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ON UNIT-REGULAR RINGS SATISFYING S-COMPARABILITY

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1. Preliminaries and notations

In [2] and [3], we studied the properties of unit-regular rings satisfying the comparability axiom. In this paper, we shall investigate unit-regular rings satisfying s-comparability which is a generalized notion of the comparability axiom. In section 2, we shall show that these rings have the property (DF), that is, $P\oplus Q$ is directly finite for every two directly finite projective modules P and Q. In section 3, we shall obtain a criterion of direct finiteness of projective modules over these rings (Proposition 4 and Theorem 7). Using this result, we can determine the types of directly finite projective modules and classify the family of all unit-regular rings satisfying s-comparability into three types; Types A, B and C (Theorem 12). In section 4, we shall give the ideal-theoretic characterization for Types A, B and C (Theorems 14, 15 and 16).

Throughout this paper, R is a ring with identity and all modules are unital right R-modules.

NOTATION. If M and N are R-modules, then the notation $N\lesssim M$ (resp. $N\lesssim\oplus\ M$) means that N is isomorphic to a submodule of M (resp. N is isomorphic to a direct summand of M). For a cardinal number α and an R-module M, αM denotes the direct sum of α-copies of M. For a set X, we denote the cardinal number of X by $|X|$. We denote by N_0 the set of all positive integers.

DEFINITION. A ring R is *directly finite* if $xy=1$ implies $yx=1$ for all $x,y\in R$. An R-module M is *directly finite* if $\mathrm{End}_R(M)$ is directly finite. A ring R (a module M) is *directly infinite* if it is not directly finite. It is well-known that M is directly finite if and only if M is not isomorphic to a proper direct summand of M itself. A ring R is said to be a *unit-regular* ring if, for each $x\in R$, there exists a unit (i.e. an invertible element) u of R such that $xux=x$. Let s be a positive integer. Then a regular ring R is said to satisfy *s-comparability* provided that for any $x,y\in R$, either $xR\lesssim s(yR)$ or $yR\lesssim s(xR)$. Note that 1-comparability is called *the comparability axiom.*

Now we shall recall some elementary properties (see [2, Lemma 1]).

Let R be a unit-regular ring. Then

(1) Every finitely generated projective R-module P has the cancellation property, and so P is directly finite.

(2) For any projective R-module X and any finitely generated projective R-modules $Y_1, Y_2, \cdots$ such that $Y_1 \oplus \cdots \oplus Y_n \lesssim X$ for all positive integers n, we have that $\oplus_{n=1}^{\infty} Y_n \lesssim X$.

(3) Let P and Q be projective R-modules such that Q is finitely generated. Then $P \oplus Q$ is directly finite if and only if so is P.

All basic results concerning regular rings can be found in a book by K. R. Goodearl [1].

2. The property (DF)

Lemma 1. *Let R be a unit-regular ring, and P be a projective R-module with a cyclic decomposition $P = \oplus_{i \in I} P_i$. Then the following conditions* (a)$\sim$(c) *are equivalent*:

(a) *P is directly infinite.*

(b) *There exists a nonzero principal right ideal X of R such that $X \lesssim \oplus_{i \in I - \{i_1, \cdots, i_n\}} P_i$ for every finite subset $\{i_1, \cdots, i_n\}$ of I.*

(c) *There exists a nonzero principal right ideal X of R such that $\aleph_0 X \lesssim \oplus$ P.*

Proof. (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are clear. We will show (a) $\Rightarrow$ (b). Suppose P is directly infinite. Then there exists a nonzero module Y such that $P \simeq P \oplus Y$, and so we can take a nonzero principal right ideal X of R such that $X \lesssim Y$ and $X \lesssim P_{n(1)} \oplus \cdots \oplus P_{m(1)}$ for some finite subset $\{n(1), \cdots, m(1)\}$ of I. Put $I' = I - \{i_1, \cdots, i_n\}$. Using that $P \simeq P \oplus Y$, we have that

$$\begin{aligned}&(P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{i_1} \oplus \cdots \oplus P_{i_n}) \oplus (\oplus_{i \in I'} P_i) \\ \simeq\, &(P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{i_1} \oplus \cdots \oplus P_{i_n}) \oplus (\oplus_{i \in I'} P_i) \oplus Y.\end{aligned}$$

Noting that every finitely generated projective module has the cancellation property, we see that $\oplus_{i \in I'} P_i \simeq (\oplus_{i \in I'} P_i) \oplus Y$, and so $X \lesssim Y \lesssim \oplus_{i \in I'} P_i$ as desired.

Proposition 2. *Let R be a unit-regular ring satisfying s-comparability, and P be a projective R-module. Then P is directly finite if and only if so is nP for every positive integer n.*

Proof. "If part" is clear. We will show "Only if part". It is sufficient to prove that if $2P$ is directly infinite, then so is P. Let $P \simeq \oplus_{i \in I} P_i$ be a principal right ideal decomposition for P. Assume that $2P$ is directly infinite. From Lemma 1, there exists a nonzero principal right ideal X of R such that

$$X \lesssim (P_{n(1)} \oplus P_{n(1)+1}) \oplus \cdots \oplus (P_{m(1)} \oplus P_{m(1)+1}),$$
$$X \lesssim (P_{n(2)} \oplus P_{n(2)+1}) \oplus \cdots \oplus (P_{m(2)} \oplus P_{m(2)+1}),$$
$$\cdots\cdots$$

for some sequence $n(1) = n(1)+1 < m(1) = m(1)+1 < n(2) = n(2)+1 < m(2) = m(2)+1 < \cdots$ of I, and so $P_{n(i)} = P_{n(i)+1}$ and $P_{m(i)} = P_{m(i)+1}$ for every positive integer i. We shall argue in steps (I), (II) and (III).

Step (I). Noting that $X \lesssim (P_{n(1)} \oplus P_{n(1)+1}) \oplus \cdots \oplus (P_{m(1)} \oplus P_{m(1)+1})$, we have a decomposition $X = \oplus_{i_1} x_{i_1} R$ such that $x_{i_1} R \lesssim P_{i_1}$ for each $i_1 = n(1), n(1)+1, \cdots, m(1), m(1)+1$ by [1, Lemma 2.7]. Using that $X \lesssim (P_{n(2)} \oplus P_{n(2)+1}) \oplus \cdots \oplus (P_{m(2)} \oplus P_{m(2)+1})$, we have a decomposition

$$X \simeq (x_{n(2)} R \oplus x_{n(2)+1} R) \oplus \cdots \oplus (x_{m(2)} R \oplus x_{m(2)+1} R)$$

for some $x_{i_2} R < \oplus \ P_{i_2}$, where $i_2 = n(2), n(2)+1, \cdots, m(2),\ m(2)+1$ and

$$x_{i_1} R \simeq \oplus_{i_2} x_{i_2, i_1} R$$

for some $x_{i_2, i_1} R \leqq x_{i_2} R < \oplus \ P_{i_2}$ by [1, Corollary 2.9].
Therefore there exists a decomposition

$$X = \oplus_{i_1, i_2} x_{i_1 i_2} R$$

such that

$$x_{i_2, i_1} R \simeq x_{i_1 i_2} R \leqq x_{i_1} R \quad \text{and}$$
$$2(x_{i_1 i_2} R) \simeq x_{i_1 i_2} R \oplus x_{i_2, i_1} R \lesssim P_{i_1} \oplus P_{i_2}$$
$$\leqq (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{n(2)} \oplus \cdots \oplus P_{m(2)}) \ (\leqq P).$$

Next, noting that $X \lesssim (P_{n(3)} \oplus P_{n(3)+1}) \oplus \cdots \oplus (P_{m(3)} \oplus P_{m(3)+1})$, we have decompositions

$$X = \oplus_{i_1, i_2} x_{i_1 i_2} R$$
$$\simeq (x_{n(3)} R \oplus x_{n(3)+1} R) \oplus \cdots \oplus (x_{m(3)} R \oplus x_{m(3)+1} R)$$

for some $x_{i_3} R < \oplus \ P_{i_3}$, where $i_3 = n(3), n(3)+1, \cdots, m(3),\ m(3)+1$ and

$$X_{i_1 i_2} R \simeq \oplus_{i_3} x_{i_3, i_1 i_2} R$$

for some $x_{i_3,i_1i_2}R \leqq x_{i_3}R < \oplus\ P_{i_3}$. Therefore there exists a decomposition

$$x_{i_1}x_{i_2}R = \oplus_{i_3} x_{i_1i_2i_3}R$$

such that

$$x_{i_1i_2i_3}R \simeq x_{i_3,i_1i_2}R$$

and hence

$$X = \oplus_{i_1,i_2} x_{i_1i_2}R = \oplus_{i_1,i_2,i_3} x_{i_1i_2i_3}R$$

such that

$$\begin{aligned}
&3(x_{i_1i_2i_3}R) \leqq 2(x_{i_1i_2i_3}R) \oplus x_{i_1i_2i_3}R \lesssim 2(x_{i_1i_2}R) \oplus x_{i_3,i_1i_2}R \\
&\leqq P_{i_1} \oplus P_{i_2} \oplus P_{i_3} \\
&\leqq (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus (P_{n(2)} \oplus \cdots \oplus P_{m(2)}) \oplus (P_{n(3)} \oplus \cdots \oplus P_{m(3)})\ (\leqq P).
\end{aligned}$$

Continuing this procedure, we have a decomposition

$$X = \oplus_{i_1,i_2,\cdots,i_s} x_{i_1i_2\cdots i_s}R$$

such that

$$\begin{aligned}
&s(x_{i_1i_2\cdots i_s}R) \\
&\lesssim (P_{n(1)} \oplus \cdots \oplus P_{m(1)}) \oplus \cdots \oplus (P_{n(s)} \oplus \cdots \oplus P_{m(s)})\ (\leqq P)
\end{aligned}$$

for each $i_1, i_2, \cdots, i_s$.

Step (II). Noting that $X \lesssim (P_{n(s+1)} \oplus P_{n(s+1)+1}) \oplus \cdots \oplus (P_{m(s+1)} \oplus P_{m(s+1)+1})$, we may assume with no loss of generality that $X \lesssim P_{n(s+1)} \oplus P_{n(s+1)+1}$, and so $X \simeq x_{n(s+1)}R \oplus x_{n(s+1)+1}R$ for some $x_{n(s+1)}R < \oplus\ P_{n(s+1)}$ and $x_{n(s+1)+1}R < \oplus\ P_{n(s+1)+1}$. We put $w_{n(s+1)}R = x_{n(s+1)}R \cap x_{n(s+1)+1}R$ in R, and so there exist principal right ideals $y_{n(s+1)}R$ and $y_{n(s+1)+1}R$ of R such that

$$\begin{aligned}
&x_{n(s+1)}R = w_{n(s+1)}R \oplus y_{n(s+1)}R, \\
&x_{n(s+1)+1}R = x_{n(s+1)}R \oplus y_{n(s+1)+1}R \quad \text{and} \\
&w_{n(s+1)}R \oplus y_{n(s+1)}R \oplus y_{n(s+1)+1}R \lesssim P_{n(s+1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
X &\simeq x_{n(s+1)}R \oplus x_{n(s+1)+1}R \\
&\simeq (w_{n(s+1)}R \oplus y_{n(s+1)}R \oplus y_{n(s+1)+1}R) \oplus w_{n(s+1)}R \\
&\lesssim P_{n(s+1)} \oplus w_{n(s+1)}R.
\end{aligned}$$

We can take a direct summand z_1R of X such that $z_1R\simeq w_{n(s+1)}R$. Next, noting that $w_{n(s+1)}R\lesssim X\lesssim(P_{n(s+2)}\oplus P_{n(s+2)+1})\oplus\cdots\oplus(P_{m(s+2)}\oplus P_{m(s+2)+1})$, we have that $w_{n(s+1)}R\simeq(x_{n(s+2)}R\oplus x_{n(s+2)+1}R)\oplus\cdots\oplus(x_{m(s+2)}R\oplus x_{m(s+2)+1}R)$ for some $x_{n(s+2)}R <\oplus\ P_{n(s+2)}$, $x_{n(s+2)+1}R <\oplus\ P_{n(s+2)+1},\cdots,x_{m(s+2)}R <\oplus\ P_{m(s+2)}$ and $x_{m(s+2)+1}R <\oplus\ P_{m(s+2)+1}$. We put $w_iR=x_iR\cap x_{i+1}R$ in R for each $i=n(s+2),\cdots,m(s+2)$, from which we have decompositions

$$x_iR=w_iR\oplus y_iR,$$

$$x_{i+1}R=w_iR\oplus y_{i+1}R \quad \text{for some} \quad y_iR<\oplus\ x_iR \quad \text{and} \quad y_{i+1}R<\oplus\ x_{i+1}R$$

and

$$x_iR\oplus x_{i+1}R\simeq(w_iR\oplus y_iR\oplus y_{i+1}R)\oplus w_iR\lesssim P_i\oplus w_iR$$

for each $i=n(s+2),\cdots,m(s+2)$. Therefore $w_{n(s+1)}R\lesssim(P_{n(s+2)}\oplus\cdots\oplus P_{m(s+2)})\oplus(w_{n(s+2)}R\oplus\cdots\oplus w_{m(s+2)}R)$ and $2(w_{n(s+2)}R\oplus\cdots\oplus w_{m(s+2)}R)\lesssim w_{n(s+1)}R$. We can take a direct summand z_2R of $w_{n(s+1)}R$ such that $z_2R\simeq w_{n(s+2)}R\oplus\cdots\oplus w_{m(s+2)}R$, and hence

$$X\lesssim(P_{n(s+1)}\oplus\cdots\oplus P_{m(s+1)})\oplus(P_{n(s+2)}\oplus\cdots\oplus P_{m(s+2)})\oplus z_2R$$

and

$$2(z_2R)\lesssim w_{n(s+1)}R\simeq z_1R.$$

Continuing this procedure, we can take a family $\{z_kR\}_{k=1}^{\infty}$ of principal right ideals of R such that

$$X\lesssim(P_{n(s+1)}\oplus\cdots\oplus P_{m(s+1)})\oplus\cdots\oplus(P_{n(s+k)}\oplus\cdots\oplus P_{m(s+k)})\oplus z_kR,$$

$$X\geqq z_1R\gtrsim z_2R\gtrsim\cdots, \quad \text{and that}$$

$2(z_{k+1}R)\lesssim z_kR$ for each positive integer k.

Step (III). We claim that $z_kR\lesssim(P_{n(1)}\oplus\cdots\oplus P_{m(1)})\oplus\cdots\oplus(P_{n(s)}\oplus\cdots\oplus P_{m(s)})$ for some positive integer k. We assume that $z_kR\not\lesssim s(x_{i_1i_2\cdots i_s}R)$ for all $i_1,i_2,\cdots,i_s$ and k, and so $z_kR\neq 0$. Using that R satisfies s-comparability, $x_{i_1i_2\cdots i_s}R\lesssim s(z_kR)$, and so we have that $X=x_{i_1,i_2,\cdots i_s}x_{i_1i_2\cdots i_s}R\lesssim s^l(z_{k'}R)\lneqq(s^l+1)(z_{k'}R)\lesssim X$ for some positive integer l and k' by step (II), which contradicts the direct finiteness of X. Hence there exist positive integers $i_1,\cdots,i_s$ and k such that $z_kR\lesssim s(x_{i_1i_2\cdots i_s}R)\lesssim(P_{n(1)}\oplus\cdots\oplus P_{m(1)})\oplus\cdots\oplus(P_{n(s)}\oplus\cdots\oplus P_{m(s)})$ by step (I).

Combining steps (II) and (III), we see that $X\lesssim(P_{n(s+1)}\oplus\cdots\oplus P_{m(s+1)})\oplus\cdots\oplus(P_{n(s+k)}\oplus\cdots\oplus P_{m(s+k)})\oplus z_kR\lesssim(P_{n(1)}\oplus\cdots\oplus P_{m(1)})\oplus\cdots\oplus(P_{n(s+k)}\oplus\cdots\oplus P_{m(s+k)})\ (\leqq P)$. Similarly, we apply the above discussion for $I-\{n(1),n(1)+1,\cdots,m(s+k),m(s+k)+1\}$. Continuing this procedure, we have that $\aleph_0X\lesssim\oplus\ P$, from which P is directly infinite. Therefore the proof of the proposition is complete.

Theorem 3. *Let R be a unit-regular ring satisfying s-comparability. Then R has the property* (DF), *that is, $P\oplus Q$ is directly finite for every two directly finite projective R-modules P and Q.*

Proof. Let $P=\oplus_{i\in I}P_i$ and $Q=\oplus_{i\in I'}Q_i$ be cyclic decompositions of P and Q. Assume that $P\oplus Q$ is directly infinite. We may assume, without loss of generality, that $I=I'$ and $|I|=\infty$ by the elementary properties (1) and (3). From Lemma 1, there exist a nonzero principal right ideal X of R and a sequence $n(1)<\cdots<m(1)<n(2)<\cdots<m(2)<\cdots$ of I such that

$$X\lesssim(P_{n(1)}\oplus Q_{n(1)})\oplus\cdots\oplus(P_{m(1)}\oplus Q_{m(1)}),$$
$$X\lesssim(P_{n(2)}\oplus Q_{n(2)})\oplus\cdots\oplus(P_{m(2)}\oplus Q_{m(2)}),$$
$$\cdots\cdots\cdots,$$

from which we have a decomposition $X=p_iR\oplus q_iR$ for each positive integer i such that $p_iR\lesssim P_{n(i)}\oplus\cdots\oplus P_{m(i)}$ and $q_iR\lesssim Q_{n(i)}\oplus\cdots\oplus Q_{m(i)}$. Set $J=\{i\in N_0 \mid p_iR\lesssim s(q_iR)\}$. If $|J|=\infty$, then $X=p_iR\oplus q_iR\lesssim(s+1)q_iR$ for all $i\in J$, and so $\aleph_0X\simeq|J|X\lesssim\oplus\ (s+1)(\oplus_{i\in J}q_iR)\lesssim\oplus\ (s+1)Q$. Therefore $(s+1)Q$ is directly infinite, and so is Q by Proposition 2. Otherwise $|J|<\infty$. We see that $|J'|=\infty$, where $J'=N_0-J$. Then $q_iR\lesssim s(p_iR)$ for all $i\in J'$, and so $X=p_iR\oplus q_iR\lesssim(s+1)(p_iR)$, hence $\aleph_0X\lesssim\oplus\ \oplus_{i\in J'}\ (s+1)(p_iR)\lesssim\oplus\ (s+1)P$. Therefore $(s+1)P$ is directly infinite, ans so is P. Thus the theorem is proved.

3. Directly finite projective modules

In this section, we shall determine the types of directly finite projective modules.

Proposition 4. *Let R be a unit-regular ring satisfying s-comparability, and P be a non-finitely and non-countably generated projective R-module with a cyclic decomposition $P=\oplus_{i\in I}P_i$, where $|I|>\aleph_0$. Then P is directly infinite.*

Proof. For each $i\in I$, put $I_i=\{j\in I \mid P_i\lesssim sP_j\}$. If $|I_i|\geqq\aleph_0$ for some $i\in I$, then we see that $\aleph_0P_i\lesssim\oplus\ (\oplus_{i\in I_i}sP_j)<\oplus\ sP$, from which sP is directly infinite, hence so is P from Proposition 2. Thus we may assume that $|I_i|$ is finite for all $i\in I$. We can take $i_1\in I$, $i_2\in I-I_{i_1}$, $i_3\in I-(I_{i_1}\cup I_{i_2})$ and so on. By the calculation of cardinal numbers, we see that $I-(I_{i_1}\cup I_{i_2}\cup\cdots)$ is a nonempty set, and so there exists $i_0\in I-(I_{i_1}\cup I_{i_2}\cup\cdots)$. Nothing that $i_0\notin I_{i_1}\cup I_{i_2}\cup\cdots$ and that R satisfies s-comparability, we see that $P_{i_0}\lesssim sP_i$ for each $i=i_1,i_2,\cdots$. Hence $\aleph_0P_{i_0}\lesssim\oplus\ s(P_{i_1}\oplus P_{i_2}\oplus\cdots)<\oplus\ sP$. Thus sP is directly infinite, hence so is P as desired.

NOTE. Let R be a unit-regular ring satisfying s-comparability. Then we see that every directly finite projective R-module is finitely generated or countably generated from the above proposition.

Lemma 5. *Let R be a unit-regular ring satisfying s-comparability, and let $P_1,P_2,\cdots,P_n$ be cyclic projective R-modules, where $n \geqq 2$. Then there is a set $\{i_1,i_2,\cdots,i_n\}=\{1,2,\cdots,n\}$ such that $sP_{i_1} \gtrsim P_{i_2}$, $s^2P_{i_1} \gtrsim P_{i_3},\cdots$ and $s^{n-1}P_{i_1} \gtrsim P_{i_n}$.*

Proof. We shall prove this lemma by the induction on n ($\geqq 2$). For cyclic projective modules P_1 and P_2, we have that $sP_1 \gtrsim P_2$ or $sP_2 \gtrsim P_1$, and so the lemma holds when $n=2$. Assume that $sP_{i_1} \gtrsim P_{i_2},\cdots$ and $s^{n-1}P_{i_1} \gtrsim P_{i_n}$. For P_{i_1} and P_{n+1}, we have that $sP_{i_1} \gtrsim P_{n+1}$ or $sP_{n+1} \gtrsim P_{i_1}$. In the first case, $s^nP_{i_1} \geqq sP_{i_1} \gtrsim P_{n+1}$, hence we can take $i_{n+1}=n+1$. In the second case, we see that

$$sP_{n+1} \gtrsim P_{i_1},$$
$$s^2P_{n+1} \gtrsim sP_{i_1} \gtrsim P_{i_2},$$
$$\cdots$$
$$s^nP_{n+1} \gtrsim s^{n-1}P_{i_1} \gtrsim P_{i_n},$$

from which we can take $j_1=n+1$, $j_2=i_1,\cdots$ and $j_{n+1}=i_n$, and so $\{j_1,\cdots,j_{n+1}\}=\{1,\cdots,n+1\}$. Therefore the induction argument works.

Lemma 6. *Let R be a unit-regular ring satisfying s-comparability. Let $P_1,P_2,\cdots,P_n$ and X be cyclic projective R-modules such that $X \not\lesssim s(P_1\oplus\cdots\oplus P_n)$. Then $P_1\oplus\cdots\oplus P_n \lesssim \bar{s}X$, where $\bar{s}=s^0+s^1+\cdots+s^{s-1}$.*

Proof. Assume that $X \not\lesssim s(P_1\oplus\cdots\oplus P_n)$. Then we see that $P_1 \lesssim sX$ by the s-comparability and $X \not\lesssim sP_1$. Then we have decompositions

$$X=X_{11}\oplus X_{11}^*=X_{12}\oplus X_{12}^*=\cdots=X_{1s}\oplus X_{1s}^*$$

such that

$$X_{11}\oplus\cdots\oplus X_{1s}\simeq P_1.$$

From Lemma 5, we may assume that $sX_{11} \gtrsim X_{12}$, $s^2X_{11} \gtrsim X_{13},\cdots$ and $s^{s-1}X_{11} \gtrsim X_{1s}$, from which $P_1 \lesssim \bar{s}X_{11}$. Note that $X_{11}^* \not\lesssim sP_2$. If not, we see that $X=X_{11}\oplus X_{11}^* \lesssim P_1\oplus sP_2 \lesssim s(P_1\oplus P_2)$, which contradicts the assumption. Hence we have that $P_2 \lesssim sX_{11}^*$, and that

$$P_1\oplus P_2 \lesssim \bar{s}X_{11}\oplus sX_{11}^* \leqq \bar{s}(X_{11}\oplus X_{11}^*) \leqq \bar{s}X.$$

Noting that $P_2 \lesssim \bar{s}X_{11}^*$, we have decompositions

$$X_{11}^*=X_{21}\oplus X_{21}^*=X_{22}\oplus X_{22}^*=\cdots=X_{2s}\oplus X_{2s}^*$$

such that

$$X_{21}\oplus\cdots\oplus X_{2s}\simeq P_2.$$

From Lemma 5, we may assume that $sX_{21}\gtrsim X_{22}$, $s^2X_{21}\gtrsim X_{23},\cdots$, and $s^{s-1}X_{21}\gtrsim X_{2s}$. Hence $P_2\lesssim\bar{s}X_{21}$. Note that $X_{21}^*\not\lesssim sP_3$. If not,

$$X=X_{11}\oplus X_{11}^*=X_{11}\oplus(X_{21}\oplus X_{21}^*)\lesssim P_1\oplus P_2\oplus sP_3\leqq s(P_1\oplus P_2\oplus P_3),$$

which contradicts the assumption. Then $P_3\lesssim\bar{s}X_{21}^*$, and so $P_1\oplus P_2\oplus P_3\lesssim\bar{s}X_{11}\oplus(\bar{s}X_{21}\oplus\bar{s}X_{21}^*)\leqq\bar{s}(X_{11}\oplus X_{11}^*)\leqq\bar{s}X$. Continuing this procedure, we see that $P_1\oplus\cdots\oplus P_n\simeq\bar{s}X$.

Henceforth we put $\bar{s}=s^0+s^1+\cdots+s^{s-1}$ for s.

Theorem 7. *Let R be a unit-regular ring satisfying s-comparability, and let P be a countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$. Then P is directly finite if and only if, for each nonzero cyclic projective R-module X there exist positive integers n and t such that $\oplus_{i=n}^{\infty}P_i\lesssim tX$.*

Proof. "Only if part". Assume that P is directly finite, hence so is sP. From Lemma 1, we see that, for each nonzero cyclic projective module X there exists a positive integer n such that $X\not\lesssim s(P_n\oplus P_{n+1}\oplus\cdots)$. We see that $P_n\oplus P_{n+1}\oplus\cdots\lesssim\bar{s}X$ from Lemma 6 and the elementary property (2).

"If part". Assume that for each nonzero cyclic projective module X there exist positive integers n and t such that $\oplus_{i=n}^{\infty}P_i\lesssim tX$, and that P is directly infinite. There exists a nonzero principal right ideal Y of R such that $Y\lesssim\oplus_{I-\{i_1,\cdots,i_n\}}P_i$ for every finite subset $\{i_1,\cdots,i_n\}$ of I from Lemma 1, and we can take positive integers n and t such that $\oplus_{i=n}^{\infty}P_i\lesssim tY$. Then we see that

$$tY\lesssim\aleph_0Y\lesssim(\oplus_{i=n}^{\infty}P_i)\lesssim tY,$$

which contradicts the direct finiteness of tY.

Corollary 8. *Let R be a simple unit-regular ring satisfying s-comparability, and let P be a countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$. Then the following conditions* (a)$\sim$(c) *are equivalent:*

(a) *P is directly finite.*

(b) *There exist positive integers n and t such that $\oplus_{i=n}^{\infty}P_i\lesssim tR$.*

(c) *$P\lesssim t'R$ for some positive integer t'.*

Proof. Note that R is simple. Then for each nonzero principal right ideal X of R, there exist positive integers t_1 and t_2 such that $X\lesssim t_1R$ and $R\lesssim t_2X$ by

[1, Corollary 2.23]. Combining this result with Theorem 7, we see that this corollary holds.

NOTE. It is known from [4] that simple directly finite regular rings satisfying s-comparability are unit-regular.

DEFINITION. Let R be a unit-regular ring satisfying s-comparability. Let CP(R) be the family of cyclic projective R-modules. For elements A and B in CP(R), we define the relation "$\sim$" as follows: $A \sim B$ provided that $A \lesssim t_1 B$ and that $B \lesssim t_2 A$ for some positive integers t_1 and t_2. It is clear that the relation "$\sim$" is an equivalence relation. Put $[A] = \{B \in \mathrm{CP}(R)\} \mid A \sim B\}$ for each $A \in \mathrm{CP}(R)$. We see that $(\{[A] \mid B \in \mathrm{CP}(R)\}, \geqq)$ is a linearly ordered set, where $[A] \geqq [B]$ means that $B \lesssim tA$ for some positive integer t. Note that this definition is well-defined. We define $[A] \gneqq [B]$ if $[A] \geqq [B]$ and $[A] \neq [B]$, and this is equivalent to saying that $\aleph_0 B \lesssim \bar{s}A$ by the following Lemma 9.

Lemma 9. *Let R be a unit-regular ring satisfying s-comparability, and let A and B be elements in* CP(R). *Then $A \lesssim tB$ for some positive integer t if and only if $\aleph_0 B \not\lesssim \bar{s}A$.*

Proof. "If part". Assume that $\aleph_0 B \not\lesssim \bar{s}A$. If $A \not\lesssim tB$ for all positive integers t, we see that $A \not\lesssim s(tB)$ and so $tB \lesssim \bar{s}A$ for all t from Lemma 6, hence $\aleph_0 B \lesssim \bar{s}A$ which contradicts the assumption.

"Only if part". Assume that $A \lesssim tB$ for some positive integer t. If $\aleph_0 B \lesssim \bar{s}A$, we see that $\bar{s}tB \lesssim \aleph_0 B \lesssim \bar{s}A \lesssim \bar{s}tB$, which contradicts the direct finiteness of $\bar{s}tB$.

Let R be a unit-regular ring satisfying s-comparability. For a countably generated projective R-module P with a cyclic decomposition $P = \oplus_{i=1}^{\infty} P_i$, we consider the following three conditions (∗), (A) and (B) in order to investigate the direct finiteness of P:

(∗) For each nonzero cyclic projective R-module X, $\{i \in N_0 \mid X \lesssim \bar{s}P_i\}$ is a finite set.

(A) (i) $[P_n] = [P_{n+1}] = \cdots$ for some positive integer n.

(ii) $\oplus_{i=n+1}^{\infty} P_i \lesssim tP_n$ for some positive integer t.

(B) There exists a sequence $i_1 < i_2 < \cdots$ of positive integers such that $[P_{i_1}] = [P_{i_1+1}] = \cdots \gneqq [P_{i_2}] = [P_{i_2+1}] = \cdots \gneqq [P_{i_3}] = [P_{i_3+1}] = \cdots$.

NOTES 1. If (∗) holds, then $\{i \in N_0 \mid X \lesssim sP_i\}$ is a finite set for each nonzero cyclic projective R-module X.

2. If P is directly finite then P has the property (∗). Because if P does not have (∗), then $\aleph_0 X \lesssim \oplus\ \bar{s}P$ and so $\bar{s}P$ is directly infinite, hence so is P.

3. For a countably generated projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that P has $(*)$, we see that either condition (A)(i) or (B) holds, but not both, by a linearly orderedness of $(\{[A] \mid A\in \mathrm{CP}(R)\}, \geqq)$.

Proposition 10. *Let R be a unit-regular ring satisfying s-comparability, and P be a countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that P has $(*)$ and* (A)(i). *Then P is directly finite if and only if* (A)(ii) *holds.*

Proof. "If part". Assume that (A)(ii) holds. For each nonzero cyclic projective module X, there exists a positive integer m $(\geqq n+1)$ such that $P_m \lesssim sX$ from the condition $(*)$ and Note 1 above. Thus, using the condition (A)(i), we have that $\oplus_{i=n+1}^{\infty}P_i \lesssim tP_n \lesssim t'P_m \lesssim st'X$ for some t'. By Theorem 7, P is directly finite.

"Only if part". Assume that P is directly finite. By Theorem 7, there exist positive integers k and t such that $\oplus_{i=k}^{\infty}P_i \lesssim tP_n$. We may assume $n+1<k$. Then $\oplus_{i=n+1}^{\infty}P_i=(P_{n+1}\oplus\cdots\oplus P_{k-1})\oplus(\oplus_{i=k}^{\infty}P_i)\lesssim(P_{n+1}\oplus\cdots\oplus P_{k-1})\oplus tP_n \lesssim t'P_n$ for some t' usig the condition (A)(i). Therefore (A)(ii) holds.

Proposition 11. *Let R be a unit-regular ring satisfying s-comparability, and P be a countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that P has $(*)$ and* (B). *Then P is directly finite.*

Proof. Assume that P is directly infinite and $P=\oplus_{i=1}^{\infty}P_i$ has $(*)$ and (B). Then there exists asequence $n(1)<m(1)<n(2)<m(2)<\cdots$ of positive integers such that

$$
\begin{aligned}
&X \lesssim P_{n(1)}\oplus\cdots\oplus P_{m(1)},\\
&X \lesssim P_{n(2)}\oplus\cdots\oplus P_{m(2)},\\
&\cdots\cdots\cdots,
\end{aligned}
$$

for some nonzero cyclic projective module X, and that $X \not\lesssim sP_{i_n}$ for some i_n, and that $[P_{i_n}]\gneqq[P_{i_{n+1}}]\gneqq\cdots$ by $(*)$ and (B). We take a positive integer $n(t+1)$ with $i_n<i_{n+1}<n(t+1)$. Then $s\bar{s}X \lesssim (P_{n(t+1)}\oplus\cdots\oplus P_{m(t+1)})\oplus\cdots\oplus(P_{n(t+s\bar{s})}\oplus\cdots\oplus P_{m(t+s\bar{s})}) \lesssim \aleph_0 P_{i_{n+1}} \lesssim \bar{s}P_{i_n} \lesssim s\bar{s}X$, which contradicts the direct finiteness of $s\bar{s}X$.

Therefore we have the following theorem from Propositions 4, 10 and 11.

Theorem 12. *Unit-regular rings R satisfying s-comparability are of the following three* types A, B *and* C, *and exclusively:*

Type A. *There exists a countably generated directly finite projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that P has $(*)$ and* (A), *and every countably generated directly finite projective R-module has $(*)$ and* (A) *for some cyclic decomposition.*

Type B. *There exists a countably generated directly finite projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that P has $(*)$ and* (B), *and every* *countably generated directly finite projective R-module has* $(*)$ *and* (B) *for some cyclic decomposition.*

Type C. *All directly finite projectgive R-modules are finitely generated.*

Proof. It is sufficient to prove that Types A and B are independent. For a unit-regular ring R satisfying s-comparability, there exist countably generated projective R-modules P and Q with cyclic decompositions $P=\oplus_{i=1}^{\infty}P_i$ and $Q=\oplus_{i=1}^{\infty}Q_i$ such that P has $(*)$ and (A) and that Q has $(*)$ and (B). Then $P_n \not\lesssim sQ_{i_k}$ for some i_k by $(*)$, and so $Q_{i_k} \lesssim sP_n$. Similarly, $P_m \lesssim sQ_{i_{k+1}}$ for some m $(\geqq n)$. Then $[P_n]=[P_m]$ and $[P_n] \geqq [Q_{i_k}] \gneqq [Q_{i_{k+1}}] \geqq [P_m]$, which contradicts the property of the order "$\geqq$".

NOTE. It is clear from the condition $(*)$ that every unit-regular ring satisfying s-comparability with $\mathrm{Soc}(R) \neq 0$ is of Type C.

4. Types A, B and C

In this section, we shall give an ideal-theoretic characterization of Types A, B and C. Some results in this section are similar to the ones of [3]. But our proofs require something extra from [3].

Let R be a unit-regular ring satisfying s-comparability. We denote the family of all ideals of R by $\mathrm{L}_2(R)$. Then $\mathrm{L}_2(R)$ is a linearly ordered set under inclusion by the proof of [1, Proposition 8.5]. We put $I_0(R)=\cap\{I \mid 0 \neq I \in \mathrm{L}_2(R)\}$.

DEFINITION. A subfamily $\{I_i\}_{i=1}^{\infty}$ of $\mathrm{L}_2(R)$ is said to be a cofinal subfamily of $\mathrm{L}_2(R)$ if all I_i are nonzero, $I_1 \gneqq I_2 \gneqq \cdots$, and if for each nonzero X in $\mathrm{L}_2(R)$ there exists a positive integer n such that $X \geqq I_n$.

For an element a of a ring R, we put $\Sigma_a=\Sigma\{xR \mid x \in R \text{ and } xR \lesssim aR\}$.

Lemma 13. *Let R be a unit-regular ring satisfying s-comparability.*

(a) *For each* $a \in R$, Σ_a *is the smallest ideal of R containing a, and hence* $\Sigma_a = RaR$.

(b) *For each* $a, b \in R$, $\Sigma_a \leqq \Sigma_b$ *if and only if* $aR \lesssim t(bR)$ *for some positive integer t.*

(c) *For* $a, b \in R$, $\Sigma_a \lneqq \Sigma_b$ *if and only if* $\aleph_0(aR) \lesssim \bar{s}(bR)$. *Therefore we see that* $\Sigma_a = \Sigma_b$ *if and only if* $[aR]=[bR]$, *and* $\Sigma_a \lneqq \Sigma_b$ *if and only if* $[aR] \lneqq [bR]$.

Proof. See the proof of [3, Lemma 3.2] and Lemma 6.

Theorem 14. *Let R be a unit-regular ring satisfying s-comparability. Then the following conditions* (a)$\sim$(c) *are equivalent:*

(a) *R is of* Type A.

(b) $\mathrm{Soc}(R)=0$ *and* $I_0(R)\neq 0$.

(c) *There exists a countably generated directly finite projective R-module P with a cyclic decomposition* $P=\oplus_{i=1}^{\infty}P_i$ *such that P satisfies the condition* (*), $[P_1]=[P_2]=\cdots$ *and that* $\oplus_{i=2}^{\infty}P_i\lesssim tP_1$ *for some positive integer t.*

Proof. (a) $\Rightarrow$ (b). Let R be of Type A. It is clear from the Note following Theorem 12 that $\mathrm{Soc}(R)=0$. Then there exists a countably generated directly finite projective R-module P with the properties (*) and (A). Put $P_i\simeq x_iR$ for some $x_i\in R$. If $I_0(R)=0$, then there exists a nonzero ideal X of R such that $X\subsetneqq\Sigma_{x_n}=\Sigma_{x_{n+1}}=\cdots$, and so we can take a nonzero element x in X, hence $xR\leqq\aleph_0(xR)\lesssim\bar{s}(x_mR)$ for each m ($\geqq n$) from Lemma 13 and $\aleph_0(xR)<\oplus\ \bar{s}P$. Therefore $\bar{s}P$ is directly infinite, hence so is P which contradicts the direct finiteness of P. Thus (b) holds.

(b) $\Rightarrow$ (c). Assume (b), and so there exists a nonzero element x_1 in $I_0(R)$ such that $\Sigma_{x_1}=I_0(R)$. Using that $\mathrm{Soc}(R)=0$, we can take nonzero elements y_1 and z_1 in R such that $x_1R=y_1R\oplus z_1R$ and $y_1R\lesssim s(z_1R)$. From this, there exists a nonzero cyclic submodule x_2R of y_1R which is subisomorphic to z_1R such that $2(x_2R)\lesssim x_1R$. Noting that $\mathrm{Soc}(R)=0$ again, we apply the above discussion to x_2R. Continuing this procedure, we obtain a nonzero submodule $x_{n+1}R$ of y_nR such that $2(x_{n+1}R)\lesssim x_nR$ for $n=1,2,\cdots$. Put $P=\oplus_{i=1}^{\infty}x_iR$. We claim that P is a desired one. Since Σ_{x_1} is the smallest ideal of R, $\Sigma_{x_1}=\Sigma_{x_2}=\cdots$, and so $[x_1R]=[x_2R]=\cdots$. For each nonzero $y\in R$, assume that $yR\lesssim\bar{s}(x_iR)$ for each $i\in J$, where J is an infinite set $\{j_1,j_2,\cdots\}$ of positive integers. Then $\Sigma_y\leqq\Sigma_{x_i}=\Sigma_{x_1}$ and so $\Sigma_y=\Sigma_{x_1}$. We can take positive integers h, t and m such that $t\bar{s}\leqq 2^h$ and $2^h(x_{j_m}R)\lesssim x_1R\lesssim t(yR)\lesssim t\bar{s}(x_{j_m}R)\leqq 2^h(x_{j_m}R)$, which is a contradiction. Thus (*) holds. It is clear that $\oplus_{i=2}^{\infty}x_iR\lesssim x_1R$. Therefore (c) holds. The implication (c) $\Rightarrow$ (a) is clear by Theorem 12.

Theorem 15. *Let R be a unit-regular ring satisfying s-comparability. Then the following conditions* (a) $\sim$ (c) *are equivalent*:

(a) *R is of* Type B.

(b) $\mathrm{Soc}(R)=0$, $I_0(R)=0$ *and* $\mathrm{L}_2(R)$ *has a cofinal subfamily.*

(c) *There exists a countably generated directly finite projective R-module with a cyclic decomposition* $P=\oplus_{i=1}^{\infty}P_i$ *such that P has* (*) *and* $[P_1]\gneqq[P_2]\gneqq\cdots$.

Proof. (a) $\Rightarrow$ (b). If (a) holds, it is clear that $\mathrm{Soc}(R)=0$. Then there exists a countably generated directly finite projective module P wihch has the properties (*) and (B). Put $P_i\simeq x_iR$ for some $x_i\in R$. Then $\cap\Sigma_{x_i}=0$. If not, there exists an nonzero element $x_0\in\cap\Sigma_{x_i}$, and so $\Sigma_{x_{i_1}}\gneqq\Sigma_{x_{i_2}}\gneqq\cdots\gneqq\Sigma_{x_0}$ by the condition (B) for P and Lemma 13. Hence we see from Lemma 13(c) that $x_0R\lesssim\oplus\ \bar{s}(x_{i_j}R)$ for $j=1,2,\cdots$ and $\aleph_0(x_0R)\lesssim\oplus\ \bar{s}(x_{i_1}R\oplus x_{i_2}R\oplus\cdots)\lesssim\oplus\ \bar{s}P$. Therefore $\bar{s}P$ is directly infinite and P is directly infinite which is a contradiction. Thus $I_0(R)=0$ and

$\{\Sigma_{x_i}\}$ is a cofinal subfamily of $L_2(R)$ from the linearly orderedness of $L_2(R)$.

(b) $\Rightarrow$ (c). If (b) holds, then there exists a cofinal subfamily $\{I_i\}_{i=1}^{\infty}$ of $L_2(R)$ such that $I_1 \supsetneqq I_2 \supsetneqq \cdots$. We can take an element $x_i \in I_i - I_{i+1}$, and so $I_i \geqq \Sigma_{x_i} \supsetneqq I_{i+1}$ by the linearly orderedness of $L_2(R)$. Then we see that $\Sigma_{x_1} \supsetneqq \Sigma_{x_2} \supsetneqq \cdots$ and $\{\Sigma_{x_i}\}$ is a cofinal subfamily of $L_2(R)$. Put $P = \oplus_{i=1}^{\infty} x_i R$. We claim that P is a desired one. It is clear that $[x_1 R] \supsetneqq [x_2 R] \supsetneqq \cdots$. If P does not satisfy $(*)$, there exists a nonzero $x_0 \in R$ such that $x_0 R \lesssim_s (x_i R)$ for $i \in J$, where J is an infinite set of positive integers, and $\Sigma_{x_i} \geqq \Sigma_{x_0}$ for $i \in J$. We see taht $\Sigma_{x_0} = 0$ (i.e. $x_0 = 0$) by using that $\{\Sigma_{x_i}\}$ is a cofinal subfamily, which contradicts that $x_0 \neq 0$. Therefore (c) holds. The implication (c) $\Rightarrow$ (a) is clear.

As a direct result from the above Theorems 14 and 15, we have the following.

Theorem 16. *Let R be a unit-regular ring satisfying s-comparability. Then the following conditions* (a) *and* (b) *are equivalent*:

(a) *R is of* Type C.

(b) $\mathrm{Soc}(R) \neq 0$, *or* $I_0(R) = 0$ *and* $L_2(R)$ *has no cofinal subfamily of* $L_2(R)$.

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