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DECOMPOSITION THEOREMS FOR THE TEMPERATURE FUNCTIONS WITH SINGULARITY

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1. Introduction

A function $u(x,t)$ defined on an open subset Ω of $\mathbb{R}^{n+1}$ is said to be a temperature function in Ω if u is twice continuously differentiable and satisfies $(\partial_t - \Delta)u = 0$ in Ω, where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ and $\partial_t = \partial/\partial t$.

In [16] it was stated that every nonnegative temperature function $u(x,t)$ in the deleted unit disc $B \setminus \{(0,0)\}$ in $\mathbb{R}^2$ can be written as

$$u(x,t) = v(x,t) + aE(x,t) \quad \text{in} \quad B \setminus \{(0,0)\} \tag{1.1}$$

for some constant $a \geq 0$ and a temperature function $v(x,t)$ in B. Here, $E(x,t)$ denotes the fundamental solution of the heat operator $(\partial_t - \Delta)$.

This characterizes the nonnegative temperature function with isolated singularity. In fact, for the harmonic function $u(x)$ in an open set $\Omega \setminus K$, K being a compact set, it is well known that $u(x)$ can be decomposed as the sum of two harmonic functions, one extending harmonically across the boundary of K, the other extending harmonically across the boundary of Ω. The purpose of this paper is to give an analogue of this for the temperature functions.

In this paper we will give several decomposition theorems which generalize the above results of [16] and [4].

At first, when K is a compact subset of an open subset Ω in $\mathbb{R}^{n+1}$ given by $K = K_0 \times \{t = t_0\}$, $K_0 \subset \mathbb{R}^n$, we characterize the temperature functions $u(x,t)$ with the distributional growth (see (2.1)) near K as follows:

$$u(x,t) = u_1(x,t) + u_0 \underset{(x)}{*} E(x, t - t_0), \tag{1.2}$$

where u_0 is a distribution in $\mathbb{R}^n$ with compact support in K_0 (see Theorem 2.1.) and $\underset{(x)}{*}$ denotes the convolution product with respect to x variable. Moreover, if $u(x,t)$ is a nonnegative temperature function, then we can choose u_0 as a positive (Radon) measure in $\mathbb{R}^n$ in (1.2) (see Theorem 2.2.).

The last section is devoted to give a decomposition theorem which characterizes the temperature functions with ultradistributional growth near K (see Theorem 3.6). This result will give a unique description of temperature function with the various growth near its singular set K (see Corollaries 3.7 and 3.8). For example, it will be shown that if $u(x,t)$ is a temperature function in $\Omega \setminus K$, as above, satisfying that for every $\varepsilon > 0$ there is a constant $C > 0$ such that

$$|u(x,t)| \le C \exp\left[\frac{\varepsilon}{|t-t_0|}\right] \quad \text{near} \quad K,$$

then there exist a temperature function $v(x,t)$ in Ω which belongs to the Gevery class $\mathcal{E}_{\{2\}}(\Omega)$ and an analytic functional $u_0 \in A'(K_0)$ in $\mathbb{R}^n$ such that

$$u(x,t) = v(x,t) + u_0 \underset{(x)}{*} E(x, t-t_0) \quad \text{in} \quad \Omega \setminus K.$$

Throughout this paper we make much use of the generalized function theory such as, distributions, analytic functionals, ultradistributions, etc. They will be used to prove the results very effectively. In fact, it is another purpose of this paper to develop our theory via only the generalized function theory.

2. Temperature functions with distribution growth

In this section we will give several decomposition theorems for the temperature functions which have the distributional growth near the singular set K.

Here for a compact subset K of an open set Ω we say that a continuous function $f(x)$ in $\Omega \setminus K$ has the distributional growth near K if $f(x)$ satisfies

$$(2.1) \qquad |f(x)| \le C[d(x,K)]^{-M} \quad \text{near} \quad K$$

for some $M > 0$ and $C > 0$, where d denotes the Euclidean distance.

First, we will state a decomposition theorem which characterizes temperature functions in $\Omega \setminus K$ with the distributional growth near K when K is contained in a hyperplane.

In what follows we denote by $E(x,t)$ the fundamental solution of the heat operator $(\partial_t - \Delta)$ given by

$$E(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp(-|x|^2/4t), & t > 0, \\ 0, & t < 0. \end{cases}$$

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^{n+1}$ and let K be a compact subset of Ω given by*

$$K = K_0 \times \{t = t_0\}$$

for some compact subset K_0 of $\mathbb{R}^n$. If $u(x,t)$ is a temperature function in $\Omega \setminus K$ satisfying that

$$|u(x,t)| \leq C|t-t_0|^{-M} \quad \textit{near} \quad K \tag{2.2}$$

for some constants $C > 0$ and $M > 0$, then there exist a unique temperature function $v(x,t)$ in Ω and a distribution u_0 in $\mathbb{R}^n$ with support in K_0 such that

$$u(x,t) = v(x,t) + u_0 \underset{(x)}{*} E(x, t-t_0) \quad \textit{in} \quad \Omega \setminus K. \tag{2.3}$$

Proof. With a little consideration we can see that the temperature function $u(x,t)$ can be extended to a distribution on the whole of Ω. Let $\tilde{u}(x,t)$ be a distributional extension of $u(x,t)$ to Ω. Then it follows that $(\partial_t - \Delta)\tilde{u} = g(x,t)$ is a distribution in $\mathbb{R}^{n+1}$ with support in K. The structure theorem for distributions with support in a hyperplane implies that there exist distributions $f_0, f_1, \cdots, f_N$ in $\mathbb{R}^n$ with support in K_0 such that

$$g(x,t) = \sum_{j=0}^{N} f_j \otimes \delta^{(j)}(t-t_0).$$

Then if we put $v(x,t) = \tilde{u} - \sum_{j=0}^{N} f_j \underset{(x)}{*} \partial_t^j E(x,t-t_0)$, then $(\partial_t - \Delta)v(x,t) = 0$ in Ω so that

$$u(x,t) = v(x,t) + \sum_{j=0}^{N} f_j \underset{(x)}{*} \partial_t^j E(x,t-t_0) \quad \text{in} \quad \Omega \setminus K,$$

and $v(x,t)$ is a temperature function in Ω in view of the hypoellipticity of the heat operator in the distributions. But, using $\partial_t^j E(x,t) = \Delta^j E(x,t)$ for $t > 0$ we have

$$\sum_{j=0}^{N} f_j \underset{(x)}{*} \partial_t^j E(x,t-t_0) = \left(\sum_{j=0}^{N} \Delta^j f_j \right) \underset{(x)}{*} E(x,t-t_0) \tag{2.4}$$

for $t > t_0$. For $t < t_0$ the both sides of (2.4) are zero. But since the hypoellipticity of $(\partial_t - \Delta)$ in the distributions implies that the both sides belong to $C^\infty(\mathbb{R}^{n+1} \setminus K)$ they coincide in $\mathbb{R}^{n+1} \setminus K$. Hence if we put $u_0 = \sum_{j=0}^{N} \Delta^j f_j$, then u_0 is a distribution in $\mathbb{R}^n$ with support in K_0 and

$$u(x,t) = v(x,t) + u_0 * E(x,t-t_0) \quad \text{in} \quad \Omega \setminus K.$$

The uniqueness is easy, so the proof is complete. □

Let B be the unit disc in $\mathbb{R}^2$. Then it is well known that every nonnegative temperature function $u(x,t)$ in $B \setminus \{(0,0)\}$ can be written as

$$u(x,t) = v(x,t) + aE(x,t) \quad \text{in} \quad B \setminus \{(0,0)\} \tag{2.5}$$

for some nonnegative constant a and a temperature function in B (see [16, Theorem 4.8]). When K is a compact set in a hyperplane, we can give a similar decomposition for nonnegative temperature function in $\Omega \setminus K$, which generalizes the result in [16].

Theorem 2.2. *Let Ω be an open subset of $\mathbb{R}^{n+1}$ and let K be a compact subset of Ω given by*

$$K = K_0 \times \{t = t_0\}$$

for some compact subset K_0 of $\mathbb{R}^n$. If $u(x,t)$ is a temperature function in $\Omega \setminus K$ satisfying that

$$u(x,t) \geq M \quad \textit{in} \quad \Omega \setminus K \tag{2.6}$$

for some real number M, then there exist a temperature function $v(x,t)$ in Ω and a positive (Radon) measure μ in $\mathbb{R}^n$ with support in K_0 such that

$$u(x,t) = v(x,t) + \int_K E(x-y, t-t_0)d\mu(y) \quad \textit{in} \quad \Omega \setminus K. \tag{2.7}$$

Proof. We first decompose $u(x,t)$ as

$$u(x,t) = u_1(x,t) + u_2(x,t) \tag{2.8}$$

as in Theorem 2.1. Here $u_2(x,t)$ is a temperature function in $\mathbb{R}^{n+1} \setminus K$ vanishing for $t < t_0$. Adding a some constant to the both sides of (2.8) we may assume that $u_2(x,t)$ is positive in some open subset $\Omega_0 \times (t_0, T)$, with $t_0 < T$, where Ω_0 is a bounded open set in $\mathbb{R}^n$ containing K_0. Then the Harnack inequality for the heat equation in a bounded set (see [5, p.195]) implies that there exists a constant $\alpha > 0$ such that

$$u(x,t) \geq \alpha u(x,t')$$

for $T > t \geq t' > t_0$ and $x \in \Omega_0$.

Then it follows that

$$0 \leq \alpha \int_{\Omega_0} u(x,t')dx \leq \int_{\Omega_0} u(x,t)dx \leq \int_{\Omega_0} u(x,T)dx < \infty$$

for $t_0 < t' \le t < T$.

From this fact we can say that $\int_{\Omega_0} u(x,t_0)dx$ and $\int_{-T/2}^{T/2}\int_{\Omega_0} u(x,t)dxdt$ are finite so that $u_2(x,t)$ is eventually locally integrable function in $\mathbb{R}^{n+1}$. This means that $(\partial_t - \Delta)u_2(x,t)$ is a distribution of order at most 2. Using the structure theorem of distribution with support in the compact polar set K we can write

$$u_2(x,t) = w(x,t) + \left(\sum_{j=0}^{2} \Delta^j f_j\right) \underset{(x)}{*} E(x,t-t_0) \quad \text{in} \quad \mathbb{R}^{n+1} \setminus K, \tag{2.9}$$

where $w(x,t)$ is a temperature function in $\mathbb{R}^{n+1}$ and f_j are distributions in $\mathbb{R}^n$ with support in K. In fact, in order to obtain the expression (2.9) we repeat the same process as in the proof of Theorem 2.3.

Since $u(x,t) \ge M$ in $\Omega \setminus K$ there exists a constant $N > 0$ such that

$$\left(N + \sum_{j=0}^{2} \Delta^j f_j\right) \underset{(x)}{*} E(x,t-t_0) \ge 0 \tag{2.10}$$

in $\Omega \setminus K$. We now define $\mu = N + \sum_{j=0}^{2} \Delta^j f_j$. Then μ is a tempered distribution. Moreover we can see that

$$\mu(\phi) = \lim_{t \to t_0^+} \int [\mu * E(x,t-t_0)]\phi(x)dx \ge 0 \tag{2.11}$$

for every infinitely differentiable nonnegative function ϕ with compact support in Ω_0. In fact, we can obtain the equality in (2.11) by some tedious calculation, using the fact $E(x,t) * \phi(x)$ converges to $\phi(x)$ as $t \to 0+$ in the topology of the test functions for distributions. Then (2.11) implies that μ is a positive distribution, so that μ is a (Radon) measure on Ω_0.

Combining (2.8) and (2.9) we obtain

$$u(x,t) = u_1(x,t) + w(x,t) - N + \mu \underset{(x)}{*} E(x,t-t_0).$$

Moreover, if we apply the heat operator to the both sides, then we can see

$$(\partial_t - \Delta)u(x,t) = \mu,$$

which implies μ has a support in K. This completes the proof.

In fact, the decomposition (2.5) makes it possible to see that every bounded temperature function in $B \setminus \{(0,0)\}$ can be continued to the whole of B as a temperature function. But, in general, this is no longer true for the temperature functions in $\Omega \setminus K$. For example, consider $u(x,t)$ in $\mathbb{R}^2$

$$u(x,t) = \int_0^1 E(x-y,t)dy.$$

Then $u(x,t)$ is a bounded temperature function in $\mathbb{R}^2 \setminus K$, where $K = [0,1] \times \{0\}$.

3. Temperature functions with ultradistributional growth

In this section we will give decomposition theorems characterizing the temperature functions with ultradistributional growth near their singular set in a hyperplane. To do this we need some preliminaries on the ultradistributions.

Let M_p, $p = 0,1,2,\cdots$, be a sequence of positive numbers and let Ω be an open subset of $\mathbb{R}^n$. An infinitely differentiable function ϕ on Ω is called an ultradifferentiable function of class (M_p) (of class $\{M_p\}$, respectively) if for any compact set K of Ω for each $h > 0$ (for some constant $h > 0$, respectively)

$$|\phi|_{M_p,K,h} = \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}M_{|\alpha|}}$$

is finite. Throughout this paper we impose the following conditions on M_p:

(M.0) There exist constants $C > 0$ and $A > 0$ such that

$$p! \le CA^p M_p, \ p = 0,1,2,\cdots .$$

(M.1) $M_p^2 \le M_{p-1}M_{p+1}, \ p = 1,2,\cdots$.
(M.2) There are constants $C > 0$ and $H > 0$ such that

$$M_{p+q} \le CH^{p+q}M_pM_q, \quad p,q = 0,1,2,\cdots .$$

We call the above sequence M_p the defining sequence and denote by $\mathcal{E}_{(M_p)}(\Omega)$ ($\mathcal{E}_{\{M_p\}}(\Omega)$, respectively) the space of all ultradifferentiable functions of class (M_p) (of class $\{M_p\}$, respectively) on Ω. In particular, if $\Omega = \mathbb{R}^n$, then we write simply $\mathcal{E}_{(M_p)}$ for $\mathcal{E}_{(M_p)}(\mathbb{R}^n)$ ($\mathcal{E}_{\{M_p\}}$ for $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n)$, respectively).

If $M_p = p!$, then by Pringsheim's theorem $\mathcal{E}_{\{p!\}}$ is the set of all (real) analytic functions on $\mathbb{R}^n$ and $\mathcal{E}_{(p!)}$is the set of all (real) entire functions on $\mathbb{R}^n$. Thus (M.0) means that $\mathcal{E}_{(p!)}$ is the smallest space to be considered in this paper.

Especially, if $M_p = p!^s$ $(s > 1)$, then $\mathcal{E}_*$ is called the Gevrey class of order s and denoted simply by $\mathcal{E}_{(s)}$ or $\mathcal{E}_{\{s\}}$ sometimes.

The condition (M.1) can be naturally fulfilled since by Gorny's theorem in [14, p.226] the sequence M_p can be rearranged to satisfy (M.1). The condition (M.2) ensures the stability of the spaces of ultradifferentiable functions under the ultradifferential operator (see (3.3)). Thus the above conditions (M.0)$\sim$(M.2) are very natural ones.

The topologies of such spaces are defined as follows:

A sequence $\phi_j \to 0$ in $\mathcal{E}_{(M_p)}(\Omega)$ ($\mathcal{E}_{\{M_p\}}(\Omega)$, respectively) if for any compact set K of Ω and for any $h > 0$ (for some $h > 0$ respectively) we have

$$\sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi_j(x)|}{h^{|\alpha|}M_{|\alpha|}} \to 0 \quad \text{as} \quad j \to \infty.$$

As usual, we denote by $\mathcal{E}'_{(M_p)}(\Omega)$ ($\mathcal{E}'_{\{M_p\}}(\Omega)$, respectively) the strong dual space of $\mathcal{E}_{(M_p)}(\Omega)$ ($\mathcal{E}_{\{M_p\}}(\Omega)$, respectively) and we call its elements ultradistributions of Beurling type (of Roumieu type, respectively) with compact support in Ω. Let $K \subset \mathbb{R}^n$ be a compact set. We denote by $\mathcal{E}'_{(M_p)}(K)$ ($\mathcal{E}'_{\{M_p\}}(K)$, respectively) the set of ultradistributions of class (M_p) (of class $\{M_p\}$, respectively) with support in K. For example, $u \in \mathcal{E}'_{(M_p)}(K)$ if and only if for any neighborhood Ω of K there exist constants $h > 0$ and $C > 0$ such that

$$|u(\phi)| \le C \sup_{\substack{x\in\Omega\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}M_{|\alpha|}}, \quad \phi \in \mathcal{E}_{(M_p)}(\mathbb{R}^n).$$

If $M_p = p!$, then $\mathcal{E}'_{\{p!\}}(K)$ is the same as the space $A'(K)$ of analytic functional carried by K.

For each defining sequence M_p we define for $t > 0$

$$\begin{aligned} M(t) &= \sup_p \log \frac{t^p M_0}{M_p}, \\ M^*(t) &= \sup_p \log \frac{p!t^p M_0}{M_p}, \\ \overline{M}(t) &= \sup_p \log \frac{p!t^p M_0^2}{M_p^2}. \end{aligned} \tag{3.1}$$

If $(M_p/M_0)^{1/p}$ is bounded below by a positive constant, then $M(t)$ is an increasing convex function in $\log t$ which vanishes for sufficiently small $t > 0$ and increases more rapidly than $\log t^p$ for any p as t tends to infinity. Moreover, (M.1) implies

$$M_p = M_0 \sup_{t>0} \frac{t^p}{\exp M(t)}, \quad p = 1, 2, 3, \dots, \tag{3.2}$$

which is shown in [8]. Since (M.0) implies that $(M_p/p!M_0)^{1/p}$ is bounded below, M^* and $\overline{M}$ have also the similar properties as M.

In what follows, $*$ denotes (M_p) or $\{M_p\}$. For a bounded open set Ω we define

$$\mathcal{D}'_*(\Omega) = \mathcal{E}'_*(\overline{\Omega})/\mathcal{E}'_*(\partial\Omega).$$

For an unbounded open set Ω an element of $\mathcal{D}'_*(\Omega)$ is defined in such a way that it is locally equivalent to an ultradistribution with compact support in Ω. For example, $\mathcal{D}'_{\{p!\}}$ is the space of hyperfunctions given by Sato and $\mathcal{D}'_{(p!^s)}$ $(s > 1)$ is the space of Gevrey ultradistributions of Beurling type. We refer to [6], [7], [8], and [9] for more details on the ultradistributions.

An operator of the form

$$P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha, \quad a_\alpha \in \mathbb{C}, \tag{3.3}$$

is called an ultradifferential operator of class (M_p) (of class $\{M_p\}$, respectively) if there are constants L and C (for every $L > 0$ there is a constant $C > 0$, respectively) such that

$$|a_\alpha| \le CL^{|\alpha|}/M_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.4}$$

It is well known that if $P(\partial)$ is an ultradifferential operator of class $*$, then

$$P(\partial) : \mathcal{E}_*(\Omega) \to \mathcal{E}_*(\Omega), \quad \mathcal{D}_*(\Omega) \to \mathcal{D}_*(\Omega) \tag{3.5}$$

and

$$P(\partial) : \mathcal{E}'_*(\Omega) \to \mathcal{E}'_*(\Omega), \quad \mathcal{D}'_*(\Omega) \to \mathcal{D}'_*(\Omega) \tag{3.6}$$

are continuous. The condition (3.4) is equivalent to the condition that

$$|P(\zeta)| \le C \exp M(L|\zeta|), \quad \zeta \in \mathbb{C}^n.$$

Only for a technical reason we need sometimes another condition on the defining sequence M_p as follows:

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le CpM_{p-1}/M_p \quad p = 1, 2, 3, \dots . \tag{M.3}$$

In fact, the condition (M.3) ensures the existence of cut off functions, so called, the nonquasianalyticity.

The following lemma is a variation of Lemma 11.4 in [8]. Hence we give here only a sketch of proof. For more details we refer the reader to [8].

Lemma 3.1. *Suppose that M_p satisfies* (M.0) $\sim$ (M.3).
Let $\ell_p \downarrow 0$ for $* = \{M_p\}$ ($\ell_p = \ell > 0$ for $* = (M_p)$, respectively) and

$$m_p = M_p/M_{p-1}. \tag{3.7}$$

Then

$$P(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} \left(1 + \frac{\ell_p \zeta}{m_p}\right), \quad \zeta \in \mathbb{C}$$

satisfies the followings :

(i) *For any $L > 0$ there exists $C > 0$ (there exist $L > 0$ and $C > 0$, respectively) such that*

$$|P(\zeta)| \leq C \exp M(L|\zeta|), \quad \zeta \in \mathbb{C},$$

i.e. $P(d/dt)$ is an ultradifferential operator of class $*$.

(ii) *For every $\epsilon > 0$ there exist $v, \omega \in C_0^\infty(\mathbb{R})$ such that*

$$P(d/dx)\, v(x) = \delta(x) + \omega(x), \quad x \in \mathbb{R},$$

$$|v(x)| \leq C \inf_p \frac{x^p M_p}{p!(\ell_1 \ell_2 \cdots \ell_p) M_0}, \quad x \in \mathbb{R},$$

and

$$\operatorname{supp} v \subset [0, \epsilon], \quad \operatorname{supp} \omega \subset [\epsilon/2, \epsilon].$$

Proof. (i) By the remark in [8, p.60], for any $L > 0$ there exists a constant $C > 0$ such that

$$\left| \prod_{p=1}^{\infty} \left(1 + \frac{\ell_p \zeta}{m_p}\right) \right| \leq C \exp M(L\zeta), \quad \zeta \in \mathbb{C}.$$

Then using the relations

$$t \leq C \exp M(t),$$

$$2M(t) \leq M(Ht) + C,$$

which are equivalent to (M.0) and (M.2), respectively, we obtain

$$|P(\zeta)| \leq C_\epsilon \exp M(\epsilon\zeta), \quad \zeta \in \mathbb{C}$$

for every $\epsilon > 0$.

(ii) First we set

$$F(z) = \frac{1}{2\pi i} \int_0^\infty e^{z\zeta}/P(\zeta) d\zeta, \quad \text{Re } z < 0.$$

Then $F(z)$ is holomorphic for Re$z < 0$, holomorphically continued to the Riemann domain $\left\{z \neq 0 \middle| -\frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right\}$ and

$$P\left(\frac{d}{dz}\right) F(z) = \frac{-1}{2\pi i z},$$

$$\left|\frac{d^p}{dz^p} F(z)\right| \leq \frac{M_p}{4\ell_1 \ell_2 \cdots \ell_p}.$$

Furthermore, set

$$u(x) = F(x + i0) - F(x - i0).$$

Then we have

$$\left|u^{(p)}(x)\right| \leqq \frac{1}{2} \inf_p \frac{x^p M_p}{p! \ell_1 \ell_2 \cdots \ell_p}, \quad x > 0,$$

$$P(d/dx)\, u(x) = \delta(x),$$

and

$$u(x) = 0 \quad \text{for } x < 0,$$

$$u(x) \geq 0 \quad \text{for } x \geq 0,$$

$$\int_{-\infty}^\infty u(x) dx = 1.$$

The function $v(x)$ is obtained by multiplying $u(x)$ by a suitable function ϕ in $\mathcal{E}_{(M_p)}$ which is equal to 1 in $\left[0, \frac{\epsilon}{2}\right]$ and equal to 0 in $(-\infty, -\epsilon] \cup [\epsilon, \infty)$. Then taking

$$\omega(x) = P(d/dx)(\phi u) - \phi P(d/dx)\, u(x)$$

we complete the proof. □

Lemma 3.2. *Let M_p be a sequence satisfying* (M.0) $\sim$ (M.3). *Suppose that $f(t)$ is an unbounded continuous function for $t > 0$ satisfying that for any $L > 0$ there exists a constant $C > 0$ such that*

$$|f(t)| \le C \exp M^*(L/t), \quad t > 0,$$

where $M^*(t)$ is given by (3.1).

Then there exist a sequence $\ell_p \downarrow 0$ and a constant $C > 0$ such that

$$|f(t)| \le C \sup_p \frac{M_0 p! \ell_1 \ell_2 \cdots \ell_p}{M_p t^p}, \quad t > 0. \tag{3.8}$$

Proof. Define a function $E : (0, \infty) \to \mathbb{R}$ by

$$\exp M^*(E(\rho)) = \sup_{0<t\le\rho} |f(1/t)|, \quad \rho > 0.$$

Then since M^* is increasing and f is unbounded, $E(\rho)$ is also increasing and tends to ∞ as $\rho \to \infty$.

We now show that $\lim_{\rho\to\infty} E(\rho)/\rho = 0$. Suppose that this is not true. Then there exist a constant $L > 0$ and a sequence $\rho_j \to \infty$ such that

$$E(\rho_j) > 2L\rho_j, \quad j = 1, 2, 3, \cdots .$$

Then it follows that

$$\begin{aligned} &\exp M^*(2L\rho_j) < \exp M^*(E(\rho_j)) \\ &= \sup_{0<t\le\rho_j} |f(1/t)| \le C \sup_{t\le\rho_j} \exp M^*(Lt) \\ &\le C \exp M^*(L\rho_j). \end{aligned}$$

Also, choosing $p_j \in \mathbb{N}_0$ so that for each j

$$\exp M^*(L\rho_j) = \frac{p_j!(L\rho_j)^{p_j} M_0}{M_{p_j}},$$

we have

$$\frac{p_j!(2L\rho_j)^{p_j} M_0}{M_{p_j}} \le C \frac{p_j!(L\rho_j)^{p_j} M_0}{M_{p_j}}.$$

This leads to a contradiction, since $p_j \to \infty$ as $j \to \infty$. Thus we conclude that

$$\lim_{\rho\to 0} E(\rho)/\rho = 0.$$

Now define a new sequence n_p such that

$$E\left(m_p/pn_p\right) = m_p/p, \quad p = 1, 2, 3, \cdots,$$

where $m_p = M_p/M_{p-1}$ as in (3.7). As was shown in [11, Proposition 1.1], in view of the condition (M.3) we may assume $m_p/p \uparrow \infty$. Thus it follows that $m_p/pn_p \uparrow \infty$ and

$$n_p = E\left(\frac{m_p}{pn_p}\right) \Big/ \left(\frac{m_p}{pn_p}\right) \to 0 \tag{3.9}$$

as $p \to \infty$.

On the other hand, for any $\rho > 0$ there exist $p_0 \in \mathbb{N}_0$ such that

$$\frac{m_{p_0}}{p_0 n_{p_0}} \leq \rho < \frac{m_{p_0+1}}{(p_0+1)n_{p_0+1}}. \tag{3.10}$$

This means that

$$\frac{m_{p_0}}{p_0} \leq E(\rho) < \frac{m_{p_0+1}}{p_0+1}. \tag{3.11}$$

It is already shown in [Ma] that (M.2) is equivalent to

$$m_{p+1} \leq H M_p^{\frac{1}{p}}, \quad p = 0, 1, 2, \cdots.$$

From this we obtain

$$m_{p+1} \leq H m_p, \quad p = 1, 2, \cdots. \tag{3.12}$$

Thus it follows from (3.9)$\sim$(3.12) that

$$\begin{aligned} E(\rho) &< \frac{m_{p_0+1}}{p_0+1} \frac{p_0}{m_{p_0}} \frac{m_{p_0}}{p_0 n_{p_0}} n_{p_0} \\ &\leq n_{p_0} \rho \left(m_{p_0+1}/m_{p_0}\right) \\ &\leq H \rho n_{p_0}. \end{aligned}$$

Taking $\ell_p = H \max_{q \geq p} n_q$ we can see that $\ell_p \downarrow 0$ and

$$\begin{aligned} \exp \tilde{M}(E(\rho)) &= \sup_p \frac{p!\{E(\rho)\}^p M_0}{M_p} \\ &= \sup_p \prod_{q=1}^{p} \frac{qE(\rho)}{m_q} \\ &= \prod_{q=1}^{p_0} \frac{qE(\rho)}{m_q} \leq \prod_{q=1}^{p_0} \frac{q(\rho \ell_{p_0})}{m_q} \end{aligned}$$

$$\leq \prod_{q=1}^{p_0} \frac{q\ell_q\rho}{m_q} \leq \sup_p \frac{M_0 p!\ell_1\ell_2\cdots\ell_p\rho^p}{M_p}.$$

Therefore, for any $t > 0$ with $t = 1/\rho$

$$\begin{aligned}|f(t)| &\leq \sup_{1/\rho\leq s} |f(s)| = \exp M^*(E(\rho)) \\ &\leq \sup_p \frac{M_0 p!\ell_1\ell_2\cdots\ell_p\rho^p}{M_p} \\ &= \sup_p \frac{M_0 p!\ell_1\ell_2\cdots\ell_p}{M_p t^p},\end{aligned}$$

which completes the proof. □

We are now in a position to state and prove the main theorem in this section. We are going to give a decomposition theorem which characterizes the temperature functions with ultradistributional growth near their singular set.

Here for a compact subset K of an open set Ω we say that a continuous function $f(x)$ in $\Omega \setminus K$, $K \subset\subset \Omega$, has the ultradistributional growth of class $\{M_p\}$ ((M_p), respectively) near K if it satisfies, for every L (for some L, respectively)

$$|f(x)| \leq C \exp \overline{M}\left[\frac{L}{d(x,K)}\right], \quad \text{near} \quad K,$$

with a proper constant $C > 0$ depending on L.

In fact, the condition (M.3) is too strong for our theories since (M.3) excludes the defining sequence $M_p = p!$ for the hyperfunctions. So we relax this by the following milder condition:

$$\text{(C)}: \quad M_p^2 \quad \text{satisfies} \quad \text{(M.3)}.$$

For example, $M_p = p!$ satisfies (C), but does not satisfy (M.3). In [2] they used this condition (C) to characterize the quasianalytic ultradistributions. It is well known that most of the defining sequences for the standard quasianalytic ultradistributions satisfy the condition (C).

DEFINITION 3.3. We say that the partial differential operator $P(D)$ is $*$-hypoelliptic if for every open subset Ω of $\mathbb{R}^n$ and every ultradistribution $u \in \mathcal{D}'_*(\Omega)$ with $P(D)u \in \mathcal{E}_*(\Omega)$ belongs to $\mathcal{E}_*(\Omega)$.

As in the distribution theory of L. Schwartz the $*$-hypoellipticity for $P(D)$ can also characterized by the regularity property of its fundamental solution as follows:

Lemma 3.4. *If the partial differential operator $P(D)$ with constant coefficients is $*$-hypoelliptic, then every fundamental solution belongs to $\mathcal{E}_*(\mathbb{R}^n \setminus \{0\})$. Conversely, if there is a fundamental solution $E \in \mathcal{D}'_*(\mathbb{R}^n)$ which belongs to $\mathcal{E}_*(\mathbb{R}^n \setminus \{0\})$, then $P(D)$ is $*$-hypoelliptic.*

Proof. The first part is easy. Now we prove the second part only for $* = \{M_p\}$.

Let E be an element of $\mathcal{D}'_{\{M_p\}}(\mathbb{R}^n)$ which belongs to $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{D}'_{\{M_p\}}(\Omega)$, $f \in \mathcal{E}_{\{M_p\}}(\Omega)$ with $P(D)u = f$, where Ω is an open subset of $\mathbb{R}^n$. Then to prove that $u \in \mathcal{E}_{\{M_p\}}(\Omega)$ we may assume that f has a compact support. Then

$$u = \delta * u = P(D)E * u = E * P(D)u = E * f.$$

Then by Theorem 6.10 of Komatsu [8] we have $u \in \mathcal{E}_{\{M_p\}}(\Omega)$. $\square$

It is well known that the fundamental solution $E(x,t)$ for the heat operator belongs to $\mathcal{E}_{\{2\}}(\mathbb{R}^n \setminus \{0\})$. Thus in view of the above lemma, the heat operator $\partial_t - \Delta$ is $\{p!^2\}$-hypoelliptic. If the defining sequence M_p satisfies $p!^2 \le AH^p M_p$, $p = 0, 1, 2, \ldots$, for some $A > 0$ and $H > 0$ (we denote it by $p! \subset M_p$), then the heat operator is $\{M_p\}$-hypoelliptic. Moreover, if the defining sequence M_p satisfies that for every $H > 0$, $p!^2 \le AH^p M_p$, $p = 0, 1, 2, \ldots$, for some $A > 0$ (we denote it by $p!^2 \prec M_p$), then the heat operator is (M_p)-hypoelliptic, since $\mathcal{E}_{\{2\}} \subset \mathcal{E}_{(M_p)}$. Thus we obtain the following:

Corollary 3.5. *If Ω is an open subset of $\mathbb{R}^{n+1}$ and $u(x,t)$ is an element in $\mathcal{D}'_{\{p!^2\}}(\Omega)$ and satisfies the heat equation $(\partial_t - \Delta)u(x,t) = 0$ in Ω, then $u(x,t)$ is a temperature function which belongs to $\mathcal{E}_{\{2\}}(\Omega)$.*

Theorem 3.6. *Let Ω be an open subset of $\mathbb{R}^{n+1}$ and let K be a compact subset of Ω given by*

$$K = K_0 \times \{t = t_0\}$$

for some compact subset K_0 of $\mathbb{R}^n$. Let M_p be a defining sequence satisfying (M.0), (M.1), (M.2) *and* (C). *If $u(x,t)$ is a temperature function in $\Omega \setminus K$ satisfying that for every $L > 0$ (for some L, respectively) there is a constant C such that*

$$|u(x,t)| \le C \exp \overline{M}\left(\frac{L}{|t - t_0|}\right) \quad \textit{near} \quad K, \tag{3.13}$$

then there exist a temperature function $v(x,t)$ on Ω which belongs to $\mathcal{E}_{\{M_p^2\}}(\Omega)$ ($\mathcal{E}_{(M_p^2)}(\Omega)$, respectively) and an ultradistribution $u_0 \in \mathcal{E}'_{\{M_p\}}(K_0)$ ($\mathcal{E}'_{(M_p)}(K_0)$, respectively) in $\mathbb{R}^n$ such that

$$u(x,t) = v(x,t) + u_0 \underset{(x)}{*} E(x, t - t_0) \quad \textit{in} \quad \Omega \setminus K. \tag{3.14}$$

Proof. For simplicity we assume $t_0 = 0$ and prove this only for the case $\{M_p\}$. In fact, a little modification will give the proof in the case of (M_p).

Let $N_p = M_p^2$. Then (C) implies that N_p satisfies (M.0)$\sim$(M.3).

First we suppose for the time being that $u(x,t)$ can be continued as an ultradistribution $\tilde{u}(x,t) \in \mathcal{D}'_{\{N_p\}}(\Omega)$ to the whole of Ω. Then $(\partial_t - \Delta)\tilde{u} = f_0(x,t)$ also belongs to $\mathcal{D}'_{\{N_p\}}(\Omega)$ and has a compact support in K. In view of the structure theorem ([9, Theorem 3.1]) for the $\mathcal{D}'_{\{N_p\}}$ there exists a sequence of ultradistributions $f_j \in \mathcal{E}'_{\{N_p\}}(K_0)$ satisfying the following condition:

For every $L > 0, h > 0$ and $\delta > 0$ there exists a constant $C > 0$ such that

$$|f_j(\phi)| \le CL^j N_j^{-1} \sup_{\substack{x \in K_\delta \\ \alpha}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} N_{|\alpha|}}, \quad \phi \in \mathcal{E}_{\{N_p\}}(\mathbb{R}^n), \tag{3.15}$$

where $K_\delta = \{x \in \mathbb{R}^n | d(x, K_0) < \delta\}$,

$$\operatorname{supp} f = \overline{\bigcup_{j=1}^{\infty} \operatorname{supp} f_j}$$

and

$$f_0(x,t) = \sum_{j=0}^{\infty} f_j(x) \otimes \delta^{(j)}(t). \tag{3.16}$$

We define

$$v(x,t) = \tilde{u}(x,t) - \sum_{j=0}^{\infty} f_j(x) \underset{(x)}{*} \partial_t^j E(x,t).$$

Then we can easily see from Lemma 3.4 that $v(x,t)$ is a temperature function in Ω which belongs to $\mathcal{E}_{\{N_p\}} = \mathcal{E}_{\{M_p^2\}}$ since $p!^2 \subset N_p$. Moreover, if we take $u_0 = \sum_{j=0}^{\infty} \Delta^j f_j$, then it follows from (3.15) that for every $L > 0$, $h > 0$ and $\delta > 0$,

$$\begin{aligned} |u_0(\phi)| &\le \sum_{j=0}^{\infty} |\Delta^j f_j(\phi)|, \quad \phi \in \mathcal{E}_{\{M_p\}} \\ &\le \sum_{j=0}^{\infty} CL^j M_j^{-2} \sup_{\substack{x \in K_\delta \\ \alpha}} \frac{|\partial^\alpha (\Delta^j \phi(x))|}{h^{|\alpha|} M_{|\alpha|}^2}. \end{aligned} \tag{3.17}$$

For $\phi \in \mathcal{E}_{\{M_p\}}$ with $|\phi|_{M_p,K_\delta,h} < +\infty$,

$$\tag{3.18} \begin{aligned} \sup_{x\in K_\delta} |\partial^\alpha(\Delta^j\phi)| &\le |\phi|_{M_p,K_\delta,h} n^j h^{|\alpha|+2j} M_{|\alpha|+2j} \\ &\le C|\phi|_{M_p,K_\delta,h}(\sqrt{n}hH^2)^{2j}(hH)^{|\alpha|}M_{|\alpha|}M_j^2. \end{aligned}$$

Here the last inequality follows from (M.2).

Then we can show that

$$\begin{aligned} |u_0(\phi)| &\le C'|\phi|_{M_p,K_\delta,h}\sum_{j=0}^{\infty}(nLh^2H^4)^j \sup_\alpha \frac{H^{|\alpha|}}{M_{|\alpha|}} \\ &\le C''|\phi|_{M_p,K_\delta,h}, \end{aligned}$$

taking $L > 0$ so small that $nLh^2H^4 < 1$. This means that $u_0 \in \mathcal{E}'_{\{M_p\}}(K_0)$.

Now it remains to show that $u(x,t)$ with the growth (3.13) can be continued to an ultradistribution $\tilde{u}(x,t) \in \mathcal{D}'_{\{N_p\}}(\Omega)$, where $N_p = M_p^2$. To do this we denote by $u_+(x,t) = u(x,t)$ for $t > 0$ and $u_-(x,t) = u(x,t)$ for $t < 0$. Then in view of (3.13), $u_+(x,t)$ satisfies that for every $L > 0$

$$\tag{3.19} |u_+(x,t)| \le C\exp N^*\left(\frac{L}{t}\right), \quad t > 0$$

near K. If $u_+(x,t)$ is bounded near K, then we can consider $u_+(x,t)$ as an element of $\mathcal{D}'_{\{N_p\}}$ by defining its value to be 0 for $t \le 0$. So we may assume that $u_+(x,t)$ is unbounded for $t > 0$. Hence Lemma 3.2 implies that there exist a sequence $\ell_p \downarrow 0$ and a constant $C > 0$ such that

$$\tag{3.20} |u_+(x,t)| \le C\sup_p \frac{N_0 p!\ell_1\ell_2\cdots\ell_p}{N_p t^p}, \quad t > 0.$$

Moreover, Lemma 3.1 implies that we can choose an ultradifferential operator P of class $\{N_p\}$ and v, $w \in C_0^\infty(\mathbb{R})$ such that

$$\tag{3.21} P\left(\frac{d}{dt}\right)v(t) = \delta(t) + \omega(t), \quad t \in \mathbb{R},$$

$$\tag{3.22} \begin{gathered} |v(t)| \le C\inf_p \frac{t^p N_p}{p!\ell_1\ell_2\cdots\ell_p N_0}, \\ \text{supp } v \subset [0,\varepsilon], \quad \text{supp } w[\varepsilon/2,\varepsilon], \end{gathered}$$

where ε is chosen to be sufficiently small. We now define, for $(x,t) \in \Omega$ with $t > 0$,

$$\tag{3.23} g(x,t) = \int_0^\infty u_+(x,t+s)v(s)ds.$$

Then the growth (3.20) and (3.22) make $g(x,t)$ to be bounded near K, so that we can consider $g(x,t)$ as an element of $\mathcal{D}'_{\{N_p\}}(\Omega)$ by defining its value to be 0 for $t<0$. Similarly, the function given by the integral

$$h(x,t)=\int_0^\infty u_+(x,t+s)w(s)ds$$

can also be considered as an element of $\mathcal{D}'_{\{N_p\}}(\Omega)$ by the same argument.

On the other hand, applying $P\left(-\frac{d}{dt}\right)$ to $g(x,t)$ we can prove from (3.21) that

$$u_+(x,t)=P\left(-\frac{d}{dt}\right)g(x,t)-h(x,t),$$

which means that $u_+(x,t)$ can be continued to Ω as an element $\tilde{u}_+$ of $\mathcal{D}'_{\{N_p\}}(\Omega)$ so that $\tilde{u}_+(x,t)=0$ for $t<0$. The similar method makes it possible for $u_-(x,t)$ to be also continued to Ω as an element $\tilde{u}_-(x,t)$ of $\mathcal{D}'_{\{N_p\}}(\Omega)$ so that $\tilde{u}_-(x,t)=0$ for $t>0$. Then $\{\tilde{u}_+(x,t)+\tilde{u}_-(x,t)\}/2$ gives an element of $\mathcal{D}'_{\{N_p\}}$ which extends $u(x,t)$ to Ω. This completes the proof. $\square$

In particular, if $M_p=p!^s(s>1)$, then M_p satisfies all condition for defining sequence and $\overline{M}(t)\simeq t^{\frac{1}{2s-1}}$. Hence, we can easily obtain the following characterization for the Gevrey ultradistributions.

Corollary 3.7. *Let Ω and K be the same as in Theorem 3.6. Then if $u(x,t)$ is a temperature function in $\Omega\setminus K$ satisfying that for every $s\geq 1$ and $L>0$ (for some $L>0$, respectively) there exists a constant $C>0$ such that*

$$|u(x,t)|\leq C\exp\left(\frac{L}{|t-t_0|}\right)^{\frac{1}{2s-1}}\quad \textit{near}\quad K,$$

then there exist a temperature function $v(x,t)$ in Ω which belongs to $\mathcal{E}_{\{2s\}}(\Omega)$ ($\mathcal{E}_{(2s)}(\Omega)$, respectively) and a Gevrey ultradistribution $u_0\in\mathcal{E}'_{\{s\}}(K_0)$ ($u_0\in\mathcal{E}'_{(s)}(K_0)$, respectively) in $\mathbb{R}^n$ such that

$$u(x,t)=v(x,t)+u_0\underset{(x)}{*}E(x,t-t_0)\quad \textit{in}\quad \Omega\setminus K.$$

Particularly, if $M_p=p!$, then $\mathcal{E}'_{\{M_p\}}(K_0)$ is the same as the space $A'(K_0)$ of analytic functionals supported by K_0. Thus we obtain the following:

Corollary 3.8. *Let Ω and K be the same as in Theorem 3.6. Then if $u(x,t)$ is a temperature function in $\Omega\setminus K$ satisfying that for every $\varepsilon>0$ there is a constant $C>0$ such that*

$$|u(x,t)|\leq C\exp\left[\frac{\varepsilon}{|t-t_0|}\right]\quad \textit{near}\quad K$$

then there exist a temperature function $v(x,t)$ *in* Ω *which belongs to* $\mathcal{E}_{\{2\}}(\Omega)$ *and an analytic functional* $u_0 \in A'(K_0)$ *in* $\mathbb{R}^n$ *such that*

$$u(x,t) = v(x,t) + u_0 \underset{(x)}{*} E(x, t - t_0) \quad \mathit{in} \quad \Omega \setminus K.$$

REMARK. In all decomposition theorems given throughout this paper the expression is uniquely determined.

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