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THE DUAL KNOTS OF DOUBLY PRIMITIVE KNOTS

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Abstract

For certain (1, 1)-knots in lens spaces with a longitudinal surgery yielding the 3-sphere, we determine a non-negative integer derived from its (1, 1)-splitting. The value will be an invariant for such knots. Roughly, it corresponds to a 'minimal' self-intersection number when one consider projections of a knot on a Heegaard torus. As an application, we give a necessary and sufficient condition for such knots to be hyperbolic.

1. Introduction

A *lens space* $L(p, q)$ is a 3-manifold obtained by the p/q-surgery on a trivial knot in the 3-sphere S^3 and is homeomorphic neither to S^3 nor to $S^2 \times S^1$. Throughout this paper, $-L(p, q)$ denotes the same manifold as $L(p, q)$ with reversed orientation.

A knot K in a closed orientable 3-manifold M is called a (1, 1)-*knot* if $(M, K) = (V_1, t_1) \cup_P (V_2, t_2)$, where $(V_1, V_2; P)$ is a genus one Heegaard splitting and t_i is a trivial arc in V_i ($i = 1$ and 2). (An arc t properly embedded in a solid torus V is said to be *trivial* if there is a disk D in V with $t \subset \partial D$ and $\partial D \backslash t \subset \partial V$.) Set $W_i = (V_i, t_i)$ ($i = 1$ and 2). We call the triplet $(W_1, W_2; P)$ a (1, 1)-*splitting* of (M, K). We regard P as a torus with two specified points $P \cap K$. Let E_1 (E_2 resp.) be a meridian disk of V_1 (V_2 resp.) disjoint from t_1 (t_2 resp.). It is known that such a disk is unique up to isotopy on $V_1 \setminus t_1$ ($V_2 \setminus t_2$ resp.) (cf. [13, Lemma 3.4]). A (1, 1)-splitting $(W_1, W_2; P)$ is said to be *monotone* if the signed intersection points of ∂E_1 and ∂E_2 have the same sign for some orientations of ∂E_1 and ∂E_2.

Berge's work [1] indicates that it is very important to study (1, 1)-knots. Which knots in S^3 admit Dehn surgeries yielding lens spaces? This problem is still open. In [1], Berge introduced the concept of doubly primitive knots and gave an integral surgery to obtain a lens space from any doubly primitive knot. In this paper, we call such a surgery *Berge's surgery*. He also gave a list of doubly primitive knots in S^3 (cf. Section 6). It is expected that Berge's list would be complete.

If a lens space M comes from a Dehn surgery on a knot K in S^3, then there is the dual knot K^* in M such that a Dehn surgery on K^* yields S^3. It has been proved in [1] that when Berge's surgery on a doubly primitive knot yields a lens space, its

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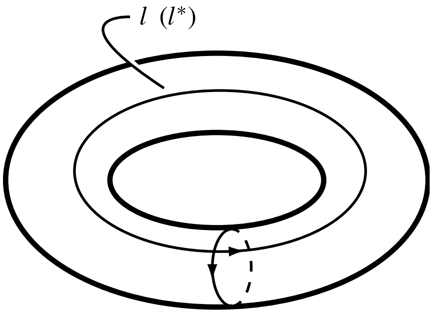


Fig. 1.

dual knot is isotopic to a (1, 1)-knot defined as follows.

DEFINITION 1.1. Let V_1 be a standard solid torus in S^3, m a meridian of V_1 and l a longitude of V_1 such that l bounds a disk in $\mathrm{cl}(S^3 \setminus V_1)$. We fix an orientation of m and l as illustrated in Fig. 1. By attaching a solid torus V_2 to V_1 so that $[\bar{m}] = p[l] + q[m]$ $(p > 0)$ in $H_1(\partial V_1; \mathbb{Z})$, we obtain a lens space $L(p, q)$, where $\bar{m}$ is a meridian of V_2. The intersection points of m and $\bar{m}$ are labelled $P_0, \dots, P_{p-1}$ successively along the positive direction of m. For an integer u with $0 < u < p$, let t_i^u be a simple arc in D_i joining P_0 to P_u $(i = 1, 2)$. Then the notation $K(L(p, q); u)$ denotes the knot $t_1^u \cup t_2^u$ in $L(p, q)$.

Set $W_i = (V_i, t_i^u)$ $(i = 1, 2)$, where V_i and t_i^u are those in Definition 1.1. Then the pair of W_1 and W_2 gives a (1, 1)-splitting of $K = K(L(p, q); u)$ which is monotone. We will prove that any (1, 1)-splitting of $(L(p, q), K)$ is monotone if K admits a longitudinal surgery yielding S^3 (see Lemma 4.1).

In this paper, we prepare the following notations.

DEFINITION 1.2. Let p and q be coprime integers with $p > 0$. Let $\{u_j\}_{1 \leq j \leq p}$ be the finite sequence such that $0 \leq u_j < p$ and $u_j \equiv q \cdot j \pmod{p}$. For an integer u with $0 < u < p$, $\Psi_{p,q}(u)$ denotes the integer j with $u_j = u$, and $\Phi_{p,q}(u)$ denotes the number of elements of the following set:

$$\{u_j \mid 1 \leq j < \Psi_{p,q}(u),\ u_j < u\}.$$

Also, $\tilde{\Phi}_{p,q}(u)$ denotes the following:

$$\tilde{\Phi}_{p,q}(u) = \min\{\Phi_{p,q}(u),\ \Phi_{p,q}(u) - \Psi_{p,q}(u) + p - u,\ \Psi_{p,q}(u) - \Phi_{p,q}(u) - 1,\ u - \Phi_{p,q}(u) - 1\}.$$

In Definition 1.1, let t'^u_1 (t'^u_2 resp.) be a projection of t^u_1 (t^u_2 resp.) on P with $t'^u_1 \subset \partial D_1$ ($t'^u_2 \subset \partial D_2$ resp.). Set $t''^u_1 = \mathrm{cl}(\partial D_1 \setminus t'^u_1)$ and $t''^u_2 = \mathrm{cl}(\partial D_2 \setminus t'^u_2)$. Each of t'^u_1 and t''^u_1 (t'^u_2 and t''^u_2 resp.) are called *monotone projections* of t^u_1 (t^u_2 resp.). There are four projections of $K = K(L(p, q); u)$: $t'^u_1 \cup t'^u_2$, $t'^u_1 \cup t''^u_2$, $t''^u_1 \cup t'^u_2$ and $t''^u_1 \cup t''^u_2$. These are called *monotone projections* of K on P. We remark that $\tilde{\Phi}_{p,q}(u)$ corresponds to a self-intersection number of a monotone projection of K on P which is minimal among the four monotone projections. We will show that $\tilde{\Phi}_{p,q}(u)$ is an invariant for K if K admits a longitudinal surgery yielding S^3 (see Corollary 4.6). Hence, in this case $\tilde{\Phi}_{p,q}(u)$ will be denoted by $\Phi(K)$.

The following is our main result.

Theorem 1.3. *Set $K = K(L(p, q); u)$. Suppose that K admits a longitudinal surgery yielding S^3. Then we have the following*:
(1) $\Phi(K) = 0$ *if and only if K is a torus knot.*
(2) $\Phi(K) = 1$ *if and only if K contains an essential torus in its exterior.*
(3) $\Phi(K) \geq 2$ *if and only if K is a hyperbolic knot.*

In Section 5, we will give formulae to obtain representations of dual knots of Berge's examples. We remark that the arguments in Section 5 are almost restatements of those by Berge [1].

2. Preliminaries

Let B be a sub-manifold of a manifold A. The notation $\eta(B; A)$ denotes a regular neighborhood of B in A. By $E(B; A)$, we mean the *exterior* of B in A, i.e., $E(B; A) = \mathrm{cl}(A \setminus \eta(B; A))$.

For two curves x and y in a *surface* (i.e., connected compact 2-manifold), the notation $\sharp(x, y)$ denotes the number of transverse intersection points and the notation $\sharp_G(x, y)$ denotes a (minimal) geometric intersection number relative to the endpoints of x and y. We say that x and y intersect *essentially* if $\sharp(x, y) = \sharp_G(x, y)$.

A triplet $(H_1, H_2; S)$ is a *genus g Heegaard splitting* of a closed orientable 3-manifold N if H_i ($i = 1$ and 2) are genus g handlebodies with $N = H_1 \cup H_2$ and $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = S$. The surface S is called a *Heegaard surface*. A properly embedded disk D in a genus g handlebody H is called a *meridian disk of H* if a 3-manifold obtained by cutting H along D is a genus $g-1$ handlebody. The boundary of a meridian disk of H is called a *meridian* of H. A collection of mutually disjoint g meridians $\{x_1, \ldots, x_g\}$ of H is called a *complete meridian system* of H if $\{x_1, \ldots, x_g\}$ bounds mutually disjoint meridian disks of H which cuts H into a 3-ball.

Let $(H_1, H_2; S)$ be a genus two Heegaard splitting of S^3. Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be complete meridian systems of H_1 and H_2 respectively. A *Heegaard diagram* of S^3 is $(S; \{x_1, x_2\}, \{y_1, y_2\})$. If x_1, x_2, y_1 and y_2 are isotoped on S so that they intersect essentially, then we call $(S; \{x_1, x_2\}, \{y_1, y_2\})$ a *normalized Heegaard diagram*. If

$\sharp(x_1, y_1) = 1$, $\sharp(x_2, y_2) = 1$, $x_2 \cap y_1 = \emptyset$ and $x_1 \cap y_2 = \emptyset$, then the Heegaard diagram is said to be *standard*. Let Σ_x (Σ_y resp.) be the 2-sphere with four holes obtained by cutting S along x_1 and x_2 (y_1 and y_2 resp.), and let x_i^+ and x_i^- (y_i^+ and y_i^- resp.) ($i = 1, 2$) be the copies of x_i (y_i resp.) in Σ_x (Σ_y resp.). A *wave* w *associated with* x_i ($i = 1$ *or* 2) is a properly embedded arc in Σ_x such that w is disjoint from $(y_1 \cup y_2) \cap \Sigma_x$, w joins x_i^+ or x_i^- to itself and w does not cut off a disk from Σ_x. Similarly, a *wave* w *associated with* y_i ($i = 1$ *or* 2) is a properly embedded arc in Σ_y such that w is disjoint from $(x_1 \cup x_2) \cap \Sigma_y$, w joins y_i^+ or y_i^- to itself and w does not cut off a disk from Σ_y. A Heegaard diagram $(S; \{x_1, x_2\}, \{y_1, y_2\})$ *contains a wave* if there is a wave associated with x_i ($i = 1$ or 2) or y_i ($i = 1$ or 2). The following has been proved by Homma, Ochiai and Takahashi [8].

Theorem 2.1 ([8, Main Theorem]). *A normalized genus two Heegaard diagram of* S^3 *is standard, or contains a wave.*

Let M be a closed orientable 3-manifold. A *trivial knot* in M is a loop bounding an embedding disk in M. It is easy to see that a Dehn surgery on a trivial knot in a lens space cannot yield S^3. A *torus knot* in M is a non-trivial knot which can be isotoped on a genus one Heegaard surface of M. The following has been proved in [13].

Theorem 2.2 ([13, Theorems 2.2–2.4]). *Let* K *be a non-trivial* $(1,1)$*-knot in* M *and* $(W_1, W_2; P)$ *a* $(1,1)$*-splitting of* (M, K) *with* $W_i = (V_i, t_i)$ $(i = 1, 2)$, *where* V_i *is a solid torus and* t_i *is a trivial arc in* V_i. *Suppose that there are projections* t_1' *and* t_2' *of* t_1 *and* t_2 *respectively and there is an essential loop* z *on* $P \setminus K$ *such that* $z \cap (t_1' \cup t_2') = \emptyset$. *Then one of the following holds.*

(1) K *is a torus knot.*

(2) $E(K; M)$ *contains an essential torus.*

(3) $K = K(\alpha, \beta; r)$ *for some* α, β *and* r.

Here, $K(\alpha, \beta; r)$ is a knot obtained by the following construction. Let $K_1 \cup K_2$ be a 2-bridge link of type (α, β). Then $K(\alpha, \beta; r)$ denotes the knot K_2 in $K_1(r)$, where $K_1(r)$ is the manifold obtained by the r-surgery on K_1 (cf. [12, Chapter 9]). By an argument similar to that in [10, Section 1], we can see that $K(\alpha, \beta; r)$ is a $(1,1)$-knot in $K_1(r)$ for any 2-bridge link and surgery coefficient r.

We remark the following which has been essentially proved in [11].

Lemma 2.3. *Set* $K = K(\alpha, \beta; r)$ *for some* α, β *and* r. *If* K *admits a Dehn surgery yielding* S^3, *then* K *is a torus knot.*

Proof. Recall that the exterior of K is obtained from the exterior of a 2-bridge link by filling a single solid torus. It has been proved in [11] that any closed 3-manifold obtained by any non-trivial Dehn surgery on a 2-bridge link is not homeomorphic to

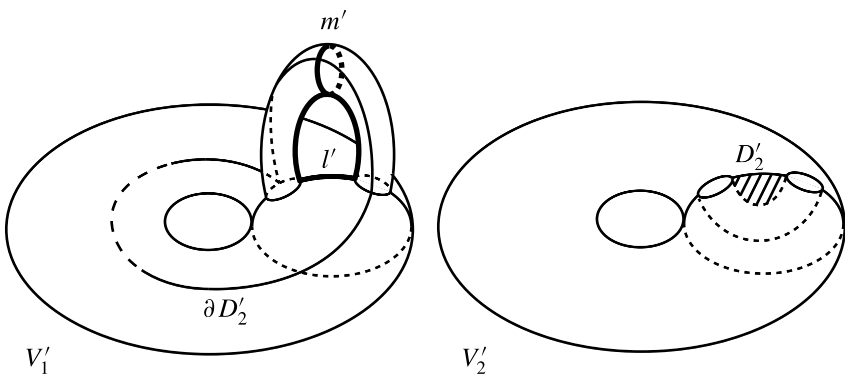


Fig. 2.

S^3 unless the 2-bridge link is a torus link (cf. [11, Theorems 2 and 3]). This implies that if K admits a Dehn surgery yielding S^3, then K is a torus knot. □

3. Dehn surgeries on $K(L(p,q);u)$

We use the notations in Definition 1.1. Let D_1 (D_2 resp.) be a meridian disk of V_1 (V_2 resp.) with $\partial D_1 = m$ and $\sharp(\partial D_1, \partial D_2) = \sharp_G(\partial D_1, \partial D_2)$. Let t'^u_1 (t'^u_2 resp.) be the monotone projection of t^u_1 (t^u_2 resp.) whose initial point is P_0 and whose endpoint is P_u passing in the positive direction of m (l resp.). Then t'^u_1 (t'^u_2 resp.) is called the *positive projection* of t^u_1 (t^u_2 resp.). Set $V_1' = V_1 \cup \eta(t^u_2; V_2)$, $V_2' = \mathrm{cl}(V_2 \setminus \eta(t^u_2; V_2))$ and $S' = \partial V_1' = \partial V_2'$. Then $(V_1', V_2'; S')$ is a genus two Heegaard splitting of $M = L(p,q)$. Let $D_2' \subset (D_2 \cap V_2')$ be a meridian disk of V_2' with $\partial D_2' \supset (t'^u_2 \cap S')$. Let m' be a meridian of $K = t^u_1 \cup t^u_2$ in the annulus $S' \cap \partial\eta(t^u_2; V_2)$. Let l' be an essential loop in S' which is a union of $t'^u_1 \cap S'$ and an essential arc in the annulus $S' \cap \partial\eta(t^u_2; V_2)$ disjoint from $\partial D_2'$ (cf. Fig. 2).

Let m^* be a meridian of K in $\partial\eta(K; V_1')$ and l^* a longitude of K in $\partial\eta(K; V_1')$ such that $l' \cup l^*$ bounds an annulus in $\mathrm{cl}(V_1' \setminus \eta(K; V_1'))$ and that $l^* \supset (\delta_1 \cap \partial\eta(K; V_1'))$, where δ_1 is the disk in V_1 bounded by $t^u_1 \cup t'^u_1$. Note that m^* and l^* are oriented as illustrated in Fig. 1. Then $\{[m^*], [l^*]\}$ is a basis of $H_1(\partial\eta(K; V_1'); \mathbb{Z})$. Let V_1'' be a genus two handlebody obtained from $\mathrm{cl}(V_1' \setminus \eta(K; V_1'))$ by attaching a solid torus $\overline{V}$ so that the boundary of a meridian disk $\overline{D}$ of $\overline{V}$ is identified with a loop represented by $r[m^*] + s[l^*]$. Set $M' = V_1'' \cup_{S'} V_2'$. Then we say that M' is obtained by the $(r/s)^*$-*surgery* on K. If r/s is an integer, the $(r/s)^*$-surgery is called a *longitudinal surgery*. A core loop of $\overline{V}$ in M' is called the *dual knot* of K in M'.

EXAMPLE 3.1. In Definition 1.2, set $p = 18$, $q = 5$ and $u = 7$. Then we have the finite sequence $\{u_j\}$ determined in Definition 1.2 as follows:

$$\{u_j\}_{1 \le j \le 18}: 5,\ 10,\ 15,\ 2,\ 7,\ 12,\ 17,\ 4,\ 9,\ 14,\ 1,\ 6,\ 11,\ 16,\ 3,\ 8,\ 13,\ 0.$$

Hence we see that $\Psi_{18,5}(7) = 5$ and $\tilde{\Phi}_{18,5}(7) = \Phi_{18,5}(7) = 2$.

Set $K = K(L(p, q); u) = K(L(18, 5); 7)$. We use the same notations as the above and in Definition 1.1. Then we can regard ∂D_2 as an (18, 5)-curve on ∂V_1. When one fixes P_0 as an initial point and follows ∂D_2 in the positive direction of l, ∂D_2 intersects ∂D_1 in the following order:

$$(P_0 \to)\ P_{u_1} \to P_{u_2} \to \cdots \to P_{u_{17}} \to P_{u_{18}} \to P_0.$$

Let E_1 (E_2 resp.) be a meridian disk of V_1 (V_2 resp.) disjoint from t_1^u (t_2^u resp.). Recall that t'^u_1 (t'^u_2 resp.) is the positive projection of t_1^u (t_2^u resp.). Then $\Psi_{p,q}(u) = \Psi_{18,5}(7)$ represents the number of intersection points of ∂E_1 and t'^u_2, and $\Phi_{p,q}(u) = \Phi_{18,5}(7)$ represents the number of intersection points of t'^u_1 and the interior of t'^u_2.

We next calculate the fundamental group of $\bar{M} = E(K; L(18, 5))$. By the argument above, we see that $(S'; \{\partial E_1\}, \{\partial E_2, \partial D'_2\})$ gives a Heegaard diagram of $E(K; L(18, 5))$. Set $\bar{x}_1 = \partial E_1$. Let y_1 and y_2 be loops on S' with $y_1 \cap \partial D'_2 = \emptyset$, $\sharp(y_1, \partial E_2) = 1$, $y_2 \cap \partial E_2 = \emptyset$, $\sharp(y_2, \partial D'_2) = 1$. Then we see that $\pi_1(\bar{M})$ has the following representation.

$$\pi_1(\bar{M}) \cong \langle y_1, y_2 \mid \bar{x}_1 = 1 \rangle.$$

By using the sequence $\{u_j\}_{1 \leq j \leq 18}$, we see

$$\begin{aligned}\pi_1(\bar{M}) &\cong \langle y_1, y_2 \mid \bar{x}_1 = 1 \rangle \\ &\cong \langle y_1, y_2 \mid y_1 y_2 y_1^3 y_2 y_1^4 y_2 y_1^3 y_2 y_1 y_2 y_1^3 y_2 y_1^3 y_2 = 1 \rangle.\end{aligned}$$

In fact, the relation is obtained by changing u_j to $y_1 y_2$ if $u_j < u$ (= 7) and changing u_j to y_1 otherwise.

We finally consider the 0^*-surgery on K. Let M' be a 3-manifold obtained by the 0^*-surgery on K^*. Set $\bar{y}_1 = \partial E_2$ and $\bar{y}_2 = \partial D'_2$. Let D'_1 be a meridian disk of V'_1 with $D'_1 \supset \bar{D}$. Let x_1 and x_2 be loops on S' with $x_1 \cap \partial D'_1 = \emptyset$, $\sharp(x_1, \partial E_1) = 1$, $x_2 \cap \partial E_1 = \emptyset$, $\sharp(x_2, \partial D'_1) = 1$. Then we see

$$\begin{aligned}\pi_1(M') &\cong \langle x_1, x_2 \mid \bar{y}_1 = 1,\ \bar{y}_2 = 1 \rangle \\ &\cong \left\langle x_1, x_2 \;\middle|\; \begin{array}{l} x_1 x_2 x_1^3 x_2 x_1^4 x_2 x_1^3 x_2 x_1 x_2 x_1^3 x_2 x_1^3 x_2 = 1, \\ x_1 x_2 x_1^3 x_2 x_1 = 1 \end{array} \right\rangle \\ &\cong \langle x_1, x_1 x_2 \mid x_1 = 1,\ x_1 x_2 = 1 \rangle.\end{aligned}$$

Since Poincaré conjecture is true for genus two 3-manifolds (cf. [3] and [5]), we see that M' is homeomorphic to S^3. We remark that $K \subset L(18, 5)$ is the dual knot of the $(-2, 3, 7)$-pretzel knot.

4. An invariant of $K(L(p,q);u)$ with a longitudinal surgery yielding S^3

We first prove the following.

Lemma 4.1. *Set $K = K(L(p,q);u)$. Suppose that K admits a longitudinal surgery yielding S^3. Then any* (1, 1)*-splitting of* (M, K) *is monotone.*

Proof. Let $(W_1, W_2; P)$ be a (1, 1)-splitting of (M, K) with $W_i = (V_i, t_i)$ $(i = 1, 2)$. Let E_1 (E_2 resp.) be a meridian disk of V_1 (V_2 resp.) disjoint from t_1 (t_2 resp.). Let D_1 (D_2 resp.) be a meridian disk of V_1 (V_2 resp.) which contains t_1 (t_2 resp.) and is disjoint from E_1 (E_2 resp.). We may assume that $\partial D_1 \setminus K$ intersects $\partial D_2 \setminus K$ essentially in $P \setminus K$.

Let t'_1 (t'_2 resp.) be a projection of t_1 (t_2 resp.) with $t'_1 \subset \partial D_1$ ($t'_2 \subset \partial D_2$ resp.). Set $V'_1 = V_1 \cup \eta(t_2; V_2)$, $V'_2 = \mathrm{cl}(V_2 \setminus \eta(t_2; V_2))$ and $S' = \partial V'_1 = \partial V'_2$. Then $(V'_1, V'_2; S')$ is a genus two Heegaard splitting of M. Let $D'_2 \subset (D_2 \cap V'_2)$ be a meridian disk of V'_2 with $\partial D'_2 \supset (t'_2 \cap S')$.

We now consider a longitudinal surgery on K. Let V''_1 be a genus two handlebody obtained from $\mathrm{cl}(V'_1 \setminus \eta(K; V'_1))$ by attaching a solid torus $\overline{V}$ so that $\partial\overline{D}$ intersects a meridian of $\eta(K; V'_1)$ transversely in a single point, where $\overline{D}$ is a meridian disk of $\overline{V}$. Let D'_1 be a meridian disk of V''_1 with $D'_1 \supset \overline{D}$. Since we consider a longitudinal surgery on K, we may assume that $\mathrm{cl}(\partial D'_1 \setminus \eta(t_2; V_2))$ is equivalent to $t'_1 \cap \partial V''_1$. Then $(S'; \{\partial D'_1, \partial E_1\}, \{\partial D'_2, \partial E_2\})$ is a Heegaard diagram of the manifold M' obtained by such a surgery on K.

Let S'_1 (S'_2 resp.) be the torus with two holes obtained by cutting S' along ∂E_1 (∂E_2 resp.). Let ∂E_1^+ and ∂E_1^- (∂E_2^+ and ∂E_2^- resp.) be the boundary components of S'_1 (S'_2 resp.).

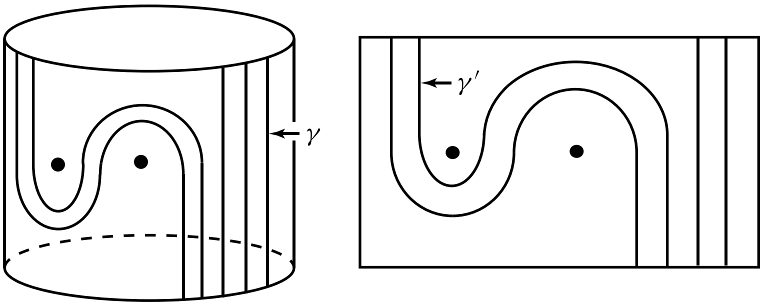
To prove Lemma 4.1, we suppose that $(W_1, W_2; P)$ is not monotone. Then there are two arc components, say γ_1 and γ'_1, of $\partial E_1 \cap S'_2$ such that γ_1 (γ'_1 resp.) joins ∂E_2^+ (∂E_2^- resp.) to itself. Since

$$\partial E_2^+ \cap (\partial E_1 \cap S'_2) = \partial E_2^- \cap (\partial E_1 \cap S'_2),$$

we see that γ_1 (γ'_1 resp.) separates the specified points in $P \setminus \partial E_2$. Similarly, there are two arc components, say γ_2 and γ'_2, of $\partial E_2 \cap S'_1$ such that γ_2 (γ'_2 resp.) joins ∂E_1^+ (∂E_1^- resp.) to itself and separates the specified points in $P \setminus \partial E_1$.

Let Σ_1 (Σ_2 resp.) be the 2-sphere with four holes obtained by cutting S'_1 (S'_2 resp.) along $\partial D'_1$ ($\partial D'_2$ resp.). Since γ_1 and γ'_1 (γ_2 and γ'_2 resp.) separates the specified points in $P \setminus \partial E_2$ ($P \setminus \partial E_1$ resp.), γ_1 and γ'_1 (γ_2 and γ'_2 resp.) assure that there are no waves in Σ_2 (Σ_1 resp.). Hence it follows from Theorem 2.1 that M' is not homeomorphic to S^3.

This completes the proof of Lemma 4.1. □



A_P D_P

Fig. 3.

Lemma 4.2. *Let K be a $(1, 1)$-knot in a lens space M and $(W_1, W_2; P)$ a $(1, 1)$-splitting of (M, K). If $(W_1, W_2; P)$ is monotone, then there is a monotone projection of K on P.*

Proof. Recall that $W_i = (V_i, t_i)$, where V_i is a solid torus and t_i is a trivial arc in V_i. Let E_1 (E_2 resp.) be a meridian disk of V_1 (V_2 resp.) disjoint from t_1 (t_2 resp.). Let D_1 be a parallel copy of E_1 which contains t_1. We suppose that $|\partial D_1 \cap \partial E_2|$ is minimal among such all meridian disks of V_1. We first prove the following.

Claim. *If ∂D_1 and ∂E_2 are oriented, then the signed intersection points of ∂D_1 and ∂E_2 have the same sign.*

Proof. Suppose that the claim does not hold. Let A_P be the annulus with two specified points $P \cap K$ which is obtained by cutting P along ∂E_1. Let γ be a component of $\partial E_2 \cap A_P$. Since $(W_1, W_2; P)$ is monotone, we see that γ joins distinct boundary components of A_P. Let D_P be the disk with the specified points which are obtained by cutting A_P along γ.

Suppose that there are no components of $\partial E_2 \cap D_P$ separating the specified points in D_P. Then this implies that each component of $\partial E_2 \cap D_P$ is parallel to γ in $A_P \setminus K$. Hence we can regard D_P as a square $[0, 1] \times [0, 1]$ such that each component of $\partial E_2 \cap D_P$ is *vertical*, i.e., each component of $\partial E_2 \cap D_P$ corresponds to $\{p\} \times [0, 1]$. We may assume that the specified points are in $[0, 1] \times \{1/2\}$. Let α be a loop on P such that α corresponds to $[0, 1] \times \{1/2\}$ in the square D_P. Then we see that α bounds a meridian disk D_α of V_1 and t_1 is isotoped into D_α relative to the endpoints (cf. [13, Section 3]). Since we suppose that the claim does not hold, we see that $|\partial D_\alpha \cap \partial E_2| < |\partial D_1 \cap \partial E_2|$. This contradicts the minimality of $|\partial D_1 \cap \partial E_2|$. Hence there is a component, say γ', of $\partial E_2 \cap D_P$ separating the specified points in D_P (cf. Fig. 3).

Let D'_P and D''_P be the disks obtained by cutting D_P along γ'. Note that each of D'_P and D''_P contains exactly one of the specified points. Then we can regard D'_P

(D''_P resp.) as a square $[0,1]\times[0,1]$ such that each component of $\partial E_2\cap D'_P$ ($\partial E_2\cap D''_P$ resp.) is vertical and that the specified point is in $[0,1]\times\{1/2\}$. Let α' be a loop on P such that $\alpha'\cap D'_P$ ($\alpha'\cap D''_P$ resp.) corresponds to $[0,1]\times\{1/2\}$ in the square D'_P (D''_P resp.). Then we see that α' bounds a meridian disk $D_{\alpha'}$ of V_1 and t_1 is isotoped into $D_{\alpha'}$ relative to the endpoints. Since we suppose that the claim does not hold, we see that $|\partial D_{\alpha'}\cap\partial E_2|<|\partial D_1\cap\partial E_2|$. This contradicts the minimality of $|\partial D_1\cap\partial E_2|$.

Hence we have the claim. □

Let D_2 be a parallel copy of E_2 with $\partial D_2\supset(P\cap K)$. Then t_2 is isotoped into D_2 relative to the endpoints. Hence D_1 and D_2 imply that there is a monotone projection of K on P.

This completes the proof of Lemma 4.2. □

The following is well known.

Lemma 4.3 (cf. [4] and [7]). *There is an orientation-preserving homeomorphism between two lens spaces $L(p,q)$ and $L(p',q')$ if and only if one of the following holds.*
(1) *$p'=p$ and $q'\equiv q \pmod p$, and*
(2) *$p'=p$ and $q'\equiv q^{-1} \pmod p$.*

We note that the following is mentioned by Berge [1] (cf. [14, Section 6]).

Lemma 4.4 ([1, Theorem 3]). *Set $K=K(L(p,q);u)$ and $K'=K(L(p',q');u')$ for some integers p, q, u, p', q' and u'. Suppose that $L(p,q)$ is homeomorphic to $L(p',q')$ and that both K and K' admit a longitudinal surgery yielding S^3. Then K is isotopic to K' if and only if $[K]=\pm[K']$ in $H_1(M;\mathbb{Z})$, where $M\cong L(p,q)\cong L(p',q')$.*

By using lemmata above, we show the following.

Proposition 4.5. *Set $K=K(L(p,q);u)$ and $K'=K(L(p',q');u')$ for some integers p, q, u, p', q' and u'. Suppose that there is an orientation-preserving homeomorphism between $L(p,q)$ and $L(p',q')$ and that both K and K' admit a longitudinal surgery yielding S^3. Then K and K' are isotopic if and only if one of the following holds.*
(1) *In case of* (1) *of* Lemma 4.3, *$u'=u$ or $u'=p-u$.*
(2) *In case of* (2) *of* Lemma 4.3, *$u'=\Psi_{p,q}(u)$ or $u'=p-\Psi_{p,q}(u)$.*

Proof. Note that it is easy to see that $K(L(p,q);u)$ and $K(L(p,q);p-u)$ are isotopic. It follows from Lemma 4.4 that K and K' are isotopic if and only if $u'=u$ or $u'=p-u$ under the assumption $q'=q$. By Lemma 4.3, we have the following two cases:

Claim 1. $q' \equiv q \pmod{p}$. *In this case, K and K' are isotopic if and only if $u' = u$ or $u' = p - u$.*

Proof. Set $q' = q + np$ for some integer n. Let $(V_1, V_2; S)$ be a Heegaard splitting of $L(p,q)$ such that the boundary of a meridian disk of V_2 is a (p,q)-curve in ∂V_1. Let $(V_1', V_2'; S')$ be a Heegaard splitting of $L(p', q')$ such that the boundary of a meridian disk of V_2' is a (p', q')-curve in $\partial V_1'$. Since genus one Heegaard surfaces of a lens space are isotopic, we may assume that $S' = S$. Moreover, since $q' = q + np$, we see that $V_1' = V_1$ and $V_2' = V_2$ (cf. [4] and [7]) and V_1' is obtained by twisting V_1 along a meridian disk of V_1. Therefore we see that $[K] = \pm[K']$ in $H_1(L(p,q); \mathbb{Z})$ if and only if $u' = u$ or $u' = p - u$. Hence it follows from Lemma 4.4 that K and K' are isotopic if and only if $u' = u$ or $u' = p - u$. Hence we have Claim 1. □

Claim 2. $q' \equiv q^{-1} \pmod{p}$. *In this case, K and K' are isotopic if and only if $u' = \Psi_{p,q}(u)$ or $u' = p - \Psi_{p,q}(u)$.*

Proof. Set $q'q = np$ for some integer n. Let $(V_1, V_2; S)$ be a Heegaard splitting of $L(p, q)$ such that the boundary of a meridian disk of V_2 is a (p, q)-curve in ∂V_1. Let $(V_1', V_2'; S')$ be a Heegaard splitting of $L(p', q')$ such that the boundary of a meridian disk of V_2' is a (p', q')-curve in $\partial V_1'$. Since genus one Heegaard surfaces of a lens space are isotopic, we may assume that $S' = S$. Moreover, since $q'q = np$ for some integer n, we see that $V_1' = V_2$ and $V_2' = V_1$ (cf. [4] and [7]).

We now isotope K so that $K \cap V_1 = t_1^u$ ($K \cap V_2 = t_2^u$ resp.) is a trivial arc in V_1 (V_2 resp.). Let t'^u_1 (t'^u_2 resp.) be a monotone projection of t_1^u (t_2^u resp.). Since $\sharp(t'^u_2, \partial E_1) = \Psi_{p,q}(u)$ or $p - \Psi_{p,q}(u)$, we see that K is isotopic to $K(L(p',q'); \Psi_{p,q}(u)) = K(L(p', q'); p - \Psi_{p,q}(u))$. Hence K and K' are isotopic if and only if $u' = \Psi_{p,q}(u)$ or $u' = p - \Psi_{p,q}(u)$. Hence we have Claim 2. □

This completes the proof of Proposition 4.5. □

As a corollary of Proposition 4.5, we have the following:

Corollary 4.6. *Set $K = K(L(p, q); u)$ and $K' = K(L(p', q'); u')$ for some integers p, q, u, p', q' and u'. Suppose that there is an orientation-preserving homeomorphism between $L(p, q)$ and $L(p', q')$ and that both K and K' admit a longitudinal surgery yielding S^3. If K and K' are isotopic, then $\tilde{\Phi}_{p,q}(u) = \tilde{\Phi}_{p',q'}(u')$.*

By this corollary we see that $\tilde{\Phi}_{p,q}(u)$ is an invariant for $K = K(L(p, q); u)$ if K admits a longitudinal surgery yielding S^3. Hence we define the following:

DEFINITION 4.7. Set $K = K(L(p, q); u)$ and suppose that K admits a longitudinal surgery yielding S^3. Then $\tilde{\Phi}_{p,q}(u)$ is denoted by $\Phi(K)$.

5. Proof of Theorem 1.3

We first remark the following.

Lemma 5.1 ([6, Theorem C] and [9, Theorem 3]). *Let K be a torus knot in M and $(W_1, W_2; P)$ a $(1,1)$-splitting of (M, K). Then there is a projection $\overline{t_1}$ ($\overline{t_2}$ resp.) of t_1 (t_2 resp.) on P such that $\overline{t_1}$ is disjoint from the interior of $\overline{t_2}$.*

Proposition 5.2. *Set $K = K(L(p, q); u)$. Suppose that K admits a longitudinal surgery yielding S^3. Then $\Phi(K) = 0$ if and only if K is a torus knot.*

Proof. Let $(W_1, W_2; P)$ be a $(1,1)$-splitting of (M, K) with $W_i = (V_i, t_i)$ $(i = 1, 2)$, where V_i is a solid torus and t_i is a trivial arc in V_i. Since K admits a longitudinal surgery yielding S^3, it follows from Lemma 4.1 that $(W_1, W_2; P)$ is monotone. Let t_1' (t_2' resp.) be a monotone projection of t_1 (t_2 resp.) such that $t_1' \cup t_2'$ gives the value $\Phi(K)$.

If $\Phi(K) = 0$, then t_1' is disjoint from the interior of t_2'. Hence we see that K is a torus knot.

Suppose that K is a torus knot. Then it follows from Lemma 5.1 that there is a projection $\overline{t_1}$ ($\overline{t_2}$ resp.) of t_1 (t_2 resp.) on P such that $\overline{t_1}$ is disjoint from the interior of $\overline{t_2}$. Let x_1 (y_1 resp.) be the boundary of a meridian disk of V_1 (V_2 resp.) disjoint from t_1 (t_2 resp.). Note that it follows from [13, Lemma 3.4] that x_1 (y_1 resp.) is unique up to isotopy on $P \setminus K$. Note also that we may assume that any projection of t_1 (t_2 resp.) on P is disjoint from x_1 (y_1 resp.). Let Σ_{x_1} (Σ_{y_1} resp.) be the component obtained by cutting P along x_1 (y_1 resp.). We may assume that $\overline{t_1}$ ($\overline{t_2}$ resp.) is isotoped so that $\overline{t_1}$ ($\overline{t_2}$ resp.) intersects y_1 (x_1 resp.) essentially. Let x_1^+ and x_1^- be the boundary of Σ_{x_1}. Since $(W_1, W_2; P)$ is monotone, we see that each component of $y_1 \cap \Sigma_{x_1}$ is an arc joining x_1^+ to x_1^-.

CASE 1. $\overline{t_2}$ is not a monotone projection of t_2.

Then there is a component, say $\overline{t_2^+}$, of $\overline{t_2} \cap \Sigma_{x_1}$ which joins x_1^+ to itself. Then since

$$x_1^+ \cap (\overline{t_2} \cap \Sigma_{x_1}) = x_1^- \cap (\overline{t_2} \cap \Sigma_{x_1}),$$

we see that there is also a component, say $\overline{t_2^-}$, of $\overline{t_2} \cap \Sigma_{x_1}$ which joins x_1^- to itself. This implies that it is impossible to obtain an arc which joins two specified points $P \cap K$ in Σ_{x_1} and is disjoint from $\overline{t_2} \cap \Sigma_{x_1}$. Since $\overline{t_1}$ is contained in A_P, this implies that $\overline{t_1} \cap \overline{t_2} \neq \emptyset$, a contradiction.

CASE 2. $\overline{t_2}$ is a monotone projection of t_2.

To obtain the conclusion $\Phi(K) = 0$, we further suppose that $\Phi(K) \neq 0$. Then there is a component, say $\overline{t_2'}$, of $\overline{t_2} \cap \Sigma_{x_1}$ which joins x_1^+ to x_1^- and intersects t_1' transversely in a single point. Also, there is a component, say $\overline{t_2''}$, of $\overline{t_2} \cap \Sigma_{x_1}$ which joins x_1^+ to x_1^- and is disjoint from t'^{u}_1. This implies that $\overline{t_2'} \cup \overline{t_2''}$ separates two specified points $P \cap K$ in Σ_{x_1}. Since $\overline{t_1}$ is contained in A_P, this implies that $\overline{t_1} \cap \overline{t_2} \neq \emptyset$, a contradiction.

This completes the proof of Proposition 5.2. □

Dehn surgeries on satellite knots in S^3 yielding lens spaces have been completely classified as the follows (cf. [2, 15, 16]).

Lemma 5.3 ([2, Theorem 1]). *Let K be a satellite knot in S^3 which admits a Dehn surgery yielding a lens space M. Then K is the $(2pq \pm 1, 2)$-cable on the (p, q)-torus knot and $M = L(4pq \pm 1, 4q^2)$.*

Here, a knot $K \subset S^3$ is called the (r, s)-*cable on a knot* $K_0 \subset S^3$ if K is isotoped into $\partial\eta(K_0; S^3)$ and is homologous to $r[l_0] + s[m_0]$ in $\partial\eta(K_0; S^3)$, where (l_0, m_0) is a standard meridian-longitude system of K_0 on $\partial\eta(K_0; S^3)$.

REMARK 5.4. (1) Let K be the $(2pq \pm 1, 2)$-cable on the (p, q)-torus knot and K' be the $(2pq \pm 1, 2)$-cable on the (q, p)-torus knot. Then K and K' are isotopic.
(2) Let p and q be coprime integers. Then we see that the following are equivalent:

$$\begin{aligned} (4pq + 1)(4pq - 1) &\equiv 0 \pmod{4pq \pm 1}, \\ 16p^2q^2 - 1 &\equiv 0 \pmod{4pq \pm 1}, \\ (4p^2)(4q^2) &\equiv 1 \pmod{4pq \pm 1}. \end{aligned}$$

Hence we see that $(4q^2)^{-1} \equiv 4p^2 \pmod{4pq \pm 1}$ and therefore

$$\begin{aligned} L(4pq \pm 1, 4q^2) &= -L(4pq \pm 1, -4q^2) \\ &= -L(4pq \pm 1, -4p^2) = L(4pq \pm 1, 4p^2). \end{aligned}$$

Lemma 5.5. *Let p and q be coprime integers. Suppose that $p > 1$ and $q \neq 0, \pm 1$. Set $K = K(L(|4pq \pm 1|, \pm 4q^2); 2|q|)$. Then K admits a longitudinal surgery yielding S^3 and $\Phi(K) = 1$.*

Proof. Since the argument is similar (cf. Remark 5.6), we give a proof in case of $1 < q < p$ and $K = K(L(4pq - 1, 4q^2); 2q)$.

Claim 1. $\tilde{\Phi}_{4pq-1,4q^2}(2q) = 1$.

Proof. For a pair of $4pq - 1$ and $4q^2$, we consider the finite sequence $\{u_j\}$ determined in Definition 1.2. Since $4q^2 \cdot p - q \equiv 0 \pmod{4pq - 1}$, we see that $u_p = q$. Suppose that there are integers p' and q' with $0 < p' < p$, $0 < q' < 2q$ and $u_{p'} = q'$. Then there is a non-negative integer n such that $4q^2 \cdot p' = n \cdot 4pq^2 + q'$. This indicates that $4q^2(p' - n \cdot p) = q'$. Since $0 < p' < p$ and $q' > 0$, we see that $n = 0$ and hence $4p'q^2 = q'$. However, this contradicts that $0 < q' < 2q$. This implies that for each integer j with $1 \leq j \leq p - 1$, we see that $u_j > 2q$. Similarly, we see that $u_{2p} = 2q$

and $u_j > 2q$ for each integer j with $p+1 \leq j \leq 2p-1$. Hence $\Phi_{4pq-1,4q^2}(2q) = 1$. Note that

$$\tilde{\Phi}_{p,q}(u) = \min\{1, 4pq - 2p - 2q, 2p - 2, 2q - 2\}.$$

Since we assume that $1 < q < p$, we see that $\tilde{\Phi}_{4pq-1,4q^2}(2q) = 1$. Therefore we have Claim 1. □

Claim 2. *The* 0^**-surgery on* K *yields* S^3.

Proof. We use an argument similar to that in Example 3.1 and hence we use the same notations as those in Example 3.1. Let M' be a 3-manifold obtained by the 0^*-surgery on K^*. Recall that x_1 and x_2 are loops on S' with $x_1 \cap \partial D_1' = \emptyset$, $\sharp(x_1, \partial E_1) = 1$, $x_2 \cap \partial E_1 = \emptyset$, $\sharp(x_2, \partial D_1') = 1$. Recall also that $\overline{y}_1 = \partial E_2$ and $\overline{y}_2 = \partial D_2'$. Then we see

$$\begin{aligned}\pi_1(M') &\cong \langle x_1, x_2 \mid \overline{y}_1 = 1, \ \overline{y}_2 = 1\rangle \\ &\cong \langle x_1, x_2' \mid \overline{y}_1 = 1, \ \overline{y}_2 = 1\rangle \quad (x_2' := x_1 x_2).\end{aligned}$$

It follows from the argument in the proof of Claim 1 that $y_2 = x_1^p x_2 x_1^p = x_1^{p-1} x_2' x_1^p$. Since $y_2 = 1$, we see that $x_2' = x_1^{1-2p}$. This implies that x_1 and x_2' are commutative with each other and hence $\pi_1(M') \cong H_1(M'; \mathbb{Z})$. We note that

$$H_1(M'; \mathbb{Z}) \cong \left\langle x_1, x_2' \;\middle|\; \begin{array}{l} ((4pq-1) - 2q) \cdot x_1 + 2q \cdot x_2' = 0, \\ (2p-1) \cdot x_1 + x_2' = 0 \end{array} \right\rangle$$

This implies that $H_1(M'; \mathbb{Z})$ is trivial and hence $\pi_1(M')$ is trivial. Since Poincaré conjecture is true for genus two 3-manifolds (cf. [3] and [5]), we see that M' is homeomorphic to S^3 and hence we have Claim 2. □

The conclusion of Lemma 5.5 follows from Claims 1 and 2. □

REMARK 5.6. To prove Lemma 5.5 in other certain cases, we need to consider the sequence obtained by reversing the order of the sequence $\{u_j\}$.

Lemma 5.7. *Let* K *be the* $(2pq \pm 1, 2)$*-cable on the* (p, q)*-torus knot with* $p > 1$ *and* $q \neq 0, \pm 1$. *Then the following holds.*
(1) *If* $q > 1$, *then* $K^* = K(L(4pq \pm 1, 4q^2); 2q)$ *is the dual knot of* K *in* $L(4pq \pm 1, 4q^2)$.
(2) *If* $q < -1$, *then* $K^* = K(L(|4pq \pm 1|, -4q^2); 2|q|)$ *is the dual knot of* K *in* $L(|4pq \pm 1|, -4q^2)$.

Proof. First we prove the case when K is the $(2pq-1,2)$-cable on the (p,q)-torus knot. The case when K is the $(2pq+1,2)$-cable on the (p,q)-torus knot will be proved similarly. Set $K^* = K(L(4pq-1, 4q^2); 2q)$. Let $(W_1, W_2; P)$ be a (1, 1)-splitting of $(L(4pq-1, 4q^2), K^*)$. Recall that $W_i = (V_i, t_i)$ $(i = 1, 2)$, where V_1 (V_2 resp.) is a solid torus and t_1 (t_2 resp.) is a trivial arc in V_1 (V_2 resp.). Let E_1 (E_2 resp.) be a meridian disk of V_1 (V_2 resp.) disjoint from t_1 (t_2 resp.). Since K^* admits a longitudinal surgery yielding S^3 (cf. Lemma 5.5), we see that $(W_1, W_2; P)$ is monotone (cf. Lemma 4.1). Hence we may assume that ∂E_2 is a $(4pq-1, 4q^2)$-curve on ∂V_1 (cf. Lemma 4.2). Let t'_1 (t'_2 resp.) be a monotone projection of t_1 (t_2 resp.) such that $t'_1 \cup t'_2$ gives the value $\Phi(K)$. It follows from Lemma 5.5 that $\Phi(K) = 1$. Let v be the self-intersection point of $t'_1 \cup t'_2$. Let $\overline{t'_1}$ ($\overline{t'_2}$ resp.) be the subarc of t'_1 (t'_2 resp.) which joins P_0 to v. Let z_1 be a loop on P obtained by moving $\overline{t'_1} \cup \overline{t'_2}$ slightly so that $\overline{t'_1} \cup \overline{t'_2}$ is disjoint from $t'_1 \cup t'_2$. Then it follows from Claim 1 in the proof of Lemma 5.5 that $\sharp_G(z_1, \partial E_1) = p$ and $\sharp_G(z_1, \partial E_2) = q$.

Let A_1 (A_2 resp.) be an annulus obtained by pushing the interior of $E(z_1; \partial V_1)$ ($E(z_2; \partial V_2)$ resp.) into the interior of V_1 (V_2 resp.) so that A_1 (A_2 resp.) is disjoint from t_1 (t_2 resp.). Then $A_1 \cup A_2$ cuts $(L(4pq-1, 4q^2), K^*)$ into (M_1, K^*) and $(M_2, \emptyset)$. Note that M_1 is a solid torus containing K^*. Since $\sharp_G(z_1, \partial E_1) = p$ and $\sharp_G(z_1, \partial E_2) = q$, we see that M_2 is homeomorphic to the exterior of the (p, q)-torus knot in S^3. Hence $A_1 \cup A_2$ is an essential torus in $E(K^*; L(4pq-1, 4q^2))$. Since K^* admits a longitudinal surgery yielding S^3 (cf. Lemma 5.5), we see that K^* is the dual knot of a cable of the (p, q)-torus knot in S^3. Hence it follows from Lemma 5.3 that K^* is the dual knot of K. □

Corollary 5.8. *Set $K = K(L(p, q); u)$. Suppose that K admits a longitudinal surgery yielding S^3. Then $\Phi(K) = 1$ if and only if $E(K; M)$ contains an essential torus.*

Proof. Suppose first that $\Phi(K) = 1$. Then by an argument similar to that in the proof of Lemma 5.7, we see that there exists a loop z_1 as in the proof of Lemma 5.7. This implies that a (1, 1)-splitting of (M, K) satisfies the assumption of Theorem 2.2. Hence it follows from Theorem 2.2 and Lemma 2.3 that K is a torus knot or $E(K; M)$ contains an essential torus. Since $\Phi(K) = 1$, K is not a torus knot (cf. Proposition 5.2) and hence $E(K; M)$ contains an essential torus.

Suppose next that $E(K; M)$ contains an essential torus. Then K is the dual knot of the $(2pq \pm 1, 2)$-cable on the (p, q)-torus knot for some integers p and q. Hence it follows from Lemmata 5.5 and 5.7 that $\Phi(K) = 1$. □

Theorem 1.3 immediately follows from Proposition 5.2 and Corollary 5.8.

6. Appendix

Here, we will recall Berge's argument [1] to obtain a relationship between Berge's examples and their dual knots. We first recall Berge's surgery on doubly primitive

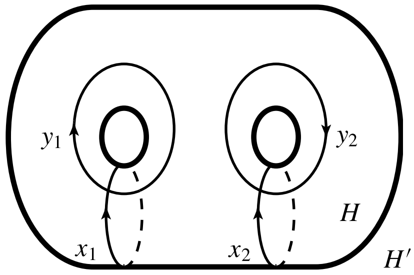


Fig. 4.

knots. Let $(H, H'; S)$ be a genus two Heegaard splitting of S^3. A knot $K \subset S$ is a *doubly primitive knot* if K represents a free generator both of $\pi_1(H)$ and of $\pi_1(H')$. If K is doubly primitive, then there are meridian disks D and E (D' and E' resp.) of H (H' resp.) with $\sharp(\partial D, K) = 1$ and $\partial E \cap K = \emptyset$ ($\sharp(\partial D', K) = 1$ and $\partial E' \cap K = \emptyset$ resp.). Then it follows from [1, Theorem 1] that a Heegaard diagram $(S; \{\partial D, \partial E\}, \{K, \partial E'\})$ represents a lens space. We call such a surgery *Berge's surgery* on K. We remark that $\partial D'$ corresponds to the dual knot of K.

Let $(H, H'; S)$ be a genus two Heegaard splitting of S^3 and $(S; \{x_1, x_2\}, \{y_1, y_2\})$ its standard Heegaard diagram with $\sharp(x_1, y_1) = 1$, $\sharp(x_2, y_2) = 1$, $x_2 \cap y_1 = \emptyset$ and $x_1 \cap y_2 = \emptyset$. We fix orientation of x_1, x_2, y_1 and y_2 as in Fig. 4.

Then $\{[x_1], [x_2], [y_1], [y_2]\}$ is a basis of $H_1(\partial H; \mathbb{Z})$. Let K be an oriented doubly primitive knot on S with $[K] = a[x_1] + b[x_2] + c[y_1] + d[y_2]$ in $H_1(\partial H; \mathbb{Z})$. Let h be an orientation-preserving homeomorphism of H' with $h(x_1) = K$. Then h induces a symplectic transformation ϕ on $H_1(\partial H'; \mathbb{Z})$ which satisfies the following:

$$\begin{aligned}
\phi(x_1) &= a[x_1] + b[x_2] + c[y_1] + d[y_2],\\
\phi(x_2) &= s[x_1] + t[x_2] + u[y_1] + v[y_2],\\
\phi(y_1) &= t[y_1] - s[y_2],\\
\phi(y_2) &= -b[y_1] + a[y_2]
\end{aligned}$$

where, s, t, u and v are integers with $at - bs = 1$ and $(au + bv) - (cs + dt) = 0$. Recall that since K is doubly primitive, $[K]$ is a free generator of $H_1(H; \mathbb{Z})$. Let $[K']$ be the other generator of $H_1(H; \mathbb{Z})$. We now consider a projection φ onto $[K']$. Then we have:

$$\begin{aligned}
\varphi(x_2) &= (cv - du)[K'],\\
\varphi(y_1) &= (-cs - dt)[K'],\\
\varphi(y_2) &= (ac + bd)[K']
\end{aligned}$$

where, we remark that $\varphi(x_1) = 0$. Let $M = L(p, q)$ be a lens space obtained by Berge's surgery on K and $K^* = K(L(p,q);u)$ the dual knot of K. Let V be a 3-manifold obtained from H by attaching a 2-handle along K. Since K is doubly primitive, we see that V is a solid torus and that V and $V' = E(V;M)$ give a genus one Heegaard splitting of M. Note that a core of V corresponds to a generator $[K']$ of $H_1(H;\mathbb{Z})$, a meridian of V' corresponds to $\phi(y_2)$, a core of V' corresponds to $\phi(x_2)$ and K^* corresponds to $\phi(y_1)$. Hence p of $K(L(p,q);u)$ satisfies that $p = ac + bd$.

We divide the rest of the arguments into the following three cases.

CASE 1. Knots of types (I)–(VI).

Each knots of types (I)–(VI) in Berge's examples satisfies that $a = \pm 1$. Since $at - bs = 1$, we see that s and t are coprime and hence we have $s = -1 + aj$ and $t = a(1-b) + bj$, where j is an integer. Hence we have $cs + dt = -c + ad(1-b)$. Also, it follows from $(au + bv) - (cs + dt) = 0$ that $au = (cs + dt - bv)$.

Let m_V be a meridian of V. Recall that $\sharp_A(m_V, \phi(y_2)) = p = ac + bd$, where $\sharp_A(\cdot,\cdot)$ means an algebraic intersection number. Note that q of $K(L(p,q);u)$ corresponds to $\sharp_A(m_V, \phi(x_2))$. Hence we need to calculate the value $cv - du$. Since we assume $a = \pm 1$, we have:

$$\begin{aligned} q &= cv - du \\ &= cv \mp d(cs + dt - bv) \\ &= (c \pm bd)v \mp d(cs + dt) \\ &\equiv -ad(cs + dt) \pmod{p = ac + bd} \\ &\equiv ad(c + ad(b-1)) \pmod{p = ac + bd}. \end{aligned}$$

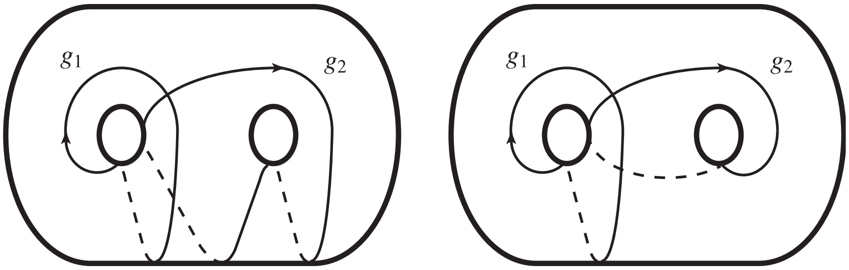
We remark that m_V is a (p,q)-curve on $\partial V'$. Hence V (V' resp.) corresponds to V_2 (V_1 resp.), where V_1 and V_2 are those in Definition 1.1. Since K^* corresponds to $\phi(y_1)$, we see that $[K^*] = (-cs - dt)[K']$. Hence we see that u of $K(L(p,q);u)$ satisfies that $u \equiv c + ad(b+1) \pmod{p = ac + bd}$ (cf. Claim 2 in the proof of Lemma 4.5). Therefore we have the following.

Theorem 6.1. *Let K be a doubly primitive knot with $[K] = a[x_1] + b[x_2] + c[y_1] + d[y_2]$ in $H_1(\partial H;\mathbb{Z})$. Let $L(p,q)$ be the lens space obtained by Berge's surgery on K and K^* the dual knot of K. If $a = \pm 1$, then K^* admits a representation $K(L(p,q);u)$ with*

$$\begin{aligned} p &= ac + bd, \\ q &\equiv ad(c + ad(b-1)) \pmod{p = ac + bd}, \\ u &\equiv c + ad(b-1) \pmod{p = ac + bd}. \end{aligned}$$

CASE 2. Knots on Seifert surfaces of genus one knots.

Let g_1 and g_2 be oriented loops on ∂H illustrated in (a) or (b) of Fig. 5. Set $K_0 = \partial\eta(g_1 \cup g_2; \partial H)$. Then K_0 is the right-hand trefoil knot in case of (a) and is the



(a) (b)

Fig. 5.

figure-eight knot in case of (b), and $\eta(g_1 \cup g_2; \partial H)$ is a genus one Seifert surface of K_0. Let K be a knot in $\eta(g_1 \cup g_2; \partial H)$ with $[K] = a[g_1] + b[g_2]$, where a and b are coprime integers.

Suppose first that K_0 is the right-hand trefoil knot. Since $[g_1] = -[x_1] + [y_1]$ and $[g_2] = -[x_1] - [x_2] + [y_2]$ in $H_1(\partial H; \mathbb{Z})$, we see that $[K] = -(a+b)[x_1] - b[x_2] + a[y_1] + b[y_2]$ in $H_1(\partial H; \mathbb{Z})$. In this case, we have $-(a+b)t + bs = 1$ and $(-(a+b)u - bv) - (as + bt) = 0$, where s, t, u and v are integers of $\phi(x_2) = s[x_1] + t[x_2] + u[y_1] + v[y_2]$. Hence we see that p of $K(L(p,q);u)$ satisfies that $p = -a^2 - ab - b^2$. Recall that u of $K(L(p,q);u)$ corresponds to the value $-as - bt$ and that q of $K(L(p,q);u)$ corresponds to the value $av - bu$. Since $-a(a+b) \equiv b^2 \pmod{p = -a^2 - ab - b^2}$, we have $-(a+b)(-as - bt) \equiv -b(-(a+b)t + bs) \pmod{p = -a^2 - ab - b^2}$. Hence we have $-(a+b)(-as - bt) \equiv -b \pmod{p = -a^2 - ab - b^2}$, because $-(a+b)t + bs = 1$. Therefore we see that $u \equiv -as - bt \equiv b(a+b)^{-1} \pmod{p = -a^2 - ab - b^2}$. For q of $K(L(p,q);u)$, we see that $q \equiv -u^2 \pmod{p = -a^2 - ab - b^2}$ by the following. (Recall that $-a(a+b) \equiv b^2 \pmod{p = -a^2 - ab - b^2}$.)

$$\begin{aligned}
(-(a+b)u - bv) &= (as + bt), \\
b(-(a+b)u - bv) &\equiv -bu \pmod{p = -a^2 - ab - b^2}, \\
(a+b)(av - bu) &\equiv -bu \pmod{p = -a^2 - ab - b^2}, \\
av - bu &\equiv -u^2 \pmod{p = -a^2 - ab - b^2}.
\end{aligned}$$

Suppose next that K_0 is the figure-eight knot. Since $[g_1] = -[x_1] + [y_1]$ and $[g_2] = -[x_1] + [x_2] + [y_2]$ in $H_1(\partial H; \mathbb{Z})$, we see that $[K] = -(a+b)[x_1] + b[x_2] + a[y_1] + b[y_2]$ in $H_1(\partial H; \mathbb{Z})$. By an argument similar to the above, we have the conclusion (2) of the following Theorem 6.2.

Theorem 6.2. *Let K be a doubly primitive knot and $L(p,q)$ a lens space obtained by Berge's surgery on K. Let K^* be the dual knot of K. In the following, a and b are coprime integers with $a > 0$ and $b > 0$.*

(1) *If* $[K]=-(a+b)[x_1]-b[x_2]+a[y_1]+b[y_2]$ *in* $H_1(\partial H;\mathbb{Z})$, *then* K^* *admits a representation* $K(L(p,q);u)$ *with*

$$\begin{aligned} &p=-a^2-ab-b^2,\\ &q\equiv -b^2(a+b)^{-2} \pmod{p=-a^2-ab-b^2},\\ &u\equiv b(a+b)^{-1} \pmod{p=-a^2-ab-b^2}. \end{aligned}$$

(2) *If* $[K]=-(a+b)[x_1]+b[x_2]+a[y_1]+b[y_2]$ *in* $H_1(\partial H;\mathbb{Z})$, *then* K^* *admits a representation* $K(L(p,q);u)$ *with*

$$\begin{aligned} &p=-a^2-ab+b^2,\\ &q\equiv -b^2(a+b)^{-2} \pmod{p=-a^2-ab+b^2},\\ &u\equiv b(a+b)^{-1} \pmod{p=-a^2-ab+b^2}. \end{aligned}$$

CASE 3. Sporadic cases.

By an argument similar to the above, we have the following.

Theorem 6.3. *Let* K *be a doubly primitive knot and* $L(p,q)$ *a lens space obtained by Berge's surgery on* K. *Let* K^* *be the dual knot of* K. *In the following,* j *is a non-negative integer.*

(1) *If* $[K]=(6j+1)[x_1]-j[x_2]+(4j+1)[y_1]+(2j+1)[y_2]$ *in* $H_1(\partial H;\mathbb{Z})$, *then* K^* *admits a representation* $K(L(p,q);u)$ *with*

$$\begin{aligned} &p=22j^2+9j+1,\\ &q\equiv -(22j+5)^2 \pmod{p=22j^2+9j+1},\\ &u\equiv 22j+5 \pmod{p=22j^2+9j+1}. \end{aligned}$$

(2) *If* $[K]=(4j+1)[x_1]-j[x_2]+(6j+2)[y_1]+(2j+1)[y_2]$ *in* $H_1(\partial H;\mathbb{Z})$, *then* K^* *admits a representation* $K(L(p,q);u)$ *with*

$$\begin{aligned} &p=22j^2+13j+2,\\ &q\equiv -(22j+7)^2 \pmod{p=22j^2+13j+2},\\ &u\equiv 22j+7 \pmod{p=22j^2+13j+2}. \end{aligned}$$

(3) *If* $[K]=(-4j-3)[x_1]+(j+1)[x_2]+(6j+4)[y_1]+(2j+1)[y_2]$ *in* $H_1(\partial H;\mathbb{Z})$, *then* K^* *admits a representation* $K(L(p,q);u)$ *with*

$$\begin{aligned} &p=22j^2+31j+11,\\ &q\equiv -(22j+15)^2 \pmod{p=22j^2+31j+11},\\ &u\equiv 22j+15 \pmod{p=22j^2+31j+11}. \end{aligned}$$

(4) *If* $[K] = (-6j-5)[x_1] + (j+1)[x_2] + (4j+3)[y_1] + (2j+1)[y_2]$ *in* $H_1(\partial H; \mathbb{Z})$, *then* K^* *admits a representation* $K(L(p, q); u)$ *with*

$$p = 22j^2 + 13j + 2,$$
$$q \equiv -(22j+17)^2 \pmod{p = 22j^2 + 13j + 2},$$
$$u \equiv 22j + 17 \pmod{p = 22j^2 + 13j + 2}.$$

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