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THE UNKNOTTING NUMBERS OF 10_{139} AND 10_{152} ARE 4

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1. Introduction and the statement of main results

A *link* is a closed oriented 1-manifold smoothly embedded in the 3-sphere S^3 and a *knot* is a link with one component. Let K be a knot. The *unknotting number* $u(K)$ of K is the minimal number of crossing changes needed to create the trivial knot.

In the 1960s, Milnor [5] conjectured that the unknotting number of any algebraic knot would be equal to the genus of the Milnor fiber. In the 1990s, Kronheimer and Mrowka [2] [3] proved this conjecture using gauge theory. In particular, the unknotting number of (p,q)-torus knot is $((p-1)(q-1))/2$. In 1995, Auckly [1] gave an alternative proof of this conjecture for certain torus knots using Seiberg-Witten theory. In this paper we shall compute unknotting numbers of certain knots using Seiberg-Witten theory.

An argument explained in Auckly's lecture notes [1] is based on the following theorem, which is the so-called generalized adjunction formula.

Theorem 1.1 (Kronheimer-Mrowka [4], Morgan-Szabó-Taubes [7]). *Let X be a smooth closed oriented four-manifold with $\dim H^2_+(X,R) > 1$. If $F \hookrightarrow X$ is a smoothly embedded closed oriented surface of genus $g \geq 1$ and K is a basic class of X, then*

$$2g-2 \geq K(F) + F \cdot F$$

where $A \cdot B$ denotes the intersection number of A and B.

By extending the argument in [1], we obtain the following Theorem.

Theorem 1.2. *Let K be an oriented knot in $S^3 \times \{1\}$ and $L_{m,n}$ the link in $S^3 \times \{0\}$ illustrated in Fig. 1. If there is a compact connected oriented surface $\hat{F}$ in $S^3 \times [0,1]$ such that $\partial \hat{F} = L_{m,n} \sqcup K$, then*

$$u(K) \geq (m-1)(n-1) - g(\hat{F})$$

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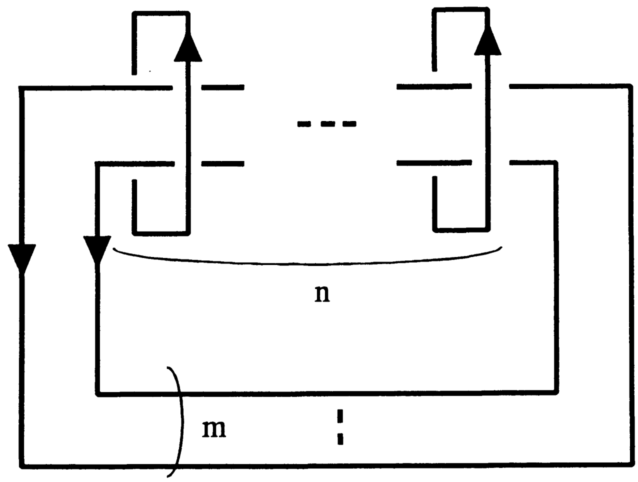

Fig. 1. Link $L_{m,n}$

where $g(\hat{F})$ *denotes the genus of* $\hat{F}$.

By means of Theorem 1.2, we can determine the unknotting numbers of certain knots. In particular, in the present paper, we show the following result.

Theorem 1.3. (1) *The unknotting number of the knot* 10_{139} *is* 4.
(2) *The unknotting number of the knot* 10_{152} *is* 4.

The unknotting numbers of 10_{139} and 10_{152} had been known to be either 3 or 4, but they had not been determined. The author was suggested by Kawauchi that this problem might be solved by Theorem 1.2, and Theorem 1.3 is the answer to it.

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2. The proof of Theorem 1.2 and its corollary

Before proving Theorem 1.2, we shall review the definition of basic class.

Let X be a smooth closed oriented 4-manifold with $b_2^+(X) > 1$. The Seiberg-Witten invariant of X is an integer valued function which is defined on the set of $Spin^c$-structures over X, (cf. for example [4] [6] [10]). This invariant is considered as a map

$$n : H^2(X; Z)/\text{Torsion} \to Z,$$

(cf. [1]). If $n(K) \neq 0$, then K is called a *basic class.*

To prove Theorem 1.2, we will use the following lemma;

Lemma 2.1 (Auckly [1]). *0 is a basic class for T^4.*

More generally, the work by Witten [10, pp. 786–789] implies that if X is a Kähler-Einstein manifold, the canonical class of X is a basic class. Lemma 2.1 can be also proved by this fact.

Auckly [1] presented a way of computing the unknotting number of $(2,5)$-torus knot applying Theorem 1.1 and Lemma 2.1 to a suitable surface in T^4. Theorem 1.2 is an extension of result in [1], and the argument in proof of Theorem 1.2 is almost same as his argument.

Proof of Theorem 1.2. We consider $T^2 = [0,1]^2/\sim$ and $T^4 = T^2 \times T^2$, where $\sim$ is the equivalent relation defined by $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$. We will construct a surface F embedded in T^4.

We define E and J by

$$E = \Bigg(\bigcup_{k=1,\cdots,m} \left\{ \left(\frac{k}{m+1}, \frac{k}{m+1} \right) \right\} \times T^2 \Bigg) \cup \Bigg(\bigcup_{k=1,\cdots,n} T^2 \times \left\{ \left(\frac{k}{n+1}, \frac{k}{n+1} \right) \right\} \Bigg)$$

$$J = \left[\frac{1}{m+2}, \frac{m+1}{m+2} \right]^2 \times \left[\frac{1}{n+2}, \frac{n+1}{n+2} \right]^2 .$$

The four-disk J includes all self-intersections of E. Then $\partial(E-J) = E \cap \partial J \subset \partial J$ is equivalent to $L_{m,n} \subset S^3$. We suppose that K can be unknotted by u-crossing changes. Then there is a surface $F' \subset D^4$ with genus u such that $\partial F' = K \subset S^3 \cong \partial D^4$.

We can regard $\partial J \times [0,1]$ as a collar of ∂J in J and $\partial J \times \{0\}$ as ∂J. We can identify $\partial J \times [0,1]$ with $S^3 \times [0,1]$ since $\partial J \cong S^3$. We can consider that $\hat{F}$ lies in $\partial J \times [0,1]$ and $\hat{F} \cap \partial J = E \cap \partial J$. We can identify the closure of $J - (\partial J \times [0,1])$ with D^4. We can consider that F' lies in the closure of $J - (\partial J \times [0,1])$ and $\hat{F} \cap \partial J \times \{1\} = \partial F'$. We define a surface F by

$$F = (E - J) \cup \hat{F} \cup F'.$$

The genus of F is equal to $m + n + g(\hat{F}) + u$. The self-intersection number $F \cdot F$ is the same as $E \cdot E$, because F is homologous to E. So $F \cdot F = E \cdot E = 2mn$.

We can apply Theorem 1.1 and Lemma 2.1 for F. Then we obtain the inequality

$$2(m + n + g(\hat{F}) + u) - 2 \geq 0 + 2mn.$$

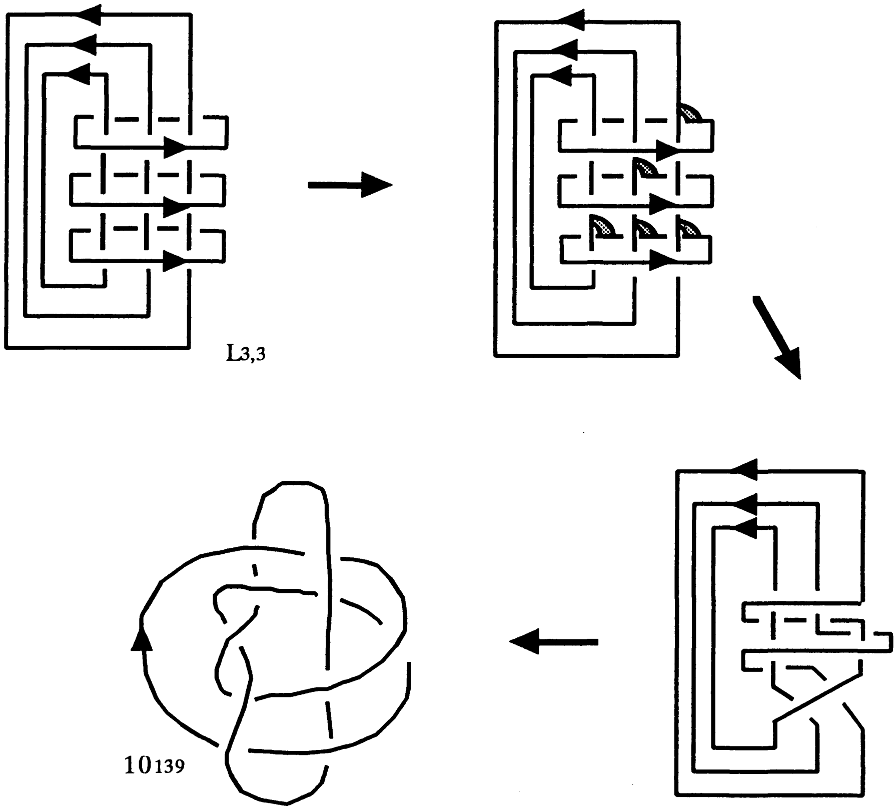

Fig. 2. Fusion procedure to obtain 10_{139} from $L_{3,3}$

Thus, we obtain the desired inequality $u \geq (m-1)(n-1) - g(\hat{F})$. $\square$

To prove Theorem 1.3, we show a corollary of Theorem 1.2. In order to state it, we shall review the definition of fusion procedure.

Let L be a μ-component oriented link. Let $B_1, \cdots B_\nu$ be mutually disjoint oriented bands in S^3 such that $B_i \cap L = \partial B_i \cap L = \alpha_i \cup \alpha_i'$, where $\alpha_1, \alpha_1', \cdots, \alpha_\nu, \alpha_\nu'$ are disjoint connected arcs. The closure of $L \cup \partial B_1 \cup \cdots \cup \partial B_\nu - \alpha_1 \cup \alpha_1' \cup \cdots \cup \alpha_\nu \cup \alpha_\nu'$ is also the link. We will write it by L'.

Definition. If L' has the orientation compatible with the orientation of $L - \bigcup_{i=1,\cdots,\nu} \alpha_i \cup \alpha_i'$ and $\bigcup_{i=1,\cdots,\nu}(\partial B_i - \alpha_i \cup \alpha_i')$, L' is called the link obtained from L by the *band surgery* along the bands $B_1, \cdots, B_\nu$. Moreover if L' has $(\mu-\nu)$-components, this transformation is called a *fusion*.

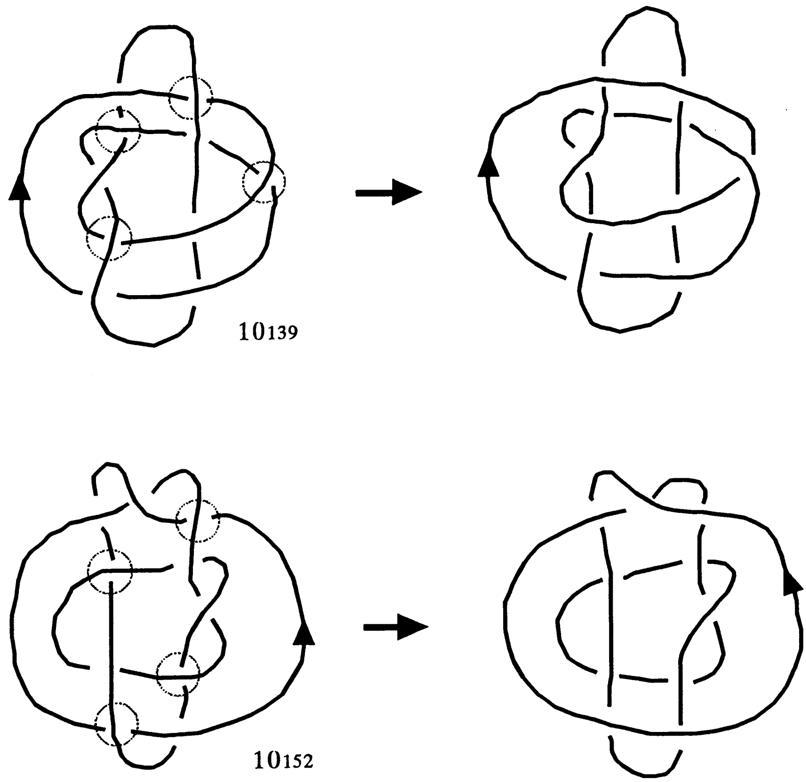

Fig. 3. Crossing changes

Corollary 2.2. *Let $L_{m,n}$ be the link illustrated in Fig. 1. If an oriented knot K in S^3 is obtained from $L_{m,n}$ by the fusion, then*

$$u(K) \geq (m-1)(n-1).$$

Proof of Corollary 2.2. To apply Theorem 1.2, we construct a suitable surface in $S^3 \times [0,1]$. Let $B_1, \cdots, B_{m+n-1}$ be the surgery bands. Identifying $\hat{F} \cap S^3 \times \{t\}$ with this band surgery in S^3 at time t ($t \in [0,1]$), we get a proper surface $\hat{F} \subset S^3 \times [0,1]$ such that $\partial \hat{F} = L_{m,n} \sqcup K$. Here we consider that K lies in $S^3 \times \{1\}$ and $L_{m,n}$ in $S^3 \times \{0\}$, and that $L_{m,n} \cup B_1 \cup \cdots \cup B_{m+n-1}$ lies in $S^3 \times \{1/2\}$. The surface $\hat{F}$ is homeomorophic to a surface which is obtained as S^2 minus $m+n+1$ disjoint open disks, so $g(\hat{F}) = 0$.

By Theorem 1.2, we have

$$u(K) \geq (m-1)(n-1). \qquad \square$$

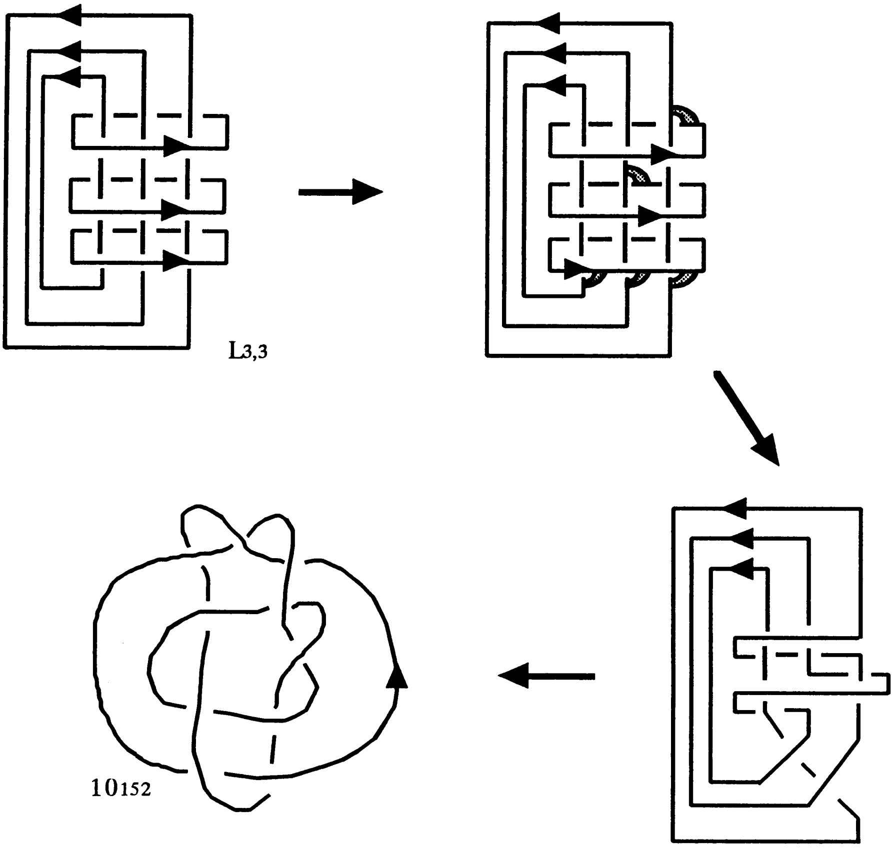

Fig. 4. Fusion procedure to obtain 10_{152} from $L_{3,3}$

3. The proof of Theorem 1.3

In this section, we prove Theorem 1.3 by using Corollary 2.2.

Proof of Theorem 1.3. (1) Fig. 2 shows that the knot 10_{139} is obtained from $L_{3,3}$ by connecting the components with 5 bands. It implies that 10_{139} is obtained from $L_{3,3}$ by a fusion.

By Corollary 2.2, we have

$$u(10_{139}) \geq (3-1)(3-1) = 4.$$

On the other hand, we observe that the knot 10_{139} can be unknotted by 4-crossing changes. By changing the crossings which are marked as in Fig. 3, we obtain the trivial knot. Therefore we conclude that $u(10_{139}) = 4$.

(2) Similarly to (1), we can show that the knot 10_{152} is obtained from $L_{3,3}$ by

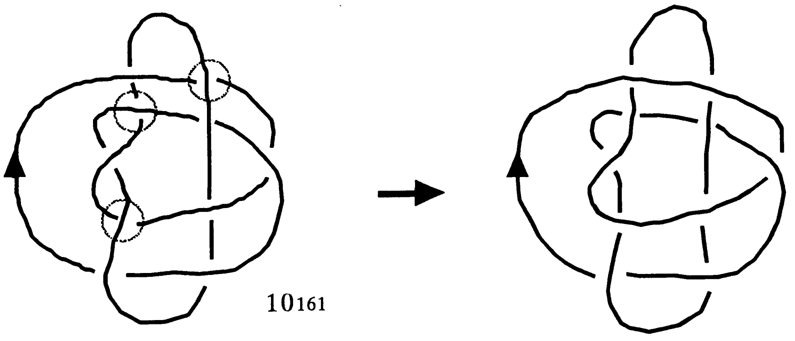

Fig. 5. Changing the crossings of 10_{161}

a fusion as in Fig. 4. By Corollary 2.2, we have

$$u(10_{152}) \geq (3-1)(3-1) = 4.$$

On the other hand, we observe that the knot 10_{152} can be unknotted by 4-crossing changes. By changing the crossings which are marked as in Fig. 3, we obtain the trivial knot. Therefore we conclude that $u(10_{152}) = 4$. □

REMARK. By Theorem 1.3(1), it can be shown that the unknotting number of 10_{161} is 3. 10_{161} is illustrated in Fig. 5, and can be obtained from 10_{139} by 1-crossing change. We see that 10_{161} cannot be unknotted by 2-crossing changes, because the unknotting number of 10_{139} is 4. On the other hand, we observe that the knot 10_{161} can be unknotted by 3-crossing changes. By changing the crossings which are marked as in Fig. 5, we obtain the trivial knot. Therefore we conclude that $u(10_{161}) = 3$. The author was informed by Shimokawa that Tanaka [9] proved the result $u(10_{161}) = 3$ using a result of Rudolph [8, pp. 56, Corollary] on a quasipositive link. That is, our arguments also give an alternative proof of a result of Tanaka. The author would like to thank Doctor Koya Shimokawa for his interest in this work.

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