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WEAKLY HYPERBOLIC EQUATION WITH FAST OSCILLATING COEFFICIENTS

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0. Introduction

The results of [1] and those of the references therein give the reader a detailed overview about the well-posedness of the Cauchy problem for the wave equation in divergence form. Among other things it was proved in [1] that the Cauchy problem for the strictly hyperbolic equation $(a(t) \geq c > 0)$,

$$u_{tt} - a(t)u_{xx} = 0, \quad u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,,$$

has in Sobolev spaces a solution satisfying

$$\sup_{[0,T^*]} ||u(t)||_{H^{m+1-\beta t}(\mathbb{R})} + ||u'(t)||_{H^{m-\beta t}(\mathbb{R})} \leq C_0\left(1 + e^{C_m T^*}\right)\left(||u_0||_{H^{m+1}(\mathbb{R})} + ||u_1||_{H^m(\mathbb{R})}\right)$$

(T^* is independent of m) if $a(t)$ is only "Log-Lipschitz". This means we have a loss of regularity.

If we weaken $a(t) \geq c > 0$ to $a(t) \geq 0$, then we obtain a weakly hyperbolic Cauchy problem. Here high regularity of $a = a(t)$ is not sufficient for the existence of the solution in Sobolev spaces. In [2] it was shown that some oscillating behaviour of the coefficient $a(t)$ leads to a nonexistence result for the solutions in Sobolev spaces although the coefficient belongs to $C^\infty([0,\infty))$.

There are different ways to exclude this counterexample, that means, to control the influence of the oscillations. One way is to prescribe a nonlocal condition for $a = a(t) \in C^1([0,T])$ of the type

$$\int_0^T \frac{|a'(t)|}{a(t)+\varepsilon}\,dt \leq C(T)|\log\varepsilon| \quad \text{for all} \quad \varepsilon \in (0,1) \tag{0.1}$$

(see [7]). Some generalization of (0.1) ($a = a(t)$ is not supposed to be from C^1 but only from C^1 without a sequence of points, decreasing or increasing, to some finite point) was used in [5]. Another way is to suppose $a(t) \geq 0$, $a(t)$ is analytic. Then the only accumulation point may be at infinity. This condition was used in [3], [4] to prove some global existence result for semilinear weakly hyperbolic equations of the type

$$u_{tt} - a(t)\Delta u = f(u).$$

Let us explain a third way by the aid of the equation

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} = 0.$$

Here $\lambda^2(t)$ is a damping term which satisfies

$$(A1) \quad \lambda(t) \in C^2([0,T_0]),\ \lambda(0) = \lambda'(0) = 0,\ \lambda(t) > 0,\ \lambda'(t) \geq 0 \quad \text{for all} \quad t > 0\,.$$

The factor $b = b(t)$ produces oscillations. We suppose

$$(A2) \qquad b \in C^2((0,T_0]), \quad 0 < c \leq b(t) \leq C.$$

Then a condition to describe *slow oscillations* is the local condition (for $t \to 0$)

$$(0.2) \qquad |D_t b(t)| \leq C\frac{\lambda'(t)}{\lambda(t)}, \quad t \in (0,T_0].$$

The C^∞-well-posedness of the Cauchy problem for second-order equations with slowly oscillating coefficients was proved in [21]. It is clear that (0.2) implies (0.1) for $a(t) = \lambda^2(t)b^2(t)$. It will be interesting to construct an example with $a(t) = \lambda^2(t)b^2(t)$ under the conditions (A1),(A2), where in opposite to (0.1) the condition (0.2) is violated.

In [19] it is shown that the Cauchy problem for

$$(0.3) \qquad u_{tt} - \exp\left(-\frac{2}{t^\alpha}\right) b^2\left(\frac{1}{t}\right) u_{xx} = 0$$

is C^∞-well-posed if and only if $\alpha \geq 1/2$. Here $b(t)$ is a non-constant, 1-periodic positive function. This example and (0.2) show that not only the frequencies of oscillations are important, but more precisely the product between the amplitude and frequency. It is easily checked that for $\lambda(t) = \exp(-t^{-\alpha})$ with $\alpha \in [1/2;1)$ the condition (0.2) is not satisfied. An interpretation of this example and a definition of *fast oscillations* by the condition

$$|D_t b(t)| \leq C\frac{\lambda'(t)}{\lambda(t)}|\ln\lambda(t)|, \quad t \in (0,T_0],$$

were given in [23]. Moreover, there is proved that this condition is sufficient for C^∞-well-posedness (necessity for (0.3) is clear). Classes of linear hyperbolic equations with countable many singular or degenerate points are studied in [24].

If one is interested in solutions in Sobolev spaces, then one has to take into consideration the phenomenon of loss of derivatives which was mentioned at the beginning for strictly hyperbolic equation which coefficients have low regularity in the time variable. Examples from [12], [14] show the precise loss of regularity in the case without oscillations, but with a degeneracy in the main part and with a lower order term of the form $h(t)u_x$. This term has an important influence on the loss of regularity.

The main goal of the present paper is to show that one can even connect fast oscillations in the coefficients with a quasilinear structure of the equation. As a model case we consider the Cauchy problem

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} + h(t)u_x = f(t,x,u_x), \quad u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,.$$

The problem becomes weakly hyperbolic if $\lambda(0) = 0$ (time degeneracy). The coefficient of u_{xx} consists of the two parts. On the one hand we have a damping term $\lambda^2(t)$. We suppose for the function $\lambda = \lambda(t)$ the assumptions (A1) and

$(A3)$ there exist positive constants $d_0 \geq 1/2$ and d_1 such that

$$d_0\frac{\lambda(t)}{\Lambda(t)} \leq \frac{\lambda'(t)}{\lambda(t)} \leq d_1\frac{\lambda(t)}{\Lambda(t)}, \quad |\lambda''(t)| \leq d_1\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2 \text{ for all } \quad t \in (0,T_0]\,,$$

where $\Lambda(t) := \int_0^t \lambda(\tau)\,d\tau.$

On the other hand we have a term $b^2(t)$ which oscillates. We suppose for the coefficient $b(t)$ the assumption (A2) and

$$(A4) \qquad |D_t^k b(t)| \leq C\left(\frac{\lambda(t)}{\Lambda(t)}|\ln\lambda(t)|\right)^k \quad \text{for all} \quad t \in (0,T_0], \quad k = 1,2\,.$$

The coefficient $h(t)$ of the lower order term has to satisfy

$$(A5) \quad h \in C^1([0,T_0]), \quad |D_t^k h(t)| \leq C\lambda'(t)\left(\frac{\lambda(t)}{\Lambda(t)}|\ln\lambda(t)|\right)^k, \quad k = 0,1,$$

for all $t \in (0,T_0]\,.$

Setting $k = 0$ this means, that $h(t)$ satisfies a C^∞-type Levi condition, which is very close to a necessary one [21]. Moreover, we allow fast oscillations, too ($k = 1$ in (A5)).

Finally we have to assume for the right-hand side:

$(A6)$ for every given $s \in \mathbb{N}$ the function $f = f(t,x,p)$ is entire in p, with values in $C([0,T_0]; W_1^s(\mathbb{R}))$, that is

$$f(t,x,p) = \sum_{k=0}^{\infty} a_k(t,x)p^k \quad \text{for all} \quad p \in \mathbb{C},$$

with coefficients $a_k(x,t) \in C([0,T_0];W_1^s(\mathbb{R}))$.

Moreover, for every positive constant D it holds

$$\sum_{k=0}^{\infty} \|a_k(x,t)\|_{W_1^s(\mathbb{R})} D^k \le o(\lambda'(t)), \quad t \in [0,T_0].$$

Theorem 0.1. *Under the assumptions (A1) to (A6) there exists a (in general sufficiently large) constant r such that the Cauchy problem for quasilinear weakly hyperbolic equations with fast oscillating coefficients*

(0.4)

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} + h(t)u_x = f(t,x,u_x), \; u(x,0) = u_0(x), \; u_t(x,0) = u_1(x),$$

and with data u_0, u_1 *belonging to* $W_1^r(\mathbb{R})$, $W_1^{r-1}(\mathbb{R})$, *respectively, has a locally defined solution*

$$u \in C([0,T^*];H^3(\mathbb{R})) \bigcap C^1([0,T^*];H^2(\mathbb{R})) \bigcap C^2([0,T^*];H^1(\mathbb{R})).$$

We give now a brief outline of our approach to attack the above weakly hyperbolic Cauchy problem with fast oscillating coefficients. In Section 1 we will motivate the construction of a Banach space $B_{M,Q,T}$ with the following *strictly hyperbolic type property*:

The Cauchy problem

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} + h(t)u_x = \lambda'(t)f(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

has a uniquely determined classical solution u with

$$u/\lambda(t),\; u_t/\lambda(t),\; u_x \in B_{M,Q,T} \quad \text{if} \quad f \in B_{M,Q,T}.$$

In the strictly hyperbolic case $\lambda(t) \ge c > 0$ this property holds if we replace $B_{M,Q,T}$ by the space $C([0,T];H^M(\mathbb{R}))$. Even in the case with slow oscillations one can prove this property by using the weighted spaces $\lambda^Q(t)C([0,T];H^M(\mathbb{R}))$ (see, for instance, [11], [15], [8], [6]), where Q is sufficiently large. The situation is more complicated in the case with fast oscillations because it seems to be impossible to use the standard energy approach. The idea of the construction of $B_{M,Q,T}$ is to describe the elements u by the behaviour of its partial Fourier transform $\hat{u}$ in the cotangent space. The approach based on a certain division of the cotangent space and special micro-local considerations was developed by the second author [22] in connection with the construction of the fundamental solution for the Cauchy problem for hyperbolic operators with characteristics of variable multiplicity. It could be used in quite different situations [23], [16], [17].

The Lemmas 1.1 and 1.2 lead to a *strictly hyperbolic type property* in the cotangent space which is essential to get an idea for the construction of the nonstandard space $B_{M,Q,T}$. Some auxiliary relations which are helpful for studying properties of $B_{M,Q,T}$ are given at the end of Section 1. It is proved in Section 2 that $B_{M,Q,T}$ is even an algebra. This allows us to include nonlinearities in (0.4). In Section 3 we prove local existence and uniqueness of the solution for (0.4). Therefore we suggest the Levi condition of (A6) which is a bit restrictive compared with (A5).

1. A strictly hyperbolic type property

In this section we shall study the Cauchy problem for the linear hyperbolic equation with fast oscillations

$$(1.1)\quad u_{tt} - \lambda^2(t)b^2(t)u_{xx} - h(t)u_x = -\lambda'(t)f(x,t), \quad u(x,0) = u_t(x,0) = 0\,.$$

Partial Fourier transform leads to the Cauchy problem for the ordinary differential equation

$$(1.2)\quad D_t^2\hat{u} - \lambda^2(t)b^2(t)\xi^2\hat{u} + i\xi h(t)\hat{u} = \lambda'(t)\hat{f}(\xi,t), \quad \hat{u}(\xi,0) = D_t\hat{u}(\xi,0) = 0\,,$$

where $\hat{f} \in C([0,T]; L^\infty(\mathbb{R}))$ and $D_t := -i\partial_t$. By $t_\xi (= t_{\langle\xi\rangle})$, $\xi \in \mathbb{R}$, we denote the solution (see [23]) of the following equation

$$(1.3)\qquad \Lambda(t_\xi)\langle\xi\rangle = N\ln\langle\xi\rangle, \quad \text{where} \quad \langle\xi\rangle := (c + |\xi|^2)^{1/2},$$

where the sufficiently large positive numbers N and c will be chosen later.

Firstly, we study (1.2) in the so-called *pseudodifferential zone* [23]:

$$Z_{pd}(c,N) := \{(t,\xi) \in \mathbb{R}\times[0,T] \,:\, \Lambda(t)\langle\xi\rangle \le N\ln\langle\xi\rangle\}.$$

According to the arguments below the Cauchy problem feels neither oscillations nor hyperbolicity in this zone. Indeed, after substitutions $w_2 := D_t\hat{u}$, $w_1 := \rho(t,\xi)\hat{u}$, where $\rho(t,\xi)$ is the positive root of $\rho^2 - 1 - \dfrac{\lambda^2(t)}{\Lambda(t)}\langle\xi\rangle\ln\langle\xi\rangle = 0$, (1.2) reads as

$$D_tW - A(t,\xi)W = F, \quad W(0,\xi) = 0,$$

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \; F = \begin{pmatrix} 0 \\ \lambda'(t)\hat{f} \end{pmatrix}, \; A(t,\xi) = \begin{pmatrix} \dfrac{D_t\rho(t,\xi)}{\rho(t,\xi)} & \rho(t,\xi) \\ \dfrac{\lambda^2(t)b^2(t)\xi^2 - i\xi h(t)}{\rho(t,\xi)} & 0 \end{pmatrix}.$$

If $X(t,s,\xi)$ is a fundamental solution, that is,

$$D_tX - A(t,\xi)X = 0, \quad X(s,s,\xi) = I\,(\text{identity matrix}), \quad 0 \le s \le t \le t_\xi\,,$$

then $W(t,\xi) = \int_0^t X(t,s,\xi)F(s,\xi)\,ds$ solves the above Cauchy problem. Using the matrizant we obtain the estimate

$$\|X(t,s,\xi)\| \le \exp\left(\int_s^t \|A(\tau,\xi)\|\,d\tau\right), \quad t \ge s\,.$$

From the definition of $\rho(t,\xi)$, $Z_{pd}(c,N)$, conditions (A3), (A4), and (A5) it follows

$$\frac{\lambda^2(t)b^2(t)\xi^2 + |\xi h(t)|}{\rho(t,\xi)} \le C\rho(t,\xi)\,.$$

Here and in the following we use C as a universal positive constant. The conditions (A3) and $d_0 \ge 1/2$ yield $\partial_t\rho(t,\xi) \ge 0$. Thus

$$\|A(\tau,\xi)\| \le C_{pd}g(\tau,\xi), \quad \text{where} \quad g(\tau,\xi) := \rho(\tau,\xi) + \frac{\rho_\tau(\tau,\xi)}{\rho(\tau,\xi)}\,.$$

For w_1 and w_2 we obtain

$$\rho(t,\xi)\hat{u}(t,\xi) = \int_0^t X_{12}(t,s,\xi)\lambda'(s)\hat{f}(s,\xi)ds, D_t\hat{u}(t,\xi) = \int_0^t X_{22}(t,s,\xi)\lambda'(s)\hat{f}(s,\xi)ds.$$

Together with the above estimates this gives

$$\begin{aligned} |\xi|\,|\hat{u}(t,\xi)| &\le \int_0^t \frac{|\xi|\lambda'(s)}{\rho(s,\xi)}|X_{12}(t,s,\xi)||\hat{f}(s,\xi)|\,ds \\ &\le C\int_0^t \rho(s,\xi)\exp\left(C_{pd}\int_s^t g(\tau,\xi)\,d\tau\right)|\hat{f}(s,\xi)|\,ds, \end{aligned} \tag{1.4}$$

$$|\hat{u}_t(t,\xi)| \le C\int_0^t \lambda'(s)\exp\left(C_{pd}\int_s^t g(\tau,\xi)\,d\tau\right)|\hat{f}(s,\xi)|\,ds. \tag{1.5}$$

These inequalities lead to the next lemma.

Lemma 1.1. *If $\hat{f}(t,\xi) \in C([0,T];L^\infty(\mathbb{R}))$ satisfies in $Z_{pd}(c,N)$ the inequality*

$$|\hat{f}(t,\xi)| \le A(\xi)\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right), \quad Q > C_{pd},$$

with C_{pd} of (1.4), (1.5), then the solution $\hat{u}(t,\xi)$ of (1.2) satisfies in $Z_{pd}(c,N)$ the estimates

$$\begin{aligned} |\xi\hat{u}(t,\xi)| &\le C\frac{A(\xi)}{Q-C_{pd}}\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right), \\ |\hat{u}_t(t,\xi)| &\le CA(\xi)\lambda(t)\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right). \end{aligned}$$

Proof. From (1.4) we have

$$
\begin{aligned}
&|\xi\hat{u}(t,\xi)| \\
&\le C\int_0^t \rho(s,\xi)\exp\left(C_{pd}\int_s^t g(\tau,\xi)\,d\tau\right)A(\xi)\exp\left(Q\int_{t_\xi}^s g(\tau,\xi)\,d\tau\right)\,ds \\
&\le CA(\xi)\exp\left(C_{pd}\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right)\int_0^t g(s,\xi)\exp\left((Q-C_{pd})\int_{t_\xi}^s g(\tau,\xi)\,d\tau\right)\,ds \\
&= CA(\xi)\exp\left(C_{pd}\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right)\frac{1}{Q-C_{pd}}\int_0^t \frac{d}{ds} \\
&\quad \exp\left((Q-C_{pd})\int_{t_\xi}^s g(\tau,\xi)\,d\tau\right)\,ds \\
&\le C\frac{A(\xi)}{Q-C_{pd}}\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right).
\end{aligned}
$$

The desired estimate for $\hat{u}_t(t,\xi)$ can be shown by (1.5) as follows:

$$
\begin{aligned}
|\hat{u}_t(t,\xi)| &\le CA(\xi)\int_0^t \lambda'(s)\exp\left(C_{pd}\int_s^t g(\tau,\xi)\,d\tau\right)\exp\left(Q\int_{t_\xi}^s g(\tau,\xi)\,d\tau\right)\,ds \\
&= CA(\xi)\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right)\int_0^t \lambda'(s)\exp\left((C_{pd}-Q)\int_s^t g(\tau,\xi)\,d\tau\right)\,ds \\
&\le CA(\xi)\lambda(t)\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right). \qquad \square
\end{aligned}
$$

Now let us devote to (1.2) in the so-called *hyperbolic zone* [23]:

$$Z_{hyp}(c,N) := \{(t,\xi)\in[0,T]\times\mathbb{R} : \Lambda(t)\langle\xi\rangle \ge N\ln\langle\xi\rangle\}.$$

In opposite to $Z_{pd}(c,N)$ the Cauchy problem feels in $Z_{hyp}(c,N)$ as well as hyperbolicity and fast oscillations. Let us denote by

$$\tau_i(t,\xi) := (-1)^i b(t)\lambda(t)|\xi|, \quad i=1,2,$$

the characteristic roots. The transformation

$$V = MH\begin{pmatrix}\hat{u}\\ D_t\hat{u}\end{pmatrix}, \quad M=\frac{1}{2b(t)}\begin{pmatrix}b(t) & -1\\ b(t) & 1\end{pmatrix}, \quad H(t,\xi)=\begin{pmatrix}\lambda(t)|\xi| & 0\\ 0 & 1\end{pmatrix}$$

reduces (1.2) to the first-order system

(1.6)

$$D_tV-\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}V+\frac{1}{2}\begin{pmatrix}\frac{D_tb(t)}{b(t)}-\frac{D_t\lambda(t)}{\lambda(t)}-\frac{ih(t)\xi}{\tau_2} & -\frac{D_tb(t)}{b(t)}-\frac{D_t\lambda(t)}{\lambda(t)}-\frac{ih(t)\xi}{\tau_2}\\ -\frac{D_tb(t)}{b(t)}-\frac{D_t\lambda(t)}{\lambda(t)}+\frac{ih(t)\xi}{\tau_2} & \frac{D_tb(t)}{b(t)}-\frac{D_t\lambda(t)}{\lambda(t)}+\frac{ih(t)\xi}{\tau_2}\end{pmatrix}V$$
$$=\frac{\lambda'(t)\hat{f}}{2b(t)}\begin{pmatrix}-1\\ 1\end{pmatrix}$$

with diagonal main part. In connection with the construction of the fundamental solution in the strictly hyperbolic case [9] and even in the weakly hyperbolic case with slow oscillations [22] one understands how to apply the perfect diagonalizer. Contrary to these cases we are not able to follow this way for equations with fast oscillations. Nevertheless in [23] it is used exactly one step of perfect diagonalization to derive C^∞-well-posedness for linear weakly hyperbolic equations with fast oscillations. We use this idea to derive a corresponding result to Lemma 1.1 in $Z_{hyp}(c,N)$.

Setting

$$\mathcal{F}(t,\xi):=\frac{1}{2}\frac{D_tb(t)}{b(t)}I,\quad \mathcal{N}(t,\xi):=I+\mathcal{N}^{(1)}(t,\xi),\quad \text{where}$$

$$\mathcal{N}^{(1)}(t,\xi):=\frac{1}{4\tau_2^2}\begin{pmatrix}0 & D_t\tau_2+ih(t)\xi\\ -D_t\tau_2+ih(t)\xi & 0\end{pmatrix},$$

then the transformation $W=\mathcal{N}^{-1}V$ reduces (1.6) to

$$D_tW-\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}W+\mathcal{F}W+\mathcal{B}W=\mathcal{N}^{-1}\frac{\lambda'(t)\hat{f}}{2b(t)}\begin{pmatrix}-1\\ 1\end{pmatrix}. \tag{1.7}$$

Due to the assumptions $(A3),(A4)$ and (1.3) we know that $\|\mathcal{N}^{(1)}(t,\xi)\|\le 1/2$ if the constant N in the definition of $Z_{hyp}(c,N)$ is large enough. Moreover, one can show ([23]) that

$$\|\mathcal{B}(t,\xi)\|\le C\left(\frac{\lambda(t)}{\Lambda(t)}+\frac{\lambda(t)|\ln\lambda(t)|^2}{\langle\xi\rangle\Lambda^2(t)}\right)\quad \text{for all}\quad (t,\xi)\in Z_{hyp}(c,N).$$

In the following we will use the notation $K(t,\xi):=\dfrac{\lambda(t)}{\Lambda(t)}+\dfrac{\lambda(t)|\ln\lambda(t)|^2}{\langle\xi\rangle\Lambda^2(t)}$ of [23] for the right-hand side of the last estimate.

If $X(t,s,\xi)$ is a fundamental matrix of (1.7), that is,

(1.8)

$$D_tX-\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}X+\mathcal{F}X+\mathcal{B}X=0,\quad X(s,s,\xi)=I\,(\text{identity matrix}),$$

then the vector-valued function

$$W(t,\xi) = \int_{t_\xi}^{t} X(t,s,\xi)F(s,\xi)\,ds + X(t,t_\xi,\xi)W(t_\xi,\xi) \tag{1.9}$$

solves (1.7) (F denotes the right-hand side of (1.7)).

The *ansatz* $X(t,s,\xi) = E(t,s,\xi)Q(t,s,\xi)$, where

$$E(t,s,\xi) = \begin{pmatrix} \exp\left(\int_s^t \left\{ i\tau_1(r,\xi) - \frac{1}{2}\partial_r \ln b(r) \right\} dr\right) & 0 \\ 0 & \exp\left(\int_s^t \left\{ i\tau_2(r,\xi) - \frac{1}{2}\partial_r \ln b(r) \right\} dr\right) \end{pmatrix}$$

reduces (1.8) to

$$D_t Q + E(s,t,\xi)\mathcal{B}(t,\xi)E(t,s,\xi)Q = 0, \quad Q(s,s,\xi) = I\,.$$

Using $||E(s,t,\xi)\mathcal{B}(t,\xi)E(t,s,\xi)|| \le ||\mathcal{B}(t,\xi)|| \le CK(t,\xi)$ gives uniformly

$$||Q(t,s,\xi)|| \le \exp\left(C_{hyp}\int_s^t K(\tau,\xi)\,d\tau\right), \quad t_\xi \le s \le t, \tag{1.10}$$

$$||X(t,s,\xi)|| \le C\exp\left(C_{hyp}\int_s^t K(\tau,\xi)\,d\tau\right), \quad t_\xi \le s \le t, \tag{1.11}$$

respectively. Consequently, by (1.9)

$$\begin{aligned} ||W(t,\xi)|| \le\; & C\int_{t_\xi}^t \lambda'(s)\exp\left(C_{hyp}\int_s^t K(\tau,\xi)\,d\tau\right)|\hat{f}(s,\xi)|\,ds \\ & + \exp\left(C_{hyp}\int_{t_\xi}^t K(\tau,\xi)\,d\tau\right)||W(t_\xi,\xi)||. \end{aligned} \tag{1.12}$$

Due to Lemma 1.1 we obtain $||W(t_\xi,\xi)|| \le C\lambda(t_\xi)A(\xi)$. Let us choose

$$A(\xi) = \langle\xi\rangle^{-M}\exp\left(Q\int_{T_0}^{t_\xi} K(\tau,\xi)\,d\tau\right).$$

Now we have all tools to prove

Lemma 1.2. *If* $\hat{f} \in C([0,T];L^\infty(\mathbb{R}))$ *satisfies in* $Z_{hyp}(c,N)$ *the inequality*

$$|\hat{f}(t,\xi)| \le C\langle\xi\rangle^{-M}\exp\left(Q\int_{T_0}^t K(\tau,\xi)\,d\tau\right)$$

with $Q > C_{hyp}$, C_{hyp} *from (1.10),(1.11), then*

$$|\hat{u}_t(t,\xi)|/\lambda(t) + |\xi\hat{u}(t,\xi)| \le C\langle\xi\rangle^{-M}\exp\left(Q\int_{T_0}^t K(\tau,\xi)\,d\tau\right).$$

Proof. Using

$$\begin{aligned} ||W(t,\xi)|| &\le \int_{t_\xi}^{t} \lambda(s)K(s,\xi)\exp\left(C_{hyp}\int_s^t K(\tau,\xi)\,d\tau\right) C\langle\xi\rangle^{-M} \\ &\times \exp\left(Q\int_{T_0}^{s} K(\tau,\xi)\,d\tau\right) ds \\ &+ C\lambda(t_\xi)\langle\xi\rangle^{-M}\exp\left(Q\int_{T_0}^{t_\xi} K(\tau,\xi)\,d\tau\right)\exp\left(C_{hyp}\int_{t_\xi}^{t} K(\tau,\xi)\,d\tau\right) \end{aligned}$$

and $\lambda(t_\xi) \le \lambda(s) \le \lambda(t)$, then the same reasoning as in the proof of Lemma 1.1 leads to

$$||W(t,\xi)|| \le C\lambda(t)\langle\xi\rangle^{-M}\exp\left(Q\int_{T_0}^{t} K(\tau,\xi)\,d\tau\right).$$

The equivalence of $||W(t,\xi)||$ with $||(\lambda(t)|\xi|\hat{u},\hat{u}_t)||$ gives immediately the statement of the lemma. □

Further, we introduce the following weight function:

$$\mathcal{N}_{Q,M}(t,\xi) := \begin{cases} \langle\xi\rangle^M \exp\left(Q\int_t^{T_0} K(\tau,\xi)\,d\tau\right), & t\in[t_\xi,T_0], \\ \langle\xi\rangle^M \exp\left(Q\int_{t_\xi}^{T_0} K(\tau,\xi)\,d\tau\right)\exp\left(Q\int_t^{t_\xi} g(\tau,\xi)\,d\tau\right), & t\in[0,t_\xi]. \end{cases} \tag{1.13}$$

Corollary 1.1. *There exists a positive constant Q such that if $\hat{f}\in C([0,T];L^\infty(\mathbb{R}))$ satisfies the estimate (P), that is,*

$$(P)\quad \mathcal{N}_{Q,M}(t,\xi)|\hat{f}(t,\xi)| \le C\,, \quad \text{for all} \quad (t,\xi)\in[0,T_0]\times\mathbb{R},$$

where $M>0$, $T\le T_0$, then the Cauchy problem (1.2) has a uniquely determined solution $\hat{u}(t,\xi)$, where $\xi\hat{u}(t,\xi)$ and $\hat{u}_t(t,\xi)/\lambda(t)$ are satisfying the estimate (P), too.

This corollary hints at suitable function spaces in which one can handle equations with fast oscillating coefficients.

DEFINITION 1.1. For a given positive number M we denote by B^M the normed linear space

$$B^M = \left\{ u\in\mathcal{S}'(\mathbb{R}) \,:\, \hat{u}\in L^\infty_{loc} \text{ is a function satisfying } ||u||_M := \sup_{\xi\in\mathbb{R}}\langle\xi\rangle^M|\hat{u}(\xi)| < \infty \right\}.$$

For given positive numbers M, $M>1$, Q and T, $T\le T_0$, we denote by $B_{M,Q,T}$ the linear space

$$B_{M,Q,T} = \{u\in C([0,T];B^{M-1}) \,:\, \hat{u}(t,\xi) \text{ satisfies } \mathcal{N}_{Q,M}(t,\xi)|\hat{u}(t,\xi)| \le C \text{ for all } (t,\xi)\in[0,T]\times\mathbb{R}\}\,.$$

It is easily checked that B^M is a Banach space.

Lemma 1.3. *The space* $B_{M,Q,T}$ *is a Banach space with the norm*

$$\|u\|_{M,Q,T} := \max_{[0,T]} \sup_{\xi\in\mathbb{R}} \mathcal{N}_{Q,M}(t,\xi)|\hat{u}(t,\xi)|\,.$$

The topology of $B_{M,Q,T}$ *is stronger than that one induced by* $C([0,T];B^{M-1})$.

Theorem 1.1. *Let us consider under the assumptions (A1) to (A5) the Cauchy problem for the linear weakly hyperbolic equation with fast oscillating coefficients*

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} + h(t)u_x = \lambda'(t)f(x,t), \quad u(x,0) = u_t(x,0) = 0\,.$$

Then there is a positive constant Q *such that for every* f *belonging to* $B_{M,Q,T}$*, there exists an uniquely determined solution* u *with the property, that* $u/\lambda(t)$, $u_t/\lambda(t)$ *and* u_x *are belonging to* $B_{M,Q,T}$*, too. Moreover, there is a constant* C_{apr}*, independent of* $T \in (0,T_0]$ *and* f*, such that*

$$\|u/\lambda(t)\|_{M,Q,T} + \|u_x\|_{M,Q,T} + \|u_t/\lambda(t)\|_{M,Q,T} \le C_{apr}\|f\|_{M,Q,T}\,. \tag{1.14}$$

Proof. From Lemma 1.1 and 1.2 we have

$$\|u_x\|_{M,Q,T} + \|u_t/\lambda(t)\|_{M,Q,T} \le C\|f\|_{M,Q,T}\,.$$

Using

$$\frac{\hat{u}(t,\xi)}{\lambda(t)}\mathcal{N}_{Q,M}(t,\xi) = \int_0^t \frac{\hat{u}_\tau(\tau,\xi)}{\lambda(\tau)}\mathcal{N}_{Q,M}(\tau,\xi)\frac{\lambda(\tau)}{\lambda(t)}\frac{\mathcal{N}_{Q,M}(t,\xi)}{\mathcal{N}_{Q,M}(\tau,\xi)}\,d\tau\,,$$
$$\mathcal{N}_Q(t,\xi) \le \mathcal{N}_Q(\tau,\xi)\,, \quad \lambda(\tau) \le \lambda(t)\,, \quad \tau \le t,$$

it follows

$$\|u/\lambda(t)\|_{M,Q,T} \le C\|u_t/\lambda(t)\|_{M,Q,T}$$

and consequently (1.14).

Further, due to Corollary 1.1 and Definition 1.1 we only have to show that $u/\lambda(t)$, u_x and $u_t/\lambda(t)$ belong to $C([0,T];B^{M-1})$. This is true if for the last two functions and for all $t_0 \in [0,T]$

$$\lim_{t\to t_0}\sup_{\xi\in\mathbb{R}}\langle\xi\rangle^{M-1}|\xi\hat{u}(t,\xi) - \xi\hat{u}(t_0,\xi)| = 0, \quad \lim_{t\to t_0}\sup_{\xi\in\mathbb{R}}\frac{\langle\xi\rangle^{M-1}}{\lambda(t)}|D_t\hat{u}(t,\xi) - D_t\hat{u}(t_0,\xi)| = 0.$$

In the pseudodifferential zone $Z_{pd}(c,N)$ we have

$$\xi\hat{u}(t,\xi) = \int_0^t X_{12}(t,s,\xi)\frac{\lambda'(s)\xi}{\rho(t,\xi)}\hat{f}(s,\xi)\,ds, \quad D_t\hat{u}(t,\xi) = \int_0^t X_{22}(t,s,\xi)\lambda'(s)\hat{f}(s,\xi)\,ds\,,$$

while in the hyperbolic zone $Z_{hyp}(c,N)$ we have

$$W(t,\xi) = \int_{t_\xi}^t X(t,s,\xi)\mathcal{N}^{-1}(s,\xi)\frac{\lambda'(s)}{2b(s)}\hat{f}(s,\xi)\,ds\begin{pmatrix}-1\\1\end{pmatrix} + X(t,t_\xi,\xi)W(t_\xi,\xi).$$

If $W = W(t,\xi)$ satisfies

$$\lim_{t\to t_0}\sup_{\xi\in\mathbb{R}}\frac{\langle\xi\rangle^{M-1}}{\lambda(t)}|W(t,\xi)-W(t_0,\xi)| = 0,$$

then after transformation we obtain the desired continuity for $\xi\hat{u}(t,\xi)$ and $D_t\hat{u}(t,\xi)/\lambda(t)$ in t.

Let us begin with $t_0 > 0$. For ξ from a compact set we obtain obviously the continuity in t. Thus, we restrict ourselves to large $|\xi|$. Then $t_\xi < t_0$, consequently, it is enough to study $W = W(t,\xi)$. We have for $t > t_0$

$$\begin{aligned}
&\frac{\langle\xi\rangle^{M-1}}{\lambda(t)}(W(t,\xi)-W(t_0,\xi))\\
&\quad= \int_{t_0}^t X(t,s,\xi)\mathcal{N}^{-1}(s,\xi)\frac{\lambda'(s)}{2b(s)}\frac{\langle\xi\rangle^{M-1}}{\lambda(t)}\hat{f}(s,\xi)\,ds\begin{pmatrix}-1\\1\end{pmatrix}\\
&\qquad+\int_{t_\xi}^{t_0}(X(t,s,\xi)-X(t_0,s,\xi))\mathcal{N}^{-1}(s,\xi)\frac{\lambda'(s)}{2b(s)}\frac{\langle\xi\rangle^{M-1}}{\lambda(t)}\hat{f}(s,\xi)\,ds\begin{pmatrix}-1\\1\end{pmatrix}\\
&\qquad+(X(t,t_\xi,\xi)-X(t_0,t_\xi,\xi))\frac{\langle\xi\rangle^{M-1}}{\lambda(t)}W(t_\xi,\xi)\,.
\end{aligned}$$

The first integral tends to 0 as $t\to t_0$ uniformly for large ξ. Thus we only have to take into consideration in the second integral and in the third term the behaviour of $X(t,s,\xi)-X(t_0,s,\xi)$. By using the matrizant we obtain the inequality

$$|X(t,s,\xi)-X(t_0,s,\xi)| \le C\langle\xi\rangle|t-t_0|\exp\left(C_{hyp}\left|\int_s^t K(\tau,\xi)\,d\tau\right|\right), s\ge t_\xi, t\ge\frac{t_0+t_\xi}{2}.$$

Indeed, the following estimate is evident

$$\begin{aligned}
|X(t,s,\xi)-X(t_0,s,\xi)| &\le |Q(t,s,\xi)||E(t,s,\xi)-E(t_0,s,\xi)|\\
&\quad+|E(t_0,s,\xi)||Q(t,s,\xi)-Q(t_0,s,\xi)|
\end{aligned}$$

while for $t_\xi \le s,t$, from the integral representation of the matrizant we obtain

$$\begin{aligned}
Q(t,s,\xi)-Q(t_0,s,\xi) &= \int_{t_0}^t \mathcal{R}(t_1,s,\xi)\,dt_1 + \int_{t_0}^t\int_s^{t_1}\mathcal{R}(t_1,s,\xi)\,\mathcal{R}(t_2,s,\xi)\,dt_1\,dt_2\\
&\quad+\int_{t_0}^t\int_s^{t_1}\int_s^{t_3}\mathcal{R}(t_1,s,\xi)\,\mathcal{R}(t_2,s,\xi)\,\mathcal{R}(t_3,s,\xi)\,dt_1\,dt_2\,dt_3+\dots
\end{aligned}$$

where notation $\mathcal{R}(t,s,\xi) := E(s,t,\xi)\mathcal{B}(t,\xi)E(t,s,\xi)$ is used. For the matrix $\mathcal{R}(t,s,\xi)$ we have proved the estimate $|\mathcal{R}(t,s,\xi)| \le K(t,\xi)$ for all $t_\xi \le s,t$ and all large ξ. This implies

$$|Q(t,s,\xi)-Q(t_0,s,\xi)|\langle\xi\rangle^{-1} \le C\left|\int_{t_0}^t \langle\xi\rangle^{-1}K(\tau,\xi)\,d\tau\right|\exp\left(C_{hyp}\left|\int_s^t K(\tau,\xi)\,d\tau\right|\right)ds$$
$$\le C|t-t_0|\exp\left(C_{hyp}\left|\int_s^t K(\tau,\xi)\,d\tau\right|\right)ds\,,\quad t_\xi \le s,t\,.$$

Further, it is also easily seen that

$$|Q(t,s,\xi)||E(t,s,\xi)-E(t_0,s,\xi)|\langle\xi\rangle^{-1} \le C|t-t_0|\exp\left(C_{hyp}\left|\int_s^t K(\tau,\xi)\,d\tau\right|\right)$$

for all $t_\xi \le s,t,$ such that $t \ge (t_0+t_\xi)/2$. Together with

$$\langle\xi\rangle^M|\hat f(s,\xi)| \le C\exp\left(Q\int_{T_0}^s K(\tau,\xi)\,d\tau\right)$$

above mentioned second integral can be estimated by

$$C|t-t_0|\int_{t_\xi}^{t_0} K(s,\xi)\exp\left(C_{hyp}\left|\int_s^t K(\tau,\xi)\,d\tau\right|\right)\exp\left(Q\int_{T_0}^s K(\tau,\xi)\,d\tau\right)ds.$$

Similar to the proof of Lemma 1.1 we choose $Q > C_{hyp}$ to get the continuity in t. The same holds for the third term if we take into consideration estimate for $W(t_\xi,\xi)$. Thus, the continuity in t is proved for $t_0>0$, u_x and $u_t/\lambda(t)$ belong to $C((0,T];B^{M-1})$.

In order to show continuity at the point $t=0$ we have to find for every given $\varepsilon>0$ a positive number $\delta=\delta(\varepsilon)$ such that

$$\langle\xi\rangle^{M-1}|\xi\hat u(t,\xi)| \le \varepsilon,\quad \langle\xi\rangle^{M-1}|D_t\hat u(t,\xi)| \le \varepsilon \quad \text{for all} \quad (t,\xi)\in[0,\delta]\times\mathbb{R}.$$

For $(t,\xi)\in[0,T]\times\mathbb{R}$ we have due to Lemmas 1.1, 1.2

$$\begin{aligned}\langle\xi\rangle^M|\xi||\hat u(t,\xi)| &\le C\exp\left(Q\int_{T_0}^t K(\tau,\xi)\,d\tau\right),\quad t\in[t_\xi,T],\\ \langle\xi\rangle^M|\xi||\hat u(t,\xi)| &\le C\exp\left(Q\int_{T_0}^{t_\xi} K(\tau,\xi)\,d\tau\right)\exp\left(Q\int_{t_\xi}^t g(\tau,\xi)\,d\tau\right),\quad t\in[0,t_\xi].\end{aligned}$$

It follows

$$\begin{aligned}\langle\xi\rangle^M|\xi||\hat u(t,\xi)| &\le C\exp\left(Q\int_{T_0}^t K(\tau,\xi)\,d\tau\right),\quad t\in[t_\xi,T],\\ \langle\xi\rangle^M|\xi||\hat u(t,\xi)| &\le C\exp\left(Q\int_{T_0}^{t_\xi} K(\tau,\xi)\,d\tau\right),\quad t\in[0,t_\xi].\end{aligned}$$

At first we choose to a given positive number ε a number $\delta_1(\varepsilon)$ such that

$$C\left(\frac{\Lambda(\delta_1(\varepsilon))}{\Lambda(T_0)}\right)^Q \le \varepsilon$$

and then the positive number $A(\varepsilon)$ with $t_{A(\varepsilon)} = \delta_1(\varepsilon)$. For $\langle\xi\rangle \ge A(\varepsilon)$ consider $t \le \delta_1(\varepsilon)$. It is clear that $t_\xi \le t_{A(\varepsilon)}$. If $t \le t_\xi$, then

$$\langle\xi\rangle^M|\xi||\hat{u}(t,\xi)| \le C\exp\left(Q\int_{T_0}^{t_\xi} K(\tau,\xi)d\tau\right) \le C\left(\frac{\Lambda(t_\xi)}{\Lambda(T_0)}\right)^Q \le C\left(\frac{\Lambda(\delta_1(\varepsilon))}{\Lambda(T_0)}\right)^Q \le \varepsilon.$$

For $\delta_1(\varepsilon) \ge t \ge t_\xi$ we have

$$\langle\xi\rangle^M|\xi||\hat{u}(t,\xi)| \le C\exp\left(Q\int_{T_0}^{t} K(\tau,\xi)\,d\tau\right) \le C\exp\left(Q\int_{T_0}^{\delta_1(\varepsilon)} K(\tau,\xi)\,d\tau\right) \le \varepsilon\,.$$

Finally, for the solution of the Cauchy problem for the ordinary differential equation (1.2) with homogeneous initial data and with a parameter taken from the compact set $\{\xi \in \mathbb{R} : \langle\xi\rangle \le A(\varepsilon)\}$, one can find a positive number $\delta_2(\varepsilon)$ such that the estimate

$$\langle\xi\rangle^M|\xi||\hat{u}(t,\xi)| \le \varepsilon$$

holds for all $t \le \delta_2(\varepsilon)$ and all ξ of that compact. Analogously we prove the statement for $\langle\xi\rangle^{M-1}|D_t\hat{u}(t,\xi)|/\lambda(t)$. But $u/\lambda(t) \in C([0,T];B^{M-1})$ follows immediately from $u_t/\lambda(t) \in C([0,T];B^{M-1})$. This completes the proof. □

At the end of this section we cite some properties of the auxiliary functions $t = t_\xi$, $g = g(t,\xi)$ and $K = K(t,\xi)$ which we need in the next section.

Lemma 1.4 ([23]).
a) The first derivative of t_ξ is equal to

$$\frac{dt_\xi}{d\langle\xi\rangle} = \frac{N(1-\ln\langle\xi\rangle)}{\lambda(t_\xi)\langle\xi\rangle^2} \le 0,$$

where $\langle\xi\rangle = (c+|\xi|^2)^{1/2}$, $c \ge e^2$.
b) For every large N there is a constant $c(N)$ such that $t_\xi \le T_0$ for all $\xi \in \mathbb{R}$ and all $c \ge c(N)$.
c) If c is sufficiently large, then there exist constants C_1 and C_2 such that $C_1\ln\langle\xi\rangle \le |\ln\lambda(t_\xi)| \le C_2\ln\langle\xi\rangle$ for all $\xi \in \mathbb{R}$.
d) There exists a constant C_N such that $\int_0^{t_\xi} g(\tau,\xi)\,d\tau \le C_N\ln\langle\xi\rangle$.
e) There exists a constant C_N such that $\int_{t_\xi}^{T_0} K(\tau,\xi)\,d\tau \le C_N\ln\langle\xi\rangle$.

Proof.
to c): Indeed, after integration of condition (A3) we obtain

$$C_1 \ln \frac{\Lambda(T_0)}{\Lambda(t_\xi)} \le \ln \frac{\lambda(T_0)}{\lambda(t_\xi)} \le C_2 \ln \frac{\Lambda(T_0)}{\Lambda(t_\xi)}\,,$$
$$C_1 \ln \frac{\Lambda(T_0)\langle\xi\rangle}{N\ln\langle\xi\rangle} \le \ln \frac{\lambda(T_0)}{\lambda(t_\xi)} \le C_2 \ln \frac{\Lambda(T_0)\langle\xi\rangle}{N\ln\langle\xi\rangle}\,,$$

respectively. But this gives the inequalities of c).

to d): It holds

$$\begin{aligned}\int_0^{t_\xi}\left(\rho(\tau,\xi)+\frac{\rho_\tau(\tau,\xi)}{\rho(\tau,\xi)}\right)d\tau &\le C+\int_0^{t_\xi}\lambda(\tau)(\Lambda(\tau))^{-1/2}\langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2}\,d\tau\\ &\quad+\ln\rho(t_\xi,\xi)-\ln\rho(0,\xi)\\ &\le C+C\left(\Lambda(t_\xi)\langle\xi\rangle\ln\langle\xi\rangle\right)^{1/2}+\ln\rho(t_\xi,\xi)\\ &\le C+CN^{1/2}\ln\langle\xi\rangle+\frac{1}{2}\ln\left(1+\frac{\lambda^2(t_\xi)}{\Lambda(t_\xi)}\langle\xi\rangle\ln\langle\xi\rangle\right).\end{aligned}$$

Together with Lemma 1.4 c) we obtain the desired estimate.

to e): It holds

$$\begin{aligned}\int_{t_\xi}^{T_0}\left(\frac{\lambda(\tau)}{\Lambda(\tau)}+\frac{\lambda(\tau)|\ln\lambda(\tau)|^2}{\langle\xi\rangle\Lambda^2(\tau)}\right)d\tau &= \int_{t_\xi}^{T_0} d\ln\Lambda(\tau)-\int_{t_\xi}^{T_0}\frac{|\ln\lambda(\tau)|^2}{\langle\xi\rangle}\,d\left(\frac{1}{\Lambda(\tau)}\right)\\ &\le \ln\Lambda(T_0)-\ln\Lambda(t_\xi)+\frac{|\ln\lambda(t_\xi)|^2}{\langle\xi\rangle\Lambda(t_\xi)}\,.\end{aligned}$$

Using definition of the function $t = t_\xi$ and Lemma 1.4 c) completes the proof. □

2. Properties of $\mathbf{B_{M,Q,T}}$

Lemma 2.1. *If* $M-l>3/2$, $l>0$ *and* $Q>0$, *then* $B_{M,Q,T}\subset C\left([0,T];H^l(\mathbb{R})\right)$.

Proof. For $u \in B_{M,Q,T}$ we have

$$\begin{aligned}\|u(t_0,x)-u(t_1,x)\|^2_{H^l(\mathbb{R})} &= \int_{\mathbb{R}}|\hat{u}(t_0,\xi)-\hat{u}(t_1,\xi)|^2\langle\xi\rangle^{2l}\,d\xi\\ &= \int_{\mathbb{R}}|\hat{u}(t_0,\xi)-\hat{u}(t_1,\xi)|^2\langle\xi\rangle^{2(M-1)}\langle\xi\rangle^{2l-2(M-1)}\,d\xi\\ &\le \varepsilon\int_{\mathbb{R}}\langle\xi\rangle^{2l-2(M-1)}\,d\xi\,.\end{aligned}$$

□

Lemma 2.2. *To given positive M and Q there exists a positive constant r_0 such that for all $r \geq r_0$ the imbedding $C([0,T]; W_1^r(\mathbb{R})) \subset B_{M,Q,T}$ holds. In particular,*

$$\|u\|_{M,Q,T} \leq C_{imb} \max_{[0,T]} \|u(x,t)\|_{W_1^r(\mathbb{R})} \quad \text{for all} \quad u \in C([0,T]; W_1^r(\mathbb{R})) . \tag{2.1}$$

Proof. Due to (1.13) and Lemma 1.4 d),e) we have

$$\mathcal{N}_{Q,M}(t,\xi) \leq C\langle\xi\rangle^{C_1} \quad \text{for all} \quad (t,\xi) \in [0,T_0] \times \mathbb{R}.$$

Using for $r \geq 0$

$$|\hat{u}(t,\xi)|\langle\xi\rangle^r \leq C \left| \int_{\mathbb{R}} e^{-ix\xi} \langle\xi\rangle^r u(t,x)\, dx \right| \leq C\|u(t,\cdot)\|_{W_1^r(\mathbb{R})}$$

we obtain for $r \geq r_0 := C_1$

$$\|u\|_{M,Q,T} = \max_{[0,T]} \sup_{\xi \in \mathbb{R}} |\hat{u}(t,\xi)| \mathcal{N}_{Q,M}(t,\xi) \leq C_{imb} \max_{[0,T]} \|u(x,t)\|_{W_1^r(\mathbb{R})} .$$

The lemma is proved. □

We need the next lemma for proving Theorem 2.1, the main result of this section.

Lemma 2.3. a) *Let $f = f(x)$ be a continuously differentiable function defined on $[0,\infty)$. Suppose that $f'(x)$ is decreasing and that $xf'(x) \leq f(x)$ on $[0,\infty)$. Then $f(x) \leq f(y) + f(x-y)$ for $y \leq x$.*
b) *Let $f = f(x)$ be a continuously differentiable function defined on $[0,\infty)$. If $|xf'(x)| \leq C$ on $[0,\infty)$, C independent of x, then*

$$|f(x) - f(y)| \leq C \quad \text{for all} \quad y \in [x/2, x] .$$

Proof.
a) The statement is equivalent to $f(x+y) \leq f(x) + f(y)$ for all $x, y \in [0,\infty)$. We have with $z \in (x, x+y)$, $y \leq x$,

$$f(x+y) = f(x) + f'(z)y \leq f(x) + f'(y)y \leq f(x) + f(y).$$

b) We have with $z \in (y,x)$, $y \in [x/2, x]$,

$$f(x) = f(y) + f'(z)(x-y) = f(y) + zf'(z)(x-y)/z .$$

Hence, $|f(x) - f(y)| \leq C(x-y)/z \leq C$. □

The next theorem contains a statement which gives the property for $B_{M,Q,T}$ to be an algebra.

Theorem 2.1. *There exist positive numbers Q and M such that $\mathcal{N}_{Q,M}(t,\xi)$ is a temperate weight, that is,*

(2.2) $\mathcal{N}_{Q,M}(t,\xi) \le C\mathcal{N}_{Q,M}(t,\eta)\mathcal{N}_{Q,M}(t,\xi-\eta)$ for all $\xi,\eta \in \mathbb{R},\ t \in [0,T_0]$,

where C is independent of t,ξ,η.

Proof. In the following we set $\sigma := \xi - \eta$. The inequality (2.2) is equivalent to

$$\ln \mathcal{N}_{Q,M}(t,\xi) \le A + \ln \mathcal{N}_{Q,M}(t,\eta) + \ln \mathcal{N}_{Q,M}(t,\sigma)\,. \tag{2.3}$$

It is enough to distinguish the following cases:
1. $|\eta| \ge 2|\xi|$,
2. $|\eta| \in [\,|\xi|/2, |\xi|\,]$,
3. $|\eta| \in [\,|\xi|, 2|\xi|\,]$, where η and ξ have the same sign.

to 1. We have to show with suitable constants A and M

$$\ln \mathcal{N}_Q(t,\xi) - \ln \mathcal{N}_Q(t,\eta) - \ln \mathcal{N}_Q(t,\sigma) \le A + M \ln \frac{\langle\sigma\rangle\langle\eta\rangle}{\langle\xi\rangle}\,, \tag{2.4}$$

where $\mathcal{N}_Q(t,\xi) := \mathcal{N}_{Q,0}(t,\xi)$. Taking account of $\mathcal{N}_Q(t,\xi) \ge 1$ this follows from $\ln \mathcal{N}_Q(t,\xi) \le A + C \ln \dfrac{\langle\sigma\rangle\langle\eta\rangle}{\langle\xi\rangle}$. Applying Lemma 1.4 gives that the left-hand side of this inequality can be estimated by $C\ln\langle\xi\rangle$. It is clear that the right-hand side of (2.4) can be minorated by $C\ln\langle\xi\rangle$ because of $\langle\xi\rangle \le \langle\sigma\rangle \le \langle\eta\rangle$. Thus after a special choice of Q we can determine a constant M such that (2.4) and (2.3) are satisfied.

to 2. The different subcases are obtained by the aid of $t_\sigma \ge t_\eta \ge t_\xi$, $|\sigma| \le |\xi|/2$. The relation (2.3) is fulfilled for $t \in [t_\eta, T_0]$ because the left-hand side of (2.4) is non-positive. Here one has to use $\langle\eta\rangle \le \langle\xi\rangle$. We need a more careful analysis for the subcases $t \in [t_\xi, t_\eta]$ and $t \in [0, t_\xi]$. For the left-hand side we have

(2.5)
$$\int_{t_\sigma}^{T_0} (K(\tau,\xi) - K(\tau,\eta) - K(\tau,\sigma))\,d\tau + \int_{t_\eta}^{t_\sigma} (K(\tau,\xi) - K(\tau,\eta))\,d\tau$$
$$+ \int_t^{t_\eta} K(\tau,\xi)\,d\tau - \int_t^{t_\eta} g(\tau,\eta)\,d\tau - \int_t^{t_\sigma} g(\tau,\sigma)\,d\tau\,, \quad t \in [t_\xi, t_\eta],$$

(2.6)
$$\int_{t_\sigma}^{T_0} (K(\tau,\xi) - K(\tau,\eta) - K(\tau,\sigma))\,d\tau + \int_{t_\eta}^{t_\sigma} (K(\tau,\xi) - K(\tau,\eta))\,d\tau$$
$$+ \int_{t_\xi}^{t_\eta} K(\tau,\xi)\,d\tau - \int_t^{t_\eta} g(\tau,\eta)\,d\tau - \int_t^{t_\sigma} g(\tau,\sigma)\,d\tau + \int_t^{t_\xi} g(\tau,\xi)\,d\tau\,, \quad t \in [0, t_\xi].$$

Let us devote to (2.5). The function $K(\tau,\xi)$ is decreasing in $|\xi|$. Thus the first two integrals are non-positive. Let us study

$$\begin{aligned}
&\int_t^{t_\eta} K(\tau,\xi)\,d\tau - \int_t^{t_\eta} g(\tau,\eta)\,d\tau \\
&\qquad = \int_t^{t_\eta} \frac{\lambda(\tau)(\ln\lambda(\tau))^2}{\Lambda^2(\tau)\langle\xi\rangle}\,d\tau + \int_t^{t_\eta} \frac{\lambda(\tau)}{\Lambda(\tau)}\,d\tau \\
&\quad - \int_t^{t_\eta} \rho(\tau,\eta)\,d\tau - \int_t^{t_\eta} \frac{\rho_\tau(\tau,\eta)}{\rho(\tau,\eta)}\,d\tau \\
&\quad \le C + \int_t^{t_\eta} \left(\frac{\lambda(\tau)(\ln\lambda(\tau))^2}{\Lambda^2(\tau)\langle\xi\rangle} - \frac{\lambda(\tau)}{\sqrt{\Lambda(\tau)}}\langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2} \right) d\tau \\
&\quad + \int_t^{t_\eta} \frac{\lambda(\tau)}{\sqrt{\Lambda(\tau)}} \left(\langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2} - \langle\eta\rangle^{1/2}(\ln\langle\eta\rangle)^{1/2} \right) d\tau. \qquad (2.7)
\end{aligned}$$

Here we have used $\rho_\tau(\tau,\eta) \ge 0$ which follows from $d_0 \ge 1/2$ of (A3).

A change of T_0 has only an influence on the constant A in (2.4). Thus we can make the constant $T_0 \le 1$ smaller such that $\lambda(\tau) \le 1$ for $\tau \in [0, T_0]$. Then

$$\begin{aligned}
\frac{(\ln\lambda(\tau))^2}{\Lambda^{3/2}(\tau)} &\le \frac{(\ln\lambda(t_\xi))^2}{\Lambda^{3/2}(t_\xi)} \le \frac{(\ln\langle\xi\rangle)^2\langle\xi\rangle^{3/2}}{N^{3/2}(\ln\langle\xi\rangle)^{3/2}} \\
&\le \frac{(\ln\langle\xi\rangle)^{1/2}\langle\xi\rangle^{3/2}}{N^{3/2}} \le (\ln\langle\xi\rangle)^{1/2}\langle\xi\rangle^{3/2} \quad (N \ge 1)\,. \qquad (2.8)
\end{aligned}$$

This shows that the integrand of the first integral is non-positive.

Now let us assume for a moment that

$$\langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2} \le \langle\eta\rangle^{1/2}(\ln\langle\eta\rangle)^{1/2} + \langle\sigma\rangle^{1/2}(\ln\langle\sigma\rangle)^{1/2}\,. \qquad (2.9)$$

Then

$$\begin{aligned}
&\int_t^{t_\eta} \frac{\lambda(\tau)}{\sqrt{\Lambda(\tau)}} \left(\langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2} - \langle\eta\rangle^{1/2}(\ln\langle\eta\rangle)^{1/2} \right) d\tau \\
&\le \int_t^{t_\sigma} \frac{\lambda(\tau)}{\sqrt{\Lambda(\tau)}}\langle\sigma\rangle^{1/2}(\ln\langle\sigma\rangle)^{1/2}\,d\tau \le \int_t^{t_\sigma} g(\tau,\sigma)\,d\tau \\
&\le C_N \ln\langle\sigma\rangle
\end{aligned}$$

by Lemma 1.4. To prove (2.9) we use Lemma 2.3 a). Let us define the function $f(\langle\xi\rangle) := \langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2}$. Then we have

$$\langle\xi\rangle\frac{df(\langle\xi\rangle)}{d\langle\xi\rangle} = \frac{f(\langle\xi\rangle)}{2}\left(1 + (\ln\langle\xi\rangle)^{-1}\right) \le f(\langle\xi\rangle), \quad \frac{d^2 f(\langle\xi\rangle)}{d\langle\xi\rangle^2} \le -\frac{(\ln\langle\xi\rangle)^{1/2}}{2\langle\xi\rangle^{3/2}} \le 0.$$

Thus the assumptions of Lemma 2.3 a) are satisfied. We obtain

$$f(\langle\xi\rangle) \le f(\langle\eta\rangle) + f(\langle\xi\rangle - \langle\eta\rangle) \le f(\langle\eta\rangle) + f(\langle\sigma\rangle)\,.$$

Consequently, (2.5) can be estimated by $C\ln\langle\sigma\rangle$. Comparing (2.5) with (2.6) then the consideration of the last one can be restricted to the additional integral

$$\begin{aligned}&\int_t^{t_\xi} (g(\tau,\xi)-g(\tau,\sigma)-g(\tau,\eta))\,d\tau\\&\le C+\int_t^{t_\xi}\frac{\lambda(\tau)}{\sqrt{\Lambda(\tau)}}\Big(\langle\xi\rangle^{1/2}(\ln\langle\xi\rangle)^{1/2}-\langle\sigma\rangle^{1/2}(\ln\langle\sigma\rangle)^{1/2}-\langle\eta\rangle^{1/2}(\ln\langle\eta\rangle)^{1/2}\Big)\,d\tau\\&+\frac{1}{2}\int_t^{t_\xi} d_\tau\left(\ln\rho^2(\tau,\xi)-\ln\rho^2(\tau,\eta)-\ln\rho^2(\tau,\sigma)\right)\,d\tau.\end{aligned}\tag{2.10}$$

Using (2.9) the first integral is non-positive. For the second one we obtain

$$\frac{1}{2}\ln\frac{\rho^2(t_\xi,\xi)\rho^2(t,\eta)\rho^2(t,\sigma)}{\rho^2(t,\xi)\rho^2(t_\xi,\eta)\rho^2(t_\xi,\sigma)}\le C\,.$$

Here we used the monotonicity of $\rho(t,\sigma)$ in t and $\langle\eta\rangle\le\langle\xi\rangle\le 2\langle\eta\rangle$.

Thus we have shown that in the second case the left-hand side of (2.4) can be majorized by $C+C\ln\langle\sigma\rangle$, C depends on Q, too. For the right-hand side we obtain

$$A+C\ln\frac{\langle\sigma\rangle\langle\eta\rangle}{\langle\xi\rangle}=A+C\ln\langle\sigma\rangle+C\ln\frac{\langle\eta\rangle}{\langle\xi\rangle}\ge A-C\ln 2+C\ln\langle\sigma\rangle\,.$$

Then to every Q we can find constants A and M such that (2.4) and (2.3) are satisfied.

to 3. The different subcases are obtained by the aid of $t_\sigma\ge t_\xi\ge t_\eta$, $|\sigma|\le|\xi|$. Using the same argument as for the second case the left-hand side of (2.4) is non-positive for $t\in[t_\sigma,T_0]$. For $t\in[t_\xi,t_\sigma]$ all integrals of the left-hand side of (2.4) are non-positive except

$$\int_t^{t_\sigma}\frac{\lambda(\tau)(\ln\lambda(\tau))^2}{\Lambda^2(\tau)}\Big(\frac{1}{\langle\xi\rangle}-\frac{1}{\langle\eta\rangle}\Big)\,d\tau\,.$$

This integral is now non-negative in opposite to the second case. Now we use

$$\frac{(\ln\lambda(\tau))^2}{\Lambda^2(\tau)\langle\xi\rangle\langle\eta\rangle}\le\frac{(\ln\lambda(\tau))^2}{\Lambda^2(\tau)\langle\xi\rangle^2}\le\frac{(\ln\lambda(t_\xi))^2}{\Lambda^2(t_\xi)\langle\xi\rangle^2}\le\frac{(\ln\langle\xi\rangle)^2}{N^2(\ln\langle\xi\rangle)^2}=\frac{1}{N^2}\,.$$

Using this inequality the above integral can be estimated in the following way:

$$\begin{aligned}\int_t^{t_\sigma}\frac{\lambda(\tau)(\ln\lambda(\tau))^2}{\Lambda^2(\tau)}\Big(\frac{1}{\langle\xi\rangle}-\frac{1}{\langle\eta\rangle}\Big)d\tau &\le \frac{1}{N^2}\int_0^{t_\sigma}\lambda(\tau)(\langle\eta\rangle-\langle\xi\rangle)d\tau\\&\le \frac{1}{N^2}\Lambda(t_\sigma)\langle\sigma\rangle=\frac{1}{N}\ln\langle\sigma\rangle\,.\end{aligned}$$

For $t \in [t_\eta, t_\xi]$ the left-hand side of (2.4) is equal to

$$\int_{t_\sigma}^{T_0} (K(\tau,\xi) - K(\tau,\eta) - K(\tau,\sigma))d\tau + \int_{t_\xi}^{t_\sigma} (K(\tau,\xi) - K(\tau,\eta))d\tau + \int_t^{t_\xi} g(\tau,\xi)d\tau - \int_t^{t_\sigma} g(\tau,\sigma)d\tau - \int_t^{t_\xi} K(\tau,\eta)d\tau \,.$$

If we prove that this sum can be majorized by $C + C\,\ln\langle\sigma\rangle$ for $t = t_\eta$, then the same is true for $t \in (t_\eta, t_\xi]$. Indeed, using the same ideas as in (2.7) this follows from

$$\begin{aligned}
&\int_{t_\eta}^{t} (K(\tau,\eta) + g(\tau,\sigma) - g(\tau,\xi))d\tau \\
&\quad \le C + \int_{t_\eta}^{t} \Big(\frac{\lambda(\tau)(\ln\lambda(\tau))^2}{\Lambda^2(\tau)\langle\eta\rangle} - \frac{\lambda(\tau)}{\sqrt{\Lambda(\tau)}}\langle\eta\rangle^{1/2}(\ln\langle\eta\rangle)^{1/2}\Big)\,d\tau + \int_{t_\eta}^{t_\xi} \frac{\lambda(\tau)}{\Lambda(\tau)}d\tau \\
&\qquad + C\ln\langle\sigma\rangle \\
&\quad \le C + \ln\frac{\Lambda(t_\xi)}{\Lambda(t_\eta)} + C\ln\langle\sigma\rangle \\
&\quad \le C + C\ln\langle\sigma\rangle \,.
\end{aligned}$$

Here we have used that the integrand of the first integral is non-positive and

$$\ln\frac{\Lambda(t_\xi)}{\Lambda(t_\eta)} = \ln\Big(\frac{\langle\eta\rangle\ln\langle\xi\rangle}{\langle\xi\rangle\ln\langle\eta\rangle}\Big) \le C \,. \tag{2.11}$$

For $t \in [0, t_\eta]$ the left-hand side of (2.4) is equal to

$$\int_{t_\xi}^{T_0} K(\tau,\xi)\,d\tau - \int_{t_\eta}^{T_0} K(\tau,\eta)\,d\tau - \int_{t_\sigma}^{T_0} K(\tau,\rho)\,d\tau + \int_t^{t_\xi} g(\tau,\xi)\,d\tau - \int_t^{t_\eta} g(\tau,\eta)\,d\tau - \int_t^{t_\sigma} g(\tau,\sigma)\,d\tau \,.$$

If we prove that this sum can be majorized by $C + C\ln\langle\sigma\rangle$ for $t = 0$, then the same is true for $t \in (0, t_\eta]$, especially for $t = t_\eta$. This follows similar to (2.10) if we replace $\int_t^{t_\xi}$ by $\int_0^t$. Consequently, the majorization for $t = 0$ implies even that one for $t \in (0, t_\xi]$. Setting

$$F(\langle\xi\rangle) = \int_{t_\xi}^{T_0} K(\tau,\xi)d\tau \quad \text{and} \quad G(\langle\xi\rangle) = \int_0^{t_\xi} g(\tau,\xi)d\tau$$

it remains to show that

$$F(\langle\xi\rangle) - F(\langle\eta\rangle) - F(\langle\sigma\rangle) + G(\langle\xi\rangle) - G(\langle\eta\rangle) - G(\langle\sigma\rangle)$$

can be estimated by $C + C\ln\langle\sigma\rangle$.

Firstly we show it for the function $G = G(\langle\xi\rangle)$. We have

$$\begin{aligned} &G(\langle\xi\rangle) - G(\langle\eta\rangle) - G(\langle\sigma\rangle) \\ &\le C + N^{1/2}\ln\frac{\langle\xi\rangle}{\langle\sigma\rangle\langle\eta\rangle} + \frac{1}{2}\ln\frac{\big(1+\lambda^2(t_\xi)\langle\xi\rangle^2/N\big)}{\big(1+\lambda^2(t_\eta)\langle\eta\rangle^2/N\big)\big(1+\lambda^2(t_\sigma)\langle\sigma\rangle^2/N\big)} \;. \end{aligned}$$

If $\lambda^2(t_\xi)\langle\xi\rangle^2 \le N$, then the last term can be estimated by $\frac{1}{2}\ln 2$. If $\lambda^2(t_\xi)\langle\xi\rangle^2 \ge N$, then

$$\begin{aligned} \ln\frac{\Big(1+\dfrac{\lambda^2(t_\xi)\langle\xi\rangle^2}{N}\Big)}{\Big(1+\dfrac{\lambda^2(t_\eta)\langle\eta\rangle^2}{N}\Big)\Big(1+\dfrac{\lambda^2(t_\sigma)\langle\sigma\rangle^2}{N}\Big)} &\le C + C\ln\frac{\lambda^2(t_\xi)\langle\xi\rangle^2}{\lambda^2(t_\eta)\langle\eta\rangle^2\lambda^2(t_\sigma)\langle\sigma\rangle^2} \\ &\le C + C\ln\frac{\langle\xi\rangle^2}{\langle\eta\rangle^2\langle\sigma\rangle^2} + C\ln\frac{\lambda(t_\xi)}{\lambda(t_\eta)\lambda(t_\sigma)} \\ &\le C + C\ln\frac{\lambda(t_\xi)}{\lambda(t_\eta)\lambda(t_\sigma)} \;. \end{aligned}$$

If we prove that $|\ln\lambda(t_\xi) - \ln\lambda(t_\eta)| \le C$, then $G(\langle\xi\rangle) - G(\langle\eta\rangle) - G(\langle\sigma\rangle)$ can be majorized by $A + C\ln\langle\sigma\rangle$. Here we have to recall $-\ln\lambda(t_\sigma) \le \ln\langle\sigma\rangle$. The above inequality is equivalent to

$$\Big|\ln\lambda(t_\eta)^{-1} - \ln\lambda(t_\xi)^{-1}\Big| \le C\;. \tag{2.12}$$

Let us consider therefore the function $f(\langle\xi\rangle) = \ln\lambda(t_\xi)^{-1}$. It holds

$$\frac{df(\langle\xi\rangle)}{d\langle\xi\rangle} \ge 0 \quad\text{and}\quad \langle\xi\rangle\frac{df(\langle\xi\rangle)}{d\langle\xi\rangle} = \frac{\Lambda(t_\xi)\lambda'(t_\xi)}{\lambda^2(t_\xi)}\frac{(\ln\langle\xi\rangle - 1)}{\ln\langle\xi\rangle} \le C$$

by condition (A3). Thus we can apply Lemma 2.3 b). This yields (2.12) and completes the calculations for the function G.

In a similar way one can handle the function F. We have

$$\begin{aligned} F(\langle\xi\rangle) &= \int_{t_\xi}^{T_0}\Big(\frac{\lambda(\tau)(\ln\lambda(\tau))^2}{\Lambda^2(\tau)\langle\xi\rangle} + \frac{\lambda(\tau)}{\Lambda(\tau)}\Big)d\tau = \frac{(\ln\lambda(t_\xi))^2}{N\ln\langle\xi\rangle} - \frac{(\ln\lambda(T_0))^2}{\Lambda(T_0)\langle\xi\rangle} \\ &\quad + \int_{t_\xi}^{T_0}\frac{2(\ln\lambda(\tau))\lambda'(\tau)}{\lambda(\tau)\Lambda(\tau)\langle\xi\rangle}d\tau + \ln\Lambda(T_0) - \ln\frac{N\ln\langle\xi\rangle}{\langle\xi\rangle}\;. \end{aligned}$$

The second term is uniformly bounded. Using $|\ln\lambda(\tau)| \le |\ln\lambda(t_\xi)|$, the same estimates as in the proof of Lemma 1.4c) and assumption (A3) one can show that the third term is bounded, too. Consequently, we have to take into consideration only the first and last term. Forming $F(\langle\xi\rangle) - F(\langle\eta\rangle) - F(\langle\sigma\rangle)$ gives on the one hand for the last terms

$$\ln\Big(\frac{\langle\xi\rangle\;\ln\langle\sigma\rangle\ln\langle\eta\rangle}{\langle\sigma\rangle\langle\eta\rangle\;\ln\langle\xi\rangle}\Big),$$

this term can be estimated by $A + C\ln\langle\sigma\rangle$, and on the other hand

$$\frac{1}{N}\Big(\frac{(\ln\lambda(t_\xi))^2}{\ln\langle\xi\rangle} - \frac{(\ln\lambda(t_\eta))^2}{\ln\langle\eta\rangle} - \frac{(\ln\lambda(t_\sigma))^2}{\ln\langle\sigma\rangle}\Big).$$

Setting $f(\langle\xi\rangle) = (\ln\lambda(t_\xi))^2/\ln\langle\xi\rangle$ we obtain for the derivative of f

$$\langle\xi\rangle\frac{df(\langle\xi\rangle)}{d\langle\xi\rangle} = \frac{f(\langle\xi\rangle)}{\ln\langle\xi\rangle}\Big(\frac{2\lambda'(t_\xi)\Lambda(t_\xi)(1-\ln\langle\xi\rangle)}{\lambda^2(t_\xi)\qquad\ln\lambda(t_\xi)} - 1\Big)$$

and consequently $\left|\langle\xi\rangle\frac{df(\langle\xi\rangle)}{d\langle\xi\rangle}\right| \le C$. After application of Lemma 2.3 b) we obtain

$$|f(\langle\eta\rangle) - f(\langle\xi\rangle)| = \left|\frac{(\ln\lambda(t_\eta))^2}{\ln\langle\eta\rangle} - \frac{(\ln\lambda(t_\xi))^2}{\ln\langle\xi\rangle}\right| \le C.$$

This relation implies the desired estimate (2.4) in the third case, too, if we use $A + C\ln\frac{\langle\sigma\rangle\langle\eta\rangle}{\langle\xi\rangle} \ge A + C\ln\langle\sigma\rangle$. All the considerations together lead to the statement of the theorem. □

Corollary 2.1. *There exist positive constants Q and M such that $B_{M,Q,T}$ is an algebra and*

(2.13) $\qquad \|uv\|_{M,Q,T} \le C_{alg}\|u\|_{M,Q,T}\|v\|_{M,Q,T}$ for all $u, v \in B_{M,Q,T}$.

Proof. Let $u, v \in B_{M,Q,T}$. Then using Lemma 2.1

$$\widehat{u\cdot v}(t,\xi) = \int_{\mathbb{R}} \frac{\hat{u}(t,\eta)\mathcal{N}_{Q,M}(t,\eta)\hat{v}(t,\xi-\eta)\mathcal{N}_{Q,M}(t,\xi-\eta)}{\mathcal{N}_{Q,M}(t,\eta)\mathcal{N}_{Q,M}(t,\xi-\eta)}\, d\eta\ .$$

Consequently,

$$\|uv\|_{M,Q,T} \le \|u\|_{M,Q,T}\|v\|_{M,Q,T}\int_{\mathbb{R}}\frac{\mathcal{N}_{Q,M}(t,\xi)}{\mathcal{N}_{Q,M}(t,\eta)\mathcal{N}_{Q,M}(t,\xi-\eta)}\, d\eta$$

$$= \|u\|_{M,Q,T}\|v\|_{M,Q,T}\int_{\mathbb{R}}\frac{\mathcal{N}_{Q,M-l}(t,\xi)}{\mathcal{N}_{Q,M-l}(t,\eta)\mathcal{N}_{Q,M-l}(t,\xi-\eta)}\left(\frac{\langle\xi\rangle}{\langle\eta\rangle\langle\xi-\eta\rangle}\right)^l d\eta\ .$$

If we choose $l > 1$ and $M - l$ equal to the constant M from Theorem 2.1, then we get (2.13). □

3. Proof of Theorem 0.1

a) *A successive approximation scheme*

Instead of (0.4) we consider the equivalent system of differential equations

$$\text{(3.1)}\quad u^{(0)}_{tt} = f(t,x,0), \qquad u^{(0)}(x,0) = u_0(x),\ \ u^{(0)}_t(x,0) = u_1(x)\ ,$$

$$v_{tt} - \lambda^2(t)b^2(t)v_{xx} + h(t)v_x = f(t,x,v_x + u^{(0)}_x) - f(t,x,0)$$

$$\text{(3.2)}\qquad\qquad + \lambda^2(t)b^2(t)u^{(0)}_{xx} - h(t)u^{(0)}_x, \qquad v(x,0) = v_t(x,0) = 0\ .$$

It is clear that $u = v + u^{(0)}$ yields a solution of (0.4) if $u^{(0)}, v$ solve (3.1), (3.2), respectively. Due to the assumptions of Theorem 0.1 and Lemma 2.2 there exists a constant r_1 such that $u^{(0)}$, $u_x^{(0)}$, $u_{xx}^{(0)} \in C([0,T]; W_1^{r_1}(\mathbb{R}))$, $B_{M,Q,T}$ respectively for $T \in (0, T_0]$.

To proceed further, we define the successive approximation scheme

$$\begin{aligned} v_{tt}^{(q+1)} - \lambda^2(t)b^2(t)v_{xx}^{(q+1)} + h(t)v_x^{(q+1)} &= f(t,x,v_x^{(q)} + u_x^{(0)}) - f(t,x,0) \\ &\quad + \lambda^2(t)b^2(t)u_{xx}^{(0)} - h(t)u_x^{(0)} \ , \\ v^{(q+1)}(x,0) = v_t^{(q+1)}(x,0) = 0 \ , \quad q = 0,1,\ldots, \qquad v^{(0)} \equiv 0 \ . \end{aligned}$$

Using (A5), (A6), Hadamard's formula, and Corollary 2.1 we obtain for $v^{(1)}$ the equation

$$\begin{aligned} v_{tt}^{(1)} - \lambda^2(t)b^2(t)v_{xx}^{(1)} + h(t)v_x^{(1)} &= \int_0^1 \partial_p f(t,x,\tau u_x^{(0)}(t,x))d\tau \ u_x^{(0)}(t,x) \\ &\quad + \lambda^2(t)b^2(t)u_{xx}^{(0)} - h(t)u_x^{(0)}, \end{aligned}$$

$$\begin{aligned} &\int_0^1 \partial_p f(t,x,\tau u_x^{(0)}(t,x))d\tau \ u_x^{(0)}(t,x) \\ &= \lambda'(t) \int_0^1 \sum_{k=0}^{\infty} (k+1)\frac{a_{k+1}(t,x)}{\lambda'(t)}(\tau u_x^{(0)}(t,x))^k d\tau \ u_x^{(0)}(t,x) \in \lambda'(t)B_{M,Q,T} \end{aligned}$$

if r_1 is large enough. Thus we can apply assumption (A5) and Theorem 1.1 to understand that $v_x^{(1)}$ belongs to $B_{M,Q,T}, T \in (0, T_0]$. The differences $w^{(q)} := v^{(q+1)} - v^{(q)}$ are satisfying

$$\begin{aligned} &w_{tt}^{(q+1)} - \lambda^2(t)b^2(t)w_{xx}^{(q+1)} + h(t)w_x^{(q+1)} = f(t,x,v_x^{(q+1)} + u_x^{(0)}) - f(t,x,v_x^{(q)} + u_x^{(0)}), \\ &w^{(q+1)}(x,0) = w_t^{(q+1)}(x,0) = 0 \ . \end{aligned}$$

Hadamard's formula gives

$$w_{tt}^{(q+1)} - \lambda^2(t)b^2(t)w_{xx}^{(q+1)} + h(t)w_x^{(q+1)} = \lambda'(t)g_q(t,x)w_x^{(q)}(t,x) \ ,$$

where for $q = 0, 1, \ldots$ we define

(3.3)

$$g_q(t,x) := \int_0^1 \sum_{k=0}^{\infty} (k+1)\frac{a_{k+1}(t,x)}{\lambda'(t)}(v_x^{(q)}(t,x) + u_x^{(0)}(t,x) + \tau w_x^{(q)}(t,x))^k d\tau.$$

The assumption (A6) guarantees that $g_0 \in B_{M,Q,T}$, $T \in (0, T_0]$. Hence, by Theorem 1.1 we have $w_x^{(1)} \in B_{M,Q,T}$. Thus we obtain step by step that $w^{(q)} \in B_{M,Q,T}$, $T \in (0, T_0]$, that is, all iterates $w^{(q)}$ are well-defined in $B_{M,Q,T}$.

b) *Estimates for the iterates*
Now we want to estimate the iterates $\{w^{(q)}\}$ and to show that there exists a constant T^* such that $\{v^{(q)}/\lambda(t)\}$, $\{v_t^{(q)}/\lambda(t)\}$ and $\{v_x^{(q)}\}$ are Cauchy sequences in B_{M,Q,T^*}. Then the limit v is obviously a solution belonging to B_{M,Q,T^*}. By Lemma 2.1 the function $v + u^{(0)}$ is a solution of our starting problem (0.4) valued in Sobolev spaces. The inequality (1.14) implies together with Corollary 2.1

$$\begin{aligned} ||w^{(q+1)}/\lambda(t)||_{M,Q,T} &+ ||w_t^{(q+1)}/\lambda(t)||_{M,Q,T} + ||w_x^{(q+1)}||_{M,Q,T} \\ &\leq C_{apr}C_{alg}||g_q||_{M,Q,T}||w_x^{(q)}||_{M,Q,T}. \end{aligned} \tag{3.4}$$

Lemma 3.1. *There exists a constant T^* such that the inequality $C_{apr}C_{alg}||g_q||_{M,Q,T^*} \leq 1/2$ holds for all $q = 1,\ldots$.*

Proof. From step a) we know that $w_x^{(0)}$ belongs to $B_{M,Q,T}, T \in (0,T_0]$. By Corollary 2.1 we have the estimates

$$\begin{aligned} ||g_q(t,x)||_{M,Q,T} &\leq \int_0^1 \sum_{k=0}^{\infty}(k+1)C_{alg}^k \left\|\frac{a_{k+1}}{\lambda'(t)}\right\|_{M,Q,T} ||v_x^{(q)} + u_x^{(0)} + \tau w_x^{(q)}||_{M,Q,T}^k d\tau \\ &\leq C_{imb}\int_0^1 \sum_{k=0}^{\infty}(k+1)C_{alg}^k \left\|\frac{a_{k+1}}{\lambda'(t)}\right\|_{C([0,T];W_1^{r_1}(\mathbb{R}))} \\ &\quad \times\left(||v_x^{(q)}||_{M,Q,T} + ||u_x^{(0)}||_{M,Q,T} + \tau||w_x^{(q)}||_{M,Q,T}\right)^k d\tau\,. \end{aligned}$$

Now let us set in assumption (A6) $D = 2(||u_x^{(0)}||_{M,Q,T_0} + ||w_x^{(0)}||_{M,Q,T_0})$. Then

$$\begin{aligned} &||w_x^{(1)}||_{M,Q,T} \\ &\leq C_{apr}C_{imb}\int_0^1 \sum_{k=0}^{\infty}(k+1)C_{alg}^k \left\|\frac{a_{k+1}}{\lambda'(t)}\right\|_{C([0,T];W_1^{r_1}(\mathbb{R}))} \tau^k \\ &\quad \times\left(||u_x^{(0)}||_{M,Q,T_0} + ||w_x^{(0)}||_{M,Q,T_0}\right)^k d\tau ||w_x^{(0)}||_{M,Q,T_0}\,. \end{aligned}$$

By (A6) we obtain $||w_x^{(1)}||_{M,Q,T} \leq C(T)$, where $C(T) \to 0$ for $T \to 0$. Thus $||w_x^{(1)}||_{M,Q,T_1} \leq ||w_x^{(0)}||_{M,Q,T_0}/2$ for a sufficiently small T_1. Analogously,

$$||g_1||_{M,Q,T} \leq C_{imb}\sum_{k=0}^{\infty}(k+1)||\frac{a_{k+1}}{\lambda'(t)}||_{C([0,T];W_1^{r_1}(\mathbb{R}))}(C_{alg}D)^k\,,$$

and finally $C_{apr}C_{alg}||g_1||_{M,Q,T^*} \leq 1/2$ for a sufficiently small $T^* \leq T_1$.

Applying (3.4) leads to $||w_x^{(2)}||_{M,Q,T^*} \leq ||w_x^{(0)}||_{M,Q,T_0}/4$. For estimating g_2 we use

$$||u_x^{(0)}||_{M,Q,T^*} + ||w_x^{(0)}||_{M,Q,T^*} + ||w_x^{(1)}||_{M,Q,T^*} + ||w_x^{(2)}||_{M,Q,T^*} \leq D\,.$$

Then the same reasoning gives the statement for $q = 2$ and so on with the same T^*. This proves the lemma. □

From (3.4) and Lemma 3.1 we conclude

$$\|w^{(q+1)}/\lambda(t)\|_{M,Q,T^*} + \|w_t^{(q+1)}/\lambda(t)\|_{M,Q,T^*} + \|w_x^{(q+1)}\|_{M,Q,T^*} \\ \leq (1/2)^{q+1}\|w_x^{(0)}\|_{M,Q,T_0} .$$

This yields the property of $\{v^{(q)}/\lambda(t)\}$, $\{v_t^{(q)}/\lambda(t)\}$ and $\{v_x^{(q)}\}$ to be Cauchy sequences in B_{M,Q,T^*}. It completes the proof of the main theorem. □

EXAMPLE 3.1. Now let us apply our main result to the Cauchy problem

$$u_{tt} - \exp\left(-\frac{2}{t^\alpha}\right) b^2\left(\frac{1}{t}\right) u_{xx} + h(t)u_x = f(t,x,\mu(t)u_x),$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $h(t)$ satisfies (A5) with $\lambda(t) = \exp(-t^{-\alpha})$ (see (0.3)). Firstly, we note that for $\alpha \geq 1/2$ the conditions (A1) to (A4) are satisfied. If $\alpha < 1/2$, then (A4) is violated.

If additionally to the assumptions (A1) to (A5) the function f is polynomial in $\mu(t)u_x$, then the above quasilinear weakly hyperbolic Cauchy problem with strong oscillations has a classical solution belonging to

$$u \in C([0,T^*]; H^3(\mathbb{R})) \cap C^1([0,T^*]; H^2(\mathbb{R})) \cap C^2([0,T^*]; H^1(\mathbb{R}))$$

if $\mu(t) = o(t^{-\alpha-1}\exp(-t^{-\alpha}))$ as $t \to 0$ and if the data u_0, u_1 belong to $W_1^{r_1}(\mathbb{R})$, $W_1^{r_1-1}(\mathbb{R})$ respectively, where r_1 is sufficiently large. It is clear that $\mu(t) = t^\beta\exp(-t^{-\alpha})$, $\beta > -(\alpha+1)$, satisfies the condition.

Up to now we have not discussed the uniqueness of solutions for (0.4).

Corollary 3.1. *The Cauchy problem (0.4) has a uniquely determined solution in $C^2([0,T]; W_1^{r_0+1}(\mathbb{R}))$, where r_0 is taken from Lemma 2.2.*

Proof. From Lemma 2.2 we know that the solution u and its derivative u_x belong to B_{M,Q,T_0}. The difference $w = u - v$ of two solutions u and v of (0.4), belonging to B_{M,Q,T_0}, satisfies

$$w_{tt} - \lambda^2(t)b^2(t)w_{xx} + h(t)w_x = \left(\int_0^1 \partial_p f(t,x,v_x + \tau w_x)d\tau\right) w_x ,$$
$$w(x,0) = w_t(x,0) = 0 .$$

As in the proof of Theorem 0.1 one can show that the right-hand side is equal to $\lambda'(t)g$, where g belonging to $B_{M,Q,T}$, $T \in (0,T_0]$, fulfils

$$||g||_{M,Q,T} \le C(T),\ C(T) \to 0 \quad \text{as} \quad T \to 0\,.$$

Thus a sufficiently small T^* gives together with Theorem 1.1 the estimate

$$||w_x||_{M,Q,T^*} \le C(T^*)||w_x||_{M,Q,T^*} \le ||w_x||_{M,Q,T^*}/2\,.$$

Consequently, the solution is uniquely determined for $t \in [0,T^*]$. For $t \in [T^*,T_0]$ the uniqueness follows from the strictly hyperbolic theory. □

REMARK 3.1. The approach of this paper to handle (0.4) can be applied to more general equations, for example,

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} + h(t)u_x = f(t,x,u,u_t,u_x)\,.$$

This follows from (1.14) and (3.5). It even allows to study Cauchy problems for the higher-dimensional case

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t)u_{x_i x_j} + \sum_{i=1}^{n} h_i(t)u_{x_i} = f(t,x,u,u_t,\nabla_x u)$$

under the assumptions

$$c\lambda^2(t)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j \le C\lambda^2(t)|\xi|^2, \quad c>0$$

$$|D_t^k a_{ij}(t)| \le C\lambda^2(t)\left(\frac{\lambda(t)|\ln\lambda(t)|}{\Lambda(t)}\right)^k, \quad k=0,1,2,$$

$$|D_t^k h_i(t)| \le C\lambda'(t)\left(\frac{\lambda(t)|\ln\lambda(t)|}{\Lambda(t)}\right)^k, \quad k=0,1.$$

The nonlinearity in right-hand side has to satisfy corresponding conditions to (A6).

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