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HITTING LAW ASYMPTOTICS FOR A FLUCTUATING BROWNIAN FUNCTIONAL

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Abstract

By using excursions from the maximum, we get asymptotic information on the hitting law of a fluctuating Brownian functional. This extends a result of Isozaki and Kotani who considered the case when the underlying Lévy process is stable.

1. Introduction

Take a Radon measure $m = m^+ + m^-$, with $m^\pm$ supported on $\mathbb{R}^\pm$ respectively, and denote by $\tilde{v}$ the positive bounded solution on the right half-plane of

$$\frac{1}{2}\frac{\partial^2\tilde{v}}{\partial y^2} + \frac{m(dy)}{dy}\operatorname{sgn}(y)\frac{\partial\tilde{v}}{\partial x} = \lambda(\tilde{v} - 1), \quad \tilde{v}(0, y)1_{(y\le 0)} \equiv 0. \tag{1.1}$$

Then, for $0 \in \overline{\operatorname{supp}}(m^-)$ and $m\{0\} = 0$, we investigate how $v(x) := \tilde{v}(x, 0)$ behaves as $x \downarrow 0$. In previous work m was assumed absolutely continuous with density a multiple of $|x|^\gamma$. There is an extensive literature dealing with the case $\gamma = 1$.

Our approach depends on the following probabilistic interpretation. Given a Brownian motion Y, with local time denoted by l, we define

$$X = x + \int l(a,\ .\,)\operatorname{sgn}(a)m(da); \quad T^X = \inf\{t > 0\colon X_t \le 0\}.$$

In McKean's [23] terminology $Z = (X, Y)$ represents a resonator driven by a white-noise, rotating clockwise about the origin, and T^X determines its half-winding time. Our question now concerns the rate of convergence for $\mathbb{E}_{x,0}[1 - e^{-\lambda T^X}]$ as $x \downarrow 0$.

We adapt a device of Isozaki-Kotani [11]. In the case $m^\pm(dx) = c_\pm|x|^\gamma\,dx$, they used properties of W, the Lévy process obtained by sampling $-X$ on the zero set of Y, to derive an integral representation for v. In like manner, assuming $\limsup_{t\uparrow\infty} W_t \overset{\text{a.s.}}{=} \infty$, we will prove the existence of $k = k_\lambda\colon \mathbb{R}^+ \to \mathbb{R}^+$ and two Radon measures $R^{\oplus/\ominus}$ satisfying

$$v(x) = v(x, \lambda) := \mathbb{E}_{x,0}[1 - e^{-\lambda T^X}] = \int_0^x R^\oplus(dy)\int_{-\infty}^0 R^\ominus(ds)k(x - y - s). \tag{1.2}$$

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Section 2 explains how (1.2) arises from applying the Wiener-Hopf method to $k := \mathbf{P}^{+}\mathcal{G}v$, where $\mathcal{G}$ denotes the generator of W. The formula includes (3.23)–(3.24) of [11] and, by estimation of the integral, it leads to the following extension of [11] (2.14).

Theorem. *Let m be a Radon measure satisfying $0 \in \overline{\operatorname{supp}}(m^-)$ and $m\{0\} = 0$. Assume that*:
(1) $\limsup_{t\uparrow\infty} W_t \overset{a.s.}{=} \infty$;
(2) $\int_{-\infty}^{0} e^{\sqrt{2\lambda}a} m(da) < \infty$;
(3) *The Lévy measure ν of W satisfies either of*:
(A) $\nu[x, \infty) = \mathrm{O}(\nu[2x, \infty))$ *as* $x \uparrow \infty$;
(B) $\int_1^\infty x\nu(dx) < \infty$.
Then there exists $0 < C(\lambda) < \infty$ such that $v(x, \lambda) \sim C(\lambda)\, R^{\oplus}[0, x]$ as $x \downarrow 0$.

REMARK 1.1. (1) Conditions (A) and (B) overlap but neither includes the other—consider $\nu[x, \infty) = x^{-1/2}, e^{-x}$.
(2) Our proof of (1.2) identifies $R^{\oplus}$ as the potential of the positive Wiener-Hopf factor of W.
(3) In [11], where $m^{\pm}(da) = c_{\pm}|a|^{\gamma}(da)$, the process W is stable of order $\alpha = 1/(2+\gamma)$ with $R^{\oplus}[0, x]$ a multiple of $x^{\alpha\rho}$ for $0 < \rho < 1$. By exploiting scaling properties, they found $C(\lambda) \sim \lambda^{\rho/2}C(1)$ as $\lambda \downarrow 0$ and showed $x^{-\alpha\rho}t^{\rho/2}\mathbb{P}_x[T^X > t] \sim C(1)/\Gamma(1 - (\rho/2))$ as $x^{2\alpha}/t \downarrow 0$.
(4) Barring $m^+ = 0$, the value of $C(\lambda)$ is known only when $m(da) = da$. See Isozaki-Watanabe [10] for a computation based on work of McKean [23].

The motivation for writing this article comes from several sources. Besides the work of Isozaki-Kotani [11], itself prompted by Sinai's [31] investigations of a similar question for random walks, there are links with David Williams' research [33] into fluctuating clock constructions for Markov chains and diffusions. We also observed that, for $m^+ = 0$ and hence X monotone, Yamazato [32] used Krein's [15] spectral theory to connect asymptotics of $\mathbb{P}[T^X > t]$ with exponents for m^-. Results in [11] suggest that (1.2) may play a similar role for fluctuating functionals. Lastly, the well-known affinity between Sturm-Liouville problems and diffusions contrasts sharply with the minimal impact of pseudo-differential operators on the theory of Lévy processes. Bertoin [1] has an interesting example in this vein. Our heuristic explanation for (1.2), in Section 2, offers another perspective.

We organize the proof of our theorem as follows.

2. Method
3. Proof of (1.2)
4. Decomposing Y^{ξ}
5. The class $\mathcal{CM}^+$
6. Regularity of v
7. Properties of k

8. Proof of Theorem

Inspired by the Greenwood-Pitman [9] approach to Rogozin's [28] Wiener-Hopf decomposition, we shall prove (1.2) by applying Maisonneuve's exit formula [22] to $\mathcal{M}^\bullet$, the set of times when W visits its maximum. Recall that $\mathcal{M}^\bullet$ is regenerative and, if regular, it has a continuous local time $l^\bullet$. Sampling W in this timescale defines $W^\oplus$, the ladder height process (cf. [2] or [6]). For information on $\mathcal{Q}^\ominus$, the measure governing excursions of W away from its maxima, we refer to [21], [22]. The triple $(W^\oplus, l^\bullet, \mathcal{Q}^\ominus)$ is sometimes called the exit system for $\mathcal{M}^\bullet$.

The main technical obstacle to proving (1.2) is not the deployment of Maisonneuve's machinery, as one might expect, but rather justifying $k := \mathbf{P}^+\mathcal{G}v$ and deriving properties thereof. Since our theorem points to non-existence of $v'(0)$, we will proceed by showing that the law of X°_ξ, the minimum, has continuous density away from zero. So in Sections 4–6 we bring to bear results of Rogers [27] and Kent [17] by invoking: a path decomposition of Brownian motion, Krein's characterization of Stieltjes transforms, Krein's correspondence for generators of gap-diffusions, and Yamazato's representation for first-passage laws of the latter—which we examine in some detail, following Knight [19], Kotani-Watanabe [20] and Yamazato [32]. However, for the crucial estimate of Section 7 we emulate [11] by exploiting a path decomposition in the Brownian excursion.

The idea of studying Z via properties of W is not new. One can use it to characterize transience/recurrence and also to show that Z doesn't hit points—by appealing to a famous result of Kesten [18]. In this note we quantify the boundary behaviour of Z in terms of fluctuation theory for W. Remark, however, that the approach fails to determine $C(\lambda)$.

NOTATION. All processes are right-continuous. Fix $\xi \overset{\mathrm{d}}{=} \exp(\lambda)$ independent and denote by $\mathbb{P}_{x,y}$ the law of $Z = (X, Y)$ started at (x, y). We write $X^\bullet_t = \sup_{0<s\le t} X_s$ (resp. $X^\circ_t = \inf_{0<s\le t} X_s$). If $\sigma = l^{-1}(0, .)$ then $W := -X_\sigma$ is a driftless Lévy process of bounded variation whose Laplace exponent we define by $\mathbb{E}[e^{-z(W_t - W_0)}] = e^{-\kappa(z)t}$; thus $\kappa(z) = \mathcal{G}(e^{z\cdot})(0) = \int[1 - e^{-zx}]\nu(dx)$ determines the generator $\mathcal{G}$ and Lévy measure ν. Writing $\nu^\pm$ for the restriction to $\mathbb{R}^\pm$ respectively, then $\nu = \nu^+ + \nu^-$ denotes additive decomposition while we write $\kappa = \kappa^\oplus\kappa^\ominus$ for the multiplicative Wiener-Hopf (WH) factorization. This convention applies throughout, as in $m = m^+ + m^-$ or $\mathcal{G} = \mathcal{G}^\oplus * \mathcal{G}^\ominus$, although for random variables we keep $U^\pm = \sup(\pm U, 0)$. The projection operator onto $(0, \infty)$ is denoted $\mathbf{P}^+$.

2. Method

This section is purely descriptive. It introduces the method of [11], gives a heuristic explanation for (1.2), and finishes with a summary of our probabilistic proof. Discussion of the major technical difficulty, proving smoothness of v, has been shunted off to Section 6.

First we explain the notation. By the strong Markov property $W := -X_\sigma$ is a (bounded variation, driftless) Lévy process. The WH factors $\mathcal{G}^{\oplus/\ominus}$ of its generator $\mathcal{G}$ are themselves generators of positive/negative subordinators $W^{\oplus/\ominus}$ ([2] p.166 or see Lemma 5.4) so their respective potentials $R^{\oplus/\ominus}$ define Radon measures on the line.

Assuming for the moment that $\mathcal{G}v$ exists, the method of [11] has three distinct parts:

(I) Inversion of $k := \mathbf{P}^+\mathcal{G}v \geq 0$ to obtain (1.2);

(II) Proving $\int_{-2}^{0} k(-y)R^\ominus(dy)$ finite by bounding k;

(III) Bounding $\int_{-\infty}^{-2} k(-y)R^\ominus(dy)$ via an estimate for k at infinity.

These steps are far from trivial. Although we write them analytically, our proofs will utilize their probabilistic interpretation.

As it happens, the formula (1.2) has a straightforward heuristic explanation. We get it by inverting $k = \mathbf{P}^+\mathcal{G}v$, a convolution equation on the half-line, using the classical WH technique. A concise description goes as follows. Dropping $\mathbf{P}^+$ gives

$$\mathcal{G}^\ominus * \mathcal{G}^\oplus v = \mathcal{G}v = k + k_1,$$

with $\mathbf{P}^+k_1 = 0$, whereupon convolution by $R^\ominus = (\mathcal{G}^\ominus)^{-1}$ yields

$$\mathcal{G}^\oplus v = \mathbf{P}^+\mathcal{G}^\oplus v = \mathbf{P}^+[R^\ominus * (k + k_1)] = \mathbf{P}^+[R^\ominus * k].$$

Applying $R^\oplus$ we now deduce $v = R^\oplus * \mathrm{P}^+[R^\ominus * k]$, alias our formula (1.2).

By dint of hard analysis, and using the explicit WH factorization of a stable process, Isozaki-Kotani [11] managed a rigorous proof of (1.2) along the lines indicated when $m^\pm(da) = c^\pm|a|^\gamma\, da$. However, the difficulty of finding analytic estimates for WH factors suggests that the above template may prove unsuited for use with general m.

We therefore offer a probabilistic approach to (I), starting from the remark that $k = \mathbf{P}^+\mathcal{G}v$ determines a martingale. This leads to a path-integral representation for v, which we then transform into (1.2) by applying Maisonneuve's formula [22] (4.3) to the excursions of W from $\mathcal{M}^\bullet$. Our proof of (II) is also probabilistic, but more straightforward, in that we work with the Brownian excursion and employ the same path decomposition used in [11]. As to the estimate (III), we prove it by formulating the various quantities probabilistically and applying results from the preceding sections. For example, $\int_{-\infty}^{-2} \nu[-x,\infty)R^\ominus(dx) < \infty$ appears as an attribute of the excursion measure $\mathcal{Q}^\ominus$. So (I) and (III) are properties of Maisonneuve's exit system $(W^\oplus, l^\bullet, \mathcal{Q}^\ominus)$ while (II) depends on the structure of the more prosaic Brownian excursion.

3. Proof of (1.2)

We first obtain a path integral representation for v in terms of $k = \mathbf{P}^+\mathcal{G}v$. To this end, let us consider the $\mathbb{P}_0$ local martingale

$$t \to v(x - W_t)1_{(W_t^\bullet < x)} + \int_0^t k(x - W_s)1_{(W_s^\bullet < x)}\, ds. \tag{3.1}$$

where we define

$$k(x) = \mathbf{P}^{+}\mathcal{G}v(x) = 1_{(x>0)} \int [v(x) - v(x-y)]\nu(dy). \tag{3.2}$$

Existence[1] of the latter requires

$$v \quad \text{increasing and continuously differentiable on} \quad (0, \infty). \tag{3.3}$$

We prove (3.3) in Section 6. However, Remark 8.4 (1) explains why $v'(0)$ doesn't exist and this, in turn, casts doubt on the finiteness of $k(0+)$. So in Section 7 we will show that

$$k \quad \text{is positive, continuous, and bounded on} \quad (0, \infty). \tag{3.4}$$

Taken together, these results imply (3.1) is a martingale and hence

$$v(x) = \mathbb{E}[v(x - W_t)1_{(W_t^\bullet < x)}] + \mathbb{E}\left[\int_0^t 1_{(W_s^\bullet < x)} k(x - W_s)\, ds\right].$$

Now let $t \uparrow \infty$. From $W_t^\bullet \uparrow \infty$ the first expectation vanishes. In the other term, $k \geq 0$ means we can pass to the limit by monotone convergence. Hence the path integral representation

$$v(x) = \mathbb{E}\left[\int_0^\infty 1_{(W_s^\bullet < x)} k(x - W_s)\, ds\right] \tag{3.5}$$

which we claim coincides with (1.2).

To prove our claim, we will decompose the path integral using the excursions of W away from the optional set $\mathcal{M}^\bullet = \{t : W_t = W_t^\bullet\}$. We therefore take W defined on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, with filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, where for every $\mathcal{F}$-stopping time T the increment $W_{T+.} - W_T$ is independent of $\mathcal{F}_T$. By the strong Markov property of $W - W^\bullet$, noted in Bingham [3], $\mathcal{M}^\bullet$ is $\mathcal{F}$-regenerative: for all $\mathcal{F}$-stopping times $[T] \subseteq \mathcal{M}^\bullet$ we have $\mathcal{M}^\bullet \overset{\mathrm{d}}{=} \mathcal{M}^\bullet \circ \theta_T$ with the latter independent of $\mathcal{F}_T$ on $(T < \infty)$. Such random sets satisfy a zero-one law [21]. Either zero is isolated in $\mathcal{M}^\bullet$, which is then (topologically) discrete, or else zero is a limit point and $\mathcal{M}^\bullet$ has no isolated points. In the latter case, assumed henceforth unless otherwise indicated, $\mathcal{M}^\bullet$ has a continuous local time $l^\bullet$. We denote its right-continuous inverse by $\sigma^\bullet$.

Maisonneuve's theory [22] applies to closed sets and, while right-continuity of W implies $\mathcal{M}^\bullet$ closed under decreasing limits, in general $\mathcal{M}^\bullet \neq \overline{\mathcal{M}}^\bullet$. Nevertheless, since

[1] Our definition of the generator simplifies comparing (3.5) with (1.2). It appears again in Sections 7–8 but the notation is not standard—unlike our convention on the Laplace exponent and Lévy measure.

both sets have the same Itô excursions, we will continue to state our results in terms of $\mathcal{M}^\bullet$.

Defining $W^\oplus = W_{\sigma^\bullet}$, the strong Markov property of W applied at the stopping times $(\sigma_t^\bullet)_{t\geq 0}$ shows that $(\sigma^\bullet, W^\oplus)$ is a bivariate subordinator—known as the ladder process. Write $\kappa^\oplus$ (resp. $R^\oplus$) for the Laplace exponent (resp. potential) of $W^\oplus$ and remark that if W, and hence $\mathcal{M}^\bullet$, is $\mathcal{F}$-adapted then $W^\oplus$ is adapted to $\mathcal{F}^\bullet := \mathcal{F}_{\sigma^\bullet}$.

The next result was extracted from [21].

Lemma 3.1. *The following relations hold almost surely.*

(1) $\left(\bigcup_{t>0}(\sigma_{t-}^\bullet, \sigma_t^\bullet)\right) \cap \mathcal{M}^\bullet = \emptyset$.

(2) *The range* $\{\sigma_t^\bullet : t \geq 0\} \subseteq \mathcal{M}^\bullet$.

(3) *If* $\sigma^\bullet$ *has drift* $b^\bullet$, *the Lebesgue measure* $|\mathcal{M}^\bullet \cap [0, \sigma_t^\bullet]| = b^\bullet t$.

(4) $W_{t-}^\oplus = W_s^\bullet > W_s$ *on* $\sigma_{t-}^\bullet < s < \sigma_t^\bullet$.

Our probabilistic description of $\kappa^\oplus$ involves sampling W on $\mathcal{M}^\bullet$. For a probabilistic interpretation of $\kappa^\ominus$ we use the excursions of W away from $\mathcal{M}^\bullet$. Introducing E as the space of strictly negative paths, and writing $\mathfrak{E}$ for the σ-algebra determined by the Skorohod topology, we define the excursion process $(\mathcal{E}_t)_{t\geq 0}$ by $\mathcal{E}_t(u) = \{W_{u+\sigma_{t-}^\bullet} - W_{t-}^\oplus : 0 < u < \Delta\sigma_t^\bullet\}$ whenever $\Delta\sigma_t^\bullet > 0$. By Lemma 3.1 (1) this takes values in $(E, \mathfrak{E})$. Then [22] shows there exists a measure $\mathcal{Q}^\ominus$ on $(E, \mathfrak{E})$ such that, for any $\mathcal{B}(\mathbb{R}^+) \otimes \mathfrak{E}$ measurable $F : \mathbb{R}^+ \times E \to \mathbb{R}^+$,

$$\mathbb{E}\left[\sum_{0<s\leq t} U_s F_s \circ \mathcal{E}_s\right] = \mathbb{E}\left[\int_0^t U_s \mathcal{Q}^\ominus[F_s]\, ds\right] \tag{3.6}$$

whenever U is positive, bounded, and $\mathcal{F}^\bullet$-predictable. In our application

$$F_s \circ \mathcal{E}_s = \int_{\sigma_{s-}^\bullet}^{\sigma_s^\bullet} k(x - W_u)\, du = \int_0^\varrho k(x - W_{s-}^\oplus - \mathcal{E}_s(u))\, du,$$

using ϱ for the excursion lifetime. By [22] (6.4) formula (3.6) applies here also.

With this in mind, let us return to our task of rearranging (3.5). We start from

$$\begin{aligned}
&\int_0^{\sigma_t^\bullet} 1_{(W_s^\bullet < x)} k(x - W_s)\, ds \\
&= \sum_{0<s\leq t} \int_{\sigma_{s-}^\bullet}^{\sigma_s^\bullet} 1_{(W_u^\bullet < x)} k(x - W_u)\, du + \int_0^{\sigma_t^\bullet} 1_{(W_s^\bullet < x)} k(x - W_s) 1_{\mathcal{M}^\bullet}(s)\, ds.
\end{aligned}$$

For the first term on the right, we invoke Lemma 3.1 (4) to replace $W_u^\bullet \to W_{s-}^\oplus$ on each excursion interval thus

$$\sum_{0<s\leq t} 1_{(W_{s-}^\oplus < x)} \int_{\sigma_{s-}^\bullet}^{\sigma_s^\bullet} k(x - W_{s-}^\oplus + \mathcal{E}_s(u))\, du.$$

Applying (3.6) with $U_t = 1_{(W^{\oplus}_{t-}<x)}$, and using Lemma 3.1 (3) for the other term, now gives

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^{\sigma_t^\bullet} 1_{(W_s^\bullet<x)}k(x-W_s)\,ds\bigg]\\&=\mathbb{E}\bigg[\int_0^t 1_{(W^{\oplus}_{s-}<x)}\bigg(\mathcal{Q}^{\ominus}\bigg[\int_0^{\varrho} k(x-W^{\oplus}_{s-}-\mathcal{E}_s(u))\,du\bigg]+b^\bullet k(x-W^{\oplus}_s)\bigg)\,ds\bigg].\end{aligned}$$

For the final step, we use continuity of the integrator to replace $W^{\oplus}_{s-}\to W^{\oplus}_s$. Then, recalling that $R^{\oplus}$ is the potential measure of $W^{\oplus}$, the limit as $t\uparrow\infty$ yields

$$\begin{aligned}v(x)&=\mathbb{E}\bigg[\int_0^{\infty} 1_{(W_s^\bullet<x)}k(x-W_s)\,ds\bigg]\\&=\int_0^x R^{\oplus}(dy)\bigg(\mathcal{Q}^{\ominus}\bigg[\int_0^{\varrho} k(x-y-\mathcal{E}(u))\,du\bigg]+b^\bullet k(x-y)\bigg),\end{aligned}\tag{3.7}$$

where we've replaced $\mathcal{E}_s$ by the generic excursion $\mathcal{E}$. Defining

$$R^{\ominus}(dy)=b^\bullet\delta_0(dy)+\mathcal{Q}^{\ominus}\bigg[\int_0^{\varrho}1_{(\mathcal{E}(u)\in dy)}\,du\bigg],\tag{3.8}$$

which by Lemma 3.1 (1) is supported on $(-\infty,0)$, we thereby identify (3.7) with (1.2). This completes the proof when $\mathcal{M}^\bullet$ has no isolated points a.s.

It remains to dispose of the discrete case. There $\mathcal{M}^\bullet=\bigcup_{n\geq 0}[T_n]$ for an increasing sequence of stopping times, and the passage from (3.5) to (1.2) becomes much simpler—the above operations reduce to manipulating i.i.d. sums. We therefore omit the details.

REMARK 3.2. (1) The local time on the set of minima $\mathcal{M}^\circ=\{t: W_t=W_t^\circ\}$ defines a negative subordinator $\hat{W}^{\ominus}:=W_{\sigma^\circ}$ whose potential $\hat{R}^{\ominus}$ can, in fact, be identified with a multiple of $R^{\ominus}$ (the present case is covered by Remark 5.5).
(2) By examining its Laplace exponent, we find $W^{\ominus}$ is compound Poisson iff $b^\bullet>0$ (cf. [6] p.31).

4. Decomposing Y^ξ

This section, and the next, prepare the ground for the proofs in Section 6. There we use a path decomposition of Y to establish smoothness of $x\to\mathbb{P}_0[X_\xi^\circ<-x]$. The idea is to split X_ξ° into independent components, which are then analysed separately—using results on Lévy processes with completely monotone jump density and related properties of gap-diffusion hitting times.

Introducing β for a generic Brownian motion started at zero, and writing

$$T^Y=\inf\{t>0: Y_t=0\};\quad L^Y=\sup\{0<t\leq\xi: Y_t=0\},$$

we therefore describe the conditional law of Y on each of the intervals $[0, T^Y]$, $[T^Y, L^Y]$, and $[L^Y, \xi]$. The results come from [13] Proposition I.2.4 (but see also [24] or [26]).

The initial excursion $[0, \xi \wedge T^Y]$ has two subcases. Using $\mathbb{P}_y[\xi > T^Y] = e^{-\sqrt{2\lambda}\,|y|}$ we find that on $(\xi > T^Y)$ the process satisfies

$$Y_t = y + \beta_t - \sqrt{2\lambda}\,\mathrm{sgn}(Y_0)t, \quad 0 < t < T^Y, \tag{4.1}$$

a.k.a. Brownian motion with constant drift stopped at zero. Write its law as $\mathbb{P}_y^{(4.1)}$. Similarly, on the set $(\xi < T^Y)$

$$Y_t = y + \beta_t + \int_0^t \frac{\sqrt{2\lambda}\,\mathrm{sgn}(Y_s)e^{-\sqrt{2\lambda}\,|Y_s|}}{1 - e^{-\sqrt{2\lambda}\,|Y_s|}}\,ds, \quad 0 < t < \xi, \tag{4.2}$$

whose law we denote by $\mathbb{P}_y^{(4.2)}$.

The interval $[T^Y, L^Y]$ is non-empty only on $(T^Y < \xi)$ so, by the strong Markov property, we can assume $Y_0 = 0$. Introducing the SDE

$$\overline{Y}_t = \beta_t - \sqrt{2\lambda}\int_0^t \mathrm{sgn}(\overline{Y}_s)\,ds, \tag{4.3}$$

then from [13] p.253 (but see also [24]) we have

$$\{Y_t : 0 \le t \le L^Y\} \overset{\text{law}}{=} \{\overline{Y}_t : 0 \le t \le \rho\} \tag{4.4}$$

for independent $\overline{l}(0, \rho) \overset{\text{d}}{=} \exp(\sqrt{2\lambda})$. Informally, the conditioned law obeys (4.3) until its local time $\overline{l}(0, \,.\,)$ hits an independent $\exp(\sqrt{2\lambda})$ variable. Jeulin's proof uses filtration enlargement. Alternatively, one can appeal to Itô's Poisson Point Process theory. For example, applying the PPP lemma of [9] to the excursion straddling ξ shows that $\overline{l}(0, \rho)$ has exponential law of parameter $\mathcal{Q}[\zeta > \xi] = \sqrt{2\lambda}$ where ζ denotes the Brownian excursion lifetime.

It remains to specify Y on $[L^Y, \xi]$. This portion is independent and is governed by $\mathbb{P}_0^{(4.2)}$ but, in order to apply a result of Kent [17], we describe it using time-reversal. Explicitly,

$$\{Y_{\xi-t} : 0 \le t \le \xi\} \quad \text{under} \quad \mathbb{P}_0^{(4.2)}[\;.\;|\; Y_\xi = y] \quad \text{has law} \quad \mathbb{P}_y^{(4.1)} \tag{4.5}$$

which follows by reversibility of the conditional law of $Y_{.\wedge\xi}$ given $\{Y_0, Y_\xi\}$. The corollary

$$\mathbb{P}_0[X_\xi - X_{L^Y} \in dx \mid Y_\xi = y] = \mathbb{P}_0^{(4.2)}[X_\xi \in dx \mid Y_\xi = y] = \mathbb{P}_y^{(4.1)}[X_{T^Y} \in dx] \tag{4.6}$$

will be needed in Section 6.

This completes our description of the conditional law of Y on the three time intervals specified above. The connection with (3.3) comes from

$$X_\xi^\circ = X_{L^Y}^\circ - (X_\xi - X_{L^Y}^\circ)^- = X_{L^Y}^\circ - (X_{L^Y} - X_{L^Y}^\circ + 1_{(Y_\xi<0)}(X_\xi - X_{L^Y}))^-, \tag{4.7}$$

where $\mathbb{P}_0[X_{L^Y}^\circ < 0] = 1$ since $0 \in \overline{\mathrm{supp}}(m^-)$. In Section 6 we will use (4.7) to prove

$$\mathbb{P}_0[X_\xi^\circ \le x] \in C^1((-\infty, 0)) \tag{4.8}$$

and (3.3) then follows via

$$\tilde{v}(x, y) = \mathbb{E}_{x,y}[1 - e^{-\lambda T^X}] = \mathbb{E}_{0,y}[1 - e^{-\lambda T^X_{-x}}] = \mathbb{P}_{0,y}[\xi < T^X_{-x}] = \mathbb{P}_{0,y}[X_\xi^\circ > -x] \tag{4.9}$$

with $T_x^X = \inf\{t > 0 \colon X_t = x\}$.

5. The class $\mathcal{CM}^+$

Itô-McKean [12] p.217 noted that the Lévy measure of $W = -X_\sigma$ has completely monotone density. They asked for a characterization. Knight [19] remarked the relevance of Krein's theory and answered their question in the context of gap-diffusions. See also Kotani-Watanabe [20].

Rogers [27] subsequently examined WH factorization for general Lévy processes with completely monotone jump density. We use his result twice: directly when proving (6.1a) and, in modified form, to justify our estimate in Section 8. Here we prove the modified version as it applies to bounded variation processes.

We therefore write $V \in \mathcal{CM}^+$ to denote a subordinator with completely monotone Lévy measure, meaning that its Laplace exponent

$$\eta^+(z) = \gamma_\infty + \gamma^+ z + \int_0^\infty (1 - e^{-zx})\mu^+(dx) = \gamma_\infty + \gamma^+ z + \int_0^\infty \frac{z}{k(z+k)}\Theta^+(dk)$$

with $\Theta^+ \ge 0$. Thus $V \in \mathcal{CM}_{bv} := \mathcal{CM}^+ - \mathcal{CM}^+$ has exponent

$$\eta(z) = \eta^+(z) + \eta^-(-z) = \gamma z + \int_{-\infty}^\infty \frac{z}{z+k}\,\Xi(dk) \tag{5.1}$$

for $\Xi \ge 0$: the killing rate is $\Xi\{0\}$ while the constraints on μ amount to $\int (1 + |k|)^{-1}\Xi(dk) < \infty$.

DEFINITION 5.1. Write $F \in \mathcal{H}'$ if holomorphic on the lower half-plane with $\Im F \ge 0$ there. If, in addition, F is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and positive on $(0, \infty)$ then $F \in \mathcal{H}$.

$\mathcal{H}'$ is related to the Pick functions of [7] while $\mathcal{H}$ appears in Krein's characterization [14] of Stieltjes transforms, to the effect that

$$F(z) = \gamma + \int_0^\infty \frac{\Xi(dk)}{k+z} \tag{5.2}$$

defines a bijection between $\mathcal{H}$ and pairs (γ, Ξ) satisfying $\gamma \geq 0$ and $\int_0^\infty (1+k)^{-1}\Xi(dk) < \infty$. Thus

$$V \in \mathcal{CM}^+ \equiv z^{-1}\eta^+(z) \in \mathcal{H} \tag{5.3}$$

and from (5.2) we easily verify

$$F \in \mathcal{H} \equiv z \to 1/[zF(z)] \in \mathcal{H}. \tag{5.4}$$

For us, WH factorization of V amounts to finding positive and negative subordinators, denoted by $V^\oplus$ and $V^\ominus$ respectively, with Laplace exponents satisfying $\eta = \eta^\oplus\eta^\ominus$. The method is well-known (e.g. [25]). It depends on identifying $\log\eta = \log\eta^\oplus + \log\eta^\ominus$ as an additive decomposition. We use this, together with the remark that

$$\text{if} \quad F \in \mathcal{H}' \quad \text{satisfies} \quad 0 \leq \Im F < \pi \quad \text{on} \quad \Im z < 0 \quad \text{then} \quad e^F \in \mathcal{H}', \tag{5.5}$$

to prove the following simplified version of Rogers' [27] result (direct implication only).

Lemma 5.2. *If $V \in \mathcal{CM}_{bv}$ then $V^\oplus$ (resp. $V^\ominus$) lies in $\mathcal{CM}^+$ (resp. $-\mathcal{CM}^+$).*

Proof. We assume $\Im z < 0$ throughout. By (5.1) $z^{-1}\eta(z) \in \mathcal{H}'$ has argument in $(0, \pi)$ so we can define a branch of $\log\eta(z) - \log z \in \mathcal{H}'$ with imaginary part in the same range. From the Herglotz representation for Pick functions [7] p.20

$$\log\eta(z) = \gamma_0 + \gamma_1 z + \int_{-\infty}^\infty \left[\frac{1}{z+k} - \frac{k}{1+k^2}\right]\Psi(dk) + \log z$$

with $\int [1+k^2]^{-1}\Psi(dk) < \infty$. Moreover, growth properties of $\log\eta$ on the imaginary axis give us $\Psi\{0\} = 0 = \gamma_1$. Now define

$$\log\eta^\oplus(z) = \gamma_0 + \int_0^\infty \left[\frac{1-kz}{z+k}\,\frac{1}{1+k^2}\right]\Psi(dk) + \log z.$$

Using $0 \leq \Im[\log\eta^\oplus(z) - \log z] \leq \Im[\log\eta(z) - \log z] < \pi$, this representation and (5.5) entail $z^{-1}\eta^\oplus(z) \in \mathcal{H}$. For the other factor, $1/\eta^\ominus(-z) \in \mathcal{H}$ because of (5.5) and

$$0 \leq \Im[-\log\eta^\ominus(-z)] = \int_{-\infty}^0 \frac{-y}{(k-x)^2+y^2}\Psi(dk) \leq \int_{-\infty}^\infty \frac{-y}{(k-x)^2+y^2}\Psi(dk) < \pi$$

for $z = x + iy$. Thus (5.4) shows $z^{-1}\eta^\ominus(-z) \in \mathcal{H}$ and we finish by noting (5.3). $\square$

REMARK 5.3. (1) The choice of $\log \eta^{\oplus}$ is unique up to an additive constant. Consequently, the factors $(\eta^{\oplus}, \eta^{\ominus})$ are unique up to constant multiple.
(2) If V has zero drift then the same holds for its WH factors. See [6] p.56.

Rogers [27] treats general Lévy processes with completely monotone Lévy measure but states his result differently. He works with Rogozin's [28] WH factorization (see also [9] Lemma 2.1), namely

$$\mathbb{E}[e^{-zV_\xi}] = \frac{\lambda}{\lambda + \eta(z)} = \frac{1}{\tilde{\eta}^{\oplus}(\lambda, z)} \frac{1}{\tilde{\eta}^{\ominus}(\lambda, z)} = \mathbb{E}[e^{-zV_\xi^\bullet}]\mathbb{E}[e^{-z(V_\xi - V_\xi^\bullet)}],$$

and describes the factors in terms of $\mathcal{ME}$. Following Sato [30] pp.388–389, we say $U \in \mathcal{ME}$ if

$$\mathbb{P}[U \in dx] = \alpha\delta_0(dx) + (1-\alpha)1_{(x>0)}\, dx \int \lambda e^{-\lambda x}\Theta(d\lambda) \tag{5.6}$$

for a probability measure Θ on $(0, \infty)$ and $0 \le \alpha \le 1$. Comparison with (5.2) shows that

$$U \in \mathcal{ME} \quad \text{iff its Laplace transform belongs to} \quad \mathcal{H}, \tag{5.7}$$

while by [27] the independent variables

$$\begin{aligned} &V_\xi^\bullet \quad \text{and} \quad V_\xi^\bullet - V_\xi \quad \text{belong to} \quad \mathcal{ME} \\ &\text{whenever} \quad V \quad \text{has completely monotone Lévy density.} \end{aligned} \tag{5.8}$$

Here $\tilde{\eta}^{\oplus/\ominus}(\lambda, 0) = 1$ guarantees uniqueness of the factors but note that Lemma 5.2 makes sense when $\lambda = 0$. Rogers' proof of (5.8) follows the pattern of Lemma 5.2. He defines $\tilde{\eta}^{\oplus}$ by additive decomposition of $\log(\lambda + \eta) \in \mathcal{H}$, verifies that $1/\tilde{\eta}^{\oplus}$ belongs to $\mathcal{H}$, whereupon the result follows from (5.7). The details in [27] are more demanding since his process may have unbounded variation.

We now invoke Krein's correspondence [15] as detailed in [8]. To any positive measure m_1 on $[0, \infty)$ this associates $D(0, z) \in \mathcal{H}$, determined from the unique positive solution of

$$dD_x(x, z) = 2zD(x, z)m_1(dx), \quad D_x(0-, z) = -1,\ D_x(\infty, 0) = 0, \tag{5.9}$$

the final condition being operative only when m_1 is Radon with compact support. From Tanaka's formula $dY_t^+ = 1_{(Y_t \ge 0)}\, dY_t + (1/2)\, dl(0, t)$ we find

$$t \to D(Y_t^+, z) \exp\left\{-z \int_{[0}^{\infty} l(a, t)m_1(da) + \frac{1}{2} l(0, t)/D(0, z)\right\}$$

is a local martingale, whereupon timechanging $t \to \sigma_t$ exhibits $2D(0, z) \in \mathcal{H}$ as the reciprocal of the Laplace exponent for $\int_{[0}^{\infty} l(a, \sigma)m_1(da)$.

Lemma 5.4. *$W \in \mathcal{CM}_{bv}$ and hence $-W^{\ominus} \in \mathcal{CM}^{+}$.*

Proof. Using (5.3)–(5.4), the first part follows by the above applied to $m^{+}(dx)$ and $m^{-}(-dx)$. To conclude we use Lemma 5.2. $\square$

REMARK 5.5. To relate Rogozin's factorization to our decomposition $W \to (W^{\oplus}, W^{\ominus})$ note that, for z purely imaginary, the excursion argument in Section 3 applied to $\mathbb{E}\big[\int_0^\infty e^{-zW_t-\lambda t}\,dt\big]$ yields

$$\frac{1}{\lambda+\kappa(z)} = \mathbb{E}\left[\int_0^\infty e^{-zW_t^{\oplus}-\lambda\sigma_t^{\bullet}}\,dt\right]\mathcal{Q}^{\ominus}\left[\int_0^{\varrho} e^{-z\mathcal{E}(u)-\lambda u}\,du + b^{\bullet}\right] := \frac{1}{\kappa^{\oplus}(z,\lambda)}\frac{1}{\kappa^{\ominus}(z,\lambda)}$$

using $\kappa^{\oplus/\ominus}(z,0)=\kappa^{\oplus/\oplus}(z)$. By Remark 5.3 (1) this differs from Rogozin's factorization by a multiple (depending on λ).

6. Regularity of v

In this section we establish (3.3) by proving (4.8). Until further notice $Z=(X,Y)$ has law $\mathbb{P}=\mathbb{P}_0$ and, for brevity, we write variously $U \in \mathcal{P}$, $\mu \in \mathcal{P}$ or $f \in \mathcal{P}$, to mean that the random variable U, of law μ or density f, satisfies property $\mathcal{P}$. In addition, $F_U(x)=\mathbb{P}[U \le x]$ while $\mathcal{ME}_0$ denotes the strictly positive elements of $\mathcal{ME}$.

Our proof of (4.8) shadows the decomposition in Section 4. There we wrote $\mathbb{P}_y^{(4.x)}$ for the law of (4.x) started at y. We begin with an outline of the main steps in our argument. First, using (4.4), Rogers' result (5.8), and Krein theory from the previous section, we show

(6.1a) $$X_{L^Y} - X^{\circ}_{L^Y} \quad \text{is independent of} \quad X^{\circ}_{L^Y} \in -\mathcal{ME}_0.$$

Denote by p_1 the density of $X^{\circ}_{L^Y}$. Next, we study properties of $\mathbb{P}_y^{(4.1)}[X_{T^Y} \in dx]$. In fact, defining $L_y^Y = \sup\{t < T^Y : Y_t = y\}$, we prove that

(6.1b) $$\text{when} \quad y<0 \quad \text{the density} \quad \mathbb{P}_y^{(4.1)}[X_{T^Y} - X_{L_y^Y} \in dx]/dx \in C_{\downarrow}^{\infty}(\mathbb{R}).$$

The proof of (6.1b) is adapted from [32] and requires $y \in \overline{\operatorname{supp}}(m^-)$. However, harmonic interpolation using the strong Markov property under $\mathbb{P}^{(4.1)}$ shows the result holds generally. By path decomposition at L_y^Y we deduce that

(6.1c) $$\text{for} \quad y<0 \quad \text{the density} \quad p(x,y) := \mathbb{P}_y^{(4.1)}[X_{T^Y} \in dx]/dx \in C^{\infty}(\mathbb{R}).$$

The role of $y \in \overline{\operatorname{supp}}(m^-)$ is to identify $x \to p(-x,y)$ as the density of a gap-diffusion first-passage law. Rösler [29] proved these are unimodal by taking the weak limit (see [30] p.396) in Keilson's [16] result for birth-death processes. Jeulin [13] p.273 has a direct treatment, as does Yamazato [32] who gave the representation $\mu_1 * \mu_2$ with $\mu_2 \in \mathcal{ME}_0$ and μ_1 strongly unimodal—convolution by unimodal gives unimodal.

Using (6.1b) and unimodality of $x \to p(x, y)$ we will prove that the density

$$p_2(x) := \mathbb{P}[X_\xi - X_{L_Y} \in dx; Y_\xi < 0]/dx \tag{6.1d}$$
$$\overset{(4.6)}{=} \int_{-\infty}^{0} p(x, y)\mathbb{P}[Y_\xi \in dy] \in C((-\infty, 0)).$$

Example 6.3 below shows that $p_2(0-)$ can be infinite so, before proving (4.8), let us note the following elementary facts.

REMARK 6.1. Assume (U, V) independent, non-negative, with $F_U \in C^1((0, \infty))$.
(1) If $F'_U(0+) < \infty$ and $F_V \in C((0, \infty))$ then $F_{U+V} \in C^1((0, \infty))$.
(2) If $F_V \in C^1((0, \infty))$ then $F_{U+V} \in C^1((0, \infty))$.
(3) $F_{(U-V)^+} \in C^1((0, \infty))$.

To deduce (4.8) from (6.1a)–(6.1d) we first apply Remark 6.1 (3), with

$$U = 1_{(Y_\xi < 0)}[X_{L^Y} - X_\xi]; \quad V = X_{L^Y} - X^\circ_{L^Y},$$

to get $\mathbb{P}[(X_\xi - X^\circ_{L^Y})^- \leq x] \in C^1((0, \infty))$. This lets us apply Remark 6.1 (2) with $U = -X^\circ_{L^Y}$ and $V = (X_\xi - X^\circ_{L^Y})^-$ to deduce $\mathbb{P}[-X^\circ_\xi \leq x] \in C^1((0, \infty))$.

So to complete the proof of (4.8) it remains to establish (6.1a)–(6.1b) and (6.1d). For (6.1a)–(6.1b) we follow closely the reasoning of [32], the essential difference being that, since we have Brownian motion with drift, adapting Yamazato's argument to our case involves changing scale (5.9). The following covers our present needs.

Take s convex, strictly increasing, and twice differentiable on $[0, \infty)$ with $s(0) = 0$. Given a measure m_2 we define $m_1[0, s(x)] = \int_0^x (1/s'(y))m_2(dy)$. Then $G(x, z) := D(s(x), z)$ satisfies

$$dG_x - \frac{s''}{s'} G_x\, dx = 2zG\, dm_2; \quad -\frac{G(0, z)}{G_x(0-, z)} = -\frac{D(0, z)}{s'(0)} \in \mathcal{H} \tag{6.2}$$

provided m_1 is Radon and D solves (5.9). Recall how $G_x(0-, z) = G_x(0, z)$ when $m_2\{0\} = 0$.

Proof of (6.1a). Denote (Y, l, σ) under $\mathbb{P}_0^{(4.3)}$ by $(\overline{Y}, \overline{l}, \overline{\sigma})$. Hence $\overline{W} := \int \overline{l}(a, \overline{\sigma})m(da)$ is a Lévy process with, by (4.4), $0 < X^\circ_{L^Y} \overset{\mathrm{d}}{=} -\overline{W}^\bullet_\chi$ for independent $\chi \overset{\mathrm{d}}{=} \exp(\sqrt{2\lambda})$. By (5.8) our result follows if $\overline{W} \in \mathcal{CM}_{bv}$. We therefore take $s(x) = e^{2\sqrt{2\lambda}x} - 1$ in (6.2), use Itô's formula to see

$$G(\overline{Y}_t^+, z) \exp\left\{-z \int_0^\infty \overline{l}(a, t)m(da) - \frac{1}{2}\overline{l}(0, t)G_x(0, z)/G(0, z)\right\}$$

is a local martingale, and follow the reasoning of Lemma 5.4. □

Proof of (6.1b). Yamazato's [32] p.155 representation $\mathbb{P}_y^{(4.1)}[-X_{T^Y} \in dx] = \mu_1 * \mu_2(dx)$, with $y \in \operatorname{supp}(m^-)$, has the pathwise interpretation

$$\mu_2(dx) = \mathbb{P}_y^{(4.1)}[-X_{L_y^Y} \in dx]; \quad \mu_1(dx) = \mathbb{P}_y^{(4.1)}[X_{L_y^Y} - X_{T^Y} \in dx].$$

By a simple calculation the $\mathbb{P}_y^{(4.1)}$ conditional law of Y on $[L_y^Y, T^Y]$ satisfies

$$Y_t = y + \beta_t + \int_0^t \sqrt{2\lambda}\coth\sqrt{2\lambda}(Y_s - y)\,ds, \quad 0 < t < T^Y - L_y^Y.$$

This diffusion has y as entrance boundary so using [17] Corollary 5.1

$$\int_0^\infty e^{zx}\mu_1(dx) = \mathbb{E}_y^{(4.1)}[\exp(-z[X_{L_y^Y} - X_{T^Y}])] = \prod_{n\geq 1}\left(\frac{a_n}{a_n + z}\right), \tag{6.3}$$

for positive $(a_n)_{n\geq 1}$ satisfying $\sum a_n^{-1} < \infty$. Crucially, since (6.1b) is determined by the final excursion from $y < 0$, these eigenvalues depend only on m restricted to $[y, 0]$.

Lemma 6.2. *In* (6.3) μ_1 *has* $C_\downarrow^\infty(\mathbb{R})$ *density if* $(a_n)_{n\geq 1}$ *is infinite.*

Proof. The characteristic function satisfies $\lim_{|t|\to\infty} |t|^n\phi(t) = 0$. By induction, using $\phi'(t) = -i\phi(t)\sum_{n\geq 1}(a_n - it)^{-1}$, we deduce $\phi \in C_\downarrow^\infty(\mathbb{R})$—which is invariant under Fourier transform. □

We claim $(a_n)_{n\geq 1}$ is infinite. If not, the corresponding Krein spectral measure has finite support. By [8] §5.8–5.9 hence also the restriction of m to $[y, 0]$. Thereby contradicting $0 \in \overline{\operatorname{supp}}(m^-)$ and $m\{0\} = 0$. □

Proof of (6.1d). By the strong Markov property of Y under $\mathbb{P}_0^{(4.2)}$ at $T_y^Y = \inf\{t > 0 \colon Y_t = y\}$

$$\begin{aligned}\mathbb{E}[e^{z(X_\xi - X_{L^Y})} \mid Y_\xi \leq y] &\overset{(4.2)}{=} \mathbb{E}_0^{(4.2)}[e^{zX_\xi} \mid Y_\xi \leq y] = \mathbb{E}_0^{(4.2)}[\exp(zX_{T_y^Y})]\,\mathbb{E}_y^{(4.2)}[e^{zX_\xi} \mid Y_\xi \leq y] \\ &\overset{(4.5)}{=} \mathbb{E}_y^{(4.1)}[\exp(z[X_{T^Y} - X_{L_y^Y}])]\,\mathbb{E}_y^{(4.2)}[e^{zX_\xi} \mid Y_\xi \leq y].\end{aligned}$$

Thus for $y_n \uparrow 0$, we infer from (6.1b) that $\mathbb{E}_0[X_\xi - X_{L^Y} \in dx \mid Y_\xi \leq y_n]$ has $C^\infty(\mathbb{R})$ density

$$\overline{p}_n(x) := \frac{\int_{-\infty}^{y_n} p(x, y)\,\mathbb{P}[Y_\xi \in dy]}{\mathbb{P}[Y_\xi \leq y_n]} \to_n p_2(x) = \int_{-\infty}^0 p(x, y)\,\mathbb{P}[Y_\xi \in dy],$$

noting (4.6) and (6.1c). We claim uniform convergence on $(-\infty, -\delta)$. Indeed, the weak convergence $\lim_{y\uparrow 0}\mathbb{P}_y^{(4.1)}[X_{T^Y} \in dx] \overset{\mathrm{d}}{=} \delta_0(dx)$ implies $\lim_{y\uparrow 0}\int_{-\infty}^{-\delta} p(x, y)\,dx =$

0 and, using unimodality of $x \to p(x, y)$ for $y \in \overline{\mathrm{supp}}(m^-)$, we deduce $\lim_{y\uparrow 0} \sup_{x \le -\delta} p(x, y) = 0$. This suffices. $\square$

EXAMPLE 6.3. If $m^-(da) = da$ then for independent $-U \stackrel{\mathrm{d}}{=} \exp(\sqrt{2\lambda})$ formula (4.6) leads to

$$\mathbb{E}[e^{z(X_\xi - X_{L^Y})} \mid Y_\xi < 0] = \mathbb{E}[\mathbb{E}_U^{(4.1)}[e^{-zT^Y}]] = \mathbb{E}[e^{(\sqrt{2\lambda+2z}-\sqrt{2\lambda})U}] = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda+2z}}.$$

Using Borodin-Salminen [5] p.223 we deduce that $2p_2(-t) = e^{-\lambda t}\sqrt{\lambda/\pi t}$, this being also the density for $\mathbb{P}_0^{(4.2)}[\xi \in dt]$.

For application in the next section, we now employ similar arguments to study $\tilde{v}$. We therefore drop our convention that $Z = (X, Y)$ has law $\mathbb{P}_0$. It is also convenient to write $\tilde{u} = 1 - \tilde{v}$ and $u = 1 - v$. The next result, on Brownian local time, is probably well-known but we were unable to find an explicit statement.

Lemma 6.4. *For m_2 a Radon measure $\mathbb{P}\big[\int l_\xi^a m_2(da) \le x\big] \in C^1((0, \infty))$.*

Proof. Let $K = \int l_\xi^a m_2(da)$ and assume first $Y_0 = 0 \in \overline{\mathrm{supp}}(m_2)$. Thus $K = K^\uparrow + K^\downarrow$, independent and contributed respectively by the positive and negative excursions. Remark 6.1 (2) shows it suffices to treat $K^\downarrow$. So we may assume $0 = \sup(\mathrm{supp}(m_2) \subseteq (-\infty, 0])$. If this is a limit point, then (4.8) applies with $m^+ = 0$ and $m^- = m_2$. On the other hand, if $\overline{\mathrm{supp}}(m_2) \setminus \{0\}$ has supremum $x_0 < 0$, the strong Markov property at first passage there implies $K = K' + \mathbf{e}$ with the latter independent exponential. We therefore apply Remark 6.1 (1), with $U = \mathbf{e}$ and $V = K'$, noting K' doesn't charge $(-\infty, 0)$ (same argument at x_0). Hence result if $Y_0 = 0 \in \overline{\mathrm{supp}}(m_2)$. In general, the strong Markov property lets us decompose K as a mixture of three independent variables: a Dirac mass at zero and K conditioned by Y positive/negative at first hit of $\mathrm{supp}(m_2)$. The above argument applies to the latter. $\square$

REMARK 6.5. Lemma 6.4 holds for other diffusion laws—such as $\mathbb{P}_y^{(4.2)}$. The decisive step is to establish (4.8) for $m^+ = 0$ and $m^- = m_2$ which, by scale and time change, reduces to the Brownian case for a different measure and more general killing functional. For the analogue of (4.4), whereby on $[T_y^Y, L_y^Y]$ the process solves an SDE stopped at an independent exponential local time, we refer to [13] p.253.

Lemma 6.6. *$x \to \tilde{u}(x, y)$ is continuously differentiable on $(0, \infty)$.*

Proof. Fix $x > 0$. Assuming $y > 0$, the strong Markov property in (4.9) gives

$$\frac{\tilde{u}(x, y)}{\mathbb{P}_y[T^Y < \xi]} = \mathbb{P}_{0,y}[X_\xi^\circ \le -x \mid T^Y < \xi] \stackrel{(4.1)}{=} \int_0^\infty \mathbb{P}_{0,y}^{(4.1)}[X_{T^Y} \in ds]u(x + s)$$

which suggests that

$$\tilde{u}_x(x, y) = \mathbb{P}_y[T^Y < \xi] \int_0^\infty \mathbb{P}_{0,y}^{(4.1)}[X_{T^Y} \in ds]u'(x+s).$$

This would hold, by the dominated convergence theorem, if u' was bounded far out—clear for p_1 by (5.6) but less so for its convolution with

$$\mathbb{P}[1_{(Y_\xi<0)}(X_\xi - X^\circ_{L^Y}) \in dx]/dx = \int_0^\infty \mathbb{P}[X_{L^Y} - X^\circ_{L^Y} \in dw]p_2(x-w), \quad x < 0.$$

Nevertheless, when proving (6.1d) we showed that $p_2 = \overline{p}_n + (p_2 - \overline{p}_n)$, respectively $C^\infty(\mathbb{R})$ and bounded far out. The former presents no difficulty, while our comment applies to the contribution from the latter. This completes the argument for $y > 0$. When $y < 0$ we have

$$\tilde{u}(x, y) = \mathbb{P}_y[T^Y < \xi] \int_{-\infty}^0 \mathbb{P}_{0,y}^{(4.1)}[X_{T^Y} \in ds]u(s+x) + \mathbb{P}_y[T^Y \geq \xi]\, \mathbb{P}_{0,y}^{(4.2)}[X_\xi \leq -x],$$

by the strong Markov property. Now use (6.1c) (resp. Remark 6.5) to get smoothness of the first (resp. second) term. □

7. Properties of k

In this section we prove (3.4) by applying the strong Markov property in the Brownian excursion. The idea comes from [11]. For $x > 0$ they write (3.2) as

$$k(x) = \mathcal{G}v(x) = \int [u(x-y) - u(x)]\nu(dy) = \mathcal{Q}[u(x + \mathcal{X}_\zeta) - u(x)],$$

where $\mathcal{Q}$ governs $\mathcal{Z} = (\mathcal{X}, \mathcal{Y})$, the excursions of $Z = (X, Y)$ from the x-axis, while ζ is the Brownian excursion lifetime. Introducing $\tau_x = \inf\{s > 0 \colon \mathcal{X}_s = -x\} \wedge \zeta$, we claim that

$$k(x) = 1_{(x>0)}\mathcal{Q}[u(x + \mathcal{X}_\zeta)[1 - e^{-\lambda\tau_x}]]. \tag{7.1}$$

This relation suffices to prove (3.4): it entails $0 \leq k \leq \mathcal{Q}[1 - e^{-\lambda\zeta}] = \sqrt{2\lambda}$ which, via the dominated convergence theorem, means that k inherits continuity from u.

To prove (7.1) we deal separately with the positive/negative excursions of Z, which travel respectively right/left. As usual, $\mathcal{Y}^\bullet_\zeta$ (resp. $\mathcal{Y}^\circ_\zeta$) denotes the maximum (resp. minimum) of $\mathcal{Y}$. On the positive excursions $\tau_x = \zeta$ so we look at these first.

Lemma 7.1. *For $x > 0$*

$$\begin{aligned}&\mathcal{Q}[\tilde{u}(x + \mathcal{X}_\zeta, 0) - \tilde{u}(x, 0); \mathcal{Y}^\bullet_\zeta > 0] - \frac{1}{2}\tilde{u}_y(x, 0+)\\ &= \mathcal{Q}[\tilde{u}(x + \mathcal{X}_\zeta, 0)[1 - e^{-\lambda\zeta}]; \mathcal{Y}^\bullet_\zeta > 0].\end{aligned}$$

Proof. Since $\mathbb{P}_{x,y}[T^Y < T^X] = 1$ when $y > 0$, the strong Markov property gives $\tilde{u}(x, y) = \mathbb{E}_{0,y}[e^{-\lambda T^Y}\tilde{u}(x + X_{T^Y}, 0)]$ and hence

$$\mathbb{E}_{0,y}[\tilde{u}(x + X_{T^Y}, 0) - \tilde{u}(x, y)] = \mathbb{E}_{0,y}[[1 - e^{-\lambda T^Y}]\tilde{u}(x + X_{T^Y}, 0)]. \tag{7.2}$$

Since $\mathcal{Q}$ defines an entrance law for Z killed on the x-axis, we deduce

$$\begin{aligned}&\mathcal{Q}[\tilde{u}(x + \mathcal{X}_\zeta - \mathcal{X}_{S_y}, 0) - \tilde{u}(x, y); \mathcal{Y}^\bullet_\zeta \geq y]\\ &= \mathcal{Q}[[1 - e^{-\lambda(\zeta - S_y)}]\tilde{u}(x + \mathcal{X}_\zeta - \mathcal{X}_{S_y}, 0); \mathcal{Y}^\bullet_\zeta \geq y]\end{aligned}$$

for $S_y = \inf\{u > 0 \colon \mathcal{Y}_u \geq y\}$. To get the result we take $y \downarrow 0$. On the right, we use $S_y \downarrow 0$ $\mathcal{Q}$ a.e. and domination of the integrand by $1 - e^{-\lambda\zeta}$. On the left, we split the integral in two. First,

$$\mathcal{Q}[u(x + \mathcal{X}_\zeta - \mathcal{X}_{S_y}) - u(x); \mathcal{Y}^\bullet_\zeta \geq y] \downarrow \mathcal{Q}[u(x + \mathcal{X}_\zeta) - u(x); \mathcal{Y}^\bullet_\zeta > 0]$$

by monotone convergence. For the other part, Williams' formula $\mathcal{Q}[\mathcal{Y}^\bullet_\zeta > y] = 1/2y$ gives

$$\mathcal{Q}[\tilde{u}(x, 0) - \tilde{u}(x, y); \mathcal{Y}^\bullet_\zeta \geq y] = \frac{\tilde{u}(x, 0) - \tilde{u}(x, y)}{2y} \to -\frac{1}{2}\tilde{u}_x(x, 0+)$$

where existence and finiteness of the limit follows from that of the other terms. □

On the negative excursions we apply the argument of Isozaki-Kotani [11]. For $y < 0$ they replaced relation (7.2) by

$$\mathbb{E}_{x,y}[\tilde{u}(X_{T^Y}, 0) - \tilde{u}(x, y)] = \mathbb{E}_{x,y}[[1 - e^{-\lambda(T^Y \wedge T^X)}]\tilde{u}(X_{T^Y}, 0)]. \tag{7.3}$$

The proof uses the strong Markov property of Y and $u(X_{T^Y}, 0) = 1$ on $T^X \leq T^Y$ to write

$$\begin{aligned}\tilde{u}(x, y) &= \mathbb{E}_{x,y}[e^{-\lambda T^X}; T^Y < T^X] + \mathbb{E}_{x,y}[e^{-\lambda T^X}; T^X \leq T^Y]\\ &= \mathbb{E}_{x,y}[e^{-\lambda T^Y}\tilde{u}(X_{T^Y}, 0); T^Y < T^X] + \mathbb{E}_{x,y}[e^{-\lambda T^X}\tilde{u}(X_{T^Y}, 0); T^X \leq T^Y].\end{aligned}$$

By passing to the excursion measure, as in Lemma 7.1, equation (7.3) yields

$$\begin{aligned}&\mathcal{Q}[\tilde{u}(x + \mathcal{X}_\zeta, 0) - \tilde{u}(x, 0); \mathcal{Y}^\circ_\zeta < 0] + \frac{1}{2}\tilde{u}_y(x, 0-)\\ &= \mathcal{Q}[\tilde{u}(x + \mathcal{X}_\zeta, 0)[1 - e^{-\lambda\tau_x}]; \mathcal{Y}^\circ_\zeta < 0]\end{aligned}$$

which, together with the result of Lemma 7.1, means (7.1) follows if $\tilde{u}_y(x, 0+) = \tilde{u}_y(x, 0-)$. For this we use (1.1) to write

$$\tilde{u}_y(x, y) - \tilde{u}_y(x, -y) = 2\int_{-y}^{y} \lambda\tilde{u}(x, s)\, ds - 2\int_{-y}^{y} \mathrm{sgn}(s)\tilde{u}_x(x, s)m(ds)$$

and take $y \downarrow 0$. Remark how $\tilde{u}_x \le 0$ guarantees finiteness of the second integral—otherwise $\tilde{u}$ would be identically infinite on each half line.

8. Proof of Theorem

We first prove $C(\lambda) := \int_{-\infty}^0 k(-s)R^{\ominus}(ds) < \infty$. Then, by an extra argument, we establish

$$\lim_{x\downarrow 0} \frac{v(x)}{R^{\oplus}[0,x]} \overset{(1.2)}{=} \lim_{x\downarrow 0} \frac{1}{R^{\oplus}[0,x]} \int_0^x R^{\oplus}(dy) \int_{-\infty}^0 k(x-y-s)R^{\ominus}(ds) = C(\lambda).$$

Here $R^{\oplus}$ has monotone decreasing density on $(0,\infty)$ (cf. results on $R^{\ominus}$ below).

To estimate $C(\lambda)$, we note first, from (3.4) and $R^{\ominus}$ Radon, that $\int_{-2}^0 k(-s)R^{\ominus}(ds) < \infty$. It remains to bound $\int_{-\infty}^{-2} k(-s)R^{\ominus}(ds)$. Consider

$$\begin{aligned} 0 \le k(s) &\overset{(3.2)}{=} 1_{(x>0)} \int_{-\infty}^{\infty} [v(s) - v(s-y)]\nu(dy) \\ &\le \int_{-\infty}^0 [v(s) - v(s-y)]\nu(dy) + \int_0^1 [v(s) - v(s-y)]\nu(dy) \\ &\quad + \int_1^{s/2} [v(s) - v(s-y)]\nu(dy) + \int_{s/2}^{\infty} [v(s) - v(s-y)]\nu(dy), \end{aligned}$$

where the first term on the right is negative. Writing the second term as $\mathcal{G}_1 v(s)$, and using obvious bounds for the others, we get

$$0 \le k(s) \le \mathcal{G}_1 v(s) + \nu[1,\infty)u(s/2) + \nu[s/2,\infty), \tag{8.1}$$

for $u := 1 - v$. So to estimate $\int_{-\infty}^{-2} k(-s)R^{\ominus}(ds)$ we will replace k by each term of (8.1) in turn.

We need extra information on $R^{\ominus}$. Being the potential of a negative subordinator started at zero, the bound $R^{\ominus}[-n,0] \le nR^{\ominus}[-1,0]$ is a well-known consequence of the strong Markov property (e.g. [2] p.74). Moreover, by Lemma 5.4

$$-W^{\ominus} \in \mathcal{CM}^+ \overset{(5.3)}{\Longrightarrow} z^{-1}\kappa^{\ominus}(-z) \in \mathcal{H} \overset{(5.4)}{\Longrightarrow} 1/\kappa^{\ominus}(-z) \in \mathcal{H}.$$

Thus its inverse Laplace transform $R^{\ominus} \in C^1((-\infty,0))$. We denote by $r^{\ominus}$ its (strictly increasing) density.

Estimate for $\mathcal{G}_1 v$. By Fubini's theorem and (3.3)

$$\begin{aligned} \mathcal{G}_1 v(s) &= \int_0^1 [v(s) - v(s-y)]\nu(dy) = \int_0^1 \nu(dy) \int_0^y v'(s-t)\,dt \\ &= \int_0^1 dt\ \nu[t,1]v'(s-t), \end{aligned}$$

for probability density v' and Lévy measure ν. Hence $s \to \mathcal{G}_1 v(s) 1_{(s>2)}$ is integrable and, by monotonicity of $r^{\ominus}$, we deduce $\int_{-\infty}^{-2} \mathcal{G}_1 v(s) r^{\ominus}(-s)\, ds < \infty$.

REMARK 8.1. The above argument yields

$$\sup_{x>0} \int_n^\infty \mathcal{G}_1 v(x+s) r^{\ominus}(-s)\, ds \le r^{\ominus}(-n) \int_0^1 dt\, \nu[t, 1].$$

We will use this in Lemma 8.3.

Estimate for $u(s/2)$. First, by the strong Markov property and (4.4)

$$\mathbb{E}_0[l(a, \xi)] = \mathbb{E}_a[l(a, \xi)]\mathbb{P}[T_a^Y < \xi] = (1/\sqrt{2\lambda}) e^{-\sqrt{2\lambda}\,|a|},$$

so $\int_{-\infty}^0 e^{\sqrt{2\lambda} a} m(da) < \infty$ implies $0 < -X_\xi^\circ \le \int_{-\infty}^0 l(a, \xi) m(da)$ is $\mathbb{P}_0$ integrable. Now consider

$$\int_2^{2n} u(s/2) r^{\ominus}(-s)\, ds = R^{\ominus}[-2n, -2] u(n) + \frac{1}{2} \int_2^{2n} v'(s/2) R^{\ominus}[-s, -2]\, ds.$$

By (4.9), subadditivity of $R^{\ominus}$, and Chebychev's inequality, the first term on the right is dominated by $R^{\ominus}[-1, 0]\, 2n\mathbb{P}[X_\xi^\circ \le -n] \le 2R^{\ominus}[-1, 0]\, \mathbb{E}[-X_\xi^\circ]$. Similarly, $2R^{\ominus}[-1, 0]\, \mathbb{E}[|X_\xi^\circ|]$ dominates the other term.

Estimate for $\nu[s/2, \infty)$. This uses hypotheses (A) and (B). If we assume (B), then in

$$\int_2^{2n} \nu[s/2, \infty) r^{\ominus}(-s)\, ds = \nu[n, \infty) R^{\ominus}[-2n, -2] + \int_1^n R^{\ominus}[-2s, -2] \nu(ds)$$

it suffices to use $R^{\ominus}[-n, 0] \le nR^{\ominus}[-1, 0]$ together with $n\nu[n, \infty) \to_n 0$. Under assumption (A), the result follows immediately from the following estimate.

Lemma 8.2. $\int_{-\infty}^{-2} \nu[-s, \infty) R^{\ominus}(ds) < \infty$.

Proof. Applying Doob's theorem at $T^W = \inf\{t > 0 \colon W_t \ge 0\}$ to the martingale

$$t \to \sum_{0<s\le t} 1_{(\Delta W_s > -W_{s-})} - \int_0^t \nu[-W_s, \infty)\, ds$$

gives $\mathbb{E}_x\left[\int_0^{T^W} \nu[-W_t, \infty) dt\right] = 1$. Next, noting (3.8) and taking $\varrho_2 = \inf\{u > 0 \colon \mathcal{E}_u \le -2\}$, we rewrite our integral as

$$\mathcal{Q}^{\ominus}\left[\int_0^{\varrho} 1_{(\mathcal{E}_u \le -2)} \nu[-\mathcal{E}_u, \infty)\, du\right]$$
$$= \int_{-\infty}^{-2} \mathcal{Q}^{\ominus}[\mathcal{E}_{\varrho_2} \in dx]\mathbb{E}_x\left[\int_0^{T^W} \nu[-W_t, \infty) 1_{(W_t \le -2)}\, dt\right],$$

since by [22] (6.3) $\mathcal{Q}^{\ominus}$ defines an entrance law for W killed at T^W. Then

$$\mathcal{Q}^{\ominus}[\mathcal{E}^{\circ}_{\varrho} \le -2] \sup_{x \le -2}\left(\mathbb{E}_x\left[\int_0^{T^W} \nu[-W_t, \infty)\, dt\right]\right) \le \mathcal{Q}^{\ominus}[\mathcal{E}^{\circ}_{\varrho} \le -2] < \infty$$

provides the required bound. □

We have now established $C(\lambda) < \infty$. Introducing $K(x) = \int_{-\infty}^{0} k(x-s)R^{\ominus}(ds)$, our theorem is an immediate consequence of the following.

Lemma 8.3. $C(\lambda) = K(0+)$.

Proof. Note that, given $\varepsilon > 0$, there exists N such that $\int_N^{\infty} k(x+s)r^{\ominus}(-s)\, ds < \varepsilon$ uniformly in $x \ge 0$. In fact, this holds for each term in (8.1): the last two are decreasing while for $\mathcal{G}_1 v$ we can apply Remark 8.1. Then, from $R^{\ominus}$ Radon and k continuous we get

$$\lim_{x \downarrow 0} \int_0^N k(x+s)r^{\ominus}(-s)\, ds = \int_0^N k(s)r^{\ominus}(-s)\, ds.$$

Thus $|K(0+) - C(\lambda)| < 2\varepsilon$. □

REMARK 8.4. (1) From (6.1a) $p_1'(0)$ does not exist, meaning v is never differentiable at zero.
(2) We have $v'(0+) < \infty$ only in the discrete case. In fact, [4] 1.7.2 says $\mathbb{R}^{\oplus}[0, x] \sim c_1 x$ as $x \downarrow 0$ iff $\kappa^{\oplus}(z) \sim 1/c_1$ as $z \uparrow \infty$. This in turn is equivalent to $W^{\oplus}$ compound Poisson which holds iff $\mathcal{M}^{\bullet}$ is countable.
(3) Bertoin has formulated, in terms of ν, a criterion for deciding when $W^{\oplus}$ is compound Poisson. See [6] Theorem 22.

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