

## ZHU'S ALGEBRA OF RANK ONE LATTICE VERTEX OPERATOR SUPERALGEBRAS

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### 1. Introduction

In this paper, we explicitly determine the algebraic structure of Zhu's algebra for a lattice vertex operator superalgebra which is constructed from a rank one odd lattice, and give a proof of rationality.

Y. Zhu introduced an associative algebra associated to a vertex operator algebra in order to study the structure of its modules [11]. Such an associative algebra is now called Zhu's algebra. Roughly speaking, the structure of such a module is determined from the action of the weight 0 component operator on the lowest weight space, which is described through the action of Zhu's algebra and the structure of the vertex operator algebra. Then, for instance, we have one to one correspondence between the classes of inequivalent admissible modules of a vertex operator algebra and the classes of inequivalent modules of Zhu's algebra [11].

The notion of Zhu's algebra has been vastly used to classify all simple modules for vertex operator algebras (affine vertex operator algebras [7], Virasoro vertex operator algebras [10] and lattice vertex operator algebras associated to even lattices of rank one [5], etc.). In this paper, we will study Zhu's algebra  $A(V_L)$  associated to a vertex operator superalgebra  $V_L$  for a rank one *odd* lattice  $L$ .

Zhu's algebra  $A(V_L)$  of  $V_L$  for an even lattice is studied in [5] and the classification of the simple modules for  $V_L$  is given, which provides another proof of the classification results known in [2]. In our odd lattice case,  $V_L$  is a vertex operator *superalgebra*, and by virtue of super symmetries in some sense, the structure of its Zhu's algebra is much simpler comparing to the even lattice case. It is a quotient algebra of the polynomial ring with one variable. By using the explicit structure of Zhu's algebra we can easily classify all simple modules for  $V_L$ . Though the classification of the simple modules for  $V_L$  for an even lattice  $L$  has been known in [2] and the method given in [2] can be applied to noneven case including our case, it is worthy to study the reason of such simpleness appeared in super case.

For an even lattice  $L$ , the rationality, more precisely, the regularity of  $V_L$  is proved in [4] by the method deeply depending on the one in [2]. In this paper, we will give a rough sketch of the proof of the regularity of  $V_L$  for a rank one odd lattice  $L$  emphasizing the differences between the super and nonsuper case.

Now we state the precise structure of Zhu’s algebra  $A(V_L)$  which is constructed from a rank one odd lattice  $L$ . Let  $L = \mathbb{Z}\alpha$  be an integral lattice of rank one generated by  $\alpha$  such that  $(\alpha|\alpha) = k$ , where  $k$  is a positive odd integer, and let  $V_L$  be the vertex operator superalgebra associated to  $L$ . Zhu’s algebra  $A(V_L)$  in our case is isomorphic to the following quotient algebra of the polynomial ring  $\mathbb{C}[x]$ :

$$A(V_L) \cong \mathbb{C}[x]/(F_k(x)),$$

where  $F_k(x) = \prod_{n \in I_k} (x - n)$  and  $I_k = \{0, \pm 1, \dots, \pm(k-1)/2\}$ . This enables us to obtain a complete list of the simple  $V_L$ -modules.

We note that  $V_L$  is isomorphic to the charged free fermions for  $k = 1$ , and to the  $N = 2$  superconformal vertex algebra with central charge  $C = 1$  for  $k = 3$  [8].

This paper is organized as follows. In Section 2.1 we define vertex operator superalgebras, their modules, and the notions of rationality and regularity. The definition of Zhu’s algebra corresponding to vertex operator superalgebras is given in Section 2.2. We construct vertex operator superalgebras  $V_L$  in Section 3.1 and determine Zhu’s algebra  $A(V_L)$  in Section 3.2. The proof of regularity of  $V_L$  is given in Section 3.3. In Applications we consider vertex operator superalgebras  $V_L$  in special cases  $k = 1$  and  $k = 3$ .

## 2. Vertex operator superalgebras and their Zhu’s algebra

**2.1. Vertex operator superalgebras and modules** A vertex operator superalgebra is a  $(1/2)\mathbb{Z}$ -graded vector space  $V = \bigoplus_{n \in (1/2)\mathbb{Z}} V_n = V_0 \oplus V_1$ , ( $V_0 = \bigoplus_{n \in \mathbb{Z}} V_n$ ,  $V_1 = \bigoplus_{n \in \mathbb{Z}+(1/2)} V_n$ ) such that  $\dim V_n < \infty$  for  $n \in (1/2)\mathbb{Z}$  and  $V_n = 0$  for sufficiently small  $n \in (1/2)\mathbb{Z}$ , equipped with a linear map

$$\begin{aligned} V &\rightarrow (\text{End } V)[[z, z^{-1}]] \\ a &\mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \end{aligned}$$

and with two distinguished homogeneous elements  $\mathbf{1} \in V_0$  (called the vacuum vector),  $\omega \in V_2$  (called the Virasoro element) satisfying the following conditions (V1) ~ (V6): for  $a, b \in V$ ,

(V1)  $a_n b = 0$  for sufficiently large  $n \in \mathbb{Z}$ ,

(V2) 
$$\begin{aligned} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1) Y(b, z_2) - (-1)^{\tilde{a}\tilde{b}} z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b, z_2) Y(a, z_1) \\ = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2), \end{aligned}$$

where  $\tilde{a} = 0$  (resp.  $\tilde{a} = 1$ ) according to  $a \in V_0$  (resp.  $a \in V_1$ ), and the formal  $\delta$  function  $\delta(z)$  is defined to be  $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$  and any binomial expressions are expanded into non-negative powers of the second variable,

(V3)  $Y(\mathbf{1}, z) = \text{id}_V, Y(a, z)\mathbf{1} \in V[[z]], \lim_{z \rightarrow 0} Y(a, z)\mathbf{1} = a,$  and

(V4) set  $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$  then

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} c_V \delta_{m+n,0} \quad (c_V \in \mathbb{C}, m, n \in \mathbb{Z}),$$

(V5)  $L_0 a = na$  for  $a \in V_n \left( n \in \frac{1}{2}\mathbb{Z} \right),$

(V6)  $\frac{d}{dz} Y(a, z) = Y(L_{-1} a, z).$

The scalar  $c_V$  is called central charge. We say an element  $a \in V_n$  homogeneous with weight  $n,$  denoted  $n = \text{wt}(a).$

The notion of a module for a vertex operator superalgebra is defined in the following way. A weak  $V$ -module  $M$  is a vector space equipped with a linear map

$$\begin{aligned} V &\rightarrow (\text{End } M)[[z, z^{-1}]] \\ a &\mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \end{aligned}$$

satisfying the following conditions (M1) ~ (M3): for any  $a, b \in V$  and  $u \in M,$

(M1)  $a_n u = 0$  for sufficiently large  $n \in \mathbb{Z},$

(M2)  $Y_M(\mathbf{1}, z) = \text{id}_M,$

(M3)  $z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(a, z_1) Y_M(b, z_2) - (-1)^{\bar{a}\bar{b}} z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(b, z_2) Y_M(a, z_1)$   
 $= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(a, z_0)b, z_2).$

An admissible  $V$ -module  $M$  is a weak  $V$ -module  $M$  which carries a  $(1/2)\mathbb{Z}_{\geq 0}$ -grading  $M = \bigoplus_{n \in (1/2)\mathbb{Z}_{\geq 0}} M_n$  subject to the conditions: for  $m \in \mathbb{Z}, n \in (1/2)\mathbb{Z}_{\geq 0}$  and homogeneous  $a \in V,$

$$a_m M_n \subseteq M_{\text{wt}(a)+n-m-1}.$$

Let  $M$  be a weak  $V$ -module such that  $L_0$  is semisimple on  $M$  and  $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$  the eigenspace decomposition with respect to  $L_0.$  If  $\dim M_\lambda < \infty$  for all  $\lambda \in \mathbb{C}$  and for fixed  $\lambda \in \mathbb{C}, M_{\lambda+n} = 0$  for sufficiently small  $n \in (1/2)\mathbb{Z}, M$  is called an (ordinary)  $V$ -module. An admissible  $V$ -module  $M$  is called simple if  $0$  and  $M$  are the only  $(1/2)\mathbb{Z}_{\geq 0}$ -graded admissible  $V$ -submodules.

DEFINITION 2.1 ([4]). A vertex operator superalgebra  $V$  is called rational if any admissible  $V$ -module is a direct sum of simple admissible  $V$ -modules. A vertex operator superalgebra  $V$  is called regular if any weak  $V$ -module is a direct sum of simple ordinary  $V$ -modules.

**2.2. Zhu’s algebra** We review the definition of Zhu’s algebra for a vertex operator superalgebra [9].

Let  $V = \bigoplus_{n \in (1/2)\mathbb{Z}} V_n$  be a vertex operator superalgebra. Let us define binary operations  $*, \circ : V \times V \rightarrow V$  as follows: for homogeneous  $a \in V$  and  $b \in V$ ,

$$a * b = \begin{cases} \operatorname{Res}_z Y(a, z) \frac{(1+z)^{\operatorname{wt}(a)}}{z} b = \sum_{i \geq 0} \binom{\operatorname{wt}(a)}{i} a_{i-1} b & (a, b \in V_0), \\ 0 & (a \text{ or } b \in V_1), \end{cases}$$

$$a \circ b = \begin{cases} \operatorname{Res}_z Y(a, z) \frac{(1+z)^{\operatorname{wt}(a)}}{z^2} b = \sum_{i \geq 0} \binom{\operatorname{wt}(a)}{i} a_{i-2} b & (a \in V_0, b \in V), \\ \operatorname{Res}_z Y(a, z) \frac{(1+z)^{\operatorname{wt}(a)-1/2}}{z} b = \sum_{i \geq 0} \binom{\operatorname{wt}(a)-1/2}{i} a_{i-1} b & (a \in V_1, b \in V). \end{cases}$$

We extend both operations  $*, \circ$  to  $V$  by linearity. Let  $O(V)$  be the linear span of elements of the form  $a \circ b$  in  $V$ . The space  $A(V)$  is defined by the quotient space  $V/O(V)$ . In the following, we set  $[a] = a + O(V) \in A(V)$  for  $a \in V$ .

For any homogeneous  $a \in V_0$  and  $b \in V$ , we have (cf. [9])

$$(2.1) \quad \operatorname{Res}_z \left( Y(a, z) \frac{(1+z)^{\operatorname{wt}(a)}}{z^{n+2}} b \right) \in O(V) \quad (n \geq 0).$$

If  $a \in V_1$ , then

$$a \circ \mathbf{1} = \operatorname{Res}_z Y(a, z) \frac{(z+1)^{\operatorname{wt}(a)-1/2}}{z} \mathbf{1} = a_{-1} \mathbf{1} = a \in O(V).$$

Since  $O(V)$  is a  $\mathbb{Z}_2$ -graded subspace, we see  $O(V) = O_0(V) + V_1$  where  $O_0(V) = O(V) \cap V_0$ . Thus we have  $A(V) = V_0/O_0(V)$ .

It follows from [9] that  $O(V)$  is a two-sided ideal of  $V$  with respect to  $*$  and the operation  $*$  induces an associative algebra structure on  $A(V)$ . Moreover the image  $[\mathbf{1}]$  of the vacuum in  $A(V)$  becomes the identity element of  $A(V)$ . We call the associative algebra  $A(V)$  Zhu’s algebra of  $V$ .

For any homogeneous  $a \in V_0$ , we denote  $o(a)$  by the weight 0 component operator  $a_{\operatorname{wt}(a)-1}$ . Clearly, for any admissible  $V$ -module  $M = \bigoplus_{n \in (1/2)\mathbb{Z}_{\geq 0}} M_n$ ,  $o(a)$  preserves each homogeneous space  $M_n$ . The action of operators  $o(a)$  on the lowest weight spaces of admissible  $V$ -modules leads to the following fundamental theorem (see Theorem 1.2, Theorem 1.3 of [9]):

- Theorem 2.1.** (1) *If  $M = \bigoplus_{n \in (1/2)\mathbb{Z}_{\geq 0}} M_n$  is an admissible  $V$ -module, then  $M_0$  is an  $A(V)$ -module under the action  $[a] \mapsto o(a)$  for  $a \in V_0$ .*  
 (2) *If  $W$  is an  $A(V)$ -module, then there exists an admissible  $V$ -module  $M = \bigoplus_{n \in (1/2)\mathbb{Z}_{\geq 0}} M_n$  such that  $M_0 \cong W$  as an  $A(V)$ -module.*  
 (3) *The map  $M \mapsto M_0$  gives a bijection between the set of inequivalent simple admissible  $V$ -modules and the set of inequivalent simple  $A(V)$ -modules.*

**3. Zhu's algebra  $A(V_L)$  for a rank one odd lattice  $L$**

**3.1. The structure of  $V_L$**  Let  $L = \mathbb{Z}\alpha$  be a rank one integral lattice with the symmetric nondegenerate bilinear form  $(\cdot|\cdot)$  given by  $(\alpha|\alpha) = k$ , where  $k$  is a positive odd integer.

Set  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$  and extend the bilinear form  $(\cdot|\cdot)$  on  $L$  to  $\mathfrak{h}$  by  $\mathbb{C}$ -linearity. Let  $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}K$  be the affinization of  $\mathfrak{h}$  regarding  $\mathfrak{h}$  as an abelian Lie algebra. Lie bracket on  $\hat{\mathfrak{h}}$  is given by

$$[t^m \otimes h, t^n \otimes h'] = m(h|h')\delta_{m+n,0}K, \quad [t^m \otimes h, K] = 0 \quad (h, h' \in \mathfrak{h}, m, n \in \mathbb{Z}).$$

Let  $\hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$ ,  $\mathfrak{b} = \mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}K$ , which are commutative subalgebras of  $\hat{\mathfrak{h}}$ . The relations  $K \cdot 1 = 1$ ,  $(\mathbb{C}[t] \otimes \mathfrak{h}) \cdot 1 = 0$  define the one dimensional module  $\mathbb{C}$  of  $\mathfrak{b}$ . We set  $M(1) = \text{Ind}_{\mathfrak{b}}^{\hat{\mathfrak{h}}} \mathbb{C}$  and denote this  $\hat{\mathfrak{h}}$ -module by  $\pi_1$ . Remark that

$$M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{b})} \mathbb{C} \cong S(\hat{\mathfrak{h}}^-) \quad \text{as a linear space,}$$

where  $U(\cdot)$  denotes the corresponding universal enveloping algebra and  $S(\cdot)$  the corresponding symmetric algebra.

Let  $\mathbb{C}[L]$  the group algebra of the additive group  $L$ . Thus  $\mathbb{C}[L]$  has a basis  $\{e^\beta\}_{\beta \in L}$ . The space  $\mathbb{C}[L]$  has a natural  $\hat{\mathfrak{h}}$ -module structure  $\pi_2$  by letting

$$\pi_2(K) = 0, \quad \pi_2(t^n \otimes h)e^\beta = \delta_{n,0}(h|\beta)e^\beta \quad (h \in \mathfrak{h}, n \in \mathbb{Z}, \beta \in L).$$

Let us define the  $\hat{\mathfrak{h}}$ -module structure on  $V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}[L]$  by  $\pi = \pi_1 \otimes 1 + 1 \otimes \pi_2$ .

Here we give the definition of the vertex operator  $Y(a, z)$  for  $a \in V_L$ . For  $h \in \mathfrak{h}$ , we set  $h_n = t^n \otimes h$  ( $n \in \mathbb{Z}$ ) and  $h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1}$ . Let  $a = \alpha_{-n_1} \cdots \alpha_{-n_r} \otimes e^\beta \in V_L$  ( $n_1, \dots, n_r \in \mathbb{Z}_{>0}, \beta \in L$ ). The vertex operator  $Y(a, z)$  is defined as

$$Y(a, z) = \circ \partial^{(n_1-1)} \alpha(z) \cdots \partial^{(n_r-1)} \alpha(z) \Gamma_\beta(z) \circ,$$

where

$$\partial^{(n)} = \frac{1}{n!} \frac{d^n}{dz^n}, \quad \Gamma_\beta(z) = e^\beta z^{\beta_0} \exp\left(\sum_{n>0} \frac{\beta_{-n} z^n}{n}\right) \exp\left(\sum_{n>0} \frac{\beta_n z^{-n}}{-n}\right),$$

and  $e^\beta$  is the operator of left multiplication by  $1 \otimes e^\beta$ . The normal ordering procedure  $\circ \cdot \circ$  follows the definition in [8].

**Theorem 3.1** ([6]).  $V_L$  is a simple vertex operator superalgebra with the vacuum vector  $\mathbf{1} = 1 \otimes 1$  and the Virasoro element  $\omega = (1/(2k)) \alpha_{-1}^2 \mathbf{1}$ .

Note that for  $a = \alpha_{-n_1} \cdots \alpha_{-n_r} \otimes e^\beta (= \alpha_{-n_1} \cdots \alpha_{-n_r} e^\beta \mathbf{1}) \in V_L$  ( $n_1, \dots, n_r \in \mathbb{Z}_{>0}$ ,  $\beta \in L$ ), its weight is

$$\text{wt}(a) = \sum_{i=1}^r n_i + \frac{1}{2}(\beta|\beta).$$

We next discuss modules for the vertex operator superalgebra  $V_L$ . Let  $L^\circ \supset L$  be the dual lattice of  $L$ . Then one has the coset decomposition

$$L^\circ = \cup_{i \in I_k} \left( L + \frac{i}{k} \alpha \right),$$

where  $I_k = \{0, \pm 1, \dots, (k-1)/2\}$ . Let  $\mathbb{C}[\mathfrak{h}]$  be the group algebra and set  $\mathbb{C}[S] = \bigoplus_{\beta \in S} \mathbb{C}e^\beta$  for any subset  $S$  of  $\mathfrak{h}$ . We define the vector space

$$V(i) = M(1) \otimes_{\mathbb{C}} \mathbb{C} \left[ L + \frac{i}{k} \alpha \right]$$

for  $i \in I_k$ .

**Theorem 3.2** ([6]).  $V(i)$  ( $i \in I_k$ ) are inequivalent simple  $V_L$ -modules.

**3.2. Zhu's algebra  $A(V_L)$**  Now we state one of the main results in this paper.

**Theorem 3.3.** *Zhu's algebra  $A(V_L)$  of the vertex operator superalgebra  $V_L$  is isomorphic to the following quotient algebra of the polynomial ring  $\mathbb{C}[x]$ :*

$$A(V_L) \cong \mathbb{C}[x]/(F_k(x)),$$

where  $F_k(x) = \prod_{n \in I_k} (x - n)$ .

As a corollary, we have

**Corollary 3.1.** *The set of the simple modules  $\{V(i)\}_{i \in I_k}$  gives the complete list of the simple  $V_L$ -modules.*

The proof of Theorem 3.3 is given after we establish several lemmas.

Let  $n \geq 0$ ,  $a \in V_L$ . Note that

$$\text{Res}_z \left( Y(\alpha_{-1} \mathbf{1}, z) \frac{1+z}{z^{n+2}} a \right) \in O(V_L),$$

by (2.1). Thus we have  $\alpha_{-n-1}a + \alpha_{-n-2}a \equiv 0 \pmod{O(V_L)}$ , and then

$$(3.2) \quad \alpha_{-n}a \equiv (-1)^{n-1}\alpha_{-1}a \pmod{O(V_L)} \quad (n \geq 1, a \in V_L).$$

Let  $p_n(x_1, x_2, \dots)$  be the elementary Schur polynomials

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{n=0}^{\infty} p_n(x_1, x_2, \dots) y^n,$$

and  $p_n(x_1, x_2, \dots) = 0$  for  $n \in \mathbb{Z}_{<0}$ . For any operator  $x$ , we define

$$\binom{x}{n} = \begin{cases} \frac{1}{n!} x(x-1)\cdots(x-n+1) & (n \in \mathbb{Z}_{>0}), \\ 1 & (n = 0), \\ 0 & (n \in \mathbb{Z}_{<0}). \end{cases}$$

Then one can easily see that

$$(3.3) \quad p_n(x, -x, x, -x, \dots) = \binom{x}{n}.$$

Let  $\beta, \gamma \in L, i \in \mathbb{Z}$  and  $\Gamma_\beta(z) = \sum_{i \in \mathbb{Z}} e_i^\beta z^{-i-1}$ . Since

$$\begin{aligned} \Gamma_\beta(z)e^\gamma \mathbf{1} &= z^{(\beta|\gamma)} \exp\left(\sum_{n=1}^{\infty} \frac{\beta_{-n}}{n} z^n\right) e^{\beta+\gamma} \mathbf{1} \\ &= \sum_{n=0}^{\infty} p_n(\beta_{-1}, \beta_{-2}, \dots) e^{\beta+\gamma} \mathbf{1} z^{n+(\beta|\gamma)}, \end{aligned}$$

(3.2) and (3.3) show that

$$(3.4) \quad \begin{aligned} e_i^\beta e^\gamma \mathbf{1} &= p_{-(\beta|\gamma)-i-1}(\beta_{-1}, \beta_{-2}, \dots) e^{\beta+\gamma} \mathbf{1} \\ &\equiv p_{-(\beta|\gamma)-i-1}(\beta_{-1}, -\beta_{-1}, \dots) e^{\beta+\gamma} \mathbf{1} \\ &\equiv \binom{\beta_{-1}}{-(\beta|\gamma)-i-1} e^{\beta+\gamma} \mathbf{1} \pmod{O(V_L)}. \end{aligned}$$

From the definition of the operation  $*$ , we have  $\alpha_{-1}\mathbf{1} * a = \alpha_{-1}a + \alpha_0a$  ( $a \in V_L$ ) and then

$$(3.5) \quad \alpha_{-1}^n \mathbf{1} = \underbrace{\alpha_{-1}\mathbf{1} * \cdots * \alpha_{-1}\mathbf{1}}_{n\text{-th}} = (\alpha_{-1}\mathbf{1})^{*n} \quad (n \geq 1).$$

- Lemma 3.1.** (1)  $\binom{\alpha_{-1}+(k-1)/2}{k} \mathbf{1} \equiv 0 \pmod{O(V_L)}$ .  
 (2)  $A(V_L)$  is spanned by vectors  $[\alpha_{-1}^n \mathbf{1}]$  where  $n \geq 0$ .

Proof. (1) Since  $e^\alpha \mathbf{1}$  is an odd element, we have

$$\begin{aligned} 0 &\equiv e^\alpha \mathbf{1} \circ e^{-\alpha} \mathbf{1} \equiv \sum_{i \geq 0} \binom{(k-1)/2}{i} \binom{\alpha_{-1}}{k-i} \mathbf{1} \\ &\equiv \binom{\alpha_{-1}+(k-1)/2}{k} \mathbf{1} \pmod{O(V_L)}, \end{aligned}$$

by (3.4) and the formula

$$\sum_{i=0}^n \binom{n}{i} \binom{x}{m-i} = \binom{x+n}{m} \quad (m \geq n \geq 0, m, n \in \mathbb{Z}_{\geq 0}, \text{ and } x: \text{ an operator}).$$

(2) Let  $2L = \{2n\alpha | n \in \mathbb{Z}\}$ . Then  $2L$  is a sublattice of  $L$ . Since the vertex operator algebra  $V_{2L}$  for the lattice  $2L$  is a vertex operator subalgebra of  $V_L$ , we have a homomorphism  $\nu : A(V_{2L}) \rightarrow A(V_L)$ . The map  $\nu$  is surjective as  $[e^{n\alpha} \mathbf{1}] = 0$  in  $A(V_L)$  for any odd integer  $n$ .

Therefore Theorem 3.2 of [5] shows  $A(V_L)$  is generated by  $[e^{2\alpha} \mathbf{1}]$ ,  $[e^{-2\alpha} \mathbf{1}]$  and  $[\alpha_{-1} \mathbf{1}]$ .

Let  $m \neq 0$ . We note that

$$\alpha_{-1} \mathbf{1} * e^{2m\alpha} \mathbf{1} = (\alpha_{-1} + \alpha_0) e^{2m\alpha} \mathbf{1} \equiv k m e^{2m\alpha} \mathbf{1} \pmod{O(V_L)}$$

as  $\alpha_{-1} e^{2m\alpha} \mathbf{1} \equiv -k m e^{2m\alpha} \mathbf{1} \pmod{O(V_L)}$ . Then, by (1), we have

$$\begin{aligned} 0 &\equiv \binom{\alpha_{-1}+(k-1)/2}{k} \mathbf{1} * e^{2m\alpha} \mathbf{1} \equiv \binom{[\alpha_{-1} \mathbf{1}] + (k-1)/2}{k} * e^{2m\alpha} \mathbf{1} \\ &\equiv \binom{k m + (k-1)/2}{k} e^{2m\alpha} \mathbf{1} \pmod{O(V_L)}. \end{aligned}$$

Thus we see that  $e^{2m\alpha} \mathbf{1} \equiv 0 \pmod{O(V_L)}$ .

Since  $e^{2\alpha} \mathbf{1} \equiv 0$  and  $e^{-2\alpha} \mathbf{1} \equiv 0 \pmod{O(V_L)}$ ,  $A(V_L)$  is generated by  $[\alpha_{-1} \mathbf{1}]$ . Therefore it implies this lemma. □

It follows from Lemma 3.1 (1) that we have a relation

$$(3.6) \quad 0 = n! \binom{[\alpha_{-1} \mathbf{1}] + (k-1)/2}{k} = F_k([\alpha_{-1} \mathbf{1}])$$

in  $A(V_L)$ . Now we can prove Theorem 3.3. Let  $\phi$  be the  $\mathbb{C}$ -linear map defined by

$$\phi : \mathbb{C}[x] \rightarrow A(V_L)$$



$$x^n \mapsto [\alpha_{-1}\mathbf{1}]^{*n}.$$

Then the map  $\phi$  is a homomorphism of an associative algebra by (3.5) and it is surjective by Lemma 3.1 (2). From (3.6), it is enough to show that  $\ker \phi$  is generated by  $F_k([\alpha_{-1}\mathbf{1}])$ . Suppose  $\ker \phi \neq (F_k([\alpha_{-1}\mathbf{1}]))$ , then  $\dim_{\mathbb{C}} \mathbb{C}[x]/(F_k(x)) < k$ . Therefore, the number of the simple modules of the associative algebra  $A(V_L)$  must be strictly less than  $k$ , which gives a contradiction as  $A(V_L)$  has the  $k$  inequivalent simple modules which correspond to the simple modules  $V(i)$  ( $i \in I_k$ ) of  $V_L$  by Theorem 2.1.  $\square$

**3.3. Regularity of  $V_L$**  The regularity of  $V_L$  can be shown in the same way given in [4], in which the regularity of lattice vertex operator algebras associated to even lattices is proved. However, in order to describe the difference between odd and even cases, we will give the outline of the proof of the regularity of  $V_L$ .

The following lemma is fundamental in our proof, whose proof is suggested by C. Dong (one can see the same statement in the proof of Lemma 3.15 of [4]. Also see [3], Proposition 11.9):

**Lemma 3.2.** *Let  $V$  be a vertex operator superalgebra and  $M$  be a nonzero weak  $V_L$ -module. If  $V$  is simple, then for any nonzero vectors  $a \in V$  and  $u \in M$ ,  $Y(a, z)u \neq 0$ .*

*Proof.* Set  $I = \{b \in V | Y(b, z)u = 0\}$ . Suppose that  $Y(a, z)u = 0$ , then  $I \neq \{0\}$ . First of all, we prove that  $I$  is an ideal of  $V$ . By the associativity, which is a result of the Jacobi identity, we have for any  $b \in V, c \in I$  and some  $m \in \mathbb{Z}_{>0}$ ,

$$(z_0 + z_2)^m Y(Y(b, z_0)c, z_2)u = (z_0 + z_2)^m Y(b, z_0 + z_2)Y(c, z_2)u = 0$$

as  $Y(c, z)u = 0$ . Thus we have  $Y(Y(b, z_0)c, z_2)u = 0$ , which implies  $b_n c \in I$  for all  $n \in \mathbb{Z}$ . Since  $V$  is simple, we see  $I = V$ . Therefore,  $Y(b, z)u = 0$  for all  $b \in V$ . This gives us a contradiction because  $Y(\mathbf{1}, z)u = u \neq 0$ .  $\square$

Now we return to the case of  $V = V_L$ . For any nonzero weak  $V_L$ -module  $M$ , let us define the vacuum space of  $M$  by

$$\Omega_M = \{u \in M | \alpha_n u = 0 \text{ for } n > 0\}.$$

Using Lemma 3.2, we can prove that  $\Omega_M \neq 0$  by argument similar to Lemma 3.15 of [4]. For  $\beta \in L$ , the following operator on  $M$  is called the Z-operator (cf. [2]):

$$\begin{aligned} Z(\beta, z) &= \exp\left(\sum_{n>0} \frac{\beta_{-n} z^n}{-n}\right) \Gamma_{\beta}(z) \exp\left(\sum_{n>0} \frac{\beta_n z^{-n}}{n}\right) \\ &= \sum_{n \in \mathbb{Z}} Z(\beta, n) z^{-n-1}. \end{aligned}$$

For the odd lattice  $L$ , the following identities are proved in the same way as in [2]:

$$(3.7) \quad [\beta_m, Z(\gamma, n)] = \delta_{m,0}(\beta|\gamma)Z(\gamma, n),$$

$$(3.8) \quad Z(\beta, n)\beta_0 = (-n - 1)Z(\beta, n)$$

for  $\beta, \gamma \in L, m, n \in \mathbb{Z}$ . Lemma 3.2, (3.7) and (3.8) show:

**Lemma 3.3.** *Let  $M$  be a nonzero weak  $V_L$ -module. Then there exist a nonzero vector  $w \in \Omega_M$  and  $\lambda \in L^\circ$  such that*

$$\beta_0 w = \lambda(\beta)w \quad \text{for any } \beta \in \mathfrak{h}.$$

*Proof.* Let  $u$  be a nonzero vector of  $\Omega_M$ . By Lemma 3.2, we have  $\Gamma_\alpha(z)u \neq 0$ . Since

$$\Gamma_\alpha(z) = \exp\left(\sum_{n>0} \frac{\alpha_{-n}z^n}{n}\right) Z(\alpha, z) \exp\left(\sum_{n>0} \frac{\alpha_n z^{-n}}{-n}\right),$$

$Z(\alpha, z)u \neq 0$ , i.e.,  $Z(\alpha, n)u \neq 0$  for some  $n \in \mathbb{Z}$ . Then, from (3.7) and (3.8), we obtain  $Z(\alpha, n)u \in \Omega_M$  and

$$\begin{aligned} \alpha_0 Z(\alpha, n)u &= ([\alpha_0, Z(\alpha, n)] + Z(\alpha, n)\alpha_0)u \\ &= ((\alpha|\alpha) - n - 1)Z(\alpha, n)u \\ &= (k - n - 1)Z(\alpha, n)u. \end{aligned}$$

Put  $w = Z(\alpha, n)u$ . Let  $\lambda \in \mathfrak{h}^*$  such that  $\alpha_0 w = \lambda(\alpha)w$ . By the nondegenerate form  $(\cdot|\cdot)$  on  $\mathfrak{h}$ , we have

$$\lambda = \frac{k - n - 1}{k} \alpha$$

under the identification  $\mathfrak{h}^*$  with  $\mathfrak{h}$ . We see that  $(\lambda|\beta) \in \mathbb{Z}$  for any  $\beta \in L$ , and then  $\lambda \in L^\circ$ .  $\square$

Let  $w$  be a nonzero vector of  $\Omega_M$  subject to the condition in Lemma 3.3. It is not difficult to prove that the  $V_L$ -submodule generated by  $w$  is simple and isomorphic to  $V(i)$  for some  $i \in I_k$ . In particular, if  $M$  is a simple weak  $V_L$ -module, then  $M$  is isomorphic to  $V(i)$  for some  $i \in I_k$ .

**REMARK 3.1.** The proof of the fact that  $\lambda \in L^\circ$  given in [2] does not work for our odd lattice (see the proof of Lemma 3.5 of [2]). However, the proof of Lemma 3.4 of [2] implicitly shows this as we have seen in Lemma 3.3.

We prove the following theorem in the same way as in Theorem 3.16 of [4].

**Theorem 3.4.** *The vertex operator superalgebra  $V_L$  is regular. In particular,  $V_L$  is rational.*

Proof. It suffices to prove that any weak  $V_L$ -module is completely reducible, since any simple  $V_L$ -module is an ordinary  $V_L$ -module. Let  $M$  be a nonzero weak  $V_L$ -module. Let  $W$  be the sum of all simple ordinary  $V_L$ -submodules in  $M$ . Suppose  $M' = M/W \neq 0$ . Let  $u (\neq 0) \in \Omega_M \setminus \Omega_W$ . It follows from Lemma 3.2 that there exists  $n \in \mathbb{Z}$  such that  $Z(\alpha, n)(u + W) \neq 0$  in  $M'$ , and then  $Z(\alpha, n)u \notin W$ . Taking  $w = Z(\alpha, n)u$ , we see that  $w (\neq 0) \in \Omega_M \setminus \Omega_W$ ,

$$\beta_0 w = \lambda(\beta)w \quad \text{for any } \beta \in \mathfrak{h},$$

and  $\lambda \in L^\circ$  from Lemma 3.3. Then the  $V_L$ -submodule generated by  $w$  is simple and is not contained in  $W$ . This gives us a contradiction.  $\square$

**3.4. Applications: Rationality of vertex operator superalgebras associated to the charged free fermions and the  $N = 2$  superconformal algebra** It is known that, if  $(\alpha|\alpha) = 1$ , then  $V_L$  is isomorphic to the charged free fermions  $F$  [8], and if  $(\alpha|\alpha) = 3$ , then  $V_L$  is isomorphic to the  $N = 2$  superconformal vertex algebra with central charge  $C = 1$  [8]. The rationality of the charged free fermions  $F$  is shown in Theorem 4.1 of [9].

Let us consider the  $N = 2$  superconformal algebra. It is a graded superalgebra spanned by the basis  $L_n, T_n, G_r^\pm, C \{n \in \mathbb{Z}, r \in \mathbb{Z} + (1/2)\}$ , and has (anti)-commutation relations given by

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{m^3 - m}{12} C \delta_{m+n,0}, \\ [L_m, G_r^\pm] &= \left(\frac{1}{2}m - r\right) G_{m+r}^\pm, \\ [L_m, T_n] &= -nT_{m+n}, \\ [T_m, T_n] &= \frac{m}{3} C \delta_{m+n,0}, \\ [T_m, G_r^\pm] &= \pm G_{m+r}^\pm, \\ \{G_r^+, G_s^-\} &= 2L_{r+s} + (r - s)T_{r+s} + \frac{C}{3} \left(r^2 - \frac{1}{4}\right) \delta_{r+s,0}, \\ [L_m, C] &= [T_n, C] = [G_r^\pm, C] = \{G_r^+, G_s^+\} = \{G_r^-, G_s^-\} = 0 \end{aligned}$$

for all  $m, n \in \mathbb{Z}$  and  $r, s \in \mathbb{Z} + (1/2)$ . Given complex numbers  $h, q$  and  $c$ , we denote the Verma module generated by the highest weight vector  $|h, q, c\rangle$  with  $L_0$  eigenvalue  $h$ ,  $T_0$  eigenvalue  $q$  and central charge  $c$  by  $\mathcal{M}(h, q, c)$ . Note that the highest weight vector  $|h, q, c\rangle$  is annihilated by  $L_n, T_n$ , and  $G_r^\pm$  for  $n \in \mathbb{Z}_{>0}, r \in \mathbb{Z}_{\geq 0} + (1/2)$ . It follows

from [8] that the vertex algebra  $\mathcal{M}(0, 0, c)$  has a unique simple quotient  $L(0, 0, c)$  and if  $(\alpha|\alpha) = 3$ , then the lattice vertex operator superalgebra  $V_L$  is isomorphic to the  $N = 2$  superconformal vertex algebra  $L(0, 0, 1)$ . The classification results of its simple modules are given by the Kac determinant of the  $N = 2$  superconformal algebra in [1].

As a consequence of a particular case  $k = 3$  of Corollary 3.1 and Theorem 3.4, we have

**Theorem 3.5.** *The  $N = 2$  superconformal vertex algebra  $L(0, 0, 1)$  is rational, and its simple modules are only  $L(0, 0, 1)$  and  $L(1/6, \pm 1/3, 1)$ .*

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