Takegoshi, K.
Osaka J. Math.
34 (1997), 783–802

ON COHOMOLOGY GROUPS OF NEF LINE BUNDLES TENSORIZED WITH MULTIPLIER IDEAL SHEAVES ON COMPACT KÄHLER MANIFOLDS

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(Received October 25, 1996)

Introduction

Let X be a compact Kähler manifold of dimension n provided with a Kähler metric ω_X and let E be a holomorphic line bundle on X. E is said to be *numerically effective*, "*nef*" for short, if the real first Chern class $c_{R,1}(E)$ of E is contained in the closure of the Kähler cone of X. If X is projective algebraic, then E is nef if and only if $C \cdot E = \int_C c_{R,1}(E) \geq 0$ for any irreducible reduced curve C of X (cf.[13], §2 and [1], §6).

If E is nef, then for a fixed smooth metric h_E of E and a given sequence of positive numbers $\{\varepsilon_k\}_{k\geq 1}$ decreasing to zero, there exists a sequence of real-valued smooth functions $\{\varphi_k\}_{k\geq 1}$ such that every form $\Theta_E + dd^c\varphi_k + \varepsilon_k\omega_X$ yields a Kähler metric. Here Θ_E is the curvature form of E relative to h_E defined by $\Theta_E = dd^c(-\log h_E)$ with $d^c = \sqrt{-1}(\bar\partial - \partial)/2$. Normalizing φ_k in such a way that $\sup_X \varphi_k = 0$, we can show that φ_k converges to an integrable function φ_∞ on X so that $\Theta_E + dd^c\varphi_\infty$ is a positive current (cf. §2, Proposition 2.5). Such an integrable function φ_∞ is said to be *almost plurisubharmonic*. In general φ_∞ has singularities and $e^{-\varphi_\infty}$ is not integrable on X (cf. [11], [18]), which implies that the nefness is strictly weaker than the semi-positivity of line bundle in the sense of Kodaira (cf. [4], Example 1.7). Hence we can define a coherent analytic sheaf of ideal $\mathcal{I}(\varphi_\infty)$ associated to φ_∞ whose zero variety (possibly empty) is the set of points in a neighborhood of which $e^{-\varphi_\infty}$ is not integrable. The sheaf $\mathcal{I}(\varphi_\infty)$ is called the *multiplier ideal sheaf* associated to φ_∞ and we obtain the canonical homomorphism $\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega^n_X(E)) \longrightarrow H^q(X, \Omega^n_X(E))$ induced by $\iota(\varphi_\infty) : \mathcal{I}(\varphi_\infty) \bigotimes \Omega^n_X(E) \hookrightarrow \Omega^n_X(E)$.

Though φ_∞ can not be uniquely determined generally, the study of $H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega^n_X(E))$ is deeply related to several interesting problems in analytic and algebraic geometry (cf. [2], [3], [11], [12], [18]). Nevertheless not much is known about the cohomology group except a vanishing theorem for multiplier ideal sheaves associated to nef and big line bundles by Nadel (cf. [11]). We study the cohomology group by establishing a certain harmonic representation theorem. In particular we can determine the structure of Image $\iota^q(\varphi_\infty)$. As a consequence we can get the following Lefschetz type theorem (cf. [5], Theorem 0.3).

Theorem 1. *Let X be a compact Kähler manifold of dimension n provided with a Kähler metric ω_X and let E be a nef line bundle on X provided with a smooth hermitian metric h_E. Let φ_∞ be an integrable function determined as above ; i.e., $\Theta_E + dd^c\varphi_\infty$ is a positive current on X, and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to φ_∞. Then the homomorphism*

$$L^q : \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E)) \longrightarrow \text{Image } \iota^q(\varphi_\infty) \subset H^q(X, \Omega_X^n(E))$$

is surjective and the Hodge star operator relative to ω_X yields a splitting homomorphism

$$\delta^q : \text{Image } \iota^q(\varphi_\infty) \longrightarrow \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E))$$

with $L^q \circ \delta^q = \mathrm{id}$ for any $q \geq 1$.

The theorem was formulated and proved by Enoki in the case where E is semi-positive, in which case the zero variety defined by $\mathcal{I}(\varphi_\infty)$ is empty and $\iota^q(\varphi_\infty)$ is isomorphic. Furthermore we can obtain certain injectivity and vanishing theorems for the cohomology groups, which are weaker than the semi-positive line bundle case and are closely linked together to study a Kawamata-Viehweg type vanishing theorem on compact Kähler manifolds (cf. §4, Theorems 4.2 and 4.3). Actually the following vanishing theorem holds (cf. [5], [9], [10], [15], [17], [19]).

Theorem 2. *Let the situation be the same as in Theorem 1. Then if $q > n - \kappa_*(E)$*

$$\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E)) \longrightarrow H^q(X, \Omega_X^n(E))$$

is the zero homomorphism. Especially if $\iota^q(\varphi_\infty)$ is surjective (resp. injective) and $q > n - \kappa_*(E)$, then

$$H^q(X, \Omega_X^n(E)) = 0 \ (\textit{resp. } H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E)) = 0),$$

where $\kappa_*(E)$ is the numerical Kodaira dimension of E defined by

$$\kappa_*(E) := \max\{l \ : \ \overset{l}{\wedge} c_{R,1}(E) \neq 0 \in H^{2l}(X, R)\}.$$

REMARK. The above vanishing theorem is a variant of Kawamata-Viehweg's vanishing theorem for nef line bundles on projective algebraic manifolds (cf. [9],

[19]). We do not know whether Kawamata-Viehweg's vanishing theorem still holds on any compact Kähler manifold even if E is *nef* and *good* (cf. §3, Comment and §4, Remark 2).

1. Harmonic representation theorem for cohomology groups of multiplier ideal sheaves

1.1. Let X be a complex manifold of dimension n and let T be a d-closed (1, 1) current on X. Setting $d^c = \sqrt{-1}(\bar\partial - \partial)/2$ we suppose that T is decomposed as follows :

$$T = \Theta + dd^c\varphi_\infty$$

for a d-closed *smooth* real (1,1) form Θ and a locally integrable function φ_∞ on X. In this article we represent the positivity of T in the sense of current by the notation "$T \gtrsim 0$" and the semi-positivity (resp. positivity) of Θ by the notation "$\Theta \geq 0$" (resp. "$\Theta > 0$"). A function φ on X is said to be *almost plurisubharmonic* if φ is locally equal to the sum of a plurisubharmonic function and of a smooth function (cf. [1], §1). If $T \gtrsim 0$ and $d\Theta = 0$, then locally there exist a plurisubharmonic function ψ and a smooth function h such that $T = dd^c\psi$, $\Theta = dd^c h$ and $h + \varphi_\infty$ is equal almost everywhere to ψ. Hence the function φ_∞ is almost plurisubharmonic. The representation $\varphi_\infty = \psi - h$ is not unique. However if $\varphi_\infty = \psi - h = \psi_* - h_*$ with $\Theta = dd^c h_*$, then $\psi - \psi_*$ is *pluriharmonic*. In particular ψ is determined uniquely whenever h is fixed. Therefore we can define the following :

DEFINITION. The multiplier ideal sheaf $\mathcal{I}(\varphi_\infty) \subset \mathcal{O}_X$ associated to φ_∞ satisfying with $T = \Theta + dd^c\varphi_\infty \gtrsim 0$ is the sheaf of germs of holomorphic functions $f_x \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-\varphi_\infty}$ is integrable with respect to the Lebesgue measure in a local coordinates around x for any point x of X.

It is known that $\mathcal{I}(\varphi_\infty)$ is a coherent analytic ideal sheaf of $\mathcal{O}_X$ (cf. [11, 1.2] and [3, Lemma 4.4]). The zero variety $V(\mathcal{I}(\varphi_\infty))$ of $\mathcal{I}(\varphi_\infty)$ is the set of points in a neighborhood of which $e^{-\varphi_\infty}$ is not integrable.

1.2.

DEFINITION. A holomorphic line bundle E on X is said to be *pseudo effective* (resp. *semi-positive, positive*) if there exists a smooth hermitian metric h_E and an almost pluri-subharmonic function φ_∞ (resp. a smooth hermitian metric h_E) such that $\Theta_E + dd^c\varphi_\infty \gtrsim 0$ (resp. $\Theta_E \geq 0$, $\Theta_E > 0$) on X for the curvature form Θ_E relative to h_E defined by $\Theta_E = dd^c(-\log h_E)$.

EXAMPLE. Let $D = \sum_{j=1}^{k} m_j D_j$ be an effective divisor on X with irreducible components D_j and non-negative integers m_j, and let $[D_j]$ be the line bundle corresponding to each D_j. Then one can verify that the line bundle $F := \bigotimes_{j=1}^{k} [D_j]^{\otimes m_j}$ is pseudo effective by the Lelong-Poincaré formula. If D is a divisor with only normal crossings, then one can take a smooth hermitian metric h_F and an almost plurisubharmonic function φ_∞ such that $\Theta_F + dd^c\varphi_\infty \gtrsim 0$ and $\mathcal{I}(\varphi_\infty) = \mathcal{O}_X(F^*)$, where F^* is the dual line bundle of F (cf. [3], §5).

1.3. To study the cohomology groups of multiplier ideal sheaves of pseudo effective line bundles we need the following Dolbeault's lemma which is formulated for our purpose (cf. [2, Proposition 4.1] and [3, (5.3) Corollary]).

Theorem. *Let S be a Stein manifold of dimension n provided with a Kähler metric ω_S defined by $\omega_S := dd^c\Phi$ by a smooth strictly plurisubharmonic function $\Phi \geq 0$ on S. Suppose E (resp. F) be a pseudo effective (resp. positive) line bundle provided with a smooth metric h_E and an almost plurisubharmonic function φ_∞ (resp. a smooth metric h_F) such that $\Theta_E + dd^c\varphi_\infty \gtrsim 0$ (resp. $\Theta_F + dd^c\Phi > 0$). Set $(G, h_G) = (E \bigotimes F, h_E \bigotimes h_F)$. Then for any $u \in L^{n,q}_{loc}(S, G)$, $q \geq 1$, with $\bar\partial u = 0$ and*

$$\int_S |u|_G^2 e^{-\varphi_\infty - 2\Phi} dv_S < \infty$$

there exists $v \in L^{n,q-1}_{loc}(S, G)$ with $\bar\partial v = u$ and

$$q \int_S |v|_G^2 e^{-\varphi_\infty - 2\Phi} dv_S \leq \int_S |u|_G^2 e^{-\varphi_\infty - 2\Phi} dv_S.$$

1.4. Let X be an n dimensional complex manifold provided with a hermitian metric ω_X. Let E be a pseudo effective line bundle provided with a smooth metric h_E and an almost plurisubharmonic function φ_∞ with $\Theta + dd^c\varphi_\infty \gtrsim 0$ and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to φ_∞. Let F be a holomorphic line bundle provided with a smooth metric h_F and set $(G, h_G) = (E \bigotimes F, h_E \bigotimes h_F)$. We denote $\| \ \|_\infty$ the L^2-norm of G-valued forms relative to ω_X and $h_G e^{-\varphi_\infty}$, and denote $\mathcal{F}^q$ the sheaf of germs of G-valued (n, q) forms u with measurable coefficients such that both u and $\bar\partial u$ are locally square integrable relative to $\| \ \|_\infty$. By applying 1.3, Theorem to arbitrary small balls one can see that the complex of sheaves $\{\mathcal{F}^\bullet, \bar\partial\}$ provides a fine resolution of the sheaf $\mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(G)$. Hence letting $\Gamma(X, \mathcal{F}^q)$ be the space of global sections with values in $\mathcal{F}^q$ and seting $\mathcal{F}^{-1} = 0$, we obtain the following :

$$H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(G)) \cong \frac{\{u \in \Gamma(X, \mathcal{F}^q) : \bar\partial u = 0\}}{\{v \in \Gamma(X, \mathcal{F}^q) : v = \bar\partial w \ with \ w \in \Gamma(X, \mathcal{F}^{q-1})\}}$$

for any $q \geq 0$.

1.5. Let $C^q(\mathcal{U}, \mathcal{S})$ be the space of q co-chains associated to the locally finite Stein open covering $\mathcal{U}$ of X with values in the sheaf $\mathcal{S} := \mathcal{I}(\varphi_\infty) \bigotimes \Omega^n_X(G)$. Combining 1.3, Theorem with the above Dolbeault's theorem in 1.4 the Čech cohomology group $H^\bullet(\mathcal{U}, \mathcal{S})$ defined by the complex $\{C^\bullet(\mathcal{U}, \mathcal{S}), \delta\}$ with the co-boundary operator δ is isomorphic to the Dolbeault cohomology group $H^\bullet(X, \mathcal{S})$ in view of Leray's theorem ; i.e., the two complexes $\{\Gamma(X, \mathcal{F}^\bullet), \bar\partial\}$ and $\{C^\bullet(\mathcal{U}, \mathcal{S}), \delta\}$ are quasi-isomorphic. In particular if X is a compact complex manifold, then the Čech cohomology group $H^\bullet(\mathcal{U}, \mathcal{S})$ has finite dimension and so it is a separeted Fréchet topological vector space (cf. [7], Appendix B, 12. Theorem).

1.6. From now on we assume that X is a compact complex manifold. Let $L^{p,q}(X, G)$ (rsep. $L^{p,q}_\infty(X, G)$) be the L^2-space of G-valued square integrable $(p,\ q)$ forms provided with the inner product $(\ ,\ )$ (resp. $(\ ,\ )_\infty$) relative to ω_X and h_G (resp. ω_X and $h_G e^{-\varphi_\infty}$). We denote $\vartheta : L^{p,q}(X, G) \to L^{p,q-1}(X, G)$ the adjoint operator of the closed densily defined operator $\bar\partial : L^{p,q}(X, G) \to L^{p,q+1}(X, G)$ relative to $(\ ,\ )$. Since φ_∞ is bounded from above, $L^{p,q}_\infty(X, G)$ can be regarded as a subspace of $L^{p,q}(X, G)$. We denote the restriction of the operator $\bar\partial : L^{n,q}(X, G) \longrightarrow L^{n,q+1}(X, G)$ onto $L^{n,q}_\infty(X, G)$ by $\bar\partial_{(\infty)}$ whose domain $\overset{n,\,q}{\mathrm{Dom}}\,(\bar\partial_{(\infty)})$ coincides with $\Gamma(X, \mathcal{F}^q) \subseteq L^{n,q}_\infty(X, G)$. We claim the following.

Lemma. *$\bar\partial_{(\infty)} : L^{n,q}_\infty(X, G) \longrightarrow L^{n,q+1}_\infty(X, G)$ is a closed densily defined operator.*

Proof. By Demailly's regularization result for almost plurisubharmonic functions on compact complex manifolds (cf. [1, Main Theorem 1.1]), there exists a sequence of smooth functions $\{\varphi_k\}$ on X and an analytic subset A of X such that φ_k decreases to φ_∞ on X as k tends to infinity and $e^{-2\varphi_\infty}$ is locally integrable outside A. Set $(\ ,\ )_k := (\ ,\ e^{-\varphi_k})$ and let $L^{n,q}_k(X, G)$ be the L^2-space relative to the inner product $(\ ,\ )_k$ for any k. Let $C^{n,q}_0(X \setminus A, G)$ be the space of G-valued smooth (n, q) forms with compact support in $X \setminus A$. Take a sequence $\{w_j\}$ in $\mathrm{Dom}(\bar\partial_{(\infty)})$ such that w_j and $\bar\partial_{(\infty)} w_j$ converge strongly to w and v respectively. By the decreasing property of φ_k, $\bar\partial w = v$ in $L^{n,q+1}_k(X, G)$ for any k. For any $u \in C^{n,q+1}_0(X \setminus A, G)$, $\langle v, u\rangle_G e^{-\varphi_\infty}$ and $\langle \bar\partial w, u\rangle_G e^{-\varphi_\infty}$ are integrable on X by Schwarz's inequality. Hence by Lebesgue's dominant convergence theorem we obtain :

$$(v, u)_\infty = \lim_{k\to\infty} (v, u)_k = \lim_{k\to\infty} (\bar\partial w, u)_k = (\bar\partial w, u)_\infty.$$

Since $C^{n,q}_0(X \setminus A, G)$ is dense in $L^{n,q}_\infty(X, G)$, $\bar\partial_{(\infty)}$ is densily defined and the above equality implies $\bar\partial_{(\infty)} w = v$ in $L^{n,q+1}_\infty(X, G)$; i.e., the closedness of $\bar\partial_{(\infty)}$. □

Hence the adjoint operator $\vartheta_{(\infty)} := \bar\partial_{(\infty)}{}^*$ of $\bar\partial_{(\infty)}$ can be defined and has the same property as $\bar\partial_{(\infty)}$ with $\bar\partial_{(\infty)} = \bar\partial_{(\infty)}{}^{**}$. The domain of $\vartheta_{(\infty)}$ is defined in the following way.

$v \in \overset{n,q}{\mathrm{Dom}}\,(\vartheta_{(\infty)})$ *if and only if there exists a positive constant* C *such that*

$$|(v, \bar{\partial}_{(\infty)} w)_\infty| \leq C\|w\|_\infty \quad \textit{for any } w \in \overset{n,q-1}{\mathrm{Dom}}\,(\bar{\partial}_{(\infty)}).$$

For a given linear operator T acting on the Hilbert spaces $L^{\bullet,\bullet}(X,G)$ and $L^{\bullet,\bullet}_\infty(X,G)$, we denote $N^{\bullet,\bullet}(T)$ (resp. $R^{\bullet,\bullet}(T)$) the null space of T (resp. the range of T). Setting $L^{n,-1}_\infty(X,G) = \{0\}$ and $L^{n,-1}(X,G) = \{0\}$ respectively, we define for any $q \geq 0$

$$H^{n,q}(X,G) := N^{n,q}(\bar{\partial}) \cap N^{n,q}(\vartheta) \quad \text{and} \quad H^{n,q}_\infty(X,G) := N^{n,q}(\bar{\partial}_{(\infty)}) \cap N^{n,q}(\vartheta_{(\infty)}).$$

$H^{n,q}(X,G)$ is the E-valued (n,q) harmonic space which is isomorphic to $H^q(X, \Omega^n_X(G))$. Usually the following weak decomposition of $L^{n,q}_\infty(X,G)$ holds (cf. [8]) :

$$L^{n,q}_\infty(X,G) = [R^{n,q}(\bar{\partial}_{(\infty)})] \bigoplus H^{n,q}_\infty(X,G) \bigoplus [R^{n,q}(\vartheta_{(\infty)})] \quad \textit{for any } q \geq 0,$$

where [ ] means the closure of space in $L^{n,q}_\infty(X,G)$. Since X is compact, for any $q \geq 0$ we note that

$$R^{n,q}(\bar{\partial}_{(\infty)}) = \bar{\partial}\Gamma(X, \mathcal{F}^{q-1}) \text{ and } [R^{n,q}(\bar{\partial}_{(\infty)})] \subset N^{n,q}(\bar{\partial}_{(\infty)}) = \Gamma(X, \mathcal{F}^q) \cap \mathrm{Ker}\bar{\partial}.$$

In view of the compactness of X, it is natural to claim the following strong decomposition.

Proposition.

$$L^{n,q}_\infty(X,G) = R^{n,q}(\bar{\partial}_{(\infty)}) \bigoplus H^{n,q}_\infty(X,G) \bigoplus R^{n,q}(\vartheta_{(\infty)}) \quad \textit{for any } q \geq 0.$$

Proof. Since the closedness of $R^{n,q}(\bar{\partial}_{(\infty)})$ is equivalent to the one of $R^{n,q-1}(\vartheta_{(\infty)})$ (cf. [8, Theorem 1.1.1]), we have only to see that $[\bar{\partial}\Gamma(X,\mathcal{F}^{q-1})] = \bar{\partial}\Gamma(X,\mathcal{F}^{q-1})$. Let $v \in [\bar{\partial}\Gamma(X,\mathcal{F}^{q-1})]$ and let $\{\bar{\partial}_{(\infty)} w_k\}_{k\geq 1}$ be a sequence in $\bar{\partial}\Gamma(X,\mathcal{F}^{q-1})$ such that $\|v - \bar{\partial}_{(\infty)} w_k\|_\infty \to 0$ *as* $k \to \infty$. We must find $w \in \Gamma(X,\mathcal{F}^{q-1})$ with $v = \bar{\partial}_{(\infty)} w$. Let $\mathcal{U}$ be a finite Stein open covering of X taken as in 1.5. Combining the L^2-estimate in 1.3, Theorem with the quasi-isomorphism theorem in 1.5, there exists a q cocycle $\sigma(v) \in Z^q(\mathcal{U},\mathcal{S})$ and a sequence of $q-1$ cochains $\{\tau(w_k)\}_{k\geq 1} \subset C^{q-1}(\mathcal{U},\mathcal{S})$ such that $\sigma(v) - \delta\tau(w_k)$ tends to zero with respect to the uniform convergence topology. From the separability of Fréchet topology induced on $H^q(\mathcal{U},\mathcal{S})$, there is a $q-1$ cochain $\tau(w) \in C^{q-1}(\mathcal{U},\mathcal{S})$ with $\delta\tau(w) = \sigma(v)$ which implies the conclusion by the compactness of X and the quasi-isomorphism theorem (cf. [17, Proposition 4.6]). □

1.7. We obtain the following theorem from the above observations :

Theorem. *Let X be a compact complex manifold of dimension n provided with a hermitian metric ω_X and let E be a pseudo effective line bundle on X provided with a smooth hermitian metric h_E and an almost plurisubharmonic function φ_∞ with $\Theta_E + dd^c\varphi_\infty \gtrsim 0$ on X for $\Theta_E = dd^c(-\log h_E)$. Let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to φ_∞. Then for any holomorphic line bundle F provided with a smooth hermitian metric h_F on X and $q \geq 0$, the space*

$$H^{n,q}_\infty(X, E \bigotimes F) := \{u \in \mathrm{Dom}(\bar\partial_{(\infty)}) \cap \mathrm{Dom}(\vartheta_{(\infty)}) : \bar\partial_{(\infty)} u = 0 \text{ and } \vartheta_{(\infty)} u = 0\}$$

defined in $L^{n,q}_\infty(X, E \otimes F)$ satisfies the following :

$$H^q(X, \mathcal{I}(\varphi_\infty) \otimes \Omega^n_X(E \otimes F)) \cong H^{n,q}_\infty(X, E \otimes F)$$

and

$$\dim_{\mathbf{C}} H^{n,q}_\infty(X, E \otimes F) < \infty.$$

Furthermore the following diagram is commutative :

$$\begin{array}{ccc} H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega^n_X(E \bigotimes F)) & \xrightarrow{\iota^q(\varphi_\infty)} & H^q(X, \Omega^n_X(E \bigotimes F)) \\ i^q_\infty \Big\downarrow & & i^q \Big\downarrow \\ H^{n,q}_\infty(X, E \bigotimes F) & \xrightarrow{H^{n,q}} & H^{n,q}(X, E \bigotimes F) \end{array}$$

where i^q_∞ and i^q (resp. $H^{n,q}$) are isomorphisms (resp. the orthogonal projection from $L^{n,q}(X, E \otimes F)$ to $H^{n,q}(X, E \otimes F)$).

2. A smoothing of almost plurisubharmonic functions associated to nef line bundles on compact Kähler manifolds

Let X be a compact Kähler manifold of dimension n provided with a Kähler metric ω_X and let E be a holomorphic line bundle provided with a smooth hermitian metric h_E on X.

DEFINITION 2.1. (E, h_E) is said to be nef if for any $\varepsilon > 0$ there exists a smooth function ψ_ε on X such that $\Theta_E + dd^c\psi_\varepsilon + \varepsilon\omega_X$ yields a Kähler metric for $\Theta_E := dd^c(-\log\ h_E)$.

The above definition depends on the choice of neither h_E nor ω_X and is equivalent to that the real first Chern class $c_{R,1}(E)$ of E is contained in the closure of the Kähler cone of X (cf. [13], §2). If E has a smooth metric whose curvature is semi-positive, then E is clearly nef. However the converse is not true in general even if X is projective algebraic (cf. [4, Example 1.7]).

We begin with the following lemma suggested by [6], Lemma 2.1 and [18], Proposition 2.1 (compare [2, Lemma 6.6]).

Lemma 2.2. *Let (X, ω_X) be a compact Kähler manifold of dimension n and let Θ be a d-closed smooth real $(1,1)$ form on X. Let $\mathcal{P}(\Theta)$ be the set of real-valued smooth functions ψ so that $\Theta + dd^c\psi \geq 0$ and $\sup_X \psi = 0$. Then any sequence $\{\psi_k\}_{k\geq 1}$, $\psi_k \in \mathcal{P}(\Theta)$, contains a Cauchy subsequence in $L^1(X)$.*

REMARK. The existence of an L^1 Cauchy subsequence in $\{\psi_k\}_{k\geq 1}$, $\psi_k \in \mathcal{P}(\Theta)$, is not trivial because a local version of such a property is never true (cf. [18, p.238, Remark] and Remark 2 below).

Proof. Let $\{\psi_k\}_{k\geq 1}$ be a sequence belonging to $\mathcal{P}(\Theta)$. Setting $\tau_X = \omega_X^{n-1}/(n-1)!$ and $dv_X = \omega_X^n/n!$, there exists a positive constant $C(\Theta, \omega_X)$ not depending on k such that

$$\begin{aligned} 0 \leq \int_X e^{\psi_k} d\psi_k \wedge d^c\psi_k \wedge \tau_X &= -\int_X e^{\psi_k} dd^c\psi_k \wedge \tau_X \qquad \text{by Stokes' theorem} \\ &= -\int_X e^{\psi_k}\{dd^c\psi_k + \Theta\} \wedge \tau_X + \int_X e^{\psi_k}\Theta \wedge \tau_X \\ &\leq \int_X |\mathrm{Trace}(\Theta, \omega_X)| dv_X \leq C(\Theta, \omega_X) < \infty. \end{aligned}$$

Since $\{e^{\psi_k/2}\}$ and their first derivatives are bounded in $L^2(X)$ from the above inequality, $\{e^{\psi_k/2}\}$ has a Cauchy subsequence in $L^2(X)$ in view of Rellich's lemma.

On the other hand there are three positive constants C_j such that $C_1\omega_X \leq C_2\omega_X + \Theta \leq C_3\omega_X$. Hence by [18], Proposition 2.1, there exist positive constants α with $0 < \alpha \ll 1$ and C_* not depending on $\psi \in \mathcal{P}(\Theta)$ such that

$$\int_X e^{-\alpha\psi} dv_X \leq C_* < \infty \tag{2.3}$$

for any $\psi \in \mathcal{P}(\Theta)$. For any $\beta > 0$ by Schwarz's inequality we obtain

$$\left(\int_X \left|e^{\beta(\psi_j - \psi_k)} - 1\right| dv_X\right)^2 \leq \left(\int_X \left|e^{\beta\psi_j} - e^{\beta\psi_k}\right|^2 dv_X\right)\left(\int_X e^{-2\beta\psi_k} dv_X\right).$$

Taking $2\beta = \alpha$ the right hand side converges to zero from the above observation and (2.3). In particular we get

$$\int_X \left|\max\left\{e^{\beta(\psi_j - \psi_k)}, 1\right\} - 1\right| dv_X \to 0 \quad \text{as } j \text{ and } k \to \infty. \tag{2.4}$$

Here we may assume $\mathrm{Vol}(X, \omega_X) = 1$ and use the following notation :

$$\log^+ t = \log \max\{t, 1\} \quad \text{and} \quad |\log t| = \log^+ t + \log^+ \left(\frac{1}{t}\right) \text{ for } t > 0.$$

By setting $\gamma = 1/\beta$ and the concavity of logarithmic functions we obtain :

$$\begin{aligned}
&\int_X |\psi_j - \psi_k| \, dv_X \\
&= \gamma \int_X \left|\log\left\{e^{\beta(\psi_j - \psi_k)}\right\}\right| dv_X \\
&= \gamma \int_X \left\{\log^+ e^{\beta(\psi_j - \psi_k)} + \log^+ e^{\beta(\psi_k - \psi_j)}\right\} dv_X \\
&\leq \gamma \log\left\{\left(\int_X \max\left\{e^{\beta(\psi_j - \psi_k)}, 1\right\} dv_X\right)\left(\int_X \max\left\{e^{\beta(\psi_k - \psi_j)}, 1\right\} dv_X\right)\right\}
\end{aligned}$$

Finally our assertion follows from the above inequality and (2.4). □

Proposition 2.5. *Let (E, h_E) be a nef line bundle on a compact Kähler manifold (X, ω_X). For a given sequence of positive numbers $\{\eta_k\}_{k\geq 1}$ decreasing to zero, let $\{\psi_k\}_{k\geq 1}$ be a sequence of smooth functions on X such that*

$$(2.5) \qquad \Theta_E + dd^c\psi_k + \eta_k\omega_X > 0 \quad \text{on } X \text{ and } \sup_X \psi_k = 0,$$

where $\Theta_E = dd^c(-\log\ h_E)$.

Then there exist an almost plurisubharmonic function φ_∞, a sequence of smooth functions $\{\varphi_k\}_{k\geq 1}$ on X, and a sequence of positive numbers $\{\varepsilon_k\}_{k\geq 1}$ decreasing to zero such that

(i) $\Theta_E + dd^c\varphi_\infty \gtrsim 0$; *i.e., E is pseudo effective on X*

(ii) $\Theta_E + dd^c\varphi_k + \varepsilon_k\omega_X > 0$ *and* $\varphi_\infty < \varphi_k \leq 1$ *on X for any $k \geq 1$*

(iii) φ_k *converges to* φ_∞ *in $L^1(X)$ and almost everywhere on X.*

Proof. By applying Lemma 2.2 to $\Theta_E + \eta_k\omega_X$, if necessary, taking a subsequence, there exists a limit $\varphi_\infty \in L^1(X)$ such that $\{\psi_k\}_{k\geq 1}$ converges to φ_∞ in $L^1(X)$. If necessary, taking a subsequence, we may assume that :

$$(1) \qquad \|\psi_k - \varphi_\infty\|_{L^1(X)} < \frac{1}{2k}$$

$$(2) \qquad \Theta_E + dd^c\varphi_\infty \gtrsim 0.$$

(2) follows from the weak continuity of $\partial\bar{\partial}$ and (2.5) immediately. Locally ω_X can be written $\omega_X = dd^c\Phi$ by a smooth strictly plurisubharmonic function Φ. By (2.5) (resp. (2)) $-\log h_E + \eta_k\Phi + \psi_k$ (resp. $-\log h_E + \varphi_\infty$) defines locally a smooth

plurisubharmonic function θ_k (resp. a plurisubharmonic function θ_∞). For every k we put

$$\lambda_k := \max\{\psi_k, \varphi_\infty\}.$$

Then λ_k satisfies the following properties for any $k \geq 1$:

$$\|\lambda_k - \varphi_\infty\|_{L^1(X)} < \frac{1}{2k} \tag{3}$$

$$\Theta_E + dd^c\lambda_k + \eta_k\omega_X \gtrsim 0. \tag{4}$$

(3) follows from (1) and (4) follows from the following local equality :

$$\lambda_k = \log h_E - \eta_k\Phi + \max\{\theta_k, \theta_\infty + \eta_k\Phi\}$$

because $\max\{\theta_k, \theta_\infty + \eta_k\Phi\}$ is plurisubharmonic. Since λ_k is *locally bounded*, the Lelong number of λ_k is zero at any point of X. Therefore by Demailly's regularization result for almost plurisubharmonic functions (cf. [1], §3. the proof of Propositions 3.1 and 3.7), there exist a sequence of smooth functions $\{\varphi_k\}_{k\geq 1}$ and a sequence of positive numbers $\{\delta_k\}_{k\geq 1}$ decreasing to zero such that

$$\varphi_\infty \leq \lambda_k < \varphi_k \leq 1 \quad on \quad X \tag{5}$$

$$\Theta_E + dd^c\varphi_k + (\eta_k + \delta_k)\omega_X \geq 0 \quad on \quad X \tag{6}$$

$$\|\varphi_k - \lambda_k\|_{L^1(X)} < \frac{1}{2k} \tag{7}$$

for any $k \geq 1$. Setting $\varepsilon_k := \eta_k + 2\delta_k$ and if necessary, taking a subsequence, we obtain the desired sequence $\{\varphi_k\}_{k\geq 1}$. This completes the proof of Proposition 2.5. □

3. On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds

Let X be a connected compact Kähler manifold of dimension n provided with a Kähler metric ω_X. Let E (resp. F) be a *nef* (resp. *semi-positive*) line bundle provided with a smooth metric h_E (resp. h_F with $\Theta_F = dd^c(-\log h_F) \geq 0$) on X. Let φ_∞ be an almost plurisubharmonic function on X with $\Theta_E + dd^c\varphi_\infty \gtrsim 0$ determined in Proposition 2.5 and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to φ_∞. For φ_∞ we fix a sequence of smooth almost plurisubharmonic functions $\{\varphi_k\}_{k\geq 1}$ taken as in Proposition 2.5. We set :

$$G = E \bigotimes F, \quad h_G = h_E \bigotimes h_F, \quad \text{and} \quad h_{G,k} = h_G e^{-\varphi_k}$$

for any k with $0 \le k \le \infty$. Here if $k = 0$, then we set $\varphi_0 \equiv 0$ and do not specify it in the notations below.

$L_k^{p,q}(X,G)$ be the L^2-space of G-valued square integrable (p,q) forms provided with the inner product $(\ ,\ )_k$ relative to ω_X and $h_{G,k}$, and let $\|\ \|_k$ denote the norm defined by the inner product. $L_\infty^{p,q}(X,G)$ can be regarded as a subspace of $L_k^{p,q}(X,G)$ for any k with $0 \le k < \infty$. Let $\vartheta_{(k)}$ denote the adjoint operator of $\bar\partial$ in $L_k^{p,q}(X,G)$ (cf. 1.6). The space $N_k^{n,q}(\bar\partial)$ of null solutions for $\bar\partial$ in $L_k^{n,q}(X,G)$ is decomposed strongly as follows :

$$N_k^{n,q}(\bar\partial) = R_k^{n,q}(\bar\partial) \bigoplus H_k^{n,q}(X,G) \tag{3.1}$$

where $H_k^{n,q}(X,G) := \{u \in L_k^{n,q}(X,G) : \bar\partial u = \vartheta_{(k)} u = 0\}$ for any $q \ge 1$ and $0 \le k \le \infty$. We denote $H_k^{n,q}$ the orthogonal projection onto $H_k^{n,q}(X,G)$ for every k with $0 \le k \le \infty$.

Setting $\mathcal{K}_\infty^{n,q}(X,G) := \mathrm{Kernel}\{H^{n,q} : H_\infty^{n,q}(X,G) \to H^{n,q}(X,G)\}$ (cf. 1.7, Theorem), we define a subspace $\mathcal{H}_\infty^{n,q}(X,G)$ of $H_\infty^{n,q}(X,G)$ by the following orthogonal decomposition relative to $(\ ,\ )_\infty$:

$$H_\infty^{n,q}(X,G) = \mathcal{H}_\infty^{n,q}(X,G) \bigoplus \mathcal{K}_\infty^{n,q}(X,G).$$

Since $\mathcal{K}_\infty^{n,q}(X,G) = H_\infty^{n,q}(X,G) \cap R^{n,q}(\bar\partial)$, the space $\mathcal{H}_\infty^{n,q}(X,G)$ is characterized as follows.

(3.2) $u \in \mathcal{H}_\infty^{n,q}(X,G)$ *if and only if* $u \in N^{n,q}(\bar\partial_\infty)$ *and* $(u, \bar\partial w)_\infty = 0$ *for any* $w \in L^{n,q-1}(X,G)$ *with* $\bar\partial w \in L_\infty^{n,q}(X,G)$.

We define a homomorphism

$$\mathcal{L}_{(\infty)}^q : \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(G)) \longrightarrow \mathcal{H}_\infty^{n,q}(X,G)$$

by the composition of the homomorphism

$$L^q : \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(G)) \longrightarrow N^{n,q}(\bar\partial_{(\infty)})$$

induced by the q-times left exterior product by ω_X with the orthogonal projection from $N^{n,q}(\bar\partial_{(\infty)})$ to $\mathcal{H}_\infty^{n,q}(X,G)$.

The following lemma is very useful (cf. [3, (4.10)]).

Lemma 3.3. *Let W be a holomorphic line bundle on X provided with a smooth hermitian metric h_W. Let Θ be a smooth real $(1,1)$ differential form on X and let $\{\lambda_j\}$ be the eigen-values of Θ relative to ω_X with $\lambda_1 \le \lambda_2 \le, \ldots, \le \lambda_n$ (which are* *continuous functions on* X) ; *i.e.*, $\Theta(x) = \sqrt{-1}\sum_{j=1}^{n} \lambda_j(x)dz^j \wedge d\bar{z}^j$ *with* $\omega_X(x) = \sqrt{-1}\sum_{j=1}^{n} dz^j \wedge d\bar{z}^j$, $x \in X$. *Then if* $v(x) = \sum v_{A_n,B_q} dz^{A_n} \wedge d\bar{z}^{B_q} \in C^{n,q}(X,W)$ *with* $q \geq 1$, *the following holds*

$$\langle \mathbf{e}(\Theta)\Lambda v, v\rangle_W(x) = \sum_{|A_n|=n,|B_q|=q} \Big(\sum_{j\in B_q} \lambda_j(x)\Big)|v_{A_n,B_q}|_W^2.$$

In particular setting $\delta_q := \sum_{j=1}^{q} \lambda_j$ *with* $q \geq 1$ *the following holds*

$$\langle \mathbf{e}(\Theta)\Lambda v, v\rangle_W \geq \delta_q\langle v, v\rangle_W \quad \textit{if } v \in C^{n,q}(X,W). \tag{3.4}$$

The nefness of E enables us to show the following theorem.

Theorem 3.5. $\mathcal{L}^q_{(\infty)}$ *is surjective and the Hodge star operator* $*$ *relative to* ω_X *yields a splitting homomorphism*

$$\delta^q_{(\infty)} : \mathcal{H}^{n,q}_{\infty}(X,G) \longrightarrow \Gamma(X,\mathcal{I}(\varphi_\infty)\bigotimes\Omega_X^{n-q}(G))$$

with $\mathcal{L}^q_{(\infty)} \circ \delta^q_{(\infty)} = \mathrm{id}$. *Furthermore* $\mathcal{L}^q_{(\infty)} = L^q$ *on* $\mathrm{Image}\delta^q_{(\infty)}$ *for any* $q \geq 1$.

Proof. If $\mathcal{H}^{n,q}_{\infty}(X,G) = \{0\}$, then we have nothing to prove. Hence we assume $\mathcal{H}^{n,q}_{(\infty)}(X,G) \neq \{0\}$ and take $u \in \mathcal{H}^{n,q}_{\infty}(X,G)$ with $\|u\|_\infty = 1$. We claim that $*u \in \Gamma(X,\mathcal{I}(\varphi_\infty)\bigotimes\Omega_X^{n-q}(G))$, which implies that $\mathcal{L}^q_{(\infty)} = L^q$ is surjective by $L^q \circ * = c(n,q)\mathrm{id}$ on the space of (n,q) forms for the uniquely determined complex number $c(n,q) \neq 0$. We have only to define $\delta^q_{(\infty)} := c(n,q)^{-1}*$.

We note that u has the following orthogonal decomposition by (3.1) :

$$u = \bar{\partial}w_k + H_k^{n,q}(u),\ \|\bar{\partial}w_k\|_k \quad \text{and} \quad \|H_k^{n,q}(u)\|_k \leq 1 \tag{3.6}$$

for any k with $0 \leq k < \infty$. Setting $u_k := H_k^{n,q}(u)$, we may assume $u_k \neq 0$ for any k. From $\|u_k\| \leq e\|u_k\|_k \leq e$, taking a subsequence, $\{u_k\}$ has a weak limit $u_\infty \in L^{n,q}(X,G)$ with $\bar{\partial}u_\infty = 0$. $\{\bar{\partial}w_k\}$ also has a weak limit v_∞. Since $R^{n,q}(\bar{\partial})$ is closed, there exists $w_* \in L^{n,q-1}(X,G)$ with $v_\infty = \bar{\partial}w_*$. Therefore we obtain

$$u = \bar{\partial}w_* + u_\infty \quad \text{in} \quad L^{n,q}(X,G). \tag{3.7}$$

We show that $*u_\infty \in \Gamma(X,\mathcal{I}(\varphi_\infty)\bigotimes\Omega_X^{n-q}(G))$ and $u_\infty \in \mathcal{H}^{n,q}_{\infty}(X,G)$, which implies $\bar{\partial}w_* = 0$ by (3.2); i.e., $u_\infty = u$.

By Calabi-Nakano-Vesentini's formula on compact Kähler manifolds (cf. [14, Proposition 1.2]), we obtain the following integral formula :

$$\|\bar{\partial}v\|_k^2 + \|\vartheta_{(k)}v\|_k^2 = \|\bar{\vartheta}v\|_k^2 + (\mathbf{e}(\Theta_G + dd^c\varphi_k)\Lambda v, v)_k$$

for any G-valued smooth (n,q) form v on X, $\Theta_G := \Theta_E + \Theta_F$ and $k \geq 1$. Since $q\|v\|_k^2 = (L\Lambda v, v)_k$, by Proposition 2.5, (ii) and the semi-positivity of Θ_F (cf. (3.4)), we obtain the following inequality :

$$\begin{aligned}\varepsilon_k q\|u_k\|_k^2 &= \|\bar{\vartheta}u_k\|_k^2 + (\mathbf{e}(\Theta_G + dd^c\varphi_k + \varepsilon_k\omega_X)\Lambda u_k, u_k)_k \\ &\geq (\mathbf{e}(\Theta_G + dd^c\varphi_k + \varepsilon_k\omega_X)\Lambda u_k, u_k)_k \geq 0.\end{aligned}$$

Therefore when k tends to infinity, we obtain

$$\|\bar{\vartheta}u_k\|_k^2 \leq \varepsilon_k q\|u_k\|_k^2 \leq \varepsilon_k q \to 0.$$

By $\bar{\vartheta} = - * \bar{\partial} *$ and $\|\bar{\partial} * u_k\|^2 \leq \|\bar{\vartheta}u_k\|_k^2$, u_∞ satisfies $\bar{\partial} * u_\infty = 0$ in the sense of distribution. Therefore $*u_\infty \in \Gamma(X, \Omega_X^{n-q}(G))$. Setting $u^k = u_k e^{-\varphi_k/2}$ and, if necessary taking a subsequence, u^k converges weakly to $u^\infty \in L^{n,q}(X,G)$ by $\|u_k\|_k \leq 1$. Let V be the analytic subset (might be empty) defined by $\mathcal{I}(\varphi_\infty)$. Since $e^{-\varphi_\infty}$ is locally integrable on $X \setminus V$, $e^{-\varphi_k}$ converges to $e^{-\varphi_\infty}$ in $L^1(K)$ for any compact subset K in $X \setminus V$ by $\varphi_\infty < \varphi_k$ and Lebesgue's dominant convergence theorem. For every E-valued smooth (n,q) form v with compact support in $X \setminus V$, by setting $K := \mathrm{Supp}(v)$ and denoting $|v|_G$ the pointwise length of v relative to ω_X and h_G, we obtain from (3.6) :

$$\begin{aligned}\lim_{k\to\infty}\left|(u_k, \{e^{-\varphi_\infty/2} - e^{-\varphi_k/2}\}v)\right| &\leq \lim_{k\to\infty} \sup |v|_G\|u_k\|\ \|e^{-\varphi_\infty/2} - e^{-\varphi_k/2}\|_{L^2(K)} \\ &\leq e\sup_K |v|_G \lim_{k\to\infty}\sqrt{\|e^{-\varphi_\infty} - e^{-\varphi_k}\|_{L^1(K)}} = 0.\end{aligned}$$

Here we have used : $(a-b)^2 < a^2 - b^2$ if $a > b > 0$. Hence we get :

$$(u^\infty, v) = \lim_{k\to\infty}(u^k, v) = \lim_{k\to\infty}(u_k, ve^{-\varphi_\infty/2}) = (u_\infty e^{-\varphi_\infty/2}, v).$$

This implies $u^\infty = u_\infty e^{-\varphi_\infty/2}$ on $X \setminus V$ as current and so $u_\infty \in L_\infty^{n,q}(X,G)$ because $u^\infty \in L^{n,q}(X,G)$. Therefore we get $*u_\infty \in \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(G))$.

Furthermore if $w \in L^{n,q-1}(X,G)$ with $\bar{\partial}w \in L_\infty^{n,q}(X,G)$, then $w \in L_k^{n,q-1}(X,G)$ with $\bar{\partial}w \in L_k^{n,q}(X,G)$ for any k with $1 \leq k < \infty$ because φ_k is smooth. Therefore by $\vartheta_k u_k = 0$ and Lebesgue's dominant convergence theorem, we obtain :

$$\begin{aligned}|(u_\infty, \bar{\partial}w)_\infty| &= \lim_{k\to\infty}\left|(u^k, \{e^{-\varphi_\infty/2} - e^{-\varphi_k/2}\}\bar{\partial}w)\right| \\ &\leq \lim_{k\to\infty}\sqrt{\|\{e^{-\varphi_\infty} - e^{-\varphi_k}\}|\bar{\partial}w|_G^2\|_{L^1(X)}} = 0.\end{aligned}$$

Therefore $u_\infty \in \mathcal{H}_\infty^{n,q}(X,G)$ by (3.2). This completes the proof of Theorem 3.5.

□

Proposition 3.8. *Every $u \in \mathcal{H}^{n,q}_{\infty}(X,G)$ with $q \geq 1$ satisfies the following :*

$$(\mathbf{e}(\Theta_G + dd^c\varphi)\Lambda u, u)_{\infty} = 0 \tag{3.9}$$

for any smooth real-valued function φ on X.

Proof. By the equations $\bar{\partial}u = \bar{\vartheta}u = 0$, we get $\bar{\partial}\vartheta_G u = \mathbf{e}(\Theta_G)\Lambda u$ and $\bar{\partial}\mathbf{e}(\bar{\partial}\varphi)^* u = \mathbf{e}(dd^c\varphi)\Lambda u$ by [14], Propositions 1.2 & 1.5. Since Θ_G and $dd^c\varphi$ are smooth on X, we obtain $\bar{\partial}\vartheta_G u$ and $\bar{\partial}e(\bar{\partial}\varphi)^* u \in L^{n,q}_{\infty}(X,G)$ by Lemma 3.3. The conclusion follows from (3.2). □

In view of the L^2-estimate (3.9), we can show the following vanishing theorem for $\mathcal{H}^{n,q}_{\infty}(X,G)$.

Theorem 3.10. *If $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$, then $\mathcal{H}^{n,q}_{\infty}(X,G) = 0$, where $\kappa_*(E)$ is defined by $\kappa_*(E) := \max\{\, l \;:\; \overset{l}{\wedge}\, c_{R,1}(E) \neq 0 \in H^{2l}(X,R)\}$ and so on.*

Proof. By (3.9), if $u \in \mathcal{H}^{n,q}_{\infty}(X,G)$, then for any smooth real-valued function φ on X and $\varepsilon > 0$ we obtain

$$0 < (\mathbf{e}(\Theta_G + dd^c\varphi + \varepsilon\omega_X)\Lambda u, u)_{\infty} = q\varepsilon\|u\|_{\infty} \tag{3.11}$$

and particularly

$$(\mathbf{e}(\Theta_F)\Lambda u, u)_{\infty} = 0. \tag{3.12}$$

If $q > n - \kappa_*(F)$, then the integrand of (3.12) is non-negative on X and positive at least one point of X by (3.4) (cf. [16], p. 277, Fact 2.7). Therefore u should vanish on X identically because $*u$ is holomorphic and X is connected.

Assume $q > n - \kappa_*(E)$ and $u \neq 0 \in \mathcal{H}^{n,q}_{\infty}(X,G)$. For any $\varepsilon > 0$ we set :

$$p(\varepsilon) := \int_X (\Theta_G + \varepsilon\omega_X)^n \Big/ \int_X \omega_X^n .$$

Since E is nef, for any $\varepsilon > 0$ there exists a smooth real-valued function φ_ε on X so that $\Theta_G + dd^c\varphi_\varepsilon + \varepsilon\omega_X$ is a Kähler metric. Furthermore by [21], there exists a smooth real-valued function ψ_ε on X such that $\gamma_\varepsilon := \Theta_G + dd^c(\varphi_\varepsilon + \psi_\varepsilon) + \varepsilon\omega_X$ is a Kähler metric on X with

$$\gamma_\varepsilon^n = p(\varepsilon)\omega^n. \tag{3.13}$$

Let $\{\lambda_{\varepsilon,j}\}$ be the eigenvalues of γ_ε relative to ω_X and let $\delta_{\varepsilon,\mu}$ be a continuous function defined as in Lemma 3.3 relative to $\{\lambda_{\varepsilon,j}\}$ for any $\varepsilon > 0$ and $1 \leq \mu \leq n$.

Set $U(\varepsilon) := \{\delta_{\varepsilon,q} < 2q\varepsilon\}$ for any $\varepsilon > 0$. By applying $\varphi_\varepsilon + \psi_\varepsilon$ to (3.11), and Lemma 3.3 we can show

$$0 < \|u\|_\infty^2 \leq 2\int_{U(\varepsilon)} |u|_G^2 e^{-\varphi_\infty} dv_X.$$

This implies $U(\varepsilon) \neq \phi$ for any $\varepsilon > 0$. We claim that there exists a positive constant C_1 not depending on ε such that $\int_{U(\varepsilon)} dv_X \geq C_1 > 0$ for any $\varepsilon > 0$. If $\int_{U(\varepsilon)} dv_X$ converges to zero, then $\int_{U(\varepsilon)} |u|^2 e^{-\varphi_\infty} dv_X$ also tends to zero because $|u|_G^2 e^{-\varphi_\infty}$ is integrable. However this contradicts to the above inequality.

Furthermore since $\int_X \mathbf{e}(\gamma_\varepsilon)\omega_X^{n-1} = \int_X \mathbf{e}(\Theta_G + \varepsilon\omega_X)\omega_X^{n-1}$ is non-negative and bounded from above, there exists positive constant C_2 and C_3 not depending on ε such that $0 < \delta_{\varepsilon,n} \leq C_2$ on an open subset $Q(\varepsilon) \subseteq U(\varepsilon)$ with $\int_{Q(\varepsilon)} dv_X \geq C_3 > 0$. Hence we obtain

$$\prod_{j=1}^{n} \lambda_{\varepsilon,j} \leq (2q)^q C_2^{n-q} \varepsilon^q \quad \text{on} \quad Q(\varepsilon) \quad \text{for any } \varepsilon > 0. \tag{3.14}$$

On the other hand since $P(\varepsilon) = \prod_{j=1}^n \lambda_{\varepsilon,j}$ is a polynomial in ε of degree n and E is nef, letting $P(\varepsilon) = \sum_{i=0}^n a_i \varepsilon^i$ we obtain : $a_i > 0$ if $i \geq n-\kappa$ and $a_i = 0$ if $i < n-\kappa$ by the definition of $\kappa = \kappa_*(E)$ and (3.13). This implies that

$$a_{n-\kappa}\varepsilon^{n-\kappa} \leq \prod_{j=1}^{n} \lambda_{\varepsilon,j} \quad \text{on} \quad X. \tag{3.15}$$

By (3.14) and (3.15) we can get $a_{n-\kappa}\varepsilon^{n-\kappa} \leq (2q)^q C_2^{n-q}\varepsilon^q$, which is a contradiction as ε tends to zero because $q > n-\kappa$. The idea of this proof is due to Enoki [5]. This completes the proof of Theorem 3.10. □

Next we show the following injectivity theorem.

Theorem 3.16.

(i) *If the j-times tensor product $E^{\otimes j}$ of E admits a non-trivial holomorphic section σ with*

$$C(\sigma) := \operatorname{ess.sup}_X |\sigma|^2_{E^{\otimes j}} e^{-j\varphi_\infty} < \infty$$

then the homomorphism

$$\mathcal{H}^{n,q}_\infty(\sigma) : \mathcal{H}^{n,q}_\infty(X, E^{\otimes i} \bigotimes F) \longrightarrow H^{n,q}_\infty(X, E^{\otimes(i+j)} \bigotimes F)$$

induced by the tensor product with σ is well defined and particularly injective for any $q \geq 0$, i and $j \geq 1$.

(ii) *If the k-times tensor product $F^{\otimes k}$ of F admits a non-trivial holomorphic section θ, then*

$$\mathcal{H}^{n,q}_{\infty}(\theta) : \mathcal{H}^{n,q}_{\infty}(X, E \bigotimes F^{\otimes j}) \longrightarrow H^{n,q}_{\infty}(X, E \bigotimes F^{\otimes (j+k)})$$

induced by the tensor product with θ is well defined and particularly injective for any $q \geq 0$, j and $k \geq 1$.

Proof of (i). For $u \in \mathcal{H}^{n,q}_{\infty}(X, E^{\otimes i} \bigotimes F)$, setting $v = \sigma \bigotimes u$ we have only to show $(v, \bar{\partial} w)_{\infty} = 0$ for any $w \in L^{n,q-1}_{\infty}(X, E^{\otimes(i+j)} \bigotimes F)$ with $\bar{\partial} w \in L^{n,q}_{\infty}(X, E^{\otimes(i+j)} \bigotimes F)$. Since $\bar{\partial} v = \bar{\vartheta} v = 0$, and Θ_F is semi-positive, by Calabi-Nakano-Vesentini's formula, Lemma 3.3 and Proposition 3.8, we can conclude :

$$\begin{aligned}
\|\vartheta_{(k)} v\|^2_k &= (\mathbf{e}((i+j)(\Theta_E + dd^c \varphi_k) + \Theta_F)\Lambda v, v)_k \\
&\leq \left(\frac{i+j}{i}\right)(\mathbf{e}(i(\Theta_E + dd^c\varphi_k + \varepsilon_k \omega_X) + \Theta_F)\Lambda v, v)_k \\
&\leq \varepsilon_k q C(\sigma)\left(\frac{i+j}{i}\right)\|u\|^2_{\infty} \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Hence by Lebesgue's dominant convergence theorem we have

$$(v, \bar{\partial} w)_{\infty} = \lim_{k \to \infty} (v, \bar{\partial} w)_k = \lim_{k\to\infty} (\vartheta_{(k)} v, w)_k = 0.$$

Proof of (ii). Since the length of θ is bounded, the proof can be done similarly. This completes the proof of Theorem 3.16. □

REMARK. If the almost plurisubharmonic function φ_{∞} is determined independently of the choice of $\{\varepsilon_k\}$, then from the above proof it can be verified that $\mathcal{H}^{n,q}_{\infty}(\sigma) : \mathcal{H}^{n,q}_{\infty}(X, E^{\otimes i} \bigotimes F) \longrightarrow \mathcal{H}^{n,q}_{\infty}(X, E^{\otimes(i+j)} \bigotimes F)$ is well defined.

Comment. In the situation of this section, setting $F =$ the trivial line bundle, Enoki claims that $H^{n,q}(X, E) = 0$ if $q > n - \kappa_*(E)$, which implies that $H^q(X, \Omega^n_X(E)) = 0$ if $q > n - \kappa_*(E)$ (cf. [5, Theorem 0.1]). His idea of the proof consists of two parts ; i.e., an L^2-estimate for the harmonic forms in $H^{n,q}(X, E)$ and the argument used to show Theorem 3.10. In fact he claims the following L^2-estimate (cf.[5, Proposition 3.1]) :

Let E be a holomorphic line bundle provided with a smooth hermitian metric h_E on a compact Kähler manifold X of dimension n provided with a Kähler metric ω_X. Then for any real-valued smooth function φ on X and $u \in H^{n,q}(X, E)$ with $q \geq 1$, setting $\eta := e^{\varphi}$ the following inequality holds

$$(\eta \mathbf{e}(\Theta_E + dd^c\varphi)\Lambda u, u) \leq 0.$$

Here we should note that any specific condition for the curvature of (E, h_E) is not assumed to show the above inequality in his proof. However the sign of the left hand side can not be always determined in the following sense.

First for any E-valued smooth (n, q) form v on X we can obtain the following integral formula (cf. [17, §1, Proposition 1.11]) :

$$\|\sqrt{\eta}(\bar{\partial}+\mathbf{e}(\bar{\partial}\varphi))v\|^2+\|\sqrt{\eta}\vartheta_h v\|^2=\|\sqrt{\eta}(\bar{\vartheta}-\mathbf{e}(\partial\varphi)^*)v\|^2+(\eta\mathbf{e}(\Theta_E+dd^c\varphi)\Lambda v, v).$$

Hence if $u \in H^{n,q}(X, E)$, by setting $w = *u$ and using $\mathbf{e}(\partial\varphi)^* = *\mathbf{e}(\bar{\partial}\varphi)*$ we can verify the following from the above formula :

$$\begin{aligned}(\eta\mathbf{e}(\Theta_E+dd^c\varphi)\Lambda u, u) &= -\|\sqrt{\eta}(\bar{\vartheta}-\mathbf{e}(\partial\varphi)^*)u\|^2+\|\sqrt{\eta}\mathbf{e}(\bar{\partial}\varphi)u\|^2\\ &= -\|\sqrt{\eta}(\bar{\partial}+\mathbf{e}(\bar{\partial}\varphi))w\|^2+\|\sqrt{\eta}\mathbf{e}(\partial\varphi)^*w\|^2.\end{aligned}$$

Here we note that $\bar{\partial}w$ is primitive ; i.e., $\Lambda\bar{\partial}w = 0$ by $\bar{\partial}u = 0$ and $\bar{\vartheta} = -\sqrt{-1}[\partial, \Lambda]$. For any E-valued smooth $(n-q, 1)$ form α, let $\alpha = \alpha_1 + \alpha_2$ be the primitive decomposition of the form ; i.e., $\Lambda\alpha_1 = 0$ and $\alpha_2 = 1/(q+1)L\Lambda\alpha$ (cf.[20, Chap.V, Theorem 1.8]). Here the coefficient $1/(q+1)$ of α_2 is crucial. Since $\mathbf{e}(\partial\varphi)^* = \sqrt{-1}[\mathbf{e}(\bar{\partial}\varphi), \Lambda]$, by applying the decomposition to $\alpha := \mathbf{e}(\bar{\partial}\varphi)w$ and the above equality it can be verified that

$$(\eta\mathbf{e}(\Theta_E+dd^c\varphi)\Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial}w+\alpha_1)\|^2 + q\|\sqrt{\eta}\alpha_2\|^2$$

and

$$\alpha_2 = 0 \quad \text{if and only if} \quad \mathbf{e}(\bar{\partial}\varphi)u = 0.$$

Therefore if $u \in H^{n,q}(X, E)$ satisfies the equality

$$(\eta\mathbf{e}(\Theta_E+dd^c\varphi)\Lambda u, u) = -\|\sqrt{\eta}(\bar{\partial}w+\alpha_1)\|^2 \leq 0$$

for *any real-valued smooth function* φ *on* X as he claims (see the last line of his proof of Proposition 3.1 in [5]), then by the above observations an E^* (the dual of E)-valued harmonic $(0, n-q)$ form $*\overline{(hu)}$ satisfies the *$\bar{\partial}$-Neumann condition* on every open ball with smooth boundary contained in any local coordinate neighborhood of X. Hence such a form should vanish on it in view of the solvability for $\bar{\partial}$ on open balls and its boundary condition (cf.[17, §4. Theorem 4.3, (iv)]), and so identically on X by a unique continuation property for harmonic forms, which implies $H^q(X, \Omega_X^n(E)) = 0$. However $H^q(X, \Omega_X^n(E))$ does not vanish without any specific condition in general.

4. On cohomology groups of nef line bundles on compact Kähler manifolds

First we state the following Lefschetz type theorem (cf. [5, Theorem 0.3]).

Theorem 4.1. *Let X be a connected compact Kähler manifold of dimension n provided with a Kähler metric ω_X. Let E (resp. F) be a nef (resp. semi-positive) line bundle provided with a smooth metric h_E (resp. h_F with $\Theta_F = dd^c(-\log h_F) \geq 0$) on X. Let φ_∞ be an almost plurisubharmonic function with $\Theta_E + dd^c\varphi_\infty \gtrsim 0$ determined in Proposition 2.5 and let $\mathcal{I}(\varphi_\infty)$ be the multiplier ideal sheaf associated to φ_∞. Then for any $q \geq 1$ the homomorphism*

$$L^q : \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E \bigotimes F)) \longrightarrow \mathrm{Image}\,\iota^q(\varphi_\infty) \subset H^q(X, \Omega_X^n(E \bigotimes F))$$

is surjective and the Hodge star operator relative to ω_X yields a splitting homomorphism

$$\delta^q : \mathrm{Image}\,\iota^q(\varphi_\infty) \longrightarrow \Gamma(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^{n-q}(E \otimes F))$$

with $L^q \circ \delta^q = \mathrm{id}$, where $\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F)) \longrightarrow H^q(X, \Omega_X^n(E \bigotimes F))$ is the canonical homomorphism induced by $\iota : \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F) \hookrightarrow \Omega_X^n(E \bigotimes F)$.

Proof. The conclusion follows from Theorem 3.5 because the image of $\iota^q(\varphi_\infty)$ can be identified with $\mathcal{H}_\infty^{n,q}(X, E \bigotimes F)$ by the commutative diagram in 1.7,Theorem. □

We denote $V(\varphi_\infty)$ the compact analytic subset of X defined by the multiplier ideal sheaf $\mathcal{I}(\varphi_\infty)$ and define $d(\varphi_\infty) := \max\{\dim_{\mathbb{C}} V(\varphi_\infty)_\alpha : V(\varphi_\infty)_\alpha$ is any irreducible component of $V(\varphi_\infty)\}$ (we set $d(\varphi_\infty) = -1$ if $V(\varphi_\infty) = \phi$; i.e., $\mathcal{I}(\varphi_\infty) \cong \mathcal{O}_X$). It is clear that $d(j\varphi_\infty) \leq d(k\varphi_\infty)$ if $1 \leq j < k$, and $\iota^q(\varphi_\infty)$ is bijective (resp. surjective) if $q > d\ (\varphi_\infty) + 1$ (resp. $q > d(\varphi_\infty)$). If the Lelong number of φ_∞ is less than one everywhere on X, then $d(\varphi_\infty) = -1$ (cf. [3, (5.6) Lemma]). Under the hypothesis of Theorem 4.1, by Theorem 3.10 we can obtain the following vanishing theorem immediately (cf. [5], [9], [15], [19]).

Theorem 4.2. *Suppose $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$. Then*

$$\iota^q(\varphi_\infty) : H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F)) \longrightarrow H^q(X, \Omega_X^n(E \bigotimes F))$$

is the zero homomorphism. Especially the following assertions hold :

(i) *If $\iota^q(\varphi_\infty)$ is surjective (resp. injective) and $q > n - \max\{\kappa_*(E), \kappa_*(F)\}$, then*

$$H^q(X, \Omega_X^n(E \bigotimes F)) = 0 \quad (\textit{resp. } H^q(X, \mathcal{I}(\varphi_\infty) \bigotimes \Omega_X^n(E \bigotimes F)) = 0)$$

(ii) *If $q > \max\{n - \max\{\kappa_*(E), \kappa_*(F)\}, d(\varphi_\infty)\}$, then*

$$H^q(X, \Omega_X^n(E \bigotimes F)) = 0$$

where $\kappa_*(E)$ (resp. $\kappa_*(F)$) is the numerical Kodaira dimension of E (resp. F).

REMARK 1. The homomorphism $\iota^q(\varphi_\infty)$ is not always injective (cf. [4, Example 1.7]).

At last we can get the following theorem from Theorem 3.16 (cf. [5, Theorem 0.2] and [10, Theorem 2.2]).

Theorem 4.3. *Under the hypothesis of Theorem 4.1 the following assertions hold*:

(i) *Suppose a non-trivial holomorphic section* σ *of* $E^{\otimes j}$ *satisfies* $\operatorname{ess.sup}_X |\sigma|^2_{E^{\otimes j}} \times e^{-j\varphi_\infty} < \infty$ *and* $q > d((i+j)\varphi_\infty) + 1$. *Then the homomorphism*

$$H^{n,q}(\sigma) : H^q(X, \Omega_X^n(E^{\otimes i} \bigotimes F)) \longrightarrow H^q(X, \Omega_X^n(E^{\otimes (i+j)} \bigotimes F))$$

induced by the tensor product with σ *is injective for any* i *and* $j \geq 1$.

(ii) *Suppose* θ *is a non-trivial holomorphic section of* $F^{\otimes j}$ *and* $q > d(\varphi_\infty) + 1$. *Then the homomorphism*

$$H^{n,q}(\theta) : H^q(X, \Omega_X^n(E \bigotimes F^{\otimes i})) \longrightarrow H^q(X, \Omega_X^n(E \bigotimes F^{\otimes (i+j)}))$$

induced by the tensor product with θ *is injective for any* i *and* $j \geq 1$.

REMARK 2. Theorems 4.2 and 4.3 yield us an indication about Kawamata-Viehweg type vanishing theorem for nef line bundles on compact Kähler manifolds ; i.e., $H^q(X, \Omega_X^n(L)) = 0$ if a holomorphic line bundle L on a compact Kähler manifold X with $\dim_C X = n$ is *nef* and *good*; i.e., $\kappa(L) = \kappa_*(L)$ and $q > n - \kappa_*(L)$, where $\kappa(L)$ is the Kodaira dimension of L. In this situation by replacing X by a bimeromorphic Kähler model of X there exist a surjective morphism $\pi : X \to Y$ with connected fibres from X to a projective algebraic manifold Y with $\dim_C Y = \kappa_*(L)$ and a nef-big $\mathbb{Q}$-divisor B on Y such that (i) $L = \pi^* B$, (ii) $kB = A + D$ with a very ample divisor A and an effective divisor D on Y for $k \gg 0$ (cf. [13, §2, Proposition 2.14]). This implies that $L^{\otimes k}$ is written by the tensor product of a semi-positive line bundle $\pi^*[A]$ and a pseudo effective one $\pi^*[D]$, and admits a non-trivial section θ which vanishes along $\pi^* D$ (cf. Theorem 4.3 and [17, §6]).

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