

LONG TIME EXISTENCE FOR VORTEX FILAMENT EQUATION IN A RIEMANNIAN MANIFOLD

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Abstract

Vortex filament equation in the Euclidean space has a long time solution for any closed initial data, because it can be converted into a standard nonlinear Schrödinger equation. While the Riemannian version of vortex filament equation is not integrable at all, we prove that it has a long time solution for any closed initial data.

1. Introduction and preliminaries

The vortex filament equation is an equation of a curve $\gamma(x, t)$ in the three-dimensional Euclidean space:

$$(V) \quad \gamma_t = \gamma_x \times \gamma_{xx}, \quad |\gamma_x| \equiv 1,$$

where \times is the exterior product. H. Hasimoto [1] proved the whole time existence of solutions of (V), provided that the solution has non-vanishing curvature. T. Nishiyama and A. Tani [5] proved the whole time existence of solutions of (V) without such an assumption. A key step in [1] is a transformation of (V) to a standard nonlinear Schrödinger equation, while [5] uses a perturbation to a 4-th order parabolic equation.

Later, the present author showed that the method of [1] can be applied without the assumption on the curvature [2], and generalized to the case of 3-dimensional space forms, i.e., the projective space $P^3(\mathbf{R})$ and the hyperbolic space H^3 with Riemannian metric of constant sectional curvature. The generalization is strongly related to the “non-linear integrable system”, and cannot be applied to the case of general 3-dimensional Riemannian manifolds. Here, vortex filament equation in oriented 3-dimensional Riemannian manifold (M, g) is given by simple replacement of differentiation to covariant differentiation:

$$(VM) \quad \gamma_t = \gamma_x \times \nabla_x \gamma_x, \quad |\gamma_x| \equiv 1.$$

Recently, the present author found a proof of short time existence for general Riemannian manifold [3], using perturbation to a parabolic equation: $\gamma_t = \gamma_x \times \nabla_x \gamma_x + \varepsilon \nabla_x \gamma_x$ ($\varepsilon > 0$). It is natural to conjecture that the solution will diverge in finite time, because it seems that (V) has infinite-time solutions solely because of its integrability.

(V) possesses infinitely many conserved quantities, while the perturbed equation (VM) cannot have such quantities, because the curvature of the Riemannian manifold varies. In fact, as we will see in Example, we can easily construct a complete Riemannian manifold where a solution of (VM) blows up at finite time.

However, in this paper, we will prove that, the conjecture is false under a curvature condition.

Theorem. *Let (M, g) be an oriented 3-dimensional complete Riemannian manifold with bounded sectional curvature. Then, equation (VM) has a unique whole time $(-\infty < t < \infty)$ solution for any C^∞ closed initial curve $\gamma_0(x)$ with $|\nabla_x \gamma_0| \equiv 1$.*

We summarize notations. We denote by $|\ast|$ the norm, by ∇ the covariant differentiation, by R the curvature tensor, and by \times the exterior product of each tangent space of M . Partial derivation is denoted by subscript or ∂_x, ∂_t . The manifold, its structure and all functions are supposed to be C^∞ .

By re-scaling, we may assume that the initial length of the curve is 1. Therefore, we may consider γ as a map from $(\mathbf{R}/\mathbf{Z}) \times \mathbf{R}_{\geq 0}$ to M .

We will take function norms only for x -direction. More precisely,

$$\langle \alpha, \beta \rangle := \int_0^1 g(\alpha, \beta) dx, \quad \|\alpha\|^2 := \langle \alpha, \alpha \rangle, \quad \|\alpha\|_n^2 = \sum_{i=0}^n \|\nabla_x^i \alpha\|^2.$$

Also, $\|\alpha\|_{C^n}$ counts only x -derivatives and is a function in t .

2. Proof of Theorem

Let γ be a solution of (VM) defined on a finite time interval $[0, T)$.

Lemma 2.1. *$\|\nabla_x \gamma_x\|$ is bounded from above.*

Proof. The quantity $\|\nabla_x \gamma_x\|$ is estimated as follows,

$$\begin{aligned} \frac{d}{dt} \|\nabla_x \gamma_x\|^2 &= 2\langle \nabla_x \gamma_x, \nabla_t \nabla_x \gamma_x \rangle = 2\langle \nabla_x \gamma_x, R(\gamma_t, \gamma_x)\gamma_x + \nabla_x \nabla_t \gamma_x \rangle \\ &= 2\langle \nabla_x \gamma_x, R(\gamma_x \times \nabla_x \gamma_x, \gamma_x)\gamma_x \rangle - 2\langle \nabla_x^2 \gamma_x, \gamma_x \times \nabla_x^2 \gamma_x \rangle \\ &\leq C_1 \|\nabla_x \gamma_x\|^2, \end{aligned}$$

and increases at most exponentially. □

Lemma 2.2. *γ is in a compact set of M . In particular, all derivatives of curvature tensor are bounded along γ .*

Proof. Since $\|\gamma_t\| = \|\gamma_x \times \nabla_x \gamma_x\|$ is bounded by a constant C_1 ,

$$\min_{x \in S^1} \int_0^T |\gamma_t| dt \leq \int_0^T \|\gamma_t\| dt \leq C_1 T.$$

Therefore, taking account of the length of γ , the distance $d(\gamma(x, t), \gamma(0, 0))$ is bounded by $C_1 T + 2$. \square

Lemma 2.3. $\|\nabla_x^2 \gamma_x\|$ is bounded from above.

Proof. Note that $|\nabla_x R| = |(\nabla R)(\gamma_x)|$ is bounded by Lemma 2.2. The quantity $\|\nabla_x^2 \gamma_x\|$ is estimated as follows:

$$\begin{aligned} \frac{d}{dt} \|\nabla_x^2 \gamma_x\|^2 &= 2\langle \nabla_x^2 \gamma_x, \nabla_t \nabla_x^2 \gamma_x \rangle \\ &= 2\langle \nabla_x^2 \gamma_x, R(\gamma_t, \gamma_x) \nabla_x \gamma_x + \nabla_x (R(\gamma_t, \gamma_x) \gamma_x) + \nabla_x^2 \nabla_t \gamma_x \rangle \\ &= 2\langle \nabla_x^2 \gamma_x, R(\gamma_x \times \nabla_x \gamma_x, \gamma_x) \nabla_x \gamma_x + \nabla_x (R(\gamma_x \times \nabla_x \gamma_x, \gamma_x) \gamma_x) + \nabla_x^2 \nabla_t \gamma_x \rangle \\ &\leq C_1 \|\nabla_x^2 \gamma_x\| (\|\nabla_x \gamma_x\| \max |\nabla_x \gamma_x| + \|\nabla_x \gamma_x\| + \|\nabla_x^2 \gamma_x\|) - 2\langle \nabla_x^3 \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle. \end{aligned}$$

Since $\|\nabla_x \gamma_x\|$ is already bounded by Lemma 2.1, we see that

$$\frac{d}{dt} \|\nabla_x^2 \gamma_x\|^2 \leq C_2 (1 + \|\nabla_x^2 \gamma_x\|^2) - 2\langle \nabla_x^3 \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle.$$

We rewrite the last term. Note that $g(\gamma_x, \nabla_x^2 \gamma_x) = -|\nabla_x \gamma_x|^2$ and $g(\gamma_x, \nabla_x^3 \gamma_x) = \partial_x (g(\gamma_x, \nabla_x^2 \gamma_x)) - g(\nabla_x \gamma_x, \nabla_x^2 \gamma_x) = -(3/2) \partial_x |\nabla_x \gamma_x|^2$. We denote by $*^\perp$ and $*^\top$ the γ_x -factor and the factor perpendicular to γ_x , respectively. The last term becomes

$$\begin{aligned} \langle \nabla_x^3 \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle &= \langle (\nabla_x^3 \gamma_x)^\perp + (\nabla_x^3 \gamma_x)^\top, \nabla_x \gamma_x \times \{(\nabla_x^2 \gamma_x)^\perp + (\nabla_x^2 \gamma_x)^\top\} \rangle \\ &= \langle (\nabla_x^3 \gamma_x)^\perp, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle + \langle \nabla_x^3 \gamma_x, \nabla_x \gamma_x \times (\nabla_x^2 \gamma_x)^\perp \rangle \\ &= \langle g(\nabla_x^3 \gamma_x, \gamma_x) \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle + \langle \nabla_x^3 \gamma_x, g(\nabla_x^2 \gamma_x, \gamma_x) \nabla_x \gamma_x \times \gamma_x \rangle \\ &= -\frac{3}{2} \langle \partial_x |\nabla_x \gamma_x|^2 \cdot \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle - \langle \nabla_x^3 \gamma_x, |\nabla_x \gamma_x|^2 \nabla_x \gamma_x \times \gamma_x \rangle \\ &= -\frac{3}{2} \langle \partial_x |\nabla_x \gamma_x|^2 \cdot \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle + \langle \nabla_x^2 \gamma_x, \partial_x |\nabla_x \gamma_x|^2 \cdot \nabla_x \gamma_x \times \gamma_x \rangle \\ &= -\frac{5}{2} \langle \partial_x |\nabla_x \gamma_x|^2 \cdot \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \|\nabla_x \gamma_x\|^2 &= 4\langle |\nabla_x \gamma_x|^2, g(\nabla_x \gamma_x, \nabla_t \nabla_x \gamma_x) \rangle = 4\langle |\nabla_x \gamma_x|^2 \nabla_x \gamma_x, R(\gamma_t, \gamma_x) \gamma_x + \nabla_x \nabla_t \gamma_x \rangle \\ &\geq -C_3 \|\nabla_x \gamma_x\|^2 \max |\nabla_x \gamma_x|^2 - 4\langle \nabla_x (|\nabla_x \gamma_x|^2 \nabla_x \gamma_x), \gamma_x \times \nabla_x^2 \gamma_x \rangle \end{aligned}$$

$$\begin{aligned} &\geq -C_4 \|\nabla_x \gamma_x\| (\|\nabla_x \gamma_x\| + \|\nabla_x^2 \gamma_x\|) - 4(\partial_x |\nabla_x \gamma_x|^2 \cdot \nabla_x \gamma_x, \gamma_x \times \nabla_x^2 \gamma_x) \\ &\geq -C_5(1 + \|\nabla_x^2 \gamma_x\|) + 4(\partial_x |\nabla_x \gamma_x|^2 \cdot \gamma_x, \nabla_x \gamma_x \times \nabla_x^2 \gamma_x). \end{aligned}$$

Therefore, we have

$$\frac{d}{dt} \{4\|\nabla_x^2 \gamma_x\|^2 - 5\|\nabla_x \gamma_x\|^2\} \leq C_6(1 + \|\nabla_x^2 \gamma_x\|^2).$$

As in the above calculation, $\|\nabla_x \gamma_x\|^2 \leq C_7(1 + \|\nabla_x^2 \gamma_x\|)$. It implies that

$$4\|\nabla_x^2 \gamma_x\|^2 - 5\|\nabla_x \gamma_x\|^2 \geq 4\|\nabla_x^2 \gamma_x\|^2 - C_8(1 + \|\nabla_x^2 \gamma_x\|) \geq 3\|\nabla_x^2 \gamma_x\|^2 - C_8(1 + C_8).$$

Thus, $X(t) := 4\|\nabla_x^2 \gamma_x\|^2 - 5\|\nabla_x \gamma_x\|^2$ satisfies $X'(t) \leq C_9(1 + X(t))$, and $X(t)$ is bounded. Hence, $\|\nabla_x^2 \gamma_x\|$ is bounded. \square

REMARK 2.4. When the manifold (M, g) is a space form, i.e., has constant sectional curvature, the quantity $4\|\nabla_x^2 \gamma_x\|^2 - 5\|\nabla_x \gamma_x\|^2$ is preserved.

Lemma 2.5. For each positive integer n , $\|\nabla_x^n \gamma_x\|$ is bounded from above.

Proof. We use induction. Take any integer $n \geq 2$ and suppose that $\|\nabla_x^k \gamma_x\|$ is bounded for any non-negative integer $k \leq n$. It implies that $|\nabla_x^k \gamma_x|$ is bounded for any non-negative integer $k < n$.

$$\begin{aligned} \frac{d}{dt} \|\nabla_x^{n+1} \gamma_x\|^2 &= 2\langle \nabla_x^{n+1} \gamma_x, \nabla_t \nabla_x^{n+1} \gamma_x \rangle \\ &= 2 \left\langle \nabla_x^{n+1} \gamma_x, \nabla_x^{n+1} \nabla_t \gamma_x + \sum_{i=0}^n \nabla_x^i (R(\gamma_x \times \nabla_x \gamma_x, \gamma_x) \nabla_x^{n-i} \gamma_x) \right\rangle. \end{aligned}$$

Here, the summation term is decomposed into contraction of $\nabla_x^* R \otimes (\bigotimes_j \nabla_x^{p_j} \gamma_x)$ where $\sum_j p_j \leq n+1$. Hence their L_2 norms are bounded by $C_1(1 + \|\nabla_x^{n+1} \gamma_x\|)$.

The term $\nabla_x^{n+1} \nabla_t \gamma_x = \nabla_x^{n+1}(\gamma_x \times \nabla_x^2 \gamma_x)$ is a linear combination of $\nabla_x^p \gamma_x \times \nabla_x^{n+3-p} \gamma_x$ ($2p < n+3$), and L_2 norm of each term is bounded by $C_2(1 + \|\nabla_x^{n+1} \gamma_x\|)$ except the cases $p=0, 1, 2, 3$. If $p=3$, then $n > 3$ and the L_2 norm of $\nabla_x^3 \gamma_x \times \nabla_x^n \gamma_x$ is bounded. If $p=2$, the term $\nabla_x^2 \gamma_x \times \nabla_x^{n+1} \gamma_x$ is perpendicular to $\nabla_x^{n+1} \gamma_x$.

We calculate the remaining terms: $p=0, 1$. As in the proof of Lemma 2.3,

$$\begin{aligned} &2\langle \nabla_x^{n+1} \gamma_x, \gamma_x \times \nabla_x^{n+3} \gamma_x + (n+1)\nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x \rangle \\ &= -2\langle \nabla_x^{n+1} \gamma_x, \nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x \rangle + 2(n+1)\langle \nabla_x^{n+1} \gamma_x, \nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x \rangle \\ &= 2n\langle \nabla_x^{n+1} \gamma_x, \nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x \rangle \\ &= 2n(\langle \nabla_x^{n+1} \gamma_x \rangle^\perp, \nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x) + 2n\langle \nabla_x^{n+1} \gamma_x, \nabla_x \gamma_x \times (\nabla_x^{n+2} \gamma_x)^\perp \rangle \end{aligned}$$

$$= 2n \langle g(\nabla_x^{n+1} \gamma_x, \gamma_x) \gamma_x, \nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x \rangle + 2n \langle \nabla_x^{n+1} \gamma_x, g(\nabla_x^{n+2} \gamma_x, \gamma_x) \nabla_x \gamma_x \times \gamma_x \rangle.$$

Here, $g(\nabla_x^{n+2} \gamma_x, \gamma_x)$ in the second term is expressed by a linear combination of $g(\nabla_x^p \gamma_x, \nabla_x^{n+2-p} \gamma_x)$ ($(n+2)/2 \leq p \leq n+1$), hence the second term is bounded by $C_3(1 + \|\nabla_x^{n+1} \gamma_x\|^2)$. For the first term, we have similar estimate by using

$$\begin{aligned} & \langle g(\nabla_x^{n+1} \gamma_x, \gamma_x) \gamma_x, \nabla_x \gamma_x \times \nabla_x^{n+2} \gamma_x \rangle \\ &= -\langle \partial_x (g(\nabla_x^{n+1} \gamma_x, \gamma_x)) \cdot \gamma_x, \nabla_x \gamma_x \times \nabla_x^{n+1} \gamma_x \rangle - \langle g(\nabla_x^{n+1} \gamma_x, \gamma_x) \gamma_x, \nabla_x^2 \gamma_x \times \nabla_x^{n+1} \gamma_x \rangle. \end{aligned}$$

Thus, we have proved that

$$\frac{d}{dt} \|\nabla_x^{n+1} \gamma_x\|^2 \leq C_4(1 + \|\nabla_x^{n+1} \gamma_x\|^2). \quad \square$$

Proof of Theorem. By Theorem 3.1 in [3], there exists a unique maximum solution γ . If γ is defined only on a finite time interval $[0, T)$, γ can be C^∞ -ly extended onto $[0, T]$ by Lemma 2.5. Hence, we can extend the solution over T again by Theorem 3.1 in [3]. This is a contradiction. Therefore, γ is defined on the interval $0 \leq t < \infty$. Since the equation is invertible, we get a unique solution on the real line $(-\infty, \infty)$. \square

3. Example

In this section, we give examples such that the equation reduces to an ordinary differential equation. Let M be a S^1 bundle over a Riemann surface B . We assume that the projection π is a Riemannian submersion from (M, g, ∇) to $(B, \bar{g}, \bar{\nabla})$. We denote by \tilde{X} the horizontal lift of a tangent vector on B . Let V be a unit vertical vector field such that $\{V, \tilde{X}_1, \tilde{X}_2\}$ becomes positive basis if $\{X_1, X_2\}$ is positive. Since $[\tilde{X}, V]$ is vertical and independent of extension of $X \in T_y B$, we can define a 1-form η on M by $\eta(\tilde{X})V = [\tilde{X}, V]$ and $\eta(V) = 0$. We assume that η is a pull back of a 1-form ξ on B , i.e., $[\tilde{X}, V] = \xi(X)V$.

Since $\nabla_V V$ is perpendicular to V and

$$\begin{aligned} 2g(\nabla_V V, \tilde{X}) &= 2V(g(V, \tilde{X})) - \tilde{X}(g(V, V)) + g([V, V], \tilde{X}) + 2g([\tilde{X}, V], V) \\ &= 2\xi(X), \end{aligned}$$

$\nabla_V V$ is the dual vector field η^\sharp of η .

Let $\{X_1, X_2\}$ be a positive orthonormal basis of $T_y B$. Then,

$$\begin{aligned} \nabla_V V &= \eta^\sharp = \xi(X_1)\tilde{X}_1 + \xi(X_2)\tilde{X}_2, \\ V \times \nabla_V V &= -\xi(X_2)\tilde{X}_1 + \xi(X_1)\tilde{X}_2, \\ [V, V \times \nabla_V V] &= -\xi(X_2)[V, \tilde{X}_1] + \xi(X_1)[V, \tilde{X}_2] = 0. \end{aligned}$$

Therefore, we can define a map $\gamma: S^1 \times \mathbf{R} = \{(x, t)\} \rightarrow M$ such that $\gamma_x = V$ and $\gamma_t = V \times \nabla_V V$. It means that if the initial data is a fiber, then the solution of the vortex filament equation is a family of fibers. The family is governed by the integral flow of the vector field $J\xi^\sharp$, where J is the almost complex structure of the base manifold.

Let (M, g^o) be an S^1 fiber bundle with geodesic fibers. (Hopf fibering: $S^3 \rightarrow S^2$ and trivial bundle $B \times S^1 = \{(y, \theta)\}$ are typical examples.) We denote by ∇^o the covariant differentiation and V^o the fiber vector field defined as above. Let f be a function on B . We define a new metric g_f on M by modifying the fiber metric to $g_f(V^o, V^o) = \exp(-2f(y))$. Then, the unit fiber vector field is given by $V = \exp f \cdot V^o$, and

$$\begin{aligned} 2g_f(\nabla_V V, \tilde{X}) &= 2V(g_f(V, \tilde{X})) - \tilde{X}(g_f(V, V)) + g_f([V, V], \tilde{X}) + 2g_f([\tilde{X}, V], V) \\ &= 2\exp(-2f)g^o([\tilde{X}, \exp f \cdot V^o], \exp f \cdot V^o) = 2X(f). \end{aligned}$$

Therefore, $\xi = df$.

Proposition 3.1. *Let (M, g_f) be as above. If the initial data is a fiber, then the solution of (VM) moves along contour lines of f . In particular, if B is compact, then the solution is periodic with respect to time, provided that the contour line does not contain critical points of f .*

EXAMPLE 3.2. Let B be the Euclidean plane $\mathbf{R}^2 = \{(u, v)\}$ and f the function $\tanh(uv^2)$. Then $df = (v^2, 0)$ along the contour line $u = 0$, and the solution with initial data $(u, v) = (0, 1)$ is governed by an ordinary differential equation $v'(t) = v^2$. This solution $v(t) = 1/(1-t)$ blows up at $t = 1$. Note that the Riemannian manifold M is complete. This example shows that we cannot omit the assumption of boundedness of curvature in Theorem.

REMARK 3.3. This behaviour of blowing up in Example 3.2 is completely different from 1-dimensional Eells-Sampson equation: $\gamma_t = \nabla_x \gamma_x$. Any solution of Eells-Sampson equation on a complete Riemannian manifold never blows up at finite time, because

$$\left(\int_0^T \|\gamma_t\| dt \right)^2 \leq T \int_0^T \|\gamma_t\|^2 dt = T \int_0^T \|\nabla_x \gamma_x\|^2 dt = -\frac{T}{2} [\|\gamma\|_x^2]_0^T$$

is bounded by the initial energy.

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