[q]}J^{[qq_0]}} \to 0. \tag{7}$$

Since J (and hence also K, since ideals with the same tight closure have the same radical) is $\mathfrak{m}$-primary, the length of the third term in (6) is the difference of the Hilbert-Kunz functions of J and K. Since these two have the same H-K multiplicity (since they have the same tight closure), the limit as $q \to \infty$ of this difference divided by q^d is 0. Hence the first and second terms are "equal in the limit".

The same comment applies to the first term of the second short exact sequence, since we have

$$\mathfrak{a}J^{[q_0]} \subseteq \mathfrak{a}(K^*)^{[q_0]} \subseteq (\mathfrak{a}K^{[q_0]})^*.$$

Thus, the second and third terms of the second short exact sequence are also "equal in the limit". Hence by transitivity,

$$\lim_{q\to\infty} \frac{\lambda(J^{[qq_0]}/\mathfrak{a}^{[q]}J^{[qq_0]})}{q^d} = \lim_{q\to\infty} \frac{\lambda(K^{[qq_0]}/\mathfrak{a}^{[q]}K^{[qq_0]})}{q^d}.$$

On the other hand, by [7, Theorem 3.5 (a)], we have

$$\lambda\left(\frac{K^{[qq_0]}}{\mathfrak{a}^{[q]}K^{[qq_0]}}\right) = \mu(K)\cdot\lambda\left(\frac{R}{\mathfrak{a}^{[q]}}\right),$$

and $\mu(K) = l^*(J)$. These two equations displayed above, then, give the result in case J is $\mathfrak{m}$-primary. The fact that (1) implies (2) in this case is just by definition of Hilbert-Kunz multiplicities.

Now we drop the assumption that J is $\mathfrak{m}$-primary. Let $\mathbf{x} = x_1, \ldots, x_n$ be R-regular elements of R whose images form a system of parameters for R/J. By Lemma 1, we can pick an integer t such that $K' := K + (\mathbf{x}^t)$ is a $*$-independent ideal. Moreover, $J' := J + (\mathbf{x}^t) \subseteq K^* + (\mathbf{x}^t) \subseteq (K + (\mathbf{x}^t))^* = K'^*$, so K' is a minimal $*$-reduction of J', both of which are, of course, $\mathfrak{m}$-primary. What remains is to connect J and K with J' and K', respectively.

For each i with $0 \le i \le n$, let $I_i = J + (x_1^t, \dots, x_i^t)$, and for $i < n$, $z_i = x_{i+1}^t$. Then applying Lemma 3 to each $I = I_i$ and $z = z_i$ with $i < n$, we have

$$\begin{aligned}\lim_{q\to\infty} \frac{\lambda\left(I_{i+1}^{[qq_0]}/\mathfrak{a}^{[q]} I_{i+1}^{[qq_0]}\right)}{q^d} &= \lim_{q\to\infty} \frac{\lambda((I_i, z_i)^{[qq_0]}/\mathfrak{a}^{[q]}(I_i, z_i)^{[qq_0]})}{q^d} \\ &= \mathrm{e_{HK}}(\mathfrak{a}) + \lim_{q\to\infty} \frac{\lambda\left(I_i^{[qq_0]}/\mathfrak{a}^{[q]} I_i^{[qq_0]}\right)}{q^d},\end{aligned}$$

so that, since $J' = I_n$ and $J = I_0$, after dividing by $\mathrm{e_{HK}}(\mathfrak{a})$ we have:

$$\frac{1}{\mathrm{e_{HK}}(\mathfrak{a})} \lim_{q\to\infty} \frac{\lambda(J'^{[qq_0]}/\mathfrak{a}^{[q]} J'^{[qq_0]})}{q^d} = n + \frac{1}{\mathrm{e_{HK}}(\mathfrak{a})} \lim_{q\to\infty} \frac{\lambda(J^{[qq_0]}/\mathfrak{a}^{[q]} J^{[qq_0]})}{q^d}.$$

However, since J' is $\mathfrak{m}$-primary, we already know that the left hand side equals $l^*(J') = \mu(K') = n + \mu(K) = n + l^*(J)$. Then subtracting n from each side gives the desired result. $\square$

4. Hilbert-Kunz multiplicity

Proposition 3. *Let $(R, \mathfrak{m})$ be an excellent analytically irreducible local ring of characteristic $p > 0$, such that $k = \kappa(\overline{R})$, where $\overline{R}$ is the normalization of R. Let I and J be $\mathfrak{m}$-primary ideals of R. Then there is some power q_0 of p such that the following conditions are equivalent*:

(a) *There exist powers q, q' of p such that $q' \ge q \ge q_0$, and $\mathrm{e_{HK}}(IJ^{[q']})$ and $\mathrm{e_{HK}}(IJ^{[q]})$ are both rational.*

(b) $\mathrm{e_{HK}}(IJ^{[q]})$ *is rational for all $q \ge q_0$.*

(c) $\mathrm{e_{HK}}(I)$ *and* $\mathrm{e_{HK}}(J)$ *are both rational.*

Moreover, there is some power q_1 of p such that

$$\mathrm{e_{HK}}(JJ^{[q]}) = (l^*(J) + q^d)\, \mathrm{e_{HK}}(J)$$

for all $q \ge q_1$, where $d = \dim R$. In particular, $\mathrm{e_{HK}}(J)$ is rational if and only if one such $\mathrm{e_{HK}}(JJ^{[q]})$ is rational if and only if all such $\mathrm{e_{HK}}(JJ^{[q]})$'s are rational.

Proof. By Theorem 1, there exists some q_0 such that for all $q \ge q_0$, we have

$$l\, \mathrm{e_{HK}}(I) + q^d\, \mathrm{e_{HK}}(J) = \mathrm{e_{HK}}(IJ^{[q]}), \tag{8}$$

where $l = l^*(J)$. Hence, if $q' \ge q$ is another power of p, then we have

$$l\, \mathrm{e_{HK}}(I) + q'^d\, \mathrm{e_{HK}}(J) = \mathrm{e_{HK}}(IJ^{[q']}), \tag{9}$$

so that subtracting Equation (8) from Equation (9), we get:

$$(q'^d - q^d)\, \mathrm{e}_{\mathrm{HK}}(J) = \mathrm{e}_{\mathrm{HK}}(IJ^{[q']}) - \mathrm{e}_{\mathrm{HK}}(IJ^{[q]}). \tag{10}$$

On the other hand, if we multiply (8), by q'^d and (9) by q^d, and then subtract, we get:

$$(q'^d - q^d) l\, \mathrm{e}_{\mathrm{HK}}(I) = q'^d\, \mathrm{e}_{\mathrm{HK}}(IJ^{[q]}) - q^d\, \mathrm{e}_{\mathrm{HK}}(IJ^{[q']}). \tag{11}$$

It is trivial that (b) $\Rightarrow$ (a). Equation (8) shows that (c) $\Rightarrow$ (b). Equations (10) and (11) show that (a) $\Rightarrow$ (c).

The second statement comes from replacing I by J in Equation (8). $\square$

The next Proposition does not refer to $*$-spread, but it is a nice base change formula for Hilbert-Kunz multiplicities that works in a very general situation.

Proposition 4. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings of prime characteristic $p > 0$, such that $S/\mathfrak{m}S$ is Cohen-Macaulay. Then for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in R and any sequence $\mathbf{z} = z_1, \dots, z_s$ of elements in S whose images form a system of parameters for $S/\mathfrak{m}S$, the following two formulas hold*:

(a)

$$\lambda_S(S/(\mathfrak{a}S, \mathbf{z})) = \lambda_S(S/(\mathfrak{m}S, \mathbf{z}))\lambda_R(R/\mathfrak{a})$$

(b)

$$\mathrm{e}^S_{\mathrm{HK}}(\mathfrak{a}S + (\mathbf{z})) = \mathrm{e}^{S/\mathfrak{m}S}(\mathbf{z})\, \mathrm{e}^R_{\mathrm{HK}}(\mathfrak{a}).$$

Proof. For part (a), we have that

$$S/(\mathfrak{a}S, \mathbf{z}) \cong \frac{S/\mathbf{z}}{\mathfrak{a}(S/\mathbf{z})} \cong S/\mathbf{z} \otimes_R R/\mathfrak{a},$$

and since $S/\mathbf{z}$ is flat over R,

$$\begin{aligned} \lambda_{S/\mathbf{z}}(S/\mathbf{z} \otimes_R R/\mathfrak{a}) &= \lambda_{S/\mathbf{z}}((S/\mathbf{z})/\mathfrak{m}(S/\mathbf{z})) \cdot \lambda_R(R/\mathfrak{a}) \\ &= \lambda_{S/\mathfrak{m}S}((S/\mathfrak{m}S)/\mathbf{z}(S/\mathfrak{m}S)) \cdot \lambda_R(R/\mathfrak{a}). \end{aligned}$$

For part (b), we replace $\mathbf{z}$ by $\mathbf{z}^{[q]}$ and $\mathfrak{a}$ by $\mathfrak{a}^{[q]}$ in (a) so that, letting $d = \dim R$, we have

$$\begin{aligned} \mathrm{e}^S_{\mathrm{HK}}(\mathfrak{a}S + (\mathbf{z})) &= \lim_{q\to\infty} \frac{\lambda_S(S/(\mathfrak{a}S, \mathbf{z})^{[q]})}{q^{d+s}} \\ &= \lim_{q\to\infty} \frac{\lambda_{S/\mathfrak{m}S}((S/\mathfrak{m}S)/\mathbf{z}^{[q]}(S/\mathfrak{m}S))}{q^s} \cdot \frac{\lambda_R(R/\mathfrak{a}^{[q]})}{q^d} \\ &= \mathrm{e}^{S/\mathfrak{m}S}_{\mathrm{HK}}(\mathbf{z})\, \mathrm{e}^R_{\mathrm{HK}}(\mathfrak{a}) = \mathrm{e}^{S/\mathfrak{m}S}(\mathbf{z})\, \mathrm{e}^R_{\mathrm{HK}}(\mathfrak{a}). \end{aligned}$$

The last equality follows from [4, Theorem 2]. □

5. $*$-Spread and flat base change

Proposition 5. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of prime characteristic $p > 0$ excellent analytically irreducible Noetherian local rings which share a test element c.*
(a) *If $x_1, \dots, x_n \in R$ are $*$-independent elements of R, they are $*$-independent in S as well.*
(b) *If I is a proper ideal of R, then $l^*(I) = l^*(IS)$.*

Proof. For part (a), suppose that $x_n \in ((x_1, \dots, x_{n-1})S)^*$. Then for all $q \geq q_0$, $cx_n^q \in (x_1^q, \dots, x_{n-1}^q)S$, so that

$$c \in \bigcap_{q \geq q_0} (x_1^q, \dots, x_{n-1}^q)S :_S x_n^q = \bigcap_{q \geq q_0} ((x_1^q, \dots, x_{n-1}^q) :_R x_n^q)S)$$
$$= \left(\bigcap_{q \geq q_0} (x_1^q, \dots, x_{n-1}^q) :_R x_n^q \right) S = (0)$$

where the first two equalities follow from flatness of S over R, and the last equality is due to the fact that $x_n \notin (x_1, \dots, x_{n-1})^*$, and R is a domain.

This contradicts the fact that c is a test element.

As for part (b), let J be a minimal $*$-reduction of I. Let $x_1, \dots, x_n$ be a minimal set of generators for J. Then since $IS \subseteq J^*S \subseteq (JS)^*$ by persistence of tight closure, IS has a $*$-reduction generated by n elements, which shows that $l^*(IS) \leq l^*(I)$.[4] On the other hand, $x_1, \dots, x_n$ are $*$-independent elements of S by part (a), so JS is a $*$-independent ideal, so $l^*(IS) \geq l^*(I)$. □

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