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ON CONGRUENT HOLOMORPHIC MAPPINGS INTO A HERMITIAN SYMMETRIC SPACE

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1. Introduction

It is one of basic problems in differential geometry to find geometric conditions that determine the congruence class of a submanifold of a given manifold. For example, a general hypersurface of n-dimensional Euclidean space $\mathbf{R}^n$ is determined up to congruence by its first and second fundamental forms. It is another classical result that for a certain generic hypersurfaces of $\mathbf{R}^n$ ($n \geq 4$), the first fundamental form alone is sufficient to determine the congruence class. On the other hand, we know some remarkable facts in the complex-analytic case. First, any complex submanifold of a complex space form has the metric rigidity, i.e., it is determined up to congruence by its first fundamental form alone (the rigidity theorem of Calabi [1]). Secondly, Green [2] asserts even if the ambient space S is a general Kähler manifold with real-analytic Kähler metric, a certain generic holomorphic mapping (a "nondegenerate" mapping in his terminology) into it has the local metric rigidity: There is actually shown the existence of such a finite-dimensional family of local real-analytic hypersurfaces of S that a holomorphic mapping into S has the local metric rigidity unless its image lies in any of them. Only one cannot find any geometric conditions that assure the nondegeneracy for a mapping.

In this article, we shall give local differential-geometric conditions that determine the congruence class of a holomorphic mapping into a Hermitian symmetric space under considerably general settings. For a technical reason, we have to restrict ourselves to the case where mappings are *infinitesimally full* in our sense. In particular we cannot deal with totally geodesic complex submanifolds. But our results extend Calabi's rigidity theorem to a direction different from [2], since our "infinitesimally full" condition is equivalent to the "full" condition used in [1] in the case of complex space forms.

2. Preliminaries

Let S be a simply connected, complex N-dimensional Hermitian symmetric space with metric tensor g. Denote by TS, T^*S, and T^r_sS the tangent bundle, cotangent bundle, and tensor bundle of type (r, s) of S respectively. Their complexifications will be denoted by $\mathbf{T}S$, $\mathbf{T}^*S$, and $\mathbf{T}^r_sS$ respectively. As usual, $\mathbf{T}S$ splits into a direct sum of its complex vector subbundles T^+S and T^-S according to the eigenvalues $\pm\sqrt{-1}$ of the complex structure J of S respectively. The set of complex vector fields on S will be denoted by $\mathbf{Z}(S)$. Throughout this paper, we always assume that the Riemannian metric g, curvature tensor R, and the Levi-Civita connection ∇ of S are complex-linearly extended. So we define the Christoffel symbols Γ^r_{st} by

$$(2.1) \qquad \left(\nabla_{\partial/\partial w^s}\partial/\partial w^t\right)_q = \sum_{r=1}^{N} \Gamma^r_{st}(q)\,(\partial/\partial w^r)_q, \quad (q \in \tilde{U}, 1 \leq s, t \leq N),$$

where $(\tilde{U}; w^1, \ldots, w^N)$ is a local holomorphic coordinate system valid in an open set $\tilde{U}$ of S.

Now let M be a complex n-dimensional connected complex manifold and f a holomorphic mapping of M into S. Let V be an open set of M. A *complex tensor field Q along f over V of type (r, s)* is a smooth cross section of the induced bundle $\pi^r_s : f|^*_V\mathbf{T}^r_sS \to V$, where $f|_V$ denotes the restriction of f to V. It may be considered in a usual way as a smooth mapping $p \mapsto Q_p$ from V into $\mathbf{T}^r_sS$ such that $\pi^r_s(Q_p) = f(p)$ for any $p \in V$. Q will be called a complex vector field (resp. complex 1-form) along f if $(r, s) = (0, 1)$ (resp. $(r, s) = (1, 0)$). The set of complex vector fields along f is denoted by $\mathbf{Z}_f(V)$, while the set of complex vector fields on V by $\mathbf{Z}(V)$. Further, we denote by $\mathbf{Z}^+_f(V)$ (resp. $\mathbf{Z}^-_f(V)$) the set of elements in $\mathbf{Z}_f(V)$ which take the values in T^+S (resp. T^-S). An element $Z \in \mathbf{Z}_f(V)$ is called holomorphic if $Z \in \mathbf{Z}^+_f(V)$ and if it is holomorphic as a mapping into T^+S. Let $(U; z^1, \ldots, z^n)$ be a local holomorphic coordinate system valid in an open set U of M. Then each $f_*(\partial/\partial z^i)$ is an element of $\mathbf{Z}^+_f(U)$ and holomorphic. If $f(U) \subset \tilde{U}$, the mapping $p \mapsto (\partial/\partial w^r)_{f(p)}$ is an holomorphic element of $\mathbf{Z}^+_f(U)$, which we denote by $(\partial/\partial w^r)_f$. Similarly $(\partial/\partial \bar{w}^r)_f$ means an element of $\mathbf{Z}^-_f(U)$. Then every $Z \in \mathbf{Z}_f(U)$ can be uniquely expressed as

$$(2.2) \qquad Z = \sum_{r=1}^{N} Z^r\,(\partial/\partial w^r)_f + \sum_{r=1}^{N} Z^{\bar{r}}\,(\partial/\partial \bar{w}^r)_f,$$

Z^r and $Z^{\bar{r}}$ being complex-valued smooth functions on U. Obviously, Z is holomorphic if and only if $Z^{\bar{r}} = 0$ and Z^r is holomorphic in U for each r.

The holomorphic mapping f gives arise to a covariant differentiation D along f that is induced from the Levi-Civita connection of S. It is given in terms of local coordinate systems as follows: Let $(U; z^1, \ldots, z^n)$ a local holomorphic coordinate system in an open set U of M such that $f(U) \subset \tilde{U}$. If $Z \in \mathbf{Z}^+_f(U)$ and $Z = \sum_{r=1}^{N} Z^r\,(\partial/\partial w^r)_f$, then

$$\left(D_{\partial/\partial z^j}Z\right)_p = \sum_{r=1}^{N}\left\{\frac{\partial Z^r}{\partial z^j}(p) + \sum_{s,t=1}^{N}\Gamma^r_{st}(f(p))\frac{\partial f^s}{\partial z^j}(p)Z^t(p)\right\}(\partial/\partial w^r)_{f(p)}$$

$$\left(D_{\partial/\partial\bar{z}^j}Z\right)_p = \sum_{r=1}^{N}\frac{\partial Z^r}{\partial\bar{z}^j}(p)(\partial/\partial w^r)_{f(p)}.$$

In the following lemma we summarize some basic properties of D that are needed in later sections.

Lemma 2.1. *Let $X, X_1, X_2 \in \mathbf{Z}(V)$ and $Z, Z_1, Z_2, Z_3 \in \mathbf{Z}_f(V)$.*

(i)

$$\begin{aligned} D_{X_1}f_*X_2 - D_{X_2}f_*X_1 - f_*[X_1, X_2] &= 0, \\ D_{X_1}D_{X_2}Z - D_{X_2}D_{X_1}Z - D_{[X_1,X_2]}Z &= R(f_*X_1, f_*X_2)Z. \end{aligned}$$

(ii) *D_X commutes with the complex structure J of S. In particular, if $Z \in \mathbf{Z}_f^+(V)$ (resp. $Z \in \mathbf{Z}_f^-(V)$), then $D_XZ \in \mathbf{Z}_f^+(V)$ (resp. $D_XZ \in \mathbf{Z}_f^-(V)$).*

(iii) *$D_{\partial/\partial\bar{z}^j}Z$ vanishes if Z is holomorphic.*

(iv) *D is real: $\overline{D_XZ} = D_{\overline{X}}\overline{Z}$, where the bars over expressions denote the complex conjugation.*

(v) *D leaves both the metric tensor g and curvature tensor R of S invariant:*

$$\begin{aligned} Xg(Z_1, Z_2) &= g(D_XZ_1, Z_2) + g(Z_1, D_XZ_2) \\ D_XR(Z_1, Z_2)Z_3 &= R(D_XZ_1, Z_2)Z_3 + R(Z_1, D_XZ_2)Z_3 + R(Z_1, Z_2)D_XZ_3. \end{aligned}$$

(vi) *If ψ is a complex 1-form along f over V, then*

$$X\psi(Z) = D_X\psi(Z) + \psi(D_XZ).$$

(vii) *Let F be a holomorphic and isometric transformation of S. Set $f' = F \circ f$ and denote by D' the covariant differentiation along f'. Then $F_*D_XZ = D'_XF_*Z$.*

Here we make a further agreement on notation: Let $(U; z^1, \ldots, z^n)$ be a local holomorphic coordinate system of M and f a holomorphic mapping of M into S. We write $f_j = f_*(\partial/\partial z^j)$, $D_j = D_{\partial/\partial z^j}$, $D_{\bar{j}} = D_{\partial/\partial\bar{z}^j}$. For a positive integer a, *a multi-index I of order a* is an a-tuple of integers $(i_1, i_2, \ldots, i_a)$ with $1 \le i_1, i_2, \ldots, i_a \le n$. The order of a multi-index I will be often denoted by $|I|$. We denote by $\mathcal{I}^a$ the set of multi-indices of order less than or equal to a. If $I = (i_1, \ldots, i_a)$, we write $D_I = D_{i_1}\cdots D_{i_a}$, $D_If = f_I = D_{i_1}\cdots D_{i_{a-1}}f_{i_a}$, $\partial_I\varphi = \partial_{i_1}\cdots\partial_{i_a}\varphi = \partial^a\varphi/\partial z^{i_1}\ldots\partial z^{i_a}$, φ being a complex-valued smooth function on U.

3. A Congruence Theorem for Holomorphic Mappings

Let o be a point of M and $(U; z^1, \dots, z^n)$ a local holomorphic coordinate system around o. We set $f(o) = \tilde{o}$. Let $T^+_{\tilde{o}}S = (\pi^+)^{-1}(\tilde{o})$, where π^+ is the projection $T^+S \to S$. For a positive integer a, we define the a-th complex osculating space $O^a_f(o)$ to f at o as the complex linear subspace of $T^+_{\tilde{o}}S$ spanned by all the $\left(D_I f\right)_o$ with $I \in \mathcal{I}^a$. We set $O^0_f(o) = 0$. If $O^d_f(o) = T^+_{\tilde{o}}S$ and $O^{d-1}_f(o) \neq T^+_{\tilde{o}}S$ for a positive integer d, f is said to be *infinitesimally full of order d at o*. Further, f is said to be *infinitesimally full* if there exists a positive integer d and a point $o \in M$ such that f is infinitesimally full of order d at o.

The following theorem is due to E. Calabi.

Theorem 3.1 (Calabi). *Let f and f' be holomorphic imbeddings of a connected complex manifold M into a complex space form S with metric tensor g. If f is full and if $f^*g = f'^*g$ on M, then f' is also full and there exists uniquely a holomorphic and isometric transformation F of S such that $F \circ f = f'$.*

The following theorem is our congruence theorem for holomorphic mappings into a simply connected Hermitian symmetric space.

Theorem 3.2. *Let f and f' be holomorphic mappings of a connected complex manifold M into a simply connected Hermitian symmetric space S with metric tensor g. Let R be the Riemannian curvature of S. Denote by D and D' the covariant differentiations along f and f' respectively. Let $(U; z^1, \dots, z^n)$ be a local holomorphic coordinate system around a point $o \in M$. Suppose that f is infinitesimally full of order d at o. Moreover, suppose that* (i) $f^*g = f'^*g$ *on U,* (ii) $R(D_i f, \overline{D_{I_1} f}, D_{I_2} f, \overline{D_{I_3} f}) = R(D'_i f', \overline{D'_{I_1} f'}, D'_{I_2} f', \overline{D'_{I_3} f'})$ *on U* $(1 \leq i \leq n, |I_1| \leq d-1, |I_2| \leq d, |I_3| \leq d-1)$, *and* (iii) $R(D_{I_1} f, \overline{D_{I_2} f}, D_{I_3} f, \overline{D_{I_4} f})_o = R(D'_{I_1} f', \overline{D'_{I_2} f'}, D'_{I_3} f', \overline{D'_{I_4} f'})_o$ $(|I_\nu| \leq d, 1 \leq \nu \leq 4)$. *Then f' is also infinitesimally full of order d at o and there exists uniquely a holomorphic and isometric transformation F such that $F \circ f = f'$.*

REMARK 3.1. When S is a complex space form, an imbedding f of M into S is said to be full if there exists no proper, totally geodesic complex submanifold of S including $f(M)$(cf. [1]). In Theorem 3.1, one can replace the condition that f is full by the one that it is infinitesimally full because both conditions are equivalent in this case, as will be shown in the last section.

Now let f and f' be holomorphic mapping of M into S. In order to prove Theorem 3.2, we may assume that $f(o) = f'(o)$, because the group of holomorphic and isometric transformations of S is transitive and the condition of the theorem does not change under the group. Let $(\tilde{U}; w^1, \dots, w^N)$ be a local holomorphic coordinate system around $f(o) = \tilde{o}$. We may assume that $f(U) \subset \tilde{U}$ and $f'(U) \subset \tilde{U}$, shrinking U if it is necessary. Set $f^r = w^r \circ f$ and $f'^r = w^r \circ f'$ for $r = 1, 2, \dots, N$.

Proposition 3.1. *The holomorphic mapping f coincides with f' on M if and only if $f(o) = f'(o)$ and $(D_I f)_o = (D'_I f')_o$ for all the multi-indices I.*

For the proof, we need lemmas.

Lemma 3.1. *Let c be a positive integer. If $f(o) = f'(o)$ and $\partial_I f^r(o) = \partial_I f'^r(o)$ for any $r = 1, \dots, N$ and $I \in \mathcal{I}^c$, then $(D_I(\partial/\partial w^r)_f)_o = (D'_I(\partial/\partial w^r)_{f'})_o$ for any $r = 1, \dots, N$ and $I \in \mathcal{I}^c$.*

Proof. By successively differentiating both sides of

$$(D_i(\partial/\partial w^r)_f)_p = \sum_{s,t} \Gamma^t_{sr}(f(p)) \frac{\partial f^s}{\partial z^i}(p)(\partial/\partial w^t)_{f(p)},$$

we see that $(D_I(\partial/\partial w^r)_f)_o$ is uniquely determined by the values

$$\frac{\partial^a \Gamma^t_{sr}}{\partial w^{s_1} \dots \partial w^{s_a}}(f(o)), \quad \frac{\partial^b f^r}{\partial z^{j_1} \dots \partial z^{j_b}}(o), \quad (\partial/\partial w^r)_{f(o)}$$

such that $0 \le a \le |I| - 1$ and $0 \le b \le |I|$. By the condition, all the values above are common for f'. This means $(D_I(\partial/\partial w^r)_f)_o = (D'_I(\partial/\partial w^r)_{f'})_o$. □

Lemma 3.2. *Let c be a positive integer. Suppose that $f(o) = f'(o)$ and $(D_I f)_o = (D'_I f')_o$ for any $I \in \mathcal{I}^c$. Then $\partial_I f^r(o) = \partial_I f'^r(o)$ for any $r = 1, \dots, N$ and $I \in \mathcal{I}^c$.*

Proof. We proceed by induction on c. The assertion is obvious if $c = 1$. Assume that we have shown our assertion for positive integer less than c. If $I = (i_1, \dots, i_c)$, we have

$$\begin{aligned} (D_I f)_o &= (D_{i_1} \cdots D_{i_{c-1}} \textstyle\sum_r f^r_{i_c} (\partial/\partial w^r)_f)_o \\ &= \textstyle\sum_r \partial_{i_1} \cdots \partial_{i_{c-1}} f^r_{i_c}(o)(\partial/\partial w^r)_{f(o)} \\ &\quad + \textstyle\sum_r \sum^* \partial_{i_{\sigma(1)}} \cdots \partial_{i_{\sigma(a)}} f^r_{i_c}(o)(D_{i_{\tau(1)}} \cdots D_{i_{\tau(b)}} (\partial/\partial w^r)_f)_o, \end{aligned} \tag{3.1}$$

where the summation $\sum^*$ is taken over certain $\sigma(1), \dots, \sigma(a)$ and $\tau(1), \dots, \tau(b)$ such that $0 \le a \le c - 2$, $1 \le b \le c - 1$, and $a + b = c - 1$. Equation (3.1) is also valid for f' and D' if we replace f by f' and D by D'.

Now suppose that $f(o) = f'(o)$ and $(D_I f)_o = (D'_I f')_o$ for any $I \in \mathcal{I}^c$. By the assumption of induction, we have in particular

$$\partial_K f^r(o) = \partial_K f'^r(o) \qquad (1 \le r \le N, K \in \mathcal{I}^{c-1}),$$

which implies by Lemma 3.1 $(D_K(\partial/\partial w^r)_f)_o = (D'_K(\partial/\partial w^r)_{f'})_o$ $(K \in \mathcal{I}^{c-1})$. Comparing the above identity (3.1) for f with the corresponding one for f', we have

$$\partial_{i_1} \cdots \partial_{i_{c-1}} f^r_{i_c}(o) = \partial_{i_1} \cdots \partial_{i_{c-1}} f'^r_{i_c}(o),$$

completing the induction. □

Proposition 3.1 is now obvious by the above two lemmas.

Lemma 3.3. (i) *For any multi-index I and integers j, k $(1 \le j, k \le n)$,*

$$\partial/\partial \bar{z}^k\, g(\overline{f_I}, f_j) = g(\overline{f_{k,I}}, f_j).$$

(ii) *For any k $(1 \le k \le n)$ and multi-index $I = (i_1, \dots, i_c)$ $(c \ge 2)$,*

$$D_{\bar{k}} f_I = \textstyle\sum_I^* R(\overline{f_k}, f_{I'}) f_{I''},$$

where the summation $\sum_I^*$ is taken over all $I' = (i_{\sigma(1)}, \cdots, i_{\sigma(a-1)}, i_{\sigma(a)})$ and $I'' = (i_{\tau(1)}, \cdots, i_{\tau(b)}, i_c)$ such that

$$\begin{aligned}
&1 \le a, \quad 0 \le b, \quad a + b = c - 1\\
&\sigma(1) < \cdots < \sigma(a) \le c - 1, \quad \tau(1) < \cdots < \tau(b) \le c - 1\\
&\{\sigma(1), \cdots, \sigma(a), \tau(1), \cdots, \tau(b)\} = \{1, \dots, c-1\}.
\end{aligned}$$

(iii) *For any k $(1 \le k \le n)$, multi-indices I $(|I| \ge 2)$ and K,*

$$\partial/\partial z^k\, g(\overline{f_I}, f_K) = g(\overline{f_I}, f_{k,K}) + \textstyle\sum_I^* g(R(f_k, \overline{f_{I'}})\overline{f_{I''}}, f_K).$$

Proof. Since f_j is holomorphic, we have (i). We shall show (ii) by induction on $|I|$. If $I = (i_1, i_2)$, we have

$$D_{\bar{k}} f_{i_1,i_2} = D_{\bar{k}} D_{i_1} f_{i_2} = R(\overline{f_k}, f_{i_1}) f_{i_2},$$

showing (ii) in the case $|I| = 2$. Assume that we have shown (ii) for any I such that $|I| \le c$. Let $I = (i_1, \cdots, i_c)$. Then for any i_0, we have

$$\begin{aligned}
D_{\bar{k}} f_{i_0,I} &= D_{\bar{k}} D_{i_0} f_{i_1,\cdots,i_c}\\
&= R(\overline{f_k}, f_{i_0}) f_{i_1,\cdots,i_c} + D_{i_0} D_{\bar{k}} f_{i_1,\cdots,i_c}\\
&= R(\overline{f_k}, f_{i_0}) f_{i_1,\cdots,i_c} + \textstyle\sum_I^* \{R(\overline{f_k}, f_{i_0,i_{\sigma(1)},\cdots,i_{\sigma(a)}}) f_{i_{\tau(1)},\cdots,i_{\tau(b)}}\\
&\qquad + R(\overline{f_k}, f_{i_{\sigma(1)},\cdots,i_{\sigma(a)}}) f_{i_0,i_{\tau(1)},\cdots,i_{\tau(b)}}\},
\end{aligned}$$

completing the induction. (iii) follows from (ii). □

Lemma 3.4. *For any multi-indices I and K, $g(\overline{f_I}, f_K)$ is uniquely determined by the functions $g(\overline{f_i}, f_j)$ and $g(R(f_i, \overline{f_{I'}})\overline{f_{I''}}, f_{K'})$ on U such that $1 \leq i, j \leq n$, $|I'|, |I''| < |I|$, and $|K'| < |K|$.*

Proof. If $I = (i_1, i_2, \ldots, i_a)$, Lemma 3.3 (i) implies that $g(\overline{f_I}, f_k)$ is uniquely determined by $g(\overline{f_{i_a}}, f_k)$. Then from Lemma 3.3 (iii), we obtain our assertion for $g(\overline{f_I}, f_K)$ by induction on $|K|$. □

For a positive intger a, we denote by $G(f, a)$ the set of functions $g(\overline{f_i}, f_j)$ and $g(R(f_i, \overline{f_{I_1}})\overline{f_{I_2}}, f_K)$ such that $1 \leq i, j \leq n$, $I_1, I_2 \in \mathcal{I}^{a-1}$, and $K \in \mathcal{I}^a$.

Lemma 3.5. *Suppose that there exist multi-indices $L_1, L_2, \ldots, L_N$ such that the set $\{f_{L_1}, f_{L_2}, \ldots, f_{L_N}\}$ forms a basis of $T^+_{f(p)}S$ for every $p \in U$. Set $d = \max_{1 \leq r \leq N} |L_r|$. Then for any multi-indices I and K, $g(\overline{f_I}, f_K)$ is uniquely determined by $G(f, d)$.*

Proof. By Lemma 3.4, the assertion is obvious if $|I| \leq d$ and $|K| \leq d$. In particular, $g(\overline{f_{L_r}}, f_{L_s})$ is uniquely determined by $G(f, d)$. We next show our assertion is true in the case where $I = L_r$ and K is an arbitrary multi-index. We proceed by induction on $|K|$. Assume $g(\overline{f_{L_r}}, f_K)$ is determined by $G(f, d)$ if $|K| \leq c$. Let $|K| = c$. If we write $f_K = \sum_s A^s_K f_{L_s}$ with certain functions A^s_K on U, then every A^s_K is determined by $g(\overline{f_{L_r}}, f_K)$ $(1 \leq r \leq N)$ and hence by $G(f, d)$ by the assumption of induction. If $|L_r| \geq 2$, then by Lemma 3.3(iii),

$$\begin{aligned} g(\overline{f_{L_r}}, f_{k,K}) &= \partial_k g(\overline{f_{L_r}}, f_K) - \textstyle\sum^*_{L_r} g(R(f_k, \overline{f_{L'}})\overline{f_{L''}}, f_K) \\ &= \partial_k g(\overline{f_{L_r}}, f_K) - \textstyle\sum^*_{L_r} \sum_s A^s_K g(R(f_k, \overline{f_{L'}})\overline{f_{L''}}, f_{L_s}), \end{aligned}$$

which shows that $g(\overline{f_{L_r}}, f_{k,K})$ is determined by $G(f, d)$. When $|L_r| = 1$, the same conclusion follows from Lemma 3.3 (i). Thus our assertion is true for $I = L_r$ and arbitrary K.

Now let I and K be arbitrary multi-indices. Then we have immediately

$$g(\overline{f_I}, f_K) = \sum_{r,s} \overline{A^r_I} A^s_K g(\overline{f_{L_r}}, f_{L_s}),$$

completing the proof. □

Proof of Theorem 3.2. Let $L_1, L_2, \ldots, L_N$ be multi-indices such that the set $\{f_{L_1}, f_{L_2}, \ldots, f_{L_N}\}$ forms a basis of $T^+_{f(p)}S$ for any $p \in U'$, U' being an open neighborhood of o included in U. By the conditions (i), (ii), and Lemma 3.4, we have $g(\overline{f_I}, f_K) = g(\overline{f'_I}, f'_K)$ for any I and $K \in \mathcal{I}^d$. In particular, $g(\overline{f_{L_r}}, f_{L_s}) = g(\overline{f'_{L_r}}, f'_{L_s})$ for any r and s. Then the Gramian $\det(g(\overline{f'_{L_r}}, f'_{L_s}))_{r,s=1,\ldots,n}$ does not vanish on U' as well as the corresponding one for f. This means that the set $\{f'_{L_1}, f'_{L_2}, \dots, f'_{L_N}\}$ also forms a basis of $T^+_{f'(p)}S$ for any $p \in U'$. So f' is infinitesimally full of order d at o.

Next we shall show the existence of F such that $F \circ f = f'$. By the conditions (i) and (ii) together with Lemma 3.5, we have

$$g(\overline{f_I}, f_K) = g(\overline{f'_I}, f'_K) \quad \text{on } U' \text{ for all } I \text{ and } K. \tag{3.2}$$

Let Φ be the unique unitary transformation of $T^+_{\tilde{o}}S$ such that $\Phi(f_{L_r}(o)) = f'_{L_r}(o)$. Φ satisfies $\Phi((f_I)_o) = (f'_I)_o$ for any I, because the coefficients of $(f_I)_o$ with respect to the $(f_{L_r})_o$ coincide with those of $(f'_I)_o$ with respect to the $(f'_{L_r})_o$ by (3.2). Moreover by the condition (iii),

$$R(Z_1, \overline{Z}_2, Z_3, \overline{Z}_4) = R(\Phi(Z_1), \overline{\Phi(Z_2)}, \Phi(Z_3), \overline{\Phi(Z_4)})$$

for any $Z_1, Z_2, Z_3, Z_4 \in T^+_{\tilde{o}}S$. Now let $\Phi^{\mathbf{R}}$ be the linear transformation of $T_{\tilde{o}}S$ induced by Φ via the natural isomorphism $\iota : T_{\tilde{o}}S \to T^+_{\tilde{o}}S$, $\iota(X) = \frac{1}{2}(X - \sqrt{-1}JX)$. It is an orthogonal transformation commuting with $J_{\tilde{o}}$ and leaving $R_{\tilde{o}}$ invariant. There exists uniquely an isometric transformation F of S such that $F(\tilde{o}) = \tilde{o}$ and $(F_*)_{\tilde{o}} = \Phi^{\mathbf{R}}$ (cf. [3], Chapter III, Lemmas 1.2 and 1.4). Appropriately modifying the lemma last cited, we see easily that F is holomorphic. Then from Lemma 2.1 (vii), it follows $((F \circ f)_I)_o = F_*(f_I)_o = (f'_I)_o$ for any I. Hence we have $F \circ f = f'$ by Proposition 3.1.

The uniqueness of F is obvious. Thus we have finished the proof of the theorem. □

4. Complex submanifolds that are infinitesimally full of order two

In this section, we shall give a more concrete expression of Theorem 3.2 for a Kähler submanifold of S that is infinitesimally full of order two. Let M be a connected Kähler manifold. Let ∇^M be the Levi-Civita connection of M. We assume that f is a holomorphic and isometric immersion of M into S. Let α be the second fundamental form of f:

$$\alpha(X_1, X_2) = D_{X_1} f_* X_2 - f_* \nabla^M_{X_1} X_2$$

for any vector fields X_1 and X_2 on M. When we say that *the normal space to f at o is spanned by the second fundamental form*, we mean by definition that the real tangent space $T_{\tilde{o}}S$ at $\tilde{o}$ is spanned by $f_*(X)$ and $\alpha(X, X')$ $(X, X' \in T_oM)$. Note that the normal space to f at o is spanned by the second fundamental form if and only if f is infinitesimally full of order two at o. In particular, when f is a complex hypersurface, f is not totally geodesic if and only if it is infinitesimally full of order two at a point.

Theorem 4.1. *Let S be a simply connected Hermitian symmetric space S and M a connected Kähler manifold. Denote by R the Riemannian curvature of S. Let f and f' be holomorphic and isometric immersions of M into S. Let α and α' be the second fundamental forms of f and f' respectively. Suppose*

(i) *the normal space to f at a point o is spanned by the second fundamental form,*

(ii)

$$\begin{aligned} &R(f_*(X_1), f_*(X_2), f_*(X_3), f_*(X_4)) \\ &\qquad = R(f'_*(X_1), f'_*(X_2), f'_*(X_3), f'_*(X_4)) \\ &R(f_*(X_1), f_*(X_2), f_*(X_3), \alpha(X_4, X_5)) \\ &\qquad = R(f'_*(X_1), f'_*(X_2), f'_*(X_3), \alpha'(X_4, X_5)) \end{aligned}$$

for any vector fields X_ν on M $(1 \leq \nu \leq 5)$,

(iii)

$$\begin{aligned} &R(f_*(X_1), f_*(X_2), \alpha(X_3, X_4), \alpha(X_5, X_6)) \\ &\qquad = R(f'_*(X_1), f'_*(X_2), \alpha'(X_3, X_4), \alpha'(X_5, X_6)) \\ &R(f_*(X_1), \alpha(X_2, X_3), \alpha(X_4, X_5), \alpha(X_6, X_7)) \\ &\qquad = R(f'_*(X_1), \alpha'(X_2, X_3), \alpha'(X_4, X_5), \alpha'(X_6, X_7)) \\ &R(\alpha(X_1, X_2), \alpha(X_3, X_4), \alpha(X_5, X_6), \alpha(X_7, X_8)) \\ &\qquad = R(\alpha'(X_1, X_2), \alpha'(X_3, X_4), \alpha'(X_5, X_6), \alpha'(X_7, X_8)) \end{aligned}$$

for any tangent vector X_ν to M at o $(1 \leq \nu \leq 8)$.

Then the normal space to f' at o is also spanned by the second fundamental form and there exists uniquely a holomorphic and isometric transformation F of S such that $f' = F \circ f$ on M.

Proof. The assertion is immediately obtained from Theorem 3.2 and the identities

$$\begin{aligned} f_{ij} &= \alpha(f_i, f_j) + \textstyle\sum_k \Lambda_{ij}^k f_k \\ f'_{ij} &= \alpha'(f'_i, f'_j) + \textstyle\sum_k \Lambda_{ij}^k f'_k, \end{aligned}$$

where Λ_{ij}^k are the Christoffel symbols of the Levi-Civita connection of M. □

5. Case where ambient space is a complex space form

A complex space form is a simply connected, complete Kähler manifold of constant holomorphic sectional curvature. According to the signature of curvature, it is holomorphically isometric to a complex projective space, a complex vector space, or its unit disc with certain metrics. In this section we consider the case where S is a complex space form. A holomorphic mapping f of a connected complex manifold M into S is said to be *full* if its image is not included in any totally geodesic complex submanifold of S(cf. [1]). We shall show that the holomorphic mapping f is full if and only if it is infinitesimally full.

Let $O_f(o)$ be the complex linear subspace of $T^+_{\tilde{o}}S$ spanned by all $(f_I)_o$. Note that $O_f(o)$ is independent upon the choice of local holomorphic coordinate system.

Proposition 5.1. *If* $\dim_{\mathbf{C}} O_f(o) < N$, *then there exists a complete, totally geodesic complex hypersurface* H *of* S *through* $\tilde{o}$ *such that* $f(M) \subset H$.

Proof. By virtue of complex space form, there exists a complete, totally geodesic complex hypersurface H of S through $\tilde{o}$ such that $O_f(o) \subset T^+_{\tilde{o}}H$. We shall prove that $f(M) \subset H$. We may assume that the coordinate system $(\tilde{U}; w^1, \ldots, w^N)$ around $\tilde{o}$ is so chosen that $H \cap \tilde{U}$ is defined by $w^N = 0$ and that

$$\Gamma^N_{r,s}(w^1, \ldots, w^{N-1}, 0) = 0 \quad (1 \leq r, s \leq N-1). \tag{5.1}$$

We have first

$$f^N_I(o) = dw^N((f_I)_o) = 0 \qquad \text{for all } I. \tag{5.2}$$

To prove $f(M) \subset H$, it will suffice to show $\partial_I f^N(o) = 0$ for every multi-indices I. We need lemmas.

Lemma 5.1. *Let* a *be a positive integer. Suppose that* $\partial_I f^N(o) = 0$ *for any* $I \in \mathcal{I}^a$. *Then for any smooth function* Γ *on* $\tilde{U}$ *such that* $\Gamma = 0$ *on* $H \cap \tilde{U}$, $\partial_I(\Gamma \circ f)$ *vanishes at* o *for any* $I \in \mathcal{I}^a$.

Proof. For any $r = 1, 2, \ldots, N-1$, $\partial\Gamma/\partial w^r$ have the same properties as Γ. The assertion can be obtained by induction on $|I|$ from the identities

$$\begin{aligned}
\partial_i(\Gamma \circ f)(p) &= \sum_{r=1}^{N-1} \frac{\partial \Gamma}{\partial w^r}(f(p))\, \partial_i f^r(p) + \frac{\partial \Gamma}{\partial w^N}(f(p))\, \partial_i f^N(p) \\
\partial_I \partial_i(\Gamma \circ f) &= \sum_{I', I''} \left\{ \sum_{r=1}^{N-1} \partial_{I'} \left(\frac{\partial \Gamma}{\partial w^r} \circ f \right) \partial_{I''} \partial_i f^r + \partial_{I'} \left(\frac{\partial \Gamma}{\partial w^N} \circ f \right) \partial_{I''} \partial_i f^N \right\}.
\end{aligned}$$

□

Let dw^N_f be the complex 1-form along f defined by the smooth assignment $p \mapsto dw^N_{f(p)}$ from U into $\mathbf{T}^*S$.

Lemma 5.2. *Let a be a positive integer. If $\partial_I f^N(o) = 0$ for any $I \in \mathcal{I}^a$, then $(D_I dw^N_f)((\partial/\partial w^r)_f) = 0$ at o for every $r = 1, \dots, N-1$ and $I \in \mathcal{I}^a$.*

Proof. We proceed by induction on a. Set $\gamma^s_{ir}(p) = \sum_{t=1}^N \Gamma^s_{t,r}(f(p))\partial_i f^t(p)$ for $p \in U$.

The assertion is obvious for $a = 1$ from

$$D_i dw^N_f = -\sum_{r=1}^{N-1} \gamma^N_{ir} dw^r_f - \gamma^N_{iN} dw^N_f \tag{5.3}$$

and $\gamma^N_{ir}(o) = 0$. Assume that our assertion is true for a. Suppose $\partial_K f^N(o) = 0$ for any $K \in \mathcal{I}^{a+1}$, and we have in particular $D_I dw^N_f((\partial/\partial w^r)_f) = 0$ at o for any $I \in \mathcal{I}^a$. Now if $|I| \le a$, then it follows from (5.3)

$$D_I D_i dw^N_f = -\sum\nolimits_I \left\{ \sum_{r=1}^{N-1} \partial_{I'}\gamma^N_{ir} D_{I''} dw^r_f + \partial_{I'}\gamma^N_{iN} D_{I''} dw^N_f \right\} \tag{5.4}$$

where the summation $\sum_I$ is taken over certain multi-indices I' and I'' such that $|I'| + |I''| = a$. On the other hand,

$$\partial_{I'}\gamma^N_{ir} = \sum\nolimits_{I'} \left\{ \sum_{t=1}^{N-1} \partial_{I_1}(\Gamma^N_{tr} \circ f)\partial_{I_2}\partial_i f^t + \partial_{I_1}(\Gamma^N_{Nr} \circ f)\partial_{I_2}\partial_i f^N \right\}, \tag{5.5}$$

where the summation $\sum_{I'}$ is taken over certain I_1 and I_2 such that $|I_1| + |I_2| = |I'|$. In (5.5), $\partial_{I_1}(\Gamma^N_{tr} \circ f)(o) = 0$ by Lemma 5.1, and $\partial_{I_2}\partial_i f^N(o) = 0$ by the assumption. We have then $\partial_{I'}\gamma^N_{ir}(o) = 0$. Hence from (5.4), $D_I D_i dw^N_f((\partial/\partial w^r)_f) = 0$ at o for any r $(1 \le r \le N-1)$, which completes the induction. □

Now we return to the proof of Proposition 5.1. We prove by induction on a that $\partial_I f^N(o) = 0$ for all I such that $|I| \le a$. It is obvious for $a = 1$. We assume that the assertion is true for a. Let I be an arbitrary multi-index of order a. From $\partial_i f^N = dw^N_f(f_i)$, we have

$$\partial_I \partial_i f^N = \sum\nolimits_I D_{I'} dw^N_f(f_{I''i}),$$

where the summation $\sum_I$ is taken over certain I' and I'' such that $|I'| + |I''| = |I|$. By Lemma 5.2 together with (5.2) and the assumption of induction, each term of the right-hand side vanishes at o, which completes the induction. Thus we have finished the proof of Proposition 5.1. □

Corollary 5.1. *A holomorphic mapping of a connected complex manifold into a complex space form is full if and only if it is infinitesimally full.*

REMARK 5.1. This corollary differs from Corollary 2.1 of [5] in two points: firstly, the present "infinitesimally full" condition is somewhat different from the corresponding one in [5], and secondly only holomorphic immersions are considered in [5].

If S is a complex space form,

$$R(f_{I_1},\overline{f_{I_2}},f_{I_3},\overline{f_{I_4}})=\kappa(g(f_{I_1},\overline{f_{I_2}})g(f_{I_3},\overline{f_{I_4}})+g(f_{I_1},\overline{f_{I_4}})g(\overline{f_{I_2}},f_{I_3}))$$

with certain constant κ. Lemma 3.4 tells us that both the conditions (ii) and (iii) of Theorem 3.2 are then consequences of (i). This means that Theorem 3.2 substantially includes Theorem 3.1.

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