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# THE CASSON-WALKER INVARIANT FOR BRANCHED CYCLIC COVERS OF $S^3$ BRANCHED OVER A DOUBLED KNOT

KATSUHIRO ISHIBE

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## 0. Introduction

In 1985, A. Casson defined an invariant $\lambda$ for oriented integral homology 3-spheres by using representations from their fundamental group into $SU(2)$ [1]. It was extended to an invariant for rational homology 3-spheres by K. Walker [11]. In 1993, C. Lescop [9] gave a formula to calculate this invariant for rational homology 3-spheres when they are presented by framed links and showed that it naturally extends to an invariant for all 3-manifolds.

Let $L$ be a link in $S^3$ and let $\Sigma_L^n$ be its n-fold cyclic branched cover. Define $\lambda_n(L)=\lambda(\Sigma_L^n)$. Then $\lambda_n$ becomes an invariant of links. For doubles of knots, torus knots and iterated torus knots, A. Davidow (see [3], [4]) calculated Casson's integer invariant for n-fold branched covers, when $\Sigma_K^n$ is an integral homology sphere. For any links, D. Mullins [10] have succeeded in calculating Casson-Walker's rational valued invariant for 2-fold branched covers, when $\Sigma_L^2$ is a rational homology sphere.

In this paper, using C. Lescop's formula and the result of D. Mullins, we will calculate the Casson-Walker invariant for branched cyclic covers of $S^3$ branched over the m-twisted double of a knot. We will show the following theorem and corollary.

**Theorem 3.1.** *Let $K$ be a knot in $S^3$ and $D_mK$ its m-twisted double. Then $\lambda_n(D_mK)$ is determined by $d/dtV_{D_mK}(-1)$ and $m$ where $d/dtV_{D_mK}(-1)$ is the derivative of the Jones polynomial of $D_mK$ at $t=-1$.*

**Corollary 3.2.** *$\lambda_n(D_mK)$ is determined by $a_1(K)$ and $m$ where $a_1(K)$ is the coefficient of $z^2$ of the Conway polynomial of $K$.*

## 1. Preliminaries

DEFINITION 1.1. The Conway polynomial $\nabla_L(z)$ of an oriented link $L$ is defined by

1. $\nabla_U(z)=1$, where $U$ is an unknot,

2. $\nabla_{L^+}(z) - \nabla_{L^-}(z) = -z\nabla_{L^o}(z)$, where $L^+$, $L^-$, $L^o$ are oriented links identical except within a ball where they are as shown in Figure 1.

It is well known that the Conway polynomial is of the form

$$\nabla_L(z) = z^{\sharp L-1}(a_0(L) + a_1(L)z^2 + \cdots + a_{d(L)}z^{2d(L)}).$$

This defines the coefficients $a_i(L)$ of $\nabla_L(z)$.

Let $K$ be a knot in $S^3$ and $D_mK$ its m-twisted double. It is easy to see that $\nabla_{D_mK}(z) = 1 - mz^2$. Thus $a_1(D_mK) = -m$ and $a_i(D_mK) = 0$ for $i \geq 2$.

DEFINITION 1.2. The Jones polynomial $V_L(t)$ of an oriented link $L$ is defined by

1. $V_U(t) = 1$, where $U$ is an unknot,
2. $t^{-1}V_{L^+}(t) - tV_{L^-}(t) = (t^{1/2} - t^{-1/2})V_{L^o}(t)$, where $L^+$, $L^-$, $L^o$ are oriented links identical except within a ball where they are as shown in Figure 1.

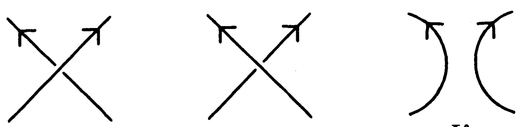



Fig. 1.

Let $W_L$ be a Seifert matrix for an oriented link $L$.

DEFINITION 1.3. The signature $\sigma(L)$ of $L$ is defined as

$$\sigma(L) = signature(W_L + W_L^T).$$

DEFINITIONS 1.4. Let $\mathcal{L} = \{(K_1, \mu_1), \ldots, (K_n, \mu_n)\}$ be a framed link in $S^3$, where each component $K_i$ is oriented and $\mu_i$ gives integer framing. The manifold obtained by surgery on $\mathcal{L}$ is denoted by $\chi(\mathcal{L})$. Let $L$ denote the underlying link of $\mathcal{L}$. Let $l_{ij}$ be the linking number $lk(K_i, K_j)$ of $K_i$ and $K_j$ if $i \neq j$ and $\mu_i$ if $i = j$.

- The linking matrix of $\mathcal{L}$ is defined by

$$E(\mathcal{L}) = (l_{ij})_{1 \leq i,j \leq n}.$$

- The sign of $\mathcal{L}$, denoted by $sign(\mathcal{L})$, is equal to $(-1)^{b_-(\mathcal{L})}$ where $b_-(\mathcal{L})$ denotes the number of negative eigen values of $E(\mathcal{L})$.
- Restriction of a framed link.

If $I$ is a subset of $N = \{1, \dots, n\}$, then $\mathcal{L}_I$ (resp. $L_I$) denotes the framed link obtained by $\mathcal{L}$ (resp. the link obtained by $L$) by forgetting the components whose subscripts do not belong to $I$.

- The circular linking of $\mathcal{L}_I$, denoted by $Lk_c(\mathcal{L}_I)$, is defined by

$$Lk_c(\mathcal{L}_I) = \sum_{\sigma \in \sigma_I} \left( \prod_{k \in I} l_{k\sigma(k)} \right),$$

where $\sigma_I$ denotes the set of cyclic permutations of $I$.

- The $\theta$-linking of $\mathcal{L}_I$.

Let $\theta_b(\mathcal{L}_I)$ be defined by

$$\theta_b(\mathcal{L}_I) = \sum_{\{(K,i,j,g) | K \subset I, (i,j) \in K^2, g \in S_{I \setminus K}\}} Lk_c(\mathcal{L}_K) l_{ig(1)} l_{g(1)g(2)} \cdots l_{g(\sharp(I\setminus K)-1)g(\sharp(I\setminus K))} l_{g(\sharp(I\setminus K))j}.$$

($S_{I\setminus K}$ denotes the set of one to one maps from $\{1, \dots, \sharp(I \setminus K)\}$ to $I \setminus K$.) This sum can be seen as the sum of the linking numbers of $\mathcal{L}_I$ with respect to the edes of one of the graphs in Figure 2 for all combinatorial ways of constructing such graphs.

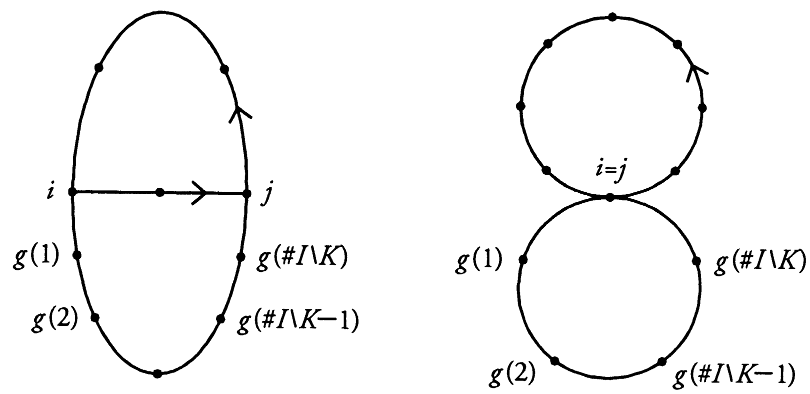



Fig. 2.

Then, the $\theta$-linking of $\mathcal{L}_I$, denoted by $\theta(\mathcal{L}_I)$, is defined by

$$\theta(\mathcal{L}_I) = \begin{cases} \theta_b(\mathcal{L}_I) & \text{if } \sharp I > 2 \\ \theta_b(\mathcal{L}_I) - 2l_{ij} & \text{if } I = \{i,j\} \\ \theta_b(\mathcal{L}_I) + 2 & \text{if } I = \{i\}. \end{cases}$$

- The modified linking matrix $E(\mathcal{L}_{N\setminus I}; I)$ is defined by

$$E(\mathcal{L}_{N\setminus I}; I) = (l_{ijI})_{i,j \in N\setminus I}$$

with

$$l_{ijI}=\begin{cases} l_{ij} & \text{if } i\neq j\\ l_{ii}+\sum\limits_{k\in I} l_{ki} & \text{if } i=j. \end{cases}$$

We state C. Lescop's formula for the Casson-Walker invariant.

**Proposition 1.5** ([9]). *Let $\mathcal{L}$ and $\chi(\mathcal{L})$ be as above. Then the Casson-Walker invariant $\lambda$ of $\chi(\mathcal{L})$ is given by*

$$\begin{aligned}\lambda(\chi(\mathcal{L}))\\ &= sign(\mathcal{L}) \sum_{\{J \mid J\neq\phi, J\subset N\}} \left(\det(E(\mathcal{L}_{N\setminus J};J))a_1(L_J)+\frac{\det(E(\mathcal{L}_{N\setminus J}))(-1)^{\sharp J}\theta(\mathcal{L}_J)}{24}\right)\\ &\qquad + sign(\mathcal{L})\det(E(\mathcal{L}))\frac{signature(E(\mathcal{L}))}{8},\end{aligned}$$

*where the determinant of an empty matrix equals to one.*

REMARK 1.6. We follow C. Lescop's normalization of the Casson-Walker invariant. If $\lambda_w$ denotes the Walker invariant as described in [11],

$$\lambda(M)=\frac{|H_1(M;\mathbf{Z})|}{2}\lambda_w(M).$$

Finally, we state the result of D. Mullins for two-fold branched covers.

**Proposition 1.7** ([10]). *Let $L$ be a link in $S^3$. Suppose the two-fold branched cover of $L$, $\Sigma_L^2$, is a rational homology sphere. Then*

$$\lambda_2(L)=-\frac{i^{\sigma(L)}}{12}\frac{d}{dt}V_L(-1)+|H_1(\Sigma_L^2;\mathbf{Z})|\frac{\sigma(L)}{8}.$$

Note that if $L$ is a knot, then $\Sigma_L^2$ is a rational homology sphere.

## 2. Surgery description of cyclic branched covers

Let $D_mK$ be the m-twisted double of a knot $K$ in $S^3$. If we introduce one surgery curve $C$ which have the framing 1 for the crossing that must be changed to obtain an unknot, we may arrive at a surgery description of $D_mK$ as shown in Figure 3, where $\overline{D_mK}$ is an unknot corresponding to $D_mK$ in other version of $S^3$.

Applying an isotopy to $S^3$, we can exchange the position of $\overline{D_mK}$ with that of $C$ (see Figure 4).

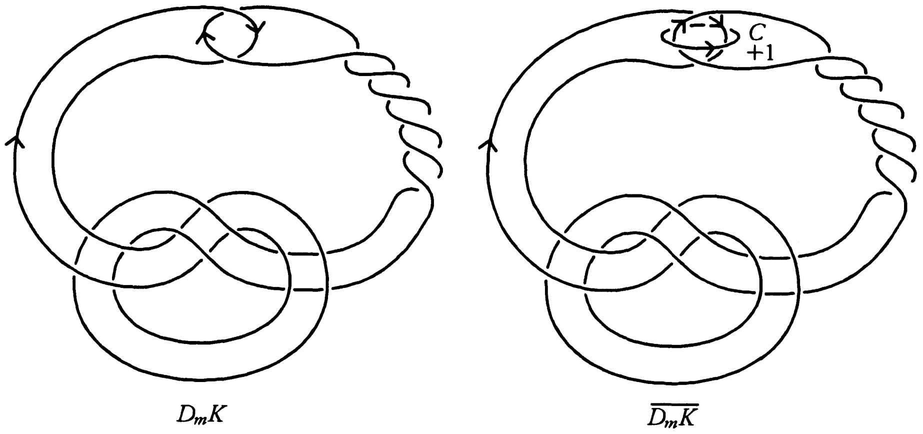



Fig. 3.

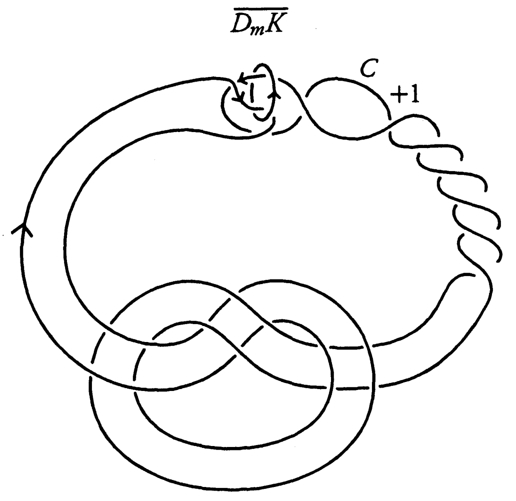
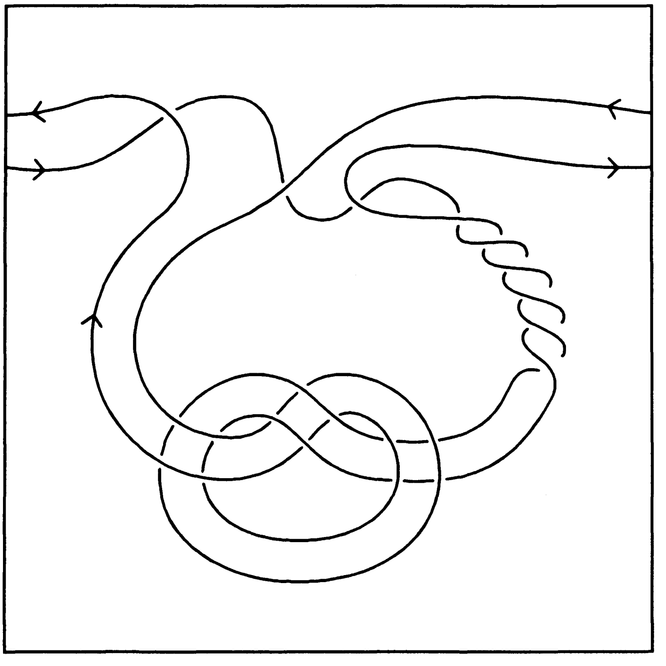



Fig. 4.

Let $T_{D_mK}$ denote the tangle which is obtained by cutting $C$ (at two points) by a spanning 2-disk for $\overline{D_mK}$ as in Figure 5.

Note that $T_{D_mK}$ has two arcs. By joining n-copies of $T_{D_mK}$ cyclically, we obtain an n-component link $L^n_{D_mK} = \{K_1, \dots, K_n\}$ as in Figure 6.

Then the n-fold cyclic branched cover of $S^3$ branched over $D_mK$, $\Sigma^n_{D_mK}$, is obtained by a surgery on the link $L^n_{D_mK}$. Note that $lk(K_1, K_2) = lk(K_2, K_3) = \cdots = lk(K_n, K_1)$ and the framing of the component $K_i$ is equal to $-2lk(K_1, K_2)+1$ if $n \geq 3$ and is equal to $-lk(K_1, K_2) + 1$ if $n = 2$.



tangle $T_{D_mK}$

Fig. 5.

**Lemma 2.1.**

$$lk(K_1, K_2) = \begin{cases} -m & if\, n \geq 3 \\ -2m & if\, n = 2. \end{cases}$$

Proof. Consider the crossing of $D_mK$ that must be changed to obtain an unknot. From the skein relation of the Conway polynomial, we get $a_1(D_mK) = -a_0(K^o)$ where $K^o$ is the 2-component link obtained by splicing the crossing of $D_mK$. On the other hand, $-a_0(K^o)$ is equal to the linking number of the components of $K^o$ (see [5]). But this is equal to $lk(K_1, K_2)$ if $n \geq 3$ and is equal to $1/2lk(K_1, K_2)$ if $n = 2$. Noting that $a_1(D_mK) = -m$, we get the conclusion. □

Then from Lemma 2.1, we can express the framing in terms of the twisting number $m$. Each $K_i$ has the framing $1+2m$. Thus $\mathcal{L}^n_{D_mK} = \{(K_1, 1+2m), \ldots, (K_n, 1+2m)\}$ is a framed link for $\Sigma^n_{D_mK}$.

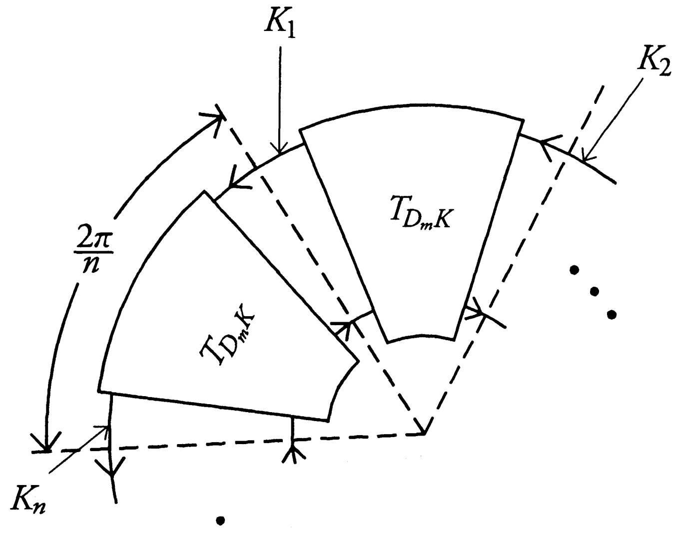



Fig. 6.

## 3. Calculation of $\lambda_n$

In this section, we will prove Theorem 3.1 and Corollary 3.2 and give a formula for $\lambda_n$.

**Theorem 3.1.** *Let $K$ be a knot in $S^3$ and $D_mK$ its m-twisted double. Then $\lambda_n(D_mK)$ is determined by $d/dtV_{D_mK}(-1)$ and $m$ where $d/dtV_{D_mK}(-1)$ is the derivative of the Jones polynomial of $D_mK$ at $t = -1$.*

**Corollary 3.2.** *$\lambda_n(D_mK)$ is determined by $a_1(K)$ and $m$ where $a_1(K)$ is the coefficient of $z^2$ of the Conway polynomial of $K$.*

### Proof of Theorem 3.1 and Corollary 3.2

The case of $n = 2$ follows from Proposition 1.7 and the fact that $|H_1(\Sigma^2_{D_mK}; \mathbf{Z})| = |1 - 4a_1(D_mK)| = |1 + 4m|$. So, assume that $n \geq 3$. We use Proposition 1.5 and the surgery description of $\Sigma^n_{D_mK}$. Let $\mathcal{L}^n_{D_mK} = \{(K_1, 1+2m), \ldots, (K_n, 1+2m)\}$ be the surgery description for $\Sigma^n_{D_mK}$ as in section 2. The linking matrix is determined by $m$ as follows;

$$E(\mathcal{L}^n_{D_mK}) = \begin{pmatrix} 1+2m & -m & & & -m \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ -m & & & -m & 1+2m \end{pmatrix}.$$

Then $sign(\mathcal{L}^n_{D_mK})$, $\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J}; J))$, $\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J}))$, $\theta((\mathcal{L}^n_{D_mK})_J)$, $\det(E(\mathcal{L}^n_{D_mK}))$ and $signature(E(\mathcal{L}^n_{D_mK}))$ are also determined by $m$. So we want to know whether or not $a_1((L^n_{D_mK})_J)$ can be expressed in terms of the original data of $D_mK$.

To do this we introduce the following notations and proposition. For given tangles $A$ and $B$, the tangle $A+B$ is defined as in Figure 7. Also, there are two operations that associate knots and links to a given tangle $A$. These are denoted $N(A)$ and $D(A)$ as in Figure 7.

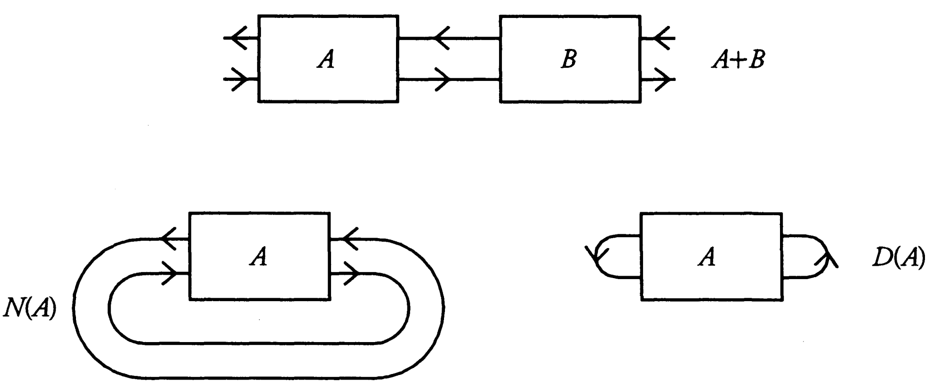
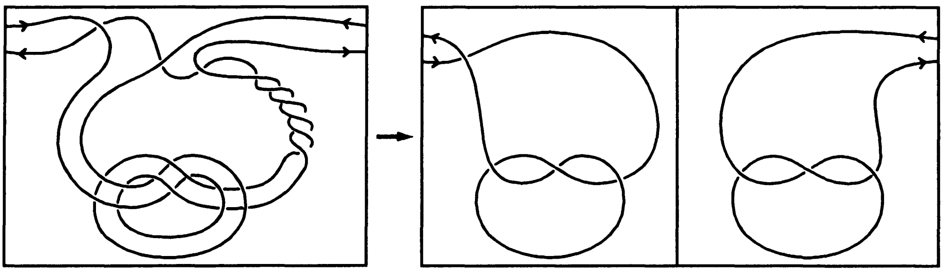



Fig. 7.

**Proposition 3.3** ([2], [7]). *Let $A$ and $B$ be tangles. Then*

$$\nabla_{N(A+B)}(z) = \nabla_{N(A)}(z)\nabla_{D(B)}(z) + \nabla_{D(A)}(z)\nabla_{N(B)}(z),$$
$$\nabla_{D(A+B)}(z) = \nabla_{D(A)}(z)\nabla_{D(B)}(z).$$

Let $T'_{D_mK}$ be the tangle which is obtained from $T_{D_mK}$ by splitting two arcs of $T_{D_mK}$ as in Figure 8.



$T_{D_mK}$ $T'_{D_mK}$

Fig. 8.

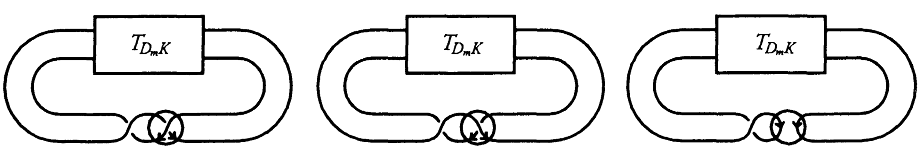
Since the Conway polynomial of a split link is zero, we only consider the case that $(L^n_{D_mK})_J$ is not split. Then $(L^n_{D_mK})_J = N(\overbrace{T_{D_mK} + \cdots + T_{D_mK}}^{\sharp J-1} + T'_{D_mK})$ if $J \neq N$ and $(L^n_{D_mK})_N = N(\overbrace{T_{D_mK} + \cdots + T_{D_mK}}^{n})$. Then it follows from Proposition 3.3 that

$$\nabla_{(L^n_{D_mK})_J}(z) = \begin{cases} \nabla_{D(T_{D_mK})}(z)^{\sharp J-1}\nabla_{N(T'_{D_mK})}(z) & \text{if } J \neq N \\ n\nabla_{D(T_{D_mK})}(z)^{n-1} & \text{if } J = N. \end{cases} \tag{1}$$

(Note that $D(T'_{D_mK})$ is a split link and $N(T_{D_mK})$ is an unknot.)
Hence

$$a_1((L^n_{D_mK})_J) = \begin{cases} a_0(D(T_{D_mK}))^{\sharp J-2}\{a_0(D(T_{D_mK}))a_1(N(T'_{D_mK})) \\ \qquad + (\sharp J - 1)a_1(D(T_{D_mK}))\} & \text{if } J \neq N \\ n(n-1)a_0(D(T_{D_mK}))^{n-2}a_1(D(T_{D_mK})) & \text{if } J = N. \end{cases} \tag{2}$$

Note that $D_mK$, $\overline{D_mK}$(= unknot) and $D(T_{D_mK})$ are related by single crossing changes as indicated in Figure 9.




$D_mK$ $\overline{D_mK}$(=unknot) $D(T_{D_mK})$

Fig. 9.

Then from the skein relation of the Conway polynomial, we get

$$a_0(D(T_{D_mK})) = -a_1(D_mK) = m \tag{3}$$

and

$$a_1(D(T_{D_mK})) = -a_2(D_mK) = 0. \tag{4}$$

(Recall that $\nabla_{D_mK}(z) = 1 - mz^2$.)

Thus only $a_1(N(T'_{D_mK}))$ has not been expressed in terms of the original data of $D_mK$ yet. To do this, we will caluculate $\lambda_2(D_mK)$ in two ways using Proposition 1.5 and Proposition 1.7.

The two-fold branched cover $\Sigma^2_{D_mK}$ is presented by the surgery description $\mathcal{L}^2_{D_mK} = \{(K_1, 1+2m), (K_2, 1+2m)\}$. Note that $lk(K_1, K_2) = -2m$. The linking matrix is

$$E(\mathcal{L}^2_{D_mK}) = \begin{pmatrix} 1+2m & -2m \\ -2m & 1+2m \end{pmatrix}.$$

The eigen values of $E(\mathcal{L}^2_{D_mK})$ are 1 and $1+4m$. So,

$$\det(E(\mathcal{L}^2_{D_mK})) = 1+4m,$$

$$sign(\mathcal{L}^2_{D_mK}) = sign(\det(E(\mathcal{L}^2_{D_mK}))) = sign(1+4m),$$

and

$$signature(E(\mathcal{L}^2_{D_mK})) = 1 + sign(1+4m).$$

Moreover we get

$$\det(E((\mathcal{L}^2_{D_mK})_{\{1,2\}\setminus J}; J)) = 1$$

and

$$\det(E((\mathcal{L}^2_{D_mK})_{\{1,2\}\setminus J})) = \begin{cases} 1+2m & \text{if } \sharp J = 1 \\ 1 & \text{if } \sharp J = 2. \end{cases}$$

Note that from (2)

$$a_1((L^2_{D_mK})_{\{j\}}) = a_1(K_j) = a_1(N(T'_{D_mK})) \quad (j = 1, 2)$$

and from (2) and (4)

$$a_1((L^2_{D_mK})_{\{1,2\}}) = a_1(\{K_1, K_2\}) = -2a_2(D_mK) = 0.$$

By considering all graphs appearing in Figure 2, we get

$$\theta((\mathcal{L}^2_{D_mK})_J) = \begin{cases} 4m^2+4m+3 & \text{if } \sharp J = 1 \\ 4m(2m+1)^2 & \text{if } \sharp J = 2. \end{cases}$$

Then according to Proposition 1.5, we get

$$\lambda_2(D_mK) = sign(1+4m)\left(2a_1(N(T'_{D_mK})) - \frac{(2m+1)(2m+3)}{12}\right.$$
$$\left. + \frac{(1+4m)(1+sign(1+4m))}{8}\right). \tag{5}$$

On the other hand, from Proposition 1.7, we get

$$\lambda_2(D_mK) = -\frac{i^{\sigma(D_mK)}}{12}\frac{d}{dt}V_{D_mK}(-1) + sign(1+4m)(1+4m)\frac{\sigma(D_mK)}{8}. \tag{6}$$

(Note that $|H_1(\Sigma^2_{D_mK};\mathbf{Z})| = |\det(E(\mathcal{L}^2_{D_mK}))| = sign(1+4m)(1+4m)$.)

Using (5), (6) and the fact that $\sigma(D_mK) = 0$ if $m \geq 0$ and $\sigma(D_mK) = -2$ if $m < 0$, we can express $a_1(N(T'_{D_mK}))$ in terms of $m$ and $d/dtV_{D_mK}(-1)$ as follows;

$$a_1(N(T'_{D_mK})) = -\frac{1}{24}\frac{d}{dt}V_{D_mK}(-1) + \frac{1}{6}m(m-1). \tag{7}$$

Thus in the case of $\mathcal{L}^n_{D_mK}$, all terms appering in Proposition 1.5 are expressed in terms of the original data $d/dtV_{D_mK}(-1)$ of $D_mK$ and $m$. This completes the proof of Theorem 3.1.

To prove Collorary 3.2, note that $N(T'_{D_mK})$ is isotopic to $K\sharp(-K)$. Since the Conway polynomial is multiplicative under connected sum, we have

$$a_1(N(T'_{D_mK})) = a_1(K\sharp(-K)) = 2a_1(K). \tag{8}$$

This proves Corollary 3.2.

REMARK 3.4. From equations (7) and (8), we can get a relation between the Conway polynomial of $K$ and the Jones polynomial of $D_mK$ as follows;

$$2a_1(K) = -\frac{1}{24}\frac{d}{dt}V_{D_mK}(-1) + \frac{1}{6}m(m-1)$$

or equivalently

$$\frac{d}{dt}V_{D_mK}(-1) = -48a_1(K) + 4m^2 - 4m.$$

**A formula for $\lambda_n$**

Note that the linking matrix $E(\mathcal{L}^n_{D_mK})$ can be diagonalized to the following matrix;

$E_n(D_mK)$

$$= \begin{pmatrix} 1 & & & & \\ & 1-2m\left(\cos\frac{2\pi}{n}-1\right) & & & \mathbf{0} \\ & \cdots & \cdots & & \\ & & 1-2m\left(\cos\frac{2k\pi}{n}-1\right) & & \\ & & \cdots & \cdots & \\ & \mathbf{0} & & & 1-2m\left(\cos\frac{2(n-1)\pi}{n}-1\right) \end{pmatrix}.$$

Then

$$sign(\mathcal{L}^n_{D_mK}) = sign(\det(E_n(D_mK))) = \begin{cases} 1 & n\text{: odd} \\ sign(1+4m) & n\text{: even.} \end{cases}$$

Let $J$ be a subset of $N = \{1, \dots, n\}$ such that $(L^n_{D_mK})_J$ is not split. We only consider such $J$ since $a_1((L^n_{D_mK})_J) = 0$ and $\theta((L^n_{D_mK})_J) = 0$ if $(L^n_{D_mK})_J$ is split.

Then from (2), (3), (4) and (7) we get

$$a_1((L^n_K)_J) = \begin{cases} m^{\sharp J-1}\left(-\dfrac{1}{24}V_{D_mK}(-1) + \dfrac{1}{6}m(m-1)\right) & \text{if } 1 \le \sharp J \le n-1 \\ 0 & \text{if } \sharp J = n. \end{cases}$$

Let $A_j(D_mK)$ be the $j \times j$ matrix

$$\begin{pmatrix} 1+m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+m \end{pmatrix}$$

and $B_j(D_mK)$ be the $j \times j$ matrix

$$\begin{pmatrix} 1+2m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+2m \end{pmatrix}.$$

Then

$$\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J}; J)) = \det(A_{n-\sharp J}(D_mK))$$

and

$$\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J})) = \det(B_{n-\sharp J}(D_mK)).$$

By considering all graphs appearing in Figure 2, we can calculate $\theta((\mathcal{L}^n_{D_mK})_J)$ as follows;

$$\theta((\mathcal{L}^n_{D_mK})_J) = \begin{cases} 4m^2+4m+3 & \text{if } \sharp J = 1 \\ 2m(3m^2+2m+1) & \text{if } \sharp J = 2 \\ 6m^4 & \text{if } \sharp J = 3, \sharp N = 3 \\ 2m^4 & \text{if } \sharp J = 3, \sharp N > 3 \\ 0 & \text{if } 4 \le \sharp J \le n-1 \\ 2n(-m)^n(2+m) & \text{if } \sharp J = n. \end{cases}$$

Then $\lambda_n(D_mK)$ can be expressed as a combination of $m$ and $d/dtV_{D_mK}(-1)$ as in the following theorem.

**Theorem 3.5.** *Let $K$ be a knot in $S^3$ and $D_mK$ its $m$-twisted double. Then*

$$\begin{aligned}\lambda_n(D_mK) = S_n(m)n\Bigg(&\varphi(D_mK)\sum_{j=1}^{n-1} m^{j-1}\det(A_{n-j}(m)) \\ &+\frac{1}{24}\sum_{j=1}^{n}(-1)^j\det(B_{n-j}(m))\psi_j(m)\Bigg) \\ &+ S_n(m)\frac{\det(E_n(m))signature(E_n(m))}{8}\end{aligned}$$

*with*

$$S_n(m) = \begin{cases} 1 & n\text{: odd or } n\text{: even, } m \ge 0 \\ -1 & n\text{: even, } m < 0, \end{cases}$$

*the $j \times j$ matrix*

$$A_j(m) = \begin{pmatrix} 1+m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+m \end{pmatrix},$$

*the $j \times j$ matrix*

$$B_j(m) = \begin{pmatrix} 1+2m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+2m \end{pmatrix},$$

$$\varphi(D_m K) = -\frac{1}{24}\frac{d}{dt}V_{D_m K}(-1) + \frac{1}{6}m(m-1),$$

$$\psi_j(m) = \begin{cases} 4m^2+4m+3 & \textit{if } j=1 \\ 2m(3m^2+2m+1) & \textit{if } j=2 \\ 6m^4 & \textit{if } j=3, n=3 \\ 2m^4 & \textit{if } j=3, n>3 \\ 0 & \textit{if } 4 \le j \le n-1 \\ 2(-m)^n(2+m) & \textit{if } j=n, \end{cases}$$

*and the $n \times n$ diagonal matrix*

$$E_n(m) = \begin{pmatrix} 1 & & & & \mathbf{0} \\ & 1-2m\left(\cos\frac{2\pi}{n}-1\right) & & & \\ & \cdots & \cdots & & \\ & & 1-2m\left(\cos\frac{2k\pi}{n}-1\right) & & \\ & & \cdots & \cdots & \\ \mathbf{0} & & & & 1-2m\left(\cos\frac{2(n-1)\pi}{n}-1\right) \end{pmatrix}.$$

REMARK 3.6. In case of $m=0$ (untwisted double), the n-fold cyclic branched cover $\Sigma^n_{D_0 K}$ is an integral homology sphere. J. Hoste [6] has caluculated $\lambda_n(D_0 K)$ in terms of $a_1(K)$ as follows;

$$\lambda_n(D_0 K) = 2na_1(K).$$

Note that $\varphi(D_m K) = 2a_1(K)$. Therefore Theorem 3.5 is a generalization of this formula.

On the other hand, $\lambda_n(D_0 K)$ is expressed in terms of $d/dt V_{D_0 K}(-1)$ as follows;

$$\lambda_n(D_0 K) = -\frac{n}{24}\frac{d}{dt}V_{D_0 K}(-1).$$

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Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi,
Tokyo, 192-03, Japan