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CROSSCAP NUMBER, RIBBON NUMBER AND ESSENTIAL TANGLE DECOMPOSITIONS OF KNOTS

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Abstract

We investigate crosscap numbers by essential tangle decompositions. We show that each of the crosscap numbers of the Kinoshita-Terasaka knot and the Conway knot is four.

1. Introduction

Let K be a knot in S^3. The *crosscap number* $\gamma(K)$ is defined as the minimal first betti number among all non-orientable spanning surfaces for K, that is, compact connected non-orientable surfaces bounded by K in S^3. (For the unknot O, $\gamma(O) = 0$.) Now it is obvious that any knot K bounds a non-orientable surface and the inequality $\gamma(K) \leq 2g(K) + 1$ holds [1], where $g(K)$ denotes the genus of K. H. Murakami and A. Yasuhara [10] showed that for the knot 7_4 the equality holds. The crosscap numbers for several classes of knots have been computed by several authors; any torus knot by M. Teragaito [11], any 2-bridge knot by M. Hirasawa and M. Teragaito [5], and any pretzel knot by K. Ichihara and S. Mizushima [6]. In this paper, we show that each of the crosscap numbers of the Kinoshita-Terasaka knot and the Conway knot is four, and we investigate a lower bound on crosscap numbers of knots with essential tangle decompositions and mutations.

A *tangle* is a pair (B, T) of a 3-ball B and a properly embedded 1-manifold T in B. If T consists of n arcs, we call (B, T) an *n-string tangle*. A tangle (B, T) is *essential* if $\partial B - \partial T$ is *incompressible in* $B - T$, that is, any non-trivial simple closed curve on $\partial B - \partial T$ does not bound a disk in $B - T$. A *Conway sphere for K* is a sphere S embedded in S^3 such that $S \cap K$ consists of four points. An *essential Conway* sphere is a Conway sphere such that $S - K$ is incompressible in $S^3 - K$.

It is well-known that a genus one hyperbolic knot does not admit essential Conway spheres. In §2, we shall show the following lemma involving this fact.

Lemma 1.1. *Let K be a knot with $g(K) = 1$ or $\gamma(K) \leq 2$. If K admits an essential* 2*-string tangle decomposition, then one of the tangles consists of two parallel arcs.*

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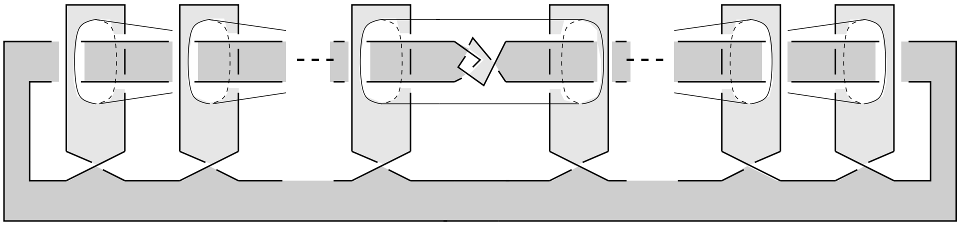
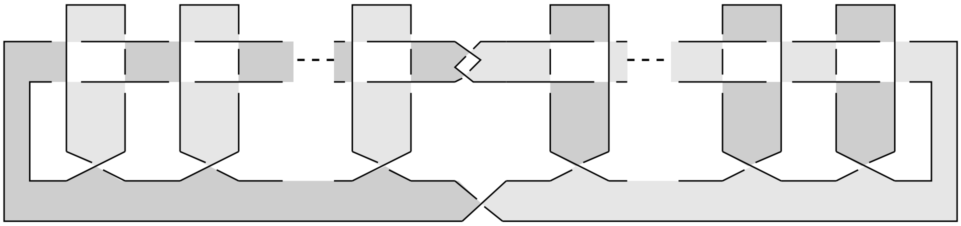


(A): $g = n + 1$.



(B): $g = 2n + 1$.

Fig. 1. Mutative knots of distinct genera.

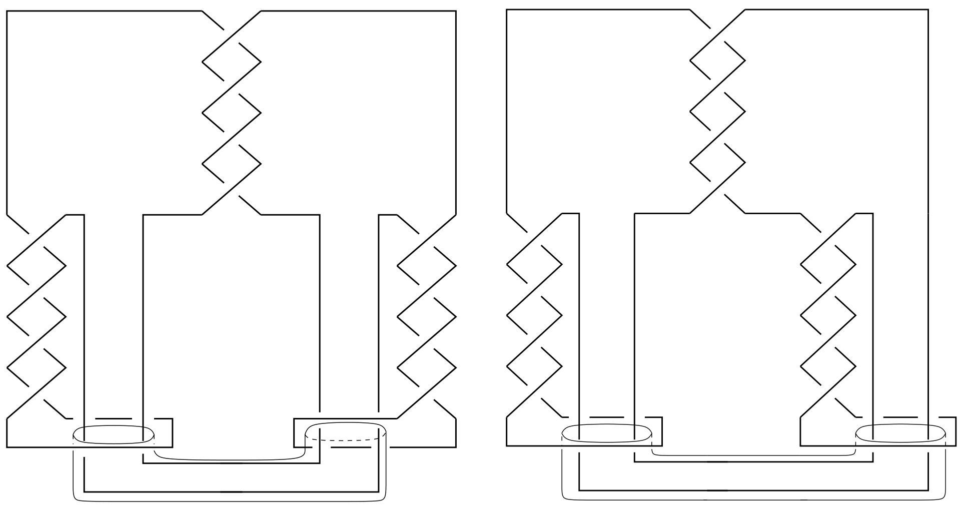
In §3, we prove that the ribbon number of the Kinoshita-Terasaka knot is three as an application of Lemma 1.1. In §4, we generalize Lemma 1.1 for knots with two disjoint essential Conway spheres as follows and determine the crosscap numbers of the Kinoshita-Terasaka knot and the Conway knot.

Theorem 1.2. *Let K be a knot with two disjoint and non-parallel essential Conway spheres S_1 and S_2. Let B_1, B_2 be the two disjoint 3-balls bounded by S_1, S_2 respectively. Let C be the $S^2 \times I$ between S_1 and S_2. Suppose none of $B_i \cap K$ consists of two parallel strings and that at least one of the four strings of $C \cap K$ is not parallel to any of the other three in C. Then $\gamma(K) \geq 4$ and $g(K) \geq 2$.*

By using a notion of a Conway sphere we define mutations for knots (see Section 5). The first example of mutative knots in the Rolfsen's table is the Kinoshita-Terasaka knot and the Conway knot (See Fig. 8). D. Gabai showed that the genus of the Kinoshita-Terasaka knot is two and that of the Conway knot is three [4]. We will give further information about it later (see Proposition 1.3).

Proposition 1.3. *For any positive integer n, there is a knot K such that $g(K^\tau) - g(K) = n$, where K^τ is a mutant of K.*

The proof is done by showing that the exteriors of the Seifert surfaces illustrated in Fig. 1 form taut sutured manifolds [4].



(A): K: $g(K) = 3$, $\gamma(K) = 4$. (B): K^τ: $g(K^\tau) = 2$, $\gamma(K^\tau) = 5$?

Fig. 2. Candidate of mutative knots of distinct crosscap numbers.

REMARK 1.4. For $n = 1$, the knots in Fig. 1 are the Kinoshita-Terasaka knot and the Conway knot.

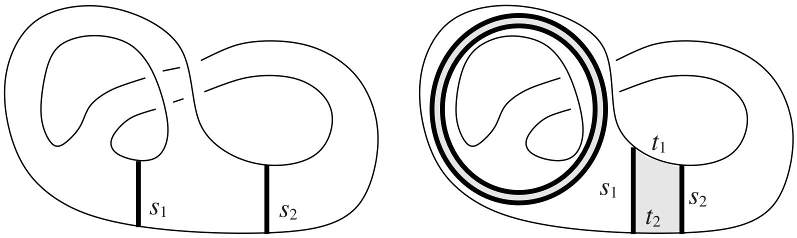
In contrast to genera, the difference of the crosscap numbers of mutative knots is at most one. More precisely, we show the following:

Proposition 1.5. *Let K be a knot in S^3 which admits an essential 2-string tangle decomposition, and K^τ a mutant of K. Suppose $\gamma(K) \leq \gamma(K^\tau)$. Then, if $\gamma(K)$ is odd, $\gamma(K^\tau) = \gamma(K)$. If $\gamma(K)$ is even, $|\gamma(K^\tau) - \gamma(K)| \leq 1$.*

At this writing, the authors do not have any concrete example of a pair of mutative knots with distinct crosscap numbers. We illustrate a candidate of such a pair in Fig. 2.

2. Proof of Lemma 1.1

Proof of Lemma 1.1. Suppose F is a surface bounded by K with the first Betti number $\beta_1(F) = 1$, and S is the Conway sphere defining the essential tangle decomposition. Suppose that F and S are in general position and $|F \cap S|$ is minimal among surfaces F bounded by K with $\beta_1(F) = 1$. Since S is essential, $F \cap S$ consists of two parallel arcs in F and the conclusion follows. Suppose F is a surface bounded by K with the first Betti number $\beta_1(F) = 2$, and S is the Conway sphere defining the essential tangle decomposition. Suppose that F and S are in general position and $|F \cap S|$ is minimal among surfaces F bounded by K with $\beta_1(F) = 2$. Then we may assume that



(A): (♯) (B)

Fig. 3. Genus one surface F and the intersection with S.

$F \cap S$ consists of two arcs and some circles. Since S is incompressible in $S^3 - K$, we may assume that each component of $F \cap S$ is essential in F. Let s_1, s_2 denote the arc components of $F \cap S$. If F is orientable, then there are two ways to draw essential arcs on F as in Fig. 3. Since s_1 is an essential arc on F, $\mathrm{cl}(F - N(s_1))$ is an annulus A_1. Then, if s_2 essential in A_1, then s_1 and s_2 are as in Fig. 3-(A). If s_2 is inessential in A_1, then s_1 and s_2 are as in Fig. 3-(B). Since S is a sphere defining a tangle decomposition, $s_1 \cup s_2$ should separate F. Hence the case of Fig. 3-(A) never occur. In the case of Fig. 3-(B), s_1 and s_2 cut off a rectangle R which gives a parallelism between the two strings t_1 and t_2 in the 3-ball of the tangle. If F is non-orientable, there are several cases as in Fig. 4. First suppose that s_1 is separating in F. Then $\mathrm{cl}(F - N(s_1))$ consists of two Möbius bands M_1 and M_2. Suppose s_2 is in M_1. If s_2 is essential in M_1, then s_1 and s_2 are as in Fig. 4-(A). If s_2 is inessential in M_1, then s_1 and s_2 are as in Fig. 4-(E). Next suppose that s_1 is non-separating in F. Suppose $\mathrm{cl}(F - N(s_1))$ is an annulus A_1. If s_2 is essential in A_1, then s_1 and s_2 are as in Fig. 4-(C). If s_2 is inessential in A_1, then s_1 and s_2 are as in Fig. 4-(F). Suppose $\mathrm{cl}(F - N(s_1))$ is a Möbius band M_1. If s_2 is essential in A_1, then s_1 and s_2 are as in Fig. 4-(B) or -(C). If s_2 is inessential in A_1, then s_1 and s_2 are as in Fig. 4-(A) or -(D). Each of the cases-(A), -(B), -(C) does not correspond to a tangle decomposition for a similar reason. □

Lemma 1.1 can be proven by computing Euler characteristics as in the proof of Theorem 1.2. The conditions on the number of strings and the betti number of spanning surfaces are essential. In Fig. 5 we illustrate a knot of genus one, a knot of crosscap number two which admit 3-string essential tangle decompositions with no parallel strings (Fig. 5-(A), -(B)) and a knot of genus two, a knot of crosscap number three which admit 2-string essential tangle decompositions with no parallel strings (Fig. 5-(C), -(D)).

More generally we have:

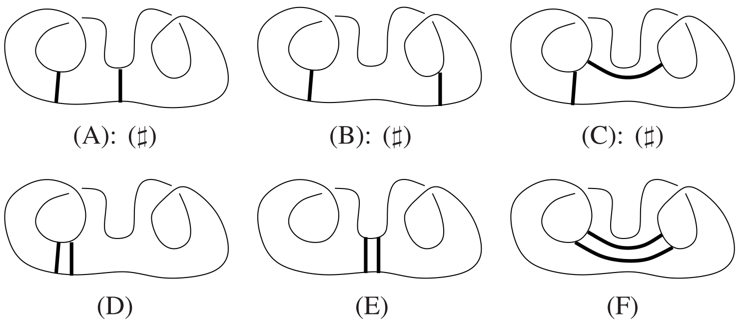


Fig. 4. Two arcs on a non-orientable surface with crosscap number two.

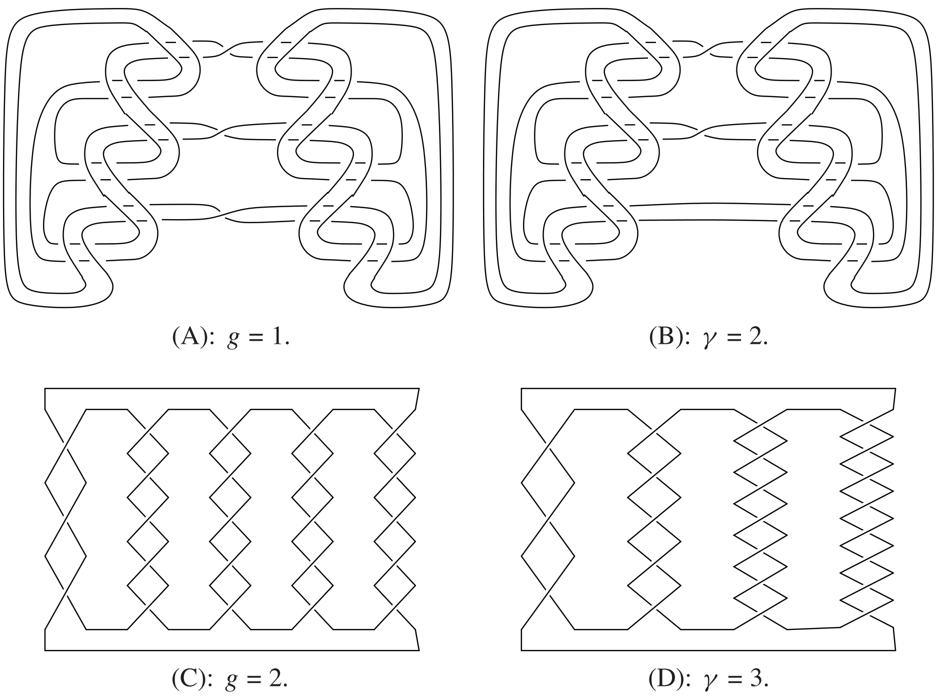


Fig. 5.

Proposition 2.1. *Suppose a knot K admits an n-string essential tangle decomposition without parallel strings. Let F be a spanning surface for K with $\beta_1(F) \geq 2$. Then $n \leq 3\beta_1(F) - 3$.*

Proof. Let S be the sphere defining the n-string essential tangle decomposition. Then, we may assume that F and S are in general position and $F \cap S$ contains mutually non-parallel essential n arcs on F.

Claim 2.2. *There are at most $3\beta_1(F)-3$ mutually disjoint, mutually non-parallel essential arcs on F if $\beta_1(F) > 1$.*

Proof. Suppose $s_1, s_2, \dots, s_n$ are n mutually disjoint, non-parallel properly embedded essential arcs in F such that if s_{n+1} is a properly embedded essential arc in F with $\left(\bigcup_{i=1}^n s_i\right) \cap s_{n+1} = \emptyset$, then s_{n+1} is parallel to some s_i. Then each component of $\mathrm{cl}\left(F - N\left(\bigcup_{i=1}^n s_i\right)\right)$ is a hexagon since F is connected and $\beta_1(F) > 1$. Let G be a graph in F such that a vertex, whose degree is three, corresponds to a component of $\mathrm{cl}\left(F - N\left(\bigcup_{i=1}^n s_i\right)\right)$ and an edge corresponds to s_i. Then there is a deformation retract $r\colon F \to G$ and hence $\chi(F) = \chi(G) = |V(G)| - |E(G)| = 2|E(G)|/3 - |E(G)| = -|E(G)|/3$. Now we have $-\beta_1(F) + 1 = -|E(G)|/3$ and $n = |E(G)| = 3\beta_1(F) - 3$. □

Now we have $n \leq 3\beta_1(F) - 3$ by Claim 2.2. □

REMARK 2.3. Special interest in the case $\beta_1(F) = 2$. If K is prime and $\beta_1(F) = 2$, then $n = 3$ by Lemma 1.1 and Proposition 2.1.

3. Ribbon number of the Kinoshita-Terasaka knot

In [9], the first author proved that the ribbon number of the Kinoshita-Terasaka knot is three by using Jones polynomial [9, Theorem 1.9]. Here we give a short proof of this theorem as an application of Lemma 1.1.

A *ribbon disk* is an immersed 2-disk of D^2 into S^3 with only transverse double points such that the singular set consists of ribbon singularities, that is, the preimage of each ribbon singularity consists of a properly embedded arc in D^2 and an embedded arc in the interior of D^2. A knot is a *ribbon knot* if it bounds a ribbon disk in S^3. (cf. [7], [8]). The *ribbon number* of a ribbon knot is defined as the minimal number of ribbon singularities needed for a ribbon disk bounded by the ribbon knot. Here we have some remarks of ribbon numbers.

REMARK 3.1. A ribbon knot whose ribbon number is zero is a trivial knot and there does not exist a ribbon knot whose ribbon number is one.

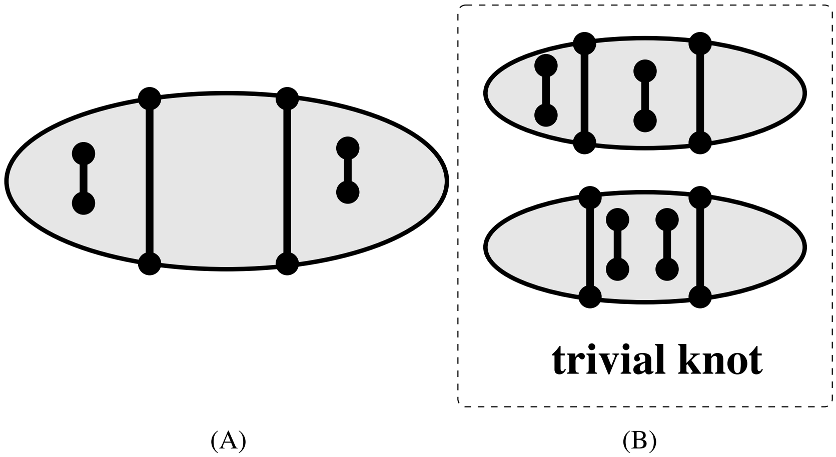


Fig. 6. Ribbon singularities for ribbon number two knots.

REMARK 3.2. The ribbon number of a ribbon knot K is greater than or equal to the genus or K ([2]). Twice the ribbon number of a ribbon knot K is greater than or equal to the crosscap number of K.

Proposition 3.3. *Let K be a ribbon knot with ribbon number two. Then $g(K) = 1$ or $\gamma(K) \leq 2$.*

Proof. The ribbon singularities of K should be as in Fig. 6-(A). By tubing the ribbon disk, we obtain a spanning surface F for K with $\beta_1(F) = 2$. This completes the proof. □

Theorem 3.4 ([9, Theorem 1.9]). *The ribbon number of the Kinoshita-Terasaka knot is three.*

Proof. Let K denote the Kinoshita-Terasaka knot. It is well-known that K admits a 2-string tangle decomposition with no parallel strings as in Fig. 7. This tangle decomposition is essential since for the double branched cover branched along K, the preimage of the Conway sphere is an incompressible torus. By Lemma 1.1 we have $\gamma(K) \geq 3$. Then by Proposition 3.3 we have that the ribbon number of K is greater than or equal to three. The diagram of Fig. 7 gives a ribbon disk with three ribbon singularities. This completes the proof. □

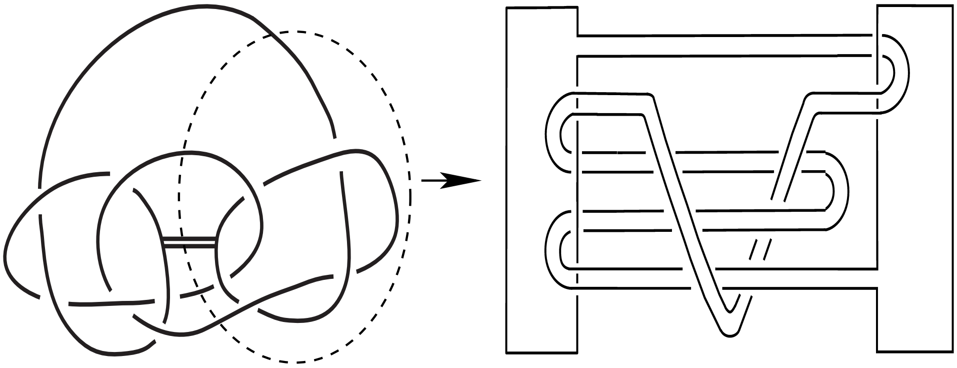


Fig. 7. The Kinoshita-Terasaka knot.

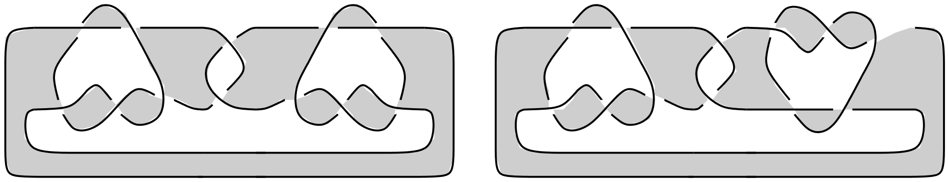
4. Crosscap numbers of the Kinoshita-Terasaka knot and the Conway knot

Proof of Theorem 1.2. By Lemma 1.1, we may assume that $\gamma(K) \geq 3$ and $g(K) \geq 2$. Suppose K bounds a spanning surface F with $\beta_1(F) = 3$. We may assume that F and $S_1 \cup S_2$ are in general position and $|F \cap (S_1 \cup S_2)|$ is minimal among spanning surfaces of crosscap number three for K. Then we may assume that $F \cap (S_1 \cup S_2)$ consists of four essential arcs and some essential circles on F since $(S_1 \cup S_2) - K$ is incompressible in $S^3 - K$. Then $F \cap S_i$ consists of two arcs and parallel circles on S_i.

Put $F_1 = F \cap B_1$, $F_2 = F \cap B_2$, $F_3 = F \cap C$. Note that $F_i \cap F_3$ ($i = 1, 2$) consists of two arcs and some circles. Then we have that $\chi(F_1) + \chi(F_3) + \chi(F_2) - 4 = \chi(F) = -2$. This implies that $\chi(F_1) + \chi(F_3) + \chi(F_2) = 2$ and one of $\chi(F_1)$, $\chi(F_3)$, $\chi(F_2)$ is positive. Suppose $\chi(F_1)$ or $\chi(F_2)$, say $\chi(F_1)$, is positive. Then F_1 has a component F_1' with $\chi(F_1') = 1$. Since each component of $F \cap S_1$ is essential in F, we see that $|F_1' \cap S| = 2$ and F_1' is a rectangle between two strings of $B_1 \cap K$, a contradiction. Now we may assume that $\chi(F_1) \leq 0$, $\chi(F_2) \leq 0$, and $\chi(F_3) \geq 2$. In this case, F_3 has two components F_3' and F_3'' such that $\chi(F_3') = 1$ and $\chi(F_3'') = 1$. Since each component of $F \cap (S_1 \cup S_2)$ is essential in F, we see that $|F_3' \cap (S_1 \cup S_2)| \geq 2$, $|F_3'' \cap (S_1 \cup S_2)| \geq 2$, and hence each of F_3' and F_3'' is a rectangle and any component of $C \cap K$ is parallel to some component of $C \cap K$, a contradiction to the assumption. This completes the proof. $\square$

Corollary 4.1. γ(*the Kinoshita-Terasaka knot*) $=$ γ(*the Conway knot*) $= 4$.

Proof. Use Theorem 1.2 and the diagram in Fig. 8. $\square$



(A): Kinoshita-Terasaka knot. (B): Conway knot.

Fig. 8.

5. Crosscap numbers of mutative knots

Let S be an essential Conway sphere for a knot K. Put $S \cap K = v_1 \cup v_2 \cup v_3 \cup v_4$. Let τ be an involution on a 3-ball bounded by S such that $\tau(v_i) \neq v_i$ for $i = 1, 2, 3, 4$ and $\tau(v_1 \cup v_2 \cup v_3 \cup v_4) = v_1 \cup v_2 \cup v_3 \cup v_4$. We denote by K^τ the mutant of K with respect to τ, that is, K^τ is obtained from K by replacing a tangle cut off by S via τ. Recall that any disjoint two arcs s_1, s_2 with $\partial s_1 \cup \partial s_2 = v_1 \cup v_2 \cup v_3 \cup v_4$ and simple closed curves in $S - (s_1 \cup s_2)$ are equivariant under τ. Now if F is a surface bounded by K such that $F \cap S$ consists of two arcs and essential simple closed curves on S, then F^τ denotes the surface bounded by K^τ obtained from F. Note that F^τ has the same Euler characteristic χ as F.

Proof of Proposition 1.5. Let F be a non-orientable surface bounded by K such that $\beta_1(F) = \gamma(K)$, and S the Conway sphere for the essential 2-tangle decomposition. If F is compressible, then by compressing F we obtain a spanning surface F' for K with $\beta_1(F') \leq \beta_1(F) - 2$. Then F' is orientable since $\beta_1(F) = \gamma(K)$. However by adding a one-sided curve as in Fig. 9 we obtain a non-orientable surface F'' with $\beta_1(F'') = \beta_1(F') + 1 \leq \beta_1(F)$, a contradiction. Hence F is incompressible. We may assume that F and S are in general position and that $|F \cap S|$ is minimal among all non-orientable surfaces for K with $\beta_1(F) = \gamma(K)$. Then $F \cap S$ consists of two essential arcs on F and some circles essential on both F and S since F and $S - K$ are incompressible in $S^3 - K$. Let F' denote the component of F^τ with $\partial F' = K^\tau$, where F^τ is the surface bounded by K^τ obtained from F by the mutation. If F^τ is disconnected, then each component of $F^\tau - F'$ consists of closed orientable surfaces. Note that $\chi(F) = \chi(F^\tau) = \chi(F') + \chi(F^\tau - F')$ and hence $\beta_1(F) \equiv \beta_1(F')$ modulo 2. If $\beta_1(F)$ is odd, then $\chi(F')$ is odd and F' is non-orientable. If $F^\tau - F'$ contains a sphere, then some circle component of $F \cap S$ bounds a disk in F, a contradiction to the essentiality of $F \cap S$. If $F^\tau - F'$ contains a torus T, then $\chi(F^\tau) = \chi(F^\tau - T)$ and we regard F^τ as $F^\tau - T$. Now we may assume that for each component F_i^τ of $F^\tau - F'$, $\chi(F_i^\tau) \leq -2$. If $F^\tau - F' \neq \emptyset$, then $\beta_1(F') \leq \beta_1(F) - 2$. By the assumption that $\gamma(K) \leq \gamma(K^\tau)$, F' is orientable and $\gamma(K)$ is even. Then we get a non-orientable surface F'' for K^τ such that $\beta_1(F'') = \beta_1(F') + 1$ by adding a one-sided loop as in

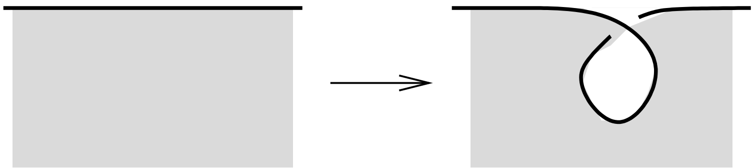


Fig. 9.

Fig. 9 and we have $\gamma(K^\tau) \leq \beta_1(F') + 1 \leq \beta_1(F) - 1 = \gamma(K) - 1$, a contradiction to $\gamma(K) \leq \gamma(K^\tau)$. Now we assume that F^τ is connected and $\beta_1(F) = \beta_1(F^\tau)$. If F^τ is non-orientable, we are done. If F^τ is orientable, we have $\gamma(K^\tau) \leq \gamma(K)+1$ by adding a one-sided loop to F^τ as in Fig. 9. This completes the proof. □

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